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A similarity solution is used to analyse the flow in the thin gap between a floating disk and an air table, the similarity solution being that for an axisymmetric stagnation point but with an upper boundary condition within the boundary layer. Viscous effects increase the height of the floating disk by 20% even at a Reynolds number of 50. Theoretical predictions are compared with experimental observations. The effect of the pressure distribution under the disk on the air flux through the table is examined.

1. Introduction

An air table consists of a horizontal porous plate above a reservoir of pressurised air, so that there is a flux of air up out of the plate. Light flat disks placed on the table will float on a cushion of air with relatively little friction. Such a device has been used to study the geometry of packings in two dimensions of disks, of mixtures of disks, of ellipses and of pentagons (Lemaître et al. 1991; Lemaître et al. 1993; and Lemaître et al. 1992), to study the dynamics of a two-dimensional gas (Annic 1994 and Annic et al. 1994), and by joining together several disks by a light thread and by having some scattered fixed heavy disks to study the diffusion of a polymer molecule in a porous medium (Tasserie, Hansen, & Bideau 1992). The construction of the air table was described by Lemaître et al. (1990). In an appendix they give a theory for the height of the floating disks in the limit of high Reynolds numbers where all effects of viscosity are neglected. The opposite limit of zero Reynolds number was studied by Petit (1986) for a similar device. The purpose of this paper is to study the effects of viscosity on the height of the floating disks at an arbitrary Reynolds number. In order to design experiments with mixtures of different disks it is necessary to be able to predict the height at which the disks float. Following the earlier work, we shall first assume that the air flux through the porous plate is fixed. Afterwards we shall examine the effect of the pressure distribution under the disk on this air flow. In the future it would be desirable to examine the transient motion and the interaction between two disks in isolation and in a dense packing.

2. Governing equations

Let the disk be of radius R and be floating at a height h above the air table. Let the velocity of the air be (u, w) in cylindrical polar coordinates r, z. Mass conservation is

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$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0$$

For thin gaps $h \ll R$, we need only consider the radial momentum equation in which we can assume that the pressure gradient is independent of height:

$$\rho\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\mathrm{d}p}{\mathrm{d}r} + \mu\frac{\partial^2 u}{\partial z^2}$$

The boundary conditions are

$$u = 0, \quad w = q, \quad \text{on } z = 0, u = 0, \quad w = 0, \quad \text{on } z = h, p = p_a \quad \text{at } r = R,$$

and

where p_a is the atmospheric pressure above the disk. The inflow q across the porous plate is assumed to be linearly related to the pressure difference across the plate,

$$q=\frac{k}{\mu}(p_r-p)$$

where k is the membrane permeability of the plate and p_r is the pressure in the reservoir under the table. We simplify our initial investigation by assuming that the inflow q is constant independent of position under the disk, which is appropriate for large reservoir pressures compared with the variations of pressure under the disk. We return to this approximation in the final two sections of the paper. We need to find the pressure distribution in the air gap p(r). The vertical force balance on the disk

$$Mg = \int_0^R (p-p_a) 2\pi r \, \mathrm{d}r$$

will then give an equation from which we will determine the height h at which the disk floats.

3. Similarity solution

Now there is a simple solution for the flow, which is the similarity solution for the boundary layer at an axisymmetric stagnation point. In this flow, diffusion of vorticity away from the upper rigid surface is balanced by the uniform advection towards it from the uniform efflux out of the porous plate, resulting in a boundary layer whose thickness is also uniform independent of the radial position. In our problem, this boundary layer may be larger or smaller than the gap. The constancy of the boundary layer thickness in the gap is however essential in permitting our similarity solution. Thus

$$u = -\frac{qr}{2h}f'(\zeta), \quad w = qf(\zeta), \quad p = p_a + \frac{\rho q^2}{2h^2} \left(R^2 - r^2\right)\beta, \quad \text{with } \zeta = \frac{z}{h},$$

with f satisfying

$$-\frac{1}{2}f'^2 + ff'' = -2\beta + \frac{1}{Re}f''',$$

with the Reynolds number $Re = \rho qh/\mu$ and with boundary conditions

$$f(0) = 1$$
, $f'(0) = 0$, $f(1) = 0$, $f'(1) = 0$

This ordinary differential equation for f must be solved to find the pressure gradient coefficient β as a function of the Reynolds number *Re*. In §6 we give results of a numerical integration, but first we examine the limits for small and large Reynolds numbers, finding the leading-order term plus one correction term.

