# Mathematical Tripos: Part II Waves

## Contents

# 0 Waves

## 0.1 Introduction

- A wave is one of those things that if you see it you know what it is!
- *Intuition:* A wave is something that goes up and down – or just up – or just down.
- *Dictionary:* A wave is a disturbance *propagating* in a medium at a finite speed ... but what about standing waves?

**Wave motion is a mechanism for *energy* communication over large distances without any necessary bulk motion of the medium.**

- Waves are not necessarily oscillatory, e.g. one-signed pulse on a string.
- Waves do not necessarily have small amplitudes, e.g. shocks, breaking waves.
- There is no implication of single-speed propagation, i.e. dispersion is possible.

## 0.2 Examples of Wave Motion

- Waves with elastic restoring forces, e.g. sound waves (aircraft noise, ultrasound), seismic waves (earthquakes, nuclear explosions, prospecting), your pulse.
- Waves arising from gravity and density inhomogeneities, e.g. water waves (including duck-generated waves), internal waves.
- Waves arising from Coriolis forces and other rotation effects, e.g. inertial waves, Rossby waves.
- Waves associated with a magnetic field 'frozen' in a fluid, e.g. Alfven waves.
- Electromagnetic waves such as light (see also *Electrodynamics*).
- Gravitational waves (see also *GR*).
- The wave-function in quantum mechanics (see *QM* courses).

## 0.3 Outline of Course

Uses *IB Methods* and little bits of *IB Complex Methods* and *IB Fluid Dynamics* (but will recap).

***Sound waves.*** Elementary properties of linear acoustic waves, i.e. linear compression waves in simple fluids (no dissipation, no dispersion) – or 'Why can you hear me?'. Nonlinear finite-amplitude effects in one dimensional gas dynamics – cumulative nonlinear effects can lead to shock waves (sonic booms); relation to hydraulic jumps, traffic flow, etc. 'Weak', i.e. discontinuous, solutions of nonlinear PDEs. See also *Differential Equations*, *Methods*, *Further Complex Methods* and *Electrodynamics*.

Formally:

> *Sound waves*. Equations of motion of an inviscid compressible fluid (without discussion of thermodynamics). Mach number. Linear acoustic waves; wave equation; wave-energy equation; plane waves; spherically symmetric waves. [4]
>
> *Non-linear waves*. One-dimensional unsteady flow of a perfect gas. Water waves. Riemann invariants; development of shocks; rarefaction waves; 'piston' problems. Rankine-Hugoniot relations for a steady shock. Shallow-water equations *and hydraulic jumps*. [5]

***Linear elastic waves.*** Similarities and differences between solids and fluids. Longitudinal (P) and transverse (S) [seismic] waves, and their interaction with interfaces.

Formally:

> *Elastic waves*. Momentum balance; stress and infinitesimal strain tensors and hypothesis of a linear relation between them for an isotropic solid. Wave equations for dilatation and rotation. Compressional and shear plane waves; simple problems of reflection and transmission; Rayleigh waves. [5]

***Dispersive waves,*** i.e. waves whose speed and direction of propagation vary with wavelength, e.g. acoustic waveguides, Love waves, surface gravity waves, waves in a density-stratified medium (see also *Electrodynamics*). Solution by Fourier analysis. Approximate solutions by stationary phase (see also *Asymptotic Methods*). The difference between the directions of crest and energy propagation.

Formally:

> *Dispersive waves.* Rectangular acoustic wave guide; Love waves; cut-off frequency. Representation of a localised initial disturbance by a Fourier integral (one-dimensional case only); modulated wave trains; stationary phase. Group velocity as energy propagation velocity; dispersing wave trains. Water waves; internal gravity waves. [6]

***Modulated wavetrains and ray theory.*** Rays, Fermat's Principle (see also *Variational Principles*), and the duck's wake.

Formally:

> *Ray theory.* Group velocity from wave-crest kinematics; ray tracing equations. Doppler effect; ship wave pattern. Cases where Fermat's Principle and Snell's law apply. [4]

## 0.4 Appropriate Books

*Fluids*

- H. Ockendon & J.R. Ockendon, *Waves and Compressible Flow*, Springer. Recommended by a supervisor.
- J. Billingham & A.C. King, *Wave Motion: Theory and Application.* Accessible.
- G.B. Whitham, *Linear and Nonlinear Waves*, Wiley-Interscience. Very good for the nonlinear part of the course. Rather expensive.
- A.R. Paterson, *A First Course in Fluid Dynamics*, CUP. Recommended by a student (many years ago) for characteristics, etc.
- L.D. Landau & E.M. Lifshitz, *Fluid Mechanics*, Pagan Press. This is where I learnt 'sound' from in 1976/77, but some students do not like the dated style.
- M.J. Lighthill, *Waves in Fluids*, CUP. This is *the* book of a former version of the *Waves* course (the version I took). Rather too wordy for my liking.
- A.P. Dowling & J.E. Ffowcs Williams, *Sound and Sources of Sound*, Ellis Horwood. Even more wordy book written for engineers.
- M. Van Dyke, *An Album of Fluid Motion*, Parabolic. An excellent book for your coffee-table.

