Standard Model | Mathematical Tripos, Part III
Dr C E Thomas | Lent Term 2019

Example Sheet 4

[You may submit questions 1 and 3 to be marked.]

1. In QCD let the strong coupling $a = g^2/(4\pi)^2$. The running coupling $a(\mu^2)$ is defined by
$$\mu^2 \frac{\mathrm{d}a}{\mathrm{d}\mu^2} = \beta(a)\,, \qquad \beta(a) = -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 + O(a^5)\,.$$
Show that, for a suitable choice of Λ,
$$\frac{1}{a(\mu^2)} = \beta_0 \log\frac{\mu^2}{\Lambda^2} + \frac{\beta_1}{\beta_0}\log\log\frac{\mu^2}{\Lambda^2} + O\left(1\Big/\log\frac{\mu^2}{\Lambda^2}\right).$$
If the coupling is redefined so that
$$\bar{a} = f(a) = a + v_1 a^2 + v_2 a^3 + O(a^4)\,,$$
show that $\bar{\beta}(\bar{a}) = \beta(a) f'(a)$ and hence $\bar{\beta}_0 = \beta_0$, $\bar{\beta}_1 = \beta_1$, $\bar{\beta}_2 = \beta_2 + \beta_0(v_2 - v_1^2) - \beta_1 v_1$. If $\bar{a}(\mu^2) = f(a(\mu^2))$ is written in terms of $\bar{\Lambda}$ in the same form as a in terms of Λ above, show that
$$\log\frac{\bar{\Lambda}^2}{\Lambda^2} = \frac{v_1}{\beta_0}\,.$$
Suppose a physical observable R has a perturbative expansion in terms of the coupling $a(\mu^2)$ of the form $R = a^N\left[r_0 + r_1 a + r_2 a^2 + \ldots\right]$. Under the above redefinition, $r_0 = \bar{r}_0$, $r_1 = \bar{r}_1 + N v_1 \bar{r}_0$, and $r_2 = \bar{r}_2 + N v_2 \bar{r}_0 + \frac{1}{2}N(N-1)v_1^2 \bar{r}_0 + (N+1)v_1\bar{r}_1$. Show that
$$\hat{r}_1 = r_1 - \beta_0 N r_0 \log\frac{\mu^2}{\Lambda^2}\,, \qquad \hat{r}_2 = r_2 + N\frac{\beta_2}{\beta_0} r_0 - \frac{N+1}{2N}\frac{r_1^2}{r_0} - \frac{\beta_1}{\beta_0} r_1\,,$$
are invariant under this redefinition of the coupling.

2. Use the interaction $\mathcal{L}_W = -(G_F/\sqrt{2})\, J_\alpha^{\text{had}\dagger}\; \bar{\nu}_\tau \gamma^\alpha (1-\gamma_5)\tau$ to show that the total decay rate for $\tau^- \to \nu_\tau$ + hadrons is
$$\Gamma_{\tau^- \to \nu_\tau + \text{hadrons}} = \frac{G_F{}^2 m_\tau^3}{16\pi}\int_0^{m_\tau^2} \mathrm{d}\sigma \left(1 - \frac{\sigma}{m_\tau^2}\right)^2 \left[\rho_0(\sigma) + \left(1 + \frac{2\sigma}{m_\tau^2}\right)\rho_1(\sigma)\right],$$
where
$$\sum_X (2\pi)^3 \delta^4(P_X - k)\langle 0|J_\alpha^{\text{hadrons}}|X\rangle\langle X|J_\beta^{\text{hadrons}\dagger}|0\rangle = k_\alpha k_\beta \rho_0(k^2) + (-g_{\alpha\beta}k^2 + k_\alpha k_\beta)\rho_1(k^2)\,,$$
and X covers all possible hadronic final states. If X is restricted to the π^- show that $\rho_0(\sigma) = 2F_\pi^2 \cos^2\theta_C\, \delta(\sigma - m_\pi^2)$ and $\rho_1(\sigma) = 0$. Hence find $\Gamma_{\tau^- \to \nu_\tau \pi^-}$.

3. Using light-cone coordinates, the longitudinal components forward/backward along the light cone in the $\mathbf{e}_3$ direction for an arbitrary 4-vector V are $V^\pm = V^0 \pm V^3$ and the transverse components are $\mathbf{V}_\perp = (V^1, V^2)$. Show that the Minkowski metric has components $g_{+-} = g_{-+} = \frac{1}{2}$ and $g_{11} = g_{22} = -1$ with the others zero.