In using a similarity solution for the flow in the thin gap, we ignore edge effects O(h/R).

4. Solution for $Re \ll 1$

Expand in a series in $Re \ll 1$:

$$f \sim f_0 + Ref_1$$
 and $\beta \sim Re^{-1}\beta_{-1} + \beta_0$

with Re > h/R in order for the correction term to be larger than the ignored edge effects. The problem for the leading-order f_0 and β_{-1} is

$$f_0''' = 2\beta_{-1}$$
 with $f_0(0) = 1$, $f_0'(0) = 0$, $f_0(1) = 0$, $f_0'(1) = 0$

Integrating twice and applying the boundary conditions on f'_0 gives

$$f_0' = \beta_{-1}(\zeta^2 - \zeta)$$

Integrating again applying the boundary condition on f_0 at $\zeta = 1$ gives

$$f_0 = \beta_{-1}(\frac{1}{3}\zeta^3 - \frac{1}{2}\zeta^2 + \frac{1}{6})$$

Finally applying the boundary condition on f_0 at $\zeta = 0$ gives

$$\beta_{-1} = 6$$
 and $f_0 = 1 - 3\zeta^2 + 2\zeta^3$

The problem for the correction f_1 and β_0 is

$$f_1''' = 2\beta_0 - \frac{1}{2}f_0'^2 + f_0f_0'' = 2\beta_0 - 6 + 12\zeta - 12\zeta^3 + 6\zeta^4$$

with $f_1(0) = f_1'(0) = f_1(1) = f_1'(1) = 0$.

Integrating once finds

$$f_1'' = 2\beta_0\zeta - 6\zeta + 6\zeta^2 - 3\zeta^4 + \frac{6}{5}\zeta^5 + a$$

and again applying the boundary conditions on f'_1

$$f_1' = \beta_0(\zeta^2 - \zeta) + \frac{7}{5}\zeta - 3\zeta^2 + 2\zeta^3 - \frac{3}{5}\zeta^5 + \frac{1}{5}\zeta^6$$

Integrating a third time applying the boundary condition on f_1 at $\zeta = 0$

$$f_1 = \beta_0(\frac{1}{3}\zeta^3 - \frac{1}{2}\zeta^2) + \frac{7}{10}\zeta^2 - \zeta^3 + \frac{1}{2}\zeta^4 - \frac{1}{10}\zeta^6 + \frac{1}{35}\zeta^7 \quad .$$

Finally the boundary condition on f_1 at $\zeta = 1$ gives

$$\beta_0 = \frac{27}{35}$$

Bringing together the results for $Re \ll 1$, we have found

$$\beta \sim 6Re^{-1} + \frac{27}{35}$$
 and $f''(1) \sim 6 + \frac{13}{35}Re$

5. Solution for $Re \gg 1$

As $Re \to \infty$ a boundary layer develops near the top boundary $\zeta = 1$ of thickness $Re^{-1/2}$. The role of this boundary layer is to permit the slip boundary condition

f' = 0 to be satisfied there. The slip boundary condition is satisfied at the other boundary $\zeta = 0$ because the fluid comes through the porous plate with zero radial momentum.