*Solids*

- J. Billingham & A.C. King, *Wave Motion: Theory and Application.* Accessible.
- J.D. Achenbach, *Wave Propagation in Elastic Solids*, North Holland. Lots of detail (i.e. written for Americans).
- D.R. Bland, *Wave Theory and Applications*, Clarendon Press.
- J.A. Hudson, *The Excitation and Propagation of Elastic Waves*, CUP. Compact, but the key points are there.

## 0.5 Extras

**Classic Film.** *Aerodynamic Generation of Sound* staring Lighthill (former Lucasian Professor) and Ffowcs-Williams (former Master of Emmanuel): `http://www.youtube.com/watch?v=8BmESsMroRM`.

**Extra Credit.** Remember that you can obtain extra credit by doing the CATAM Project on the *Waves* course, i.e. Project 2.10 on *Phase and Group Velocity*.

# 1 Acoustics/Sound Waves

The theory of compression/rarefaction waves in compressible fluids (e.g. air, water).

## 1.1 Examples of Applications

- The noise generated by machinery, e.g. jet engines, for which an understanding of the power radiated is important. Note that:
  - (i) the acoustic power of a large jet aircraft $\equiv$ all the world shouting at once;
  - (ii) the acoustic power generated by a Wembley crowd during a match $\equiv$ the power to fry one egg!
- Sonar – both navigation and detection.
- Ultrasonics – both medical (e.g. imaging of tissue, destruction of kidney stones) and industrial (e.g. testing for cracks in pressure vessels).
- Design of concert halls, 'hi-fi'.
- Explanation of how musical instruments work.
- Snoring.

## 1.2 The Equations of Motion for Compressible Fluids

### 1.2.1 Solids, liquids and gases

*Solids:* A material behaves as a solid if it suffers only a finite deformation on application of a finite force. The solid is said to be elastic if the deformation tends to zero when the applied force is released.

*Fluids:* A material is said to behave as a fluid if its deformation increases indefinitely when a *deforming* force, i.e. not a purely compressive force, is applied.

Certain materials can behave sometimes like a fluid and sometimes like a solid.

In *solids* there are strong intermolecular forces. Fluids can be divided into liquids and gases. In *liquids* the intermolecular forces are still fairly strong. The intermolecular distance is similar to that in solids, but the molecules can move relative to each other. In *gases* the intermolecular forces are much weaker, and the intermolecular spacing is much greater. There is a random arrangement of molecules. Gases are much more compressible than liquids or solids.

We begin by studying fluids. In particular we consider fluids that are:

(a) homogeneous, i.e. the same at all points,

(b) dissipationless, i.e. we ignore viscous effects, etc., on the basis that dissipation is small over many characteristic wavelengths (c.f. waves in water and treacle); e.g. middle C has a wavelength of approximately 4 feet while dissipation occurs over hundreds of feet.

Suitable revision/introductory books include:

- Acheson, D.J., *Elementary Fluid Dynamics*
- Batchelor, G.K., *An Introduction to Fluid Dynamics*
- Paterson, A.R., *A First Course in Fluid Dynamics*

### 1.2.2 Notation

We make the continuum assumption. Since it is not possible to describe the motion of each molecule separately we assume that fluids are continuous, and that fluid properties, e.g. the density $\rho$, can be defined at a point $\mathbf{x}$. This is usually possible as long as the molecular spacing is much less than the macroscopic lengthscale (e.g. the wavelength of a sound wave). Hence we define

$$\left.\begin{array}{lll} \rho(\mathbf{x},t) & : & \text{The density of the fluid at } \mathbf{x},t \\ p(\mathbf{x},t) & : & \text{The pressure in the fluid at } \mathbf{x},t \\ \mathbf{u}(\mathbf{x},t) & : & \text{The velocity of the fluid at } \mathbf{x},t \end{array}\right\} \quad \text{5 variables in three dimensions}$$

*Remark.* We need a closed system of governing equations for these five variables, i.e. we need five equations.

### 1.2.3 Conservation of mass

We consider an arbitrary *fixed* volume $\mathcal{V}$ with surface $\mathcal{S}$ (a pretty standard approach if we are going to apply a conservation argument). Then

$$\begin{aligned} \text{Rate of increase of mass in } \mathcal{V} = \quad & \text{Rate of flow of mass across } \mathcal{S} \text{ into } \mathcal{V} \\ & + \text{Source strength inside } \mathcal{V} \end{aligned}$$

i.e.

$$\frac{d}{dt}\int \rho\, dV = -\int \rho\,\mathbf{u}.\mathbf{dS} + \int Q dV\ ,$$

where $Q(\mathbf{x},t)$ is the rate of creation of mass per unit volume at $\mathbf{x},t$.

Since the volume $\mathcal{V}$ is fixed, we can differentiate under the integral sign. Then using the divergence theorem we conclude that

$$\int \frac{\partial\rho}{\partial t} dV = -\int \nabla.(\rho\mathbf{u})\, dV + \int Q dV\ .$$

Since the volume is arbitrary it follows that

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{u}) = Q(\mathbf{x},t)\,, \tag{1.1a}$$

or equivalently

$$\frac{D\rho}{Dt} \equiv \frac{\partial\rho}{\partial t} + \mathbf{u}.\nabla\rho = -\rho\nabla.\mathbf{u} + Q\,. \tag{1.1b}$$

*Remarks*

(i) This is the *first* equation that we require.