   Consider deep inelastic scattering off a hadron where P is the initial-state hadron momentum and q is the photon momentum. Using a frame where $\mathbf{P}_\perp = \mathbf{q}_\perp = 0$ show that,
$$Q^2 = -q^+ q^- \,, \qquad \nu = \tfrac{1}{2}(q^+ P^- + q^- P^+) \,.$$
   Taking the DIS limit to be $q^- \to \infty$ with $q^+ = O(P^+)$ show that,
$$\nu \sim \tfrac{1}{2} q^- P^+ \qquad x \sim -\frac{q^+}{P_+} \,.$$

   Obtain the following expressions,
$$W_H^{+-}(q,P) = -W_1 + \left(M^2 + \frac{\nu^2}{Q^2} \right) W_2 \equiv F_L(x,Q^2) \,,$$
$$W_H^{++}(q,P) = \frac{(q^+)^2}{Q^2} F_L(x,Q^2) \,, \qquad W_H^{--}(q,P) = \frac{(q^-)^2}{Q^2} F_L(x,Q^2) \,,$$
   and show that in the DIS limit,
$$F_L(x,Q^2) \sim -F_1(x,Q^2) + \frac{1}{2x} F_2(x,Q^2) \,.$$

4. For a fundamental complex scalar field ϕ the electromagnetic current has the form $J^\mu = i[\phi^* \,(\partial^\mu \phi) - (\partial^\mu \phi^*)\, \phi]$. Assuming that this field corresponds to the charged constituents of a hadron H and treating the constituents as if they were free, obtain
$$W_H{}^{\mu\nu}(q,P) \sim \int d^4k \; W_\phi{}^{\mu\nu}(q,k) \big[\Gamma_H(P,k) + \overline{\Gamma}_H(P,k) \big] \,,$$
   where
$$W_\phi{}^{\mu\nu}(q,k) = \tfrac{1}{2}(2k+q)^\mu (2k+q)^\nu \, \delta\big((k+q)^2\big) \,,$$
   the momentum of the initial-state hadron is P and the photon momentum is q. Hence obtain for $F_2(x,Q^2) = \nu\, W_2(\nu,Q^2)$ and $F_1(x,Q^2) = W_1(\nu,Q^2)$, where $x = \frac{Q^2}{2\nu}$ and $\nu = P \cdot q$, the asymptotic forms in the deep inelastic limit $Q^2 = -q^2 \to \infty$
$$F_2(x,Q^2) \sim x\big[f(x) + \overline{f}(x) \big] \,, \qquad F_1(x,Q^2) \sim 0 \,,$$
   where, for $0 < x < 1$ and taking $k = \xi P + k'$ with $k' \cdot q$ bounded
$$f(x) = x \int d^4k \, \delta(\xi - x) \, \Gamma_H(P,k) \,, \quad \overline{f}(x) = x \int d^4k \, \delta(\xi - x) \, \overline{\Gamma}_H(P,k) \,.$$

5. [From Donoghue, Golowich, Holstein, Chapter IV.] In describing the decay $\mu \to e\gamma$, one may try to use an effective Lagrangian $\mathcal{L}_{3,4}$ which contains terms of dimension 3 and 4,

$$\mathcal{L}_{3,4} \;=\; a_3(\bar{e}\mu + \bar{\mu}e) \;+\; ia_4(\bar{e}\not{D}\mu \;+\; \bar{\mu}\not{D}e)\,,$$

where $D_\mu = \partial_\mu + ig_e Q A_\mu$, a_3 and a_4 are constants, and g_e is the electric charge (which would usually be denoted e but would then be difficult to distinguish from the electron field). Show by direct calculation that $\mathcal{L}_{3,4}$ leads to a $\mu \to e\gamma$ decay amplitude which vanishes at $O(a_3, a_4)$. If $\mathcal{L}_{3,4}$ is added to the QED Lagrangian for muons and electrons, show that one can define new fields μ' and e' to yield a Lagrangian which is diagonal in flavour. Thus, even in the presence of $\mathcal{L}_{3,4}$ there are two conserved fermion numbers. Finally, at dimension 5, show that $\mu \to e\gamma$ can be described by including in the Lagrangian

$$\mathcal{L}_5 \;=\; \bar{e}\sigma^{\alpha\beta}(c + d\gamma^5)\mu F_{\alpha\beta} \;+\; \text{h.c.}$$

where c and d are constants and $\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha, \gamma^\beta]$.

Comments/corrections to `C.E.Thomas@damtp.cam.ac.uk`.