Outside the boundary layer, in the majority of the flow, we may expand

$$f \sim f_0 + Re^{-1/2}f_1$$
 and $\beta = \beta_0 + Re^{-1/2}\beta_1$

with $Re^{-1/2} > h/R$ in order for the correction terms to be larger than the ignored edge effects. The problem for the leading order is

$$-\frac{1}{2}f_0^{\prime 2} + f_0 f_0^{\prime \prime} = -2\beta_0$$

satisfying all the boundary conditions except the top slip condition

$$f_0(0) = 1, \quad f'_0(0) = 0, \quad f_0(1) = 0$$

The governing equation can be solved by any quadratic in ζ . Choosing that quadratic which also satisfies the boundary conditions yields

$$f_0 = 1 - \zeta^2 \quad \text{and} \quad \beta_0 = 1$$

This leading-order result is given in the appendix of Lemaître *et al.* (1990). The solution has a slip on the top wall $f'_1(1) = -2$.

For the boundary layer on the top surface we rescale both f and ζ in such a way as to maintain the same order of magnitude of the slip velocity f', i.e.

$$f(\zeta) \sim Re^{-1/2}F(\zeta)$$
 with $\zeta = Re^{1/2}(1-\zeta)$

The governing equation then becomes

$$-\frac{1}{2}F'^2 + FF'' = -2 - F'''$$

using $\beta \sim 1$. We need to apply the two boundary conditions on the top boundary $\zeta = 1$, i.e. $\xi = 0$,

$$F(0) = 0$$
 and $F'(0) = 0$

and the matching condition

$$F' \to 2$$
 as $\xi \to \infty$

This problem must be solved numerically. Using a fourth-order Runge-Kutta method and shooting from $\xi = 0$ with a guess for F''(0) which is adjusted until $F' \to 2$ as $\xi \to \infty$, we find

$$F''(0) = 2.624$$
 and $F \rightarrow 2\xi - 1.138$ as $\xi \rightarrow \infty$

In the boundary layer the radial flow is reduced, and so less fluid needs to be supplied into the boundary layer. Thus outside the boundary layer it appears that there is a uniform flow of 2 up to an effective surface at $\zeta = 1 - \frac{1}{2}Re^{-1/2}1.138$. This drives a correction f_1 in the flow outside the boundary layer, governed by

$$-2\beta_1 = -f'_0f'_1 + f_0f''_1 + f_1f''_0 = (1-\zeta^2)f''_1 + 2\zeta f'_1 - 2f_1$$

which satisfies the boundary conditions

$$f_1(0) = 0, \quad f'_1(0) = 0, \quad f_1(1) = -1.138$$

Again the f_1 equation is solved by any quadratic. Choosing the quadratic which satisfies the boundary condition, we have

$$f_1 = -1.138\zeta^2$$
, and $\beta_1 = 1.138$

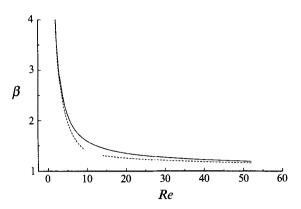


FIGURE 1. The pressure coefficient β as a function of the Reynolds number Re, with the asymptotic results of low and high Re.

Bringing together the results for $Re \gg 1$, we have found

$$\beta \sim 1 + 1.138 Re^{-1/2}$$
 and $f''(1) \sim 2.624 Re^{1/2}$

6. Numerical results

When the Reynolds number Re is neither large nor small, one must solve the ordinary differential equation governing the similarity solution by numerical methods. A fourth-order Runge-Kutta method was used, shooting from $\zeta = 1$ to $\zeta = 0$ with a guess for the values of f''(1) and β . These values were adjusted until f(0) = 1 and f'(0) = 0. It was decided to shoot from $\zeta = 1$ because at $Re \gg 1$ there is a part of the solution which decays exponentially away from $\zeta = 1$. A step size of $\Delta \zeta = 0.01$ was found to be sufficiently for the displayed accuracy.

The numerical solution for the pressure gradient factor β is plotted in figure 1, where it is compared with the asymptotic solutions for $Re \ll 1$ and $Re \gg 1$. It is found that the low Reynolds number solution is good up to Re = 4, whereas the high Reynolds number solution is not so accurate until quite large values of the Reynolds number, having a 5% error at Re = 40.