(ii) Note that since the fluid is not incompressible, $\nabla.\mathbf{u}$ is not necessarily zero (cf. *IB Fluid Dynamics*).

### 1.2.4 Conservation of momentum

Apply conservation of momentum to the same fixed volume $\mathcal{V}$. Then (remember that we are neglecting any small viscous effects)

$$\frac{d}{dt}\int \rho\mathbf{u}\, dV = -\int (\rho\mathbf{u})\mathbf{u}.\mathbf{dS} - \int p\,\mathbf{dS} + \int Q\mathbf{u}\, dV + \int \rho\mathbf{F}\, dV\ ,$$

where $\mathbf{F}(\mathbf{x},t)$ is the body force per unit mass (e.g. gravity). Since the volume is fixed, and again using the divergence theorem, it follows that

$$\int \frac{\partial}{\partial t}(\rho u_i)\, dV = \int \left(-\frac{\partial}{\partial x_j}(\rho u_i u_j) - \frac{\partial p}{\partial x_i} + Qu_i + \rho F_i\right) dV\ .$$

Because the volume is arbitrary

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + Q u_i + \rho F_i \; . \tag{1.2a}$$

An alternative form of this equation, namely **Euler's Equation**, can be obtained by means of the manipulation [(1.2a) − (1.1b)$u_i$]:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}.\nabla)\mathbf{u} = -\nabla p + \rho \mathbf{F} \, . \tag{1.2b}$$

*Kinematic boundary condition.* At a stationary rigid boundary, the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ is satisfied. Alternatively, if the boundary is moving with speed $\mathbf{U}$, then we require that $\mathbf{u} \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n}$. Since we are assuming that the fluid is inviscid there are no conditions on $\mathbf{u} \times \mathbf{n}$.

*Dynamic boundary condition.* At a *free boundary*, e.g. the atmosphere, $p =$ const., or wlog, $p = 0$. At an *interface* between fluids, assuming that there is no surface tension, $[p]_-^+ = 0$. If there is surface tension, $\sigma$, then $[p]_-^+ = \sigma\kappa$, where $\kappa$ is the curvature.

*Remark.* Euler's equation provides three more of the equations that we require. Together, (1.1b) and (1.2b) constitute four scalar equations for the five scalar variables $\rho, p$ and $\mathbf{u}$. To complete the system of governing equations we need another relation between $\rho, p$ and $\mathbf{u}$. We obtain this relation by considering the **thermodynamics** of the flow (but the *Schedules* state that I am to do this 'without discussion of thermodynamics' ☹).

## 1.3 Thermodynamics (on a *need-to-know* basis)

From school physics we know that Boyle's law for a *perfect gas* states that

$$pv = n\bar{R}T \; , \qquad \text{i.e.} \qquad p = \rho R T \; , \tag{1.3}$$

where $v$ is the volume, $T$ is the temperature, $\rho$ is the density, $n$ is the number of moles of gas, $\bar{R}$ is the universal gas constant and $R = \frac{n}{\rho v}\bar{R}$ is the specific gas constant. If $T$ is known to be constant, i.e. if the flow is isothermal, then we have the required fifth equation relating $\rho$, $p$ and $\mathbf{u}$. Unfortunately, $T$ usually varies.

01/16 *Remark.* We are no further forward: we now have five scalar equations for six scalar variables.

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### 1.3.1 Isentropic flows and a closed system

In general, as in (1.3), $p \equiv p(\rho, T)$. Hence to close the system we need another equation for $T$, e.g. by considering heat transport by conduction/advection and/or thermodynamics.

We will take an easy way out and assume that

$$\text{timescale of motion} \;\ll\; \text{timescale of thermal diffusion} \, ,$$

so that the *heat content* of each fluid parcel/particle is constant (this is not the same at the temperature of each fluid parcel/particle being constant). This is a good approximation for sound in air where typically $f = 1kHz$, period=$10^{-3}$s, $\lambda = 30$cm, so that

$$t_{\text{diffusion}} = \frac{\lambda^2}{k/\rho c_P} \approx 10^4 s \gg 10^{-3} s = t_{\text{period}} \; ,$$

where $k$ is the thermal conductivity, $c_P$ is the specific heat capacity at constant pressure, and $k/\rho c_P$ is the thermal diffusivity.

Technically, we will assume *reversible adiabatic motion*, also referred to as *isentropic motion.* For those that are interested, the thermodynamic theory behind this assumption is expounded in Appendix 1.A. In short, a motion is *isentropic* if the entropy, $S$, of a fluid parcel/particle is constant, i.e. if

$$\frac{DS}{Dt} = 0 \, . \tag{1.4a}$$

The *Second Law of Thermodynamics* states that the entropy, $S$, exists, and, like the temperature $T$, pressure $p$ and density $\rho$, is a *parameter of state*. Each of these quantities can be viewed as a function of two of the others, e.g. $p \equiv p(\rho, T)$, or more pertinently,

$$p \equiv p(\rho, S)\,. \tag{1.4b}$$

*A closed system.* The conservation of mass equation (1.1b), Euler's equation (1.2b), the isentropic motion equation (1.4a) and the equation of state (1.4b) together constitute six equations for the six variables $p$, $\rho$, $S$ and $\mathbf{u}$. Hence we now have a closed system.