7. Prediction of the height

Now that we have an expression for the pressure distribution under the disk, we can calculate the net lift force, and this will lead to a prediction of the height at which the disk floats. The vertical force balance on the disk is

$$Mg = \int_0^R 2\pi r(p-p_a) dr = \frac{\pi \rho q^2 \beta R^4}{4h^2}$$

It is convenient to introduce a non-dimensional reservoir pressure p_r^* and a nondimensional suspension height h^* . The pressure is scaled with the weight of the disk divided by its area, i.e.

$$p_r = p_a + \frac{Mg}{\pi R^2} p_r^*$$

This pressure scale sets via the momentum equation a scale for the radial velocity $(Mg/\rho\pi R^2)^{1/2}$, and via the porous plate law a scale for the vertical velocity $Mgk/\mu\pi R^2$.

Mass conservation then gives that the ratio of these two velocity scales must equal that for the geometry with radius R and height h. Thus we obtain a scale for the height

$$h = \frac{k(Mg\rho)^{1/2}}{\mu\pi^{1/2}}h^*$$
.

In terms of these non-dimensional variables, the vertical force balance gives the prediction for the height of the disks floating above the air table

$$h^* = \frac{1}{2}\beta^{1/2}p_r^*$$

where $\beta = \beta(Re)$. This assumes that the height is small $h \ll R$, i.e. $k(Mg\rho)^{1/2}h^*/\mu \ll R$. The earlier result of Lemaître *et al.* (1990), $h^* = \frac{1}{2}p_r^*$, was for the high Reynolds number limit with $\beta = 1$.

The Reynolds number for the air flow can also be written in terms of the nondimensional variables

$$Re = \frac{\rho q h}{\mu} = Re^* p_r^* h^*$$
 where $Re^* = \frac{k^2 M^{3/2} g^{3/2} \rho^{3/2}}{R^2 \mu^3 \pi^{3/2}}$

In the two series of experiments discussed below, it is the non-dimensional group Re^* rather than the true Reynolds number Re which is held constant.

The above analysis has assumed that the flow of air across the porous plate q is independent of position under the disk, i.e. that the pressure distribution under the disk has no effect on the inflow. Now the pressure variations under the disk are $\rho q^2 R^2 \beta / 2h^2 = 2Mg/\pi R^2 \beta$. Hence the variations are small compared with the difference between the reservoir pressure and atmospheric pressure $p_r^* Mg/\pi R^2$ only if $p_r^* > 2$. Clearly when $p_r - p_a < Mg/\pi R^2$ the reservoir pressure cannot lift the disk, so we should have $h^* = 0$ when $p_r^* < 1$. Predictions for the suspension height when p_r^* is just greater than one are given in §9 for low Reynolds numbers and in §10 for high Reynolds numbers.

8. Comparison with experiments

Experiments were performed with two large disks of diameter 7.9 cm and 11.9 cm with masses 14.1 g and 31.7 g respectively. These large disks hovered relatively stably on the air table.

The height at which the disks floated above the table was measured at various air flows. Heights less than 1 mm were difficult to measure accurately. To measure the pressure in the reservoir for each air flow, small masses were added on top of the disk until the air pressure just failed to lift up the disk. The non-dimensional reservoir pressure p_r^* is then simply the total mass (disk plus that added) divided by the mass of the disk.

The membrane permeability of the porous plate was found previously to be $k = 2.61 \times 10^{-8}$ m (Lemaître *et al.* 1990). Other parameters are gravity g = 9.81 m s⁻², the density of air $\rho = 1.2$ kg m⁻³ and the viscosity of air $\mu = 1.8 \times 10^{-5}$ kg m⁻¹ s⁻¹. Thus for the two disks the non-dimensional group of parameters $Re^* = 0.984$ and 1.46 respectively.

Figure 2 gives the comparison between the theory and the experiments. We have plotted h^*/p_r^* against p_r^* for the experiments, and for the theory $\frac{1}{2}(\beta(Re))^{1/2}$ against $[2Re/Re^*(\beta(Re))^{1/2}]^{1/2}$.

We observe that the viscosity increases h^*/p_r^* by typically 20% from the high

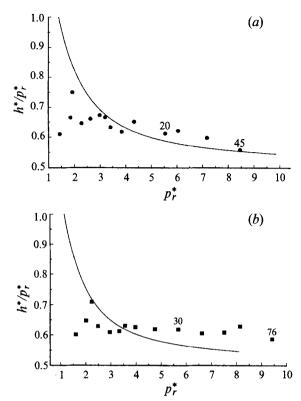


FIGURE 2. Comparison between theory and experiments: (a) for the 7.9 cm disk and (b) for the 11.9 cm disk. The numbers give the values of the Reynolds number *Re*.

Reynolds number limit value of 0.5. The Reynolds numbers are 20 and 45 for $p_r^* = 5.5$ and 8.5 for the 7.9 cm disk, and 30 & 76 for $p_r^* = 5.7$ and 9.4 for the 11.9 cm disk. The agreement at low reservoir pressure $1 < p_r^* < 2$ is poor, due no doubt to significant variations in the air inflow across the porous plate. We do not understand why the theory and experiments agree less well at the high reservoir pressures for the large disk.

9. Near lift-off at $Re \ll 1$

When the pressure in the reservoir is just above the critical value to lift up the disk, the air flow through the porous plate is not uniform but is affected by the pressure distribution under the disk. The experimental results do show a significant deviation from the theory in the range $1 < p_r^* < 2$. A calculation of the effect of the pressure variations on the air flow at an arbitrary Reynolds number would require a numerical solution of the axisymmetric Navier–Stokes equations, a major task which is not justified by the normal operations conditions of the air table which avoid this regime. Some easier progress can however be made analytically in the limits of low and high Reynolds number, now working only to leading order.

For low Reynolds numbers we can use lubrication theory. This has a depthintegrated flux

$$\int_0^h u \, \mathrm{d}z = -\frac{h^3}{12\mu} \frac{\mathrm{d}p}{\mathrm{d}r}$$

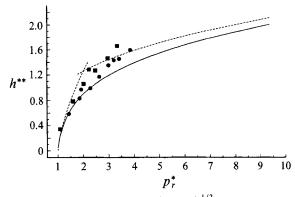


FIGURE 3. The non-dimensional height $h^{**} = h/\left(\frac{3}{2}\mu kR^2\right)^{1/3}$ as a function of the non-dimensional reservoir pressure p_r^* in the low Reynolds number limit. The solid curve is for the solution of §9 at arbitrary p_r^* , while the dashed curves are the near lift-off and the constant air flow approximations. The experimental data: •, 7.9 cm disk; **II**, 11.9 cm disk.

Equating the divergence of this flux to the inflow across the boundary, we obtain an equation governing the pressure distribution under the disk:

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{rh^3}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = \frac{k}{\mu}(p_r-p) \quad .$$

This has a solution in terms of Bessel functions satisfying the condition $p = p_a$ at r = R:

$$p = p_r - (p_r - p_a) \frac{I_0(\kappa r)}{I_0(\kappa R)}$$
 with $\kappa^2 = 12k/h^3$

Integrating this pressure distribution over the disk we obtain the vertical force balance

$$Mg = (p_r - p_a)\pi R^2 \left(1 - \frac{2I'_0(\kappa R)}{\kappa R I_0(\kappa R)}\right)$$

This equation must be solved for κR , which then gives the suspension height $h = (\frac{3}{2}kR^2)^{1/3}2(\kappa R)^{-2/3}$. The full solution is given by the solid curve in figure 3. For large reservoir pressures, $p_r^* \gg 1$ and so $\kappa R \ll 1$, we recover the low Reynolds number limit of §7, which assumed that the air flow through the porous plate was uniform:

$$h = \left(\frac{3}{2}kR^2\right)^{1/3} \left(p_r^*\right)^{1/3}$$

For near lift-off, $p_r^* \to 1$ and so $\kappa R \gg 1$, we find

$$h = (3kR^2)^{1/3} (p_r^* - 1)^{2/3}$$

The cross-over between these two formulae occurs at $p_r^* = 2$. Figure 3 shows that the near lift-off result is applicable when $1 < p_r^* < 1.5$, and that the constant air flow approximation has an error less than 10% once $p_r^* > 6$.