### 1.3.2 Homentropic flows

In fact, other than in shocks, we make the further assumption that the entropy takes a uniform constant value everywhere, i.e. that $S$ is a constant. The flow is then said to be *homentropic*, and equations (1.1b), (1.2b), and (1.4b) constitute a closed system of five equations for the five effective variables $p$, $\rho$ and $\mathbf{u}$.

**Perfect gas.** If the flow of a perfect gas is homentropic, i.e. reversible, adiabatic and uniform, then it can be shown, e.g. see Appendix 1.B, that

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{1.5a}$$

where $p_0$ and $\rho_0$ are initial/reference values, $\gamma = c_P/c_V$ is the constant ratio of specific heats, and $c_V$ is the specific heat capacity at constant volume. This equation, together with the conservation of mass equation (1.1b) and Euler's equation (1.2b), constitute a closed system of five equations for the five unknowns $p$, $\rho$, and $\mathbf{u}$.

**The speed of sound.** Subsequently we shall see that the square of the speed of sound, $c^2$, is given by

$$c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_S = \gamma \frac{p}{\rho} = \gamma R T. \tag{1.5b}$$

Newton made a mistake of assuming an isothermal change $(\partial p/\partial \rho)_T = RT$, so missing the factor of $\gamma$ (which is 1.4 for dry air).

**Internal energy and enthalpy.** For an isentropic flow (of which a homentropic flow is a special case), it follows from (1.A.5a) that

$$de = -p dV = \frac{p}{\rho^2} d\rho\,, \tag{1.5c}$$

where $e$ is internal energy per unit mass. Alternatively view this equation as 'God given' based on conservation of [mechanical] energy for an adiabatic reversible process. Thence from (1.5a)

$$\frac{de}{d\rho} = \frac{p}{\rho^2} = \frac{p_0}{\rho^2}\left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{1.5d}$$

and

$$e(p, \rho) = \frac{1}{\gamma - 1}\frac{p}{\rho}\,, \tag{1.5e}$$

where the constant of integration is normalised to be zero. As shown in Appendix 1.B, the internal energy of a perfect gas, when viewed as a function of $T$ and $\rho$, can be expressed as

$$e(T, \rho) = c_V T\,. \tag{1.5f}$$

The combination $h = e + pV$, is known as *enthalpy*. For a perfect gas it is given by

$$h(p, \rho) \equiv e + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}\,. \tag{1.5g}$$

Finally, when studying shocks we will need the entropy of a perfect gas. This can be calculated as, see (1.B.6d),

$$S(p, \rho) = S_0 + c_V \ln\left[\frac{p}{p_0}\left(\frac{\rho_0}{\rho}\right)^{\gamma}\right]. \tag{1.5h}$$

## 1.4 Linear Acoustic Waves

### 1.4.1 Linearised equations

Conservation of mass from (1.1b):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = Q(\mathbf{x}, t)\,. \tag{1.1b}$$

Conservation of momentum from (1.2a):

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + Q u_i + \rho F_i\,. \tag{1.2a}$$

$\frac{\partial}{\partial t}(1.1\text{b}) - \frac{\partial}{\partial x_i}(1.2\text{a})$ :

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2}{\partial x_i \partial x_i} p = \frac{\partial Q}{\partial t} - \frac{\partial}{\partial x_i}(\rho F_i + Q u_i) + \frac{\partial^2}{\partial x_i \partial x_j}(\rho u_i u_j)\,. \tag{1.6}$$

An exact solution is a uniform state of rest:

$$\rho = \rho_0,\ p = p_0,\ \mathbf{u} = 0,\ Q = 0,\ \mathbf{F} = 0. \tag{1.7}$$

Consider small perturbations to this state:[1]

$$\rho = \rho_0 + \tilde{\rho}, \qquad |\tilde{\rho}| \ll \rho_0 \tag{1.8a}$$

$$p = p_0 + \tilde{p}, \qquad |\tilde{p}| \ll p_0 \tag{1.8b}$$

$$\mathbf{u},\, Q,\, \mathbf{F} \qquad \text{`small'}$$

Then

$$\rho F_i = \underset{\substack{\uparrow \\ \text{small}}}{\rho_0 F_i} + \underset{\substack{\uparrow \\ \text{doubly small}}}{\tilde{\rho} F_i} \approx \rho_0 F_i$$

$Q u_i$ and $\rho u_i u_j$ are also doubly small.

Hence at leading order (1.6) linearises to

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - \nabla^2 \tilde{p} = \frac{\partial Q}{\partial t} - \rho_0 \nabla \cdot \mathbf{F}\,. \tag{1.9}$$

From thermodynamics, e.g. (1.4b),

$$p \equiv p(\rho, S)\,. \tag{1.10a}$$

Assume that the state of rest is homentropic, i.e. $S = S_0$, and remains so under perturbation. Then

$$\begin{aligned} p_0 + \tilde{p} &= p(\rho_0 + \tilde{\rho}, S_0) \\ &= p(\rho_0, S_0) + \tilde{\rho}\frac{\partial p}{\partial \rho}(\rho_0, S_0) + \dots\,. \end{aligned}$$

Hence

$$\tilde{p} = c_0^2 \tilde{\rho}\,, \tag{1.10b}$$

where

$$c_0^2 = \frac{\partial p}{\partial \rho}(\rho_0, S_0)\,. \tag{1.10c}$$

[1] Some students do not like approximations, however we make them all the time. For instance, we ignore relativistic effects when modelling the flight of a cricket ball (although not in the design of GPS systems).



 

### 1.4.2 Wave equation

Substitute (1.10b) into (1.9), then

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\tilde{p} = -\frac{\partial Q}{\partial t} + \rho_0 \nabla \cdot \mathbf{F}\,. \tag{1.11}$$

Assume no mass sources ($Q = 0$) and no body forces ($\mathbf{F} = 0$). Then $\tilde{p}$, and $\tilde{\rho}$ (from (1.10b)), satisfy the 3D wave equation with wave speed $c_0$, while the linearised versions of (1.1b) and (1.2a) are

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = 0\,, \tag{1.12a}$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla \tilde{p}\,. \tag{1.12b}$$

From $\frac{\partial}{\partial x_i}$(1.11), $\frac{\partial}{\partial t}$(1.11), (1.10b), (1.12a) and (1.12b), it follows that

$$\Box^2 \left(\frac{\partial \mathbf{u}}{\partial t}\right) = 0 \quad \text{and} \quad \Box^2(\nabla \cdot \mathbf{u}) = 0\,, \tag{1.13}$$

where

$$\Box^2 = \left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\,. \tag{1.14}$$

From integrating the first equation of (1.13), and then using the second equation of (1.13),

$$\Box^2 \mathbf{u} = \boldsymbol{\alpha}(\mathbf{x}) \quad \text{where} \quad \nabla \cdot \boldsymbol{\alpha} = 0\,. \tag{1.15}$$

Assume the motion starts from rest, then $\boldsymbol{\alpha} = 0$ and we conclude that

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$$\Box^2(\tilde{p}, \tilde{\rho}, \mathbf{u}) = 0\,. \tag{1.16}$$

### 1.4.3 The acoustic velocity potential

Curl the linearised momentum equation (1.12b), then

$$\frac{\partial \omega}{\partial t} = 0 \quad \text{where} \quad \omega = \operatorname{curl} \mathbf{u} = \text{vorticity.}$$

Hence within linear theory

$$\mathbf{u} = \boldsymbol{\nabla}\varphi(\mathbf{x}, t) + \mathbf{u}_{rot}(\mathbf{x})\,. \tag{1.17a}$$

For motions with no vorticity at $t = 0$, $\mathbf{u}_{rot} \equiv 0$. Then using (1.17a) and (1.10b) to integrate (1.12b) we obtain

$$\tilde{\rho} = \frac{\tilde{p}}{c_0^2} = -\frac{\rho_0}{c_0^2}\left(\frac{\partial \varphi}{\partial t}(\mathbf{x}, t) + \beta(t)\right)\,. \tag{1.17b}$$

Wlog we may assume that $\beta \equiv 0$ (the transformation $\varphi \to \varphi - \int \beta\, dt$ does not change $\nabla\varphi$), so

$$\tilde{p} = -\rho_0 \frac{\partial \varphi}{\partial t}, \quad \tilde{\rho} = -\frac{\rho_0}{c_0^2}\frac{\partial \varphi}{\partial t}, \quad \mathbf{u} = \boldsymbol{\nabla}\varphi. \tag{1.17c}$$

Substitute (1.17b) into (1.12a) to conclude that

$$\Box^2 \varphi = 0\,. \tag{1.18a}$$

Hence 'everything' satisfies the wave equation with wave speed

$$c_0 = \left(\frac{\partial p}{\partial \rho}\Big|_S\right)^{1/2}\,. \tag{1.18b}$$

*Sound speed.*

- For dry air at $293^o$K, $c_0 = 340\,\mathrm{ms}^{-1}$, i.e. at $20^o$C, $c_0 = 760\,$mph. (Homework: repeat Newton's experiment in Nevile's Court.)
- For water $c_0 \approx 1500\,\mathrm{m\,s}^{-1}$.

*Linear equations.* The equations for $\tilde{p}$, $\tilde{\rho}$, $\varphi$ and $\mathbf{u}$ are linear, so we can superimpose solutions, e.g. if $\varphi_1$ and $\varphi_2$ are solutions then so is $\lambda\varphi_1 + \mu\varphi_2$ for constant $\lambda$ and $\mu$.