Also plotted in figure 3 are the experimental results with Re < 10. There is reasonable agreement in the range $1 < p_r^* < 2$. By $p_r^* = 2$ the Reynolds numbers are no longer small, being 3 for the 7.9 cm disk and 4 for 11.9 cm disk.

Looking in more detail at the solution near to lift-off, we see that the pressure under the disk is constant except for a small region of width $O(\kappa^{-1})$ near to the rim of the disk. The value of this constant pressure is that of the reservoir, and so there is The effect of viscosity on the height of disks floating above an air table 321 no air flow under most of the disk. In the rim region there is a pressure distribution

$$p = p_r - (p_r - p_a)e^{-\kappa(R-r)}$$

and this is where all the air flow takes place.

10. Near lift-off at $Re \gg 1$

The high Reynolds number limit is more difficult and a solution for arbitrary reservoir pressure has not been found. The near lift-off condition can however be tackled with a similarity solution for the active rim region, with most of the air under the disk being stationary at the reservoir pressure.

In the rim region near r = R we seek a solution in the separable form

$$u(r,z) = \left(\frac{p_r - p_a}{\rho}\right)^{1/2} G(r)F'(z/h) \text{ and } w(r,z) = -\left(\frac{p_r - p_a}{\rho}\right)^{1/2} hG'(r)F(z/h)$$

with two unknown functions F and G (with F < 0). This satisfies the planar mass conservation, which is appropriate in the small region near r = R.

Substituting into the horizontal momentum equation we have

$$(p_r - p_a)GG' \left(F'^2 - FF''\right) = -\frac{\mathrm{d}p}{\mathrm{d}r}$$

In order for the pressure gradient to be independent of the vertical position, we take

$$F^{\prime 2} - FF^{\prime\prime} = 1$$

where a constant has arbitrarily been set equal to unity. This equation for F must be solved subject to boundary conditions of no flux through the top boundary F(1) = 0 and no horizontal momentum of new fluid entering through the porous plate F'(0) = 0. At high Reynolds numbers, a boundary layer on the top plate will adjust any slip there, $F'(1) \neq 0$. The solution for F is

$$F(z/h) = -\frac{2}{\pi} \cos \frac{\pi z}{2h}$$

We can therefore evaluate a constant in the inflow across the porous plate $-F(0) = 2/\pi$.

With the above vertical structure function F, the horizontal momentum equation can be satisfied if we set the pressure distribution to

$$p(r) = p_r - \frac{1}{2}(p_r - p_a)G^2(r)$$

The pressure being atmospheric at the outer edge gives

$$G(R)=\sqrt{2}$$

Substituting into the porous-plate law the results for the pressure distribution and for the air inflow, we obtain an equation for the radial structure function

$$G' = \kappa G^2$$
 with $\kappa = \frac{\pi k [\rho(p_r - p_a)]^{1/2}}{4\mu h}$

with solution

$$G = \frac{\sqrt{2}}{1 + \sqrt{2}\kappa(R-r)}$$

Finally the vertical force balance gives

$$Mg = (p_r - p_a)\pi R^2 \left(1 - \frac{\sqrt{2}}{\kappa R}\right)$$

and so

$$h^* = \frac{\pi}{4\sqrt{2}} (p_r^* - 1)$$
 as $p_r^* \to 1$

This result for $p_r^* \to 1$ does not intersect that for $p_r^* \to \infty$ $(h^* = \frac{1}{2}p_r^*)$ until $p_r^* = 10$, although one would expect it to be limited more to $1 < p_r^* < 2$.

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