*Linearisation: can we do it?* When linearising the momentum equations we neglected $\rho_0(\mathbf{u}.\nabla)\mathbf{u}$ and kept $\rho_0 \frac{\partial \mathbf{u}}{\partial t}$. Suppose that $|\mathbf{u}|$ has a typical magnitude $U$, and varies on a lengthscale $L$. Then in an order of magnitude sense (and using the wave equation)

$$\nabla \sim \frac{1}{L} \sim \frac{1}{c_0}\frac{\partial}{\partial t}\,.$$

Hence

$$\frac{\rho_0(\mathbf{u}.\nabla)\mathbf{u}}{\rho_0 \frac{\partial \mathbf{u}}{\partial t}} \sim \frac{\rho_0 U^2/L}{\rho_0 c_0 U/L} = \frac{U}{c_0} \equiv M\,, \tag{1.18c}$$

where $M$ is the *Mach number*. Hence for linearisation we need

$$M \ll 1 \quad \text{i.e. fluid speed} \ll \text{wave speed.}$$

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### 1.4.4 Wave–energy density and wave–energy flux

There are difficulties in defining energy density and energy flux for many media, especially non-uniformly moving media where second-order terms in $\mathbf{u}$, etc. need to be retained in the equations.

For perturbations to a state of rest we can 'cheat'. Form

$$\mathbf{u}\,\boldsymbol{.}\,\text{Momentum Eqn (1.12b)} + \frac{\tilde{p}}{\rho}\,\text{Mass Eqn (1.12a)}\,,$$

and use $\tilde{p} = c_0^2\tilde{\rho}$ to obtain the *Wave-Energy Equation*

$$\frac{\partial}{\partial t}(E_k + E_p) + \nabla\,\boldsymbol{.}\,\mathbf{I} = 0\,, \tag{1.19}$$

where

$$\left.\begin{array}{ll} E_k = \frac{1}{2}\rho_0\mathbf{u}^2 & : \text{K.E. density} \\ E_p = \frac{1}{2}\dfrac{c_0^2\tilde{\rho}^2}{\rho_0} & : \text{P.E. density due to compression} \end{array}\right\} \tag{1.20a}$$

$$\mathbf{I} = \tilde{p}\mathbf{u} \quad : \text{Wave-energy flux or 'acoustic intensity'.} \tag{1.20b}$$

Integrate over a fixed volume $\mathcal{V}$ to obtain

$$\frac{d}{dt}\int_{\mathcal{V}}(E_k + E_p)\,dV = -\int_{\mathcal{V}} \nabla\,\boldsymbol{.}\,\mathbf{I}\,dV = -\int_{\mathcal{S}} \tilde{p}\,\mathbf{u}\,\boldsymbol{.}\,\mathbf{dS}$$

Rate of increase of acoustic energy density        Rate of working of pressure over $\mathcal{S}$

*Loudness.* Define the *intensity* or *loudness* in decibels as

$$\text{loudness} \equiv 120 + 10\log_{10}|\mathbf{I}|\,, \quad \text{with } I = |\mathbf{I}| \text{ measured in Wm}^{-2}.$$

$$I = \left\{\begin{array}{lll} 10^{-12}\,\mathrm{Wm}^{-2} & \to 0\,dB & : \text{which can just be heard at 3000Hz;} \\ 1\,\mathrm{Wm}^{-2} & \to 120\,dB & : \text{the threshold of pain.} \end{array}\right.$$

The acoustic power of a voice in conversation $\approx 10^{-5}$W.

*Remark.* The intensity depends on distance like $1/r^2$ (see later).

### 1.4.5 * Energy density and energy flux: respectable method *(unlectured)* *

The total energy of the system is given by

$$E = \rho\left(\tfrac{1}{2}\mathbf{u}^2 + \int^{\rho} \frac{p(\hat{\rho},S)}{\hat{\rho}^2}\,d\hat{\rho}\right), \tag{1.21a}$$

where the second term on the RHS is the integral of $(-pdV)$ at constant entropy (where $V$ is the volume per unit mass and $\rho = V^{-1}$).

Using conservation of mass (1.1b) with $Q = 0$, and Euler's equation (1.2b) with $\mathbf{F} = 0$, it follows that

$$\begin{aligned}\frac{\partial E}{\partial t} + \nabla.\,(\mathbf{u}E + p\mathbf{u}) =& \rho\mathbf{u}\frac{\partial \mathbf{u}}{\partial t} + \frac{p}{\rho}\frac{\partial \rho}{\partial t} + \frac{E}{\rho}\frac{\partial \rho}{\partial t} + E\nabla.\,\mathbf{u} + \frac{E}{\rho}(\mathbf{u}.\,\boldsymbol{\nabla})\rho \\ &+ \rho(\mathbf{u}.\,\boldsymbol{\nabla})(\tfrac{1}{2}\mathbf{u}^2) + \frac{p}{\rho}(\mathbf{u}.\,\boldsymbol{\nabla})\rho + (\mathbf{u}.\,\boldsymbol{\nabla})p + p\nabla.\,\mathbf{u} \\ =& 0\,,\end{aligned} \tag{1.21b}$$

or, in integral form for a fixed volume $\mathcal{V}$, we have the exact result

$$\begin{array}{ccccccc} \dfrac{d}{dt}\displaystyle\int_{\mathcal{V}} E\,dV & = & -\displaystyle\int_{\mathcal{S}} E\mathbf{u}\cdot\mathbf{n}\,dS & - & \displaystyle\int_{\mathcal{S}} p\mathbf{u}\cdot\mathbf{n}\,dS \\ \text{Change of energy in } \mathcal{V} & = & \text{Advection of energy into } \mathcal{V} & + & \text{Rate of working by } p \text{ at surface } \mathcal{S} \end{array}$$

The following derivation for small-amplitude waves is slightly tricky because of the need to include quadratic terms. So write $\rho = \rho_0 + \tilde{\rho}$, $p = p_0 + \tilde{p}$, etc., and expand (1.21a) *including* quadratic terms:

$$\begin{aligned} E &= \tfrac{1}{2}\rho_0\mathbf{u}^2 + \rho\textstyle\int^{\rho_0}\frac{p}{\rho^2}d\rho + \rho\frac{p_0}{\rho_0^2}\tilde{\rho} + \tfrac{1}{2}\rho\frac{\partial}{\partial\rho}\left(\frac{p}{\rho^2}\right)_0\tilde{\rho}^2 + \ldots \\ &= \tfrac{1}{2}\rho_0\mathbf{u}^2 + \rho\textstyle\int^{\rho_0}\frac{p}{\rho^2}d\rho + \frac{p_0}{\rho_0}\tilde{\rho} + \tfrac{1}{2}\frac{c_0^2}{\rho_0}\tilde{\rho}^2 + \ldots \\ &= E_k + E_0 + E_r + E_p + \ldots\,, \end{aligned}$$

where the quadratic terms $E_k$ and $E_p$ are as in (1.20a), and

$$\begin{array}{ccc} E_0 = \rho_0\int^{\rho_0}\frac{p}{\rho^2}d\rho & \text{and} & E_r = \left(\int^{\rho_0}\frac{p}{\rho^2}d\rho + \frac{p_0}{\rho_0}\right)\tilde{\rho}\,. \\ \text{Internal energy per unit volume when fluid at rest} & & \text{Change in energy per unit volume due to change in mass} \end{array} \tag{1.21c}$$

From the conservation of mass equation (1.1b) it follows, after some very careful algebra, that $(E_0 + E_r)$ **exactly** satisfies

$$\frac{\partial}{\partial t}(E_0 + E_r) + \nabla.\,(\mathbf{u}(E_0 + E_r) + p_0\mathbf{u}) = 0\,, \tag{1.21d}$$

Hence from (1.21b) and (1.21d), it follows that consistent to second order we can write, as in (1.19),

$$\frac{\partial}{\partial t}(E_k + E_p) + \nabla(\tilde{p}\mathbf{u}) \approx 0\,.$$

## 1.5 Plane Waves

A solution to (1.18a), i.e.

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\varphi = 0\,, \tag{1.18a}$$

is

$$\varphi = f(lx + my + nz - \omega t) \tag{1.22a}$$

for any function $f$ if

$$(l^2 + m^2 + n^2)c_0^2 = \omega^2\,. \tag{1.22b}$$

The name *plane waves* follows from the observation that at any fixed time, $\varphi$ is constant on the planes

$$lx + my + nz = \text{constant}\,.$$

Take axes OXYZ with OX $\parallel$ to $\mathbf{k} = (l, m, n)$. Then

$$\varphi = f(kX - \omega t) = f(\mathbf{k}.\mathbf{x} - \omega t)\,, \tag{1.23a}$$

where

$$k^2 = |\mathbf{k}|^2 = l^2 + m^2 + n^2\,, \quad \text{and} \quad \omega = \pm k c_0\,. \tag{1.23b}$$

*Definition and remark.* The combination $\theta = (\mathbf{k}.\mathbf{x} - \omega t)$ is the *phase* of the wave. $\varphi$ is constant on lines of constant $\theta$, and hence solution (1.23a) describes a wave propagating in the $\mathbf{k}$ direction with speed $c_0$, whatever the choice of $f$.

By superposition another solution to (1.18a) is

$$\varphi = f(k(X - c_0 t)) + g(k(X + c_0 t))\,, \tag{1.24a}$$

i.e. two plane waves propagating in opposite directions with sound speeds $\pm c_0$, whatever the choice of $f$ and $g$.

*Characteristics.* Recall from Part IB that D'Alembert's general solution to $\varphi_{tt} - c_0^2 \varphi_{XX} = 0$ is

$$\varphi = F(X - c_0 t) + G(X + c_0 t) \tag{1.24b}$$

for any functions $F$ and $G$. The natural coordinates here are the *characteristics*

$$\xi = X - c_0 t \quad \text{and} \quad \eta = X + c_0 t\,. \tag{1.24c}$$

*Remarks*

(i) The characteristics are straight, but are a *non-orthogonal* co-ordinate system (unless $c_0 = 1$).

(ii) The characteristics $\xi$ and $\eta$ represent right and left travelling waves respectively.

(iii) In the case of a single 'simple' linear wave, crests and troughs, and indeed any points of constant phase, propagate along the relevant characteristic.

### 1.5.1 Properties of plane waves

*Longitudinal waves.* From (1.22a) and (1.24a)

$$\mathbf{u} = \nabla\varphi = \mathbf{k}\left(f'(\mathbf{k}.\mathbf{x} - kc_0 t) + g'(\mathbf{k}.\mathbf{x} + kc_0 t)\right). \tag{1.25a}$$

Hence sound consists of *longitudinal* waves since $\mathbf{u}$ is $\parallel$ to the propagation vector $\mathbf{k}$.

*Acoustic Impedance.* From (1.17c)

$$\tilde{p} = -\rho_0 \varphi_t = \rho_0 k c_0 (f'(\mathbf{k}.\mathbf{x} - kc_0 t) - g'(\mathbf{k}.\mathbf{x} + kc_0 t))\,. \tag{1.25b}$$

Hence, for a single wave with $\mathbf{u} = \hat{\mathbf{k}} u$,

$$\left.\begin{array}{ll} \tilde{p} = \rho_o c_0 u & (g = 0) \\ \tilde{p} = -\rho_0 c_0 u & (f = 0) \end{array}\right\} \tag{1.25c}$$

Hence $\tilde{p}$ and $\mathbf{u}$ are in phase. $\rho_0 c_0$ is referred to as the *acoustic impedance* of the medium.

*Equi-partition of energy.* For a single wave (e.g. $g = 0$) it follows from (1.10b), (1.20a), (1.25a), (1.25b) and (1.25c) that

$$\left.\begin{aligned} E_k &= \tfrac{1}{2}\rho_0 k^2 f'^2 \\ E_p &= \tfrac{1}{2}\rho_0 k^2 f'^2 = E_k \end{aligned}\right\} \tag{1.26a}$$

Hence there is instantaneous *equi-partition of energy* between kinetic and elastic modes (equi-partition of energy is often only true after time averaging). Further, from (1.20b), (1.25a), (1.25b) and (1.25c)

$$\mathbf{I} = \rho_0 c_0 k^2 f'^2 \hat{\mathbf{k}} = (E_k + E_p) c_0 \hat{\mathbf{k}}\,. \tag{1.26b}$$

Hence $\mathbf{I}$ is generated by the transport of acoustic energy at the *energy propagation velocity*

$$\mathbf{c}_g = c_0 \hat{\mathbf{k}}. \tag{1.26c}$$

### 1.5.2 Harmonic plane waves

Harmonic plane waves correspond to the choice:

$$f(\alpha) = A e^{i\alpha}\,, \tag{1.27a}$$

i.e.

$$\varphi = A \exp(i\mathbf{k}.\,\mathbf{x} - i\omega t)\ , \tag{1.27b}$$

where $A$ is the complex amplitude, and it is understood that $\varphi$ is actually equal to the **real part**. Equation (1.27b) is a solution to (1.18a) if the *dispersion relation* (1.23b) is satisfied, i.e. if

$$\omega = \pm|\mathbf{k}| c_0\ . \tag{1.27c}$$

*Remarks*

(i) Harmonic waves play an important role because, by Fourier decomposition in $\mathbf{k}$, a general disturbance can be written as the sum/integral of harmonic waves. Note that since there are two roots, $\omega = \pm|\mathbf{k}|c_0$, it is necessary to specify both $\varphi$ and $\varphi_t$ in an IVP.

(ii) The dispersion relation (1.27c) is *isotropic*, i.e. $\omega$ does not depend on the direction of $\mathbf{k}$.

*Key notation*

| | | |
|---|---|---|
| Wavenumber: | $\lvert\mathbf{k}\rvert \equiv k$ | |
| Wavevector: | $\mathbf{k} = \dfrac{\lvert\omega\rvert}{c_0}\widehat{\mathbf{k}}$ | |
| Wavelength: | $\begin{cases} \lambda = \frac{2\pi c_0}{\lvert\omega\rvert} & \parallel\ \widehat{\mathbf{k}} \\ \lambda = \infty & \perp^r\ \widehat{\mathbf{k}} \end{cases}$ | 3cm-30m for audible sound |
| Radian frequency: | $\lvert\omega\rvert$ | cycles per radian of time |
| Frequency: | $F = \dfrac{1}{T} = \dfrac{\lvert\omega\rvert}{2\pi}$ | cycles per unit of time |
| Period: | $T = \dfrac{2\pi}{\lvert\omega\rvert}$ | $10^{-4}$-$10^{-1}$s for audible sound |
| Phase: | $\theta(\mathbf{x}, t) = \mathbf{k}.\mathbf{x} - \omega t$ | crests given by $\theta(\mathbf{x}, t) =$ constant |
| Normal to crests: | $\mathbf{n} = \dfrac{\nabla\theta}{\lvert\nabla\theta\rvert} = \dfrac{\mathbf{k}}{\lvert\mathbf{k}\rvert} = \widehat{\mathbf{k}}$ | $\nabla\theta = \mathbf{k}$ |
| Phase velocity: | $\mathbf{c} = \left(\left.\dfrac{d\mathbf{x}}{dt}\right\rvert_\theta .\mathbf{n}\right)\mathbf{n} = \dfrac{\omega}{\lvert\mathbf{k}\rvert^2}\mathbf{k} = \dfrac{\omega}{k}\widehat{\mathbf{k}}$ | velocity of crests $\perp$ to crests |

*Remark.* The phase velocity of sound waves,

$$\mathbf{c} \equiv \frac{\omega}{k}\,\widehat{\mathbf{k}} = \pm c_0\,\widehat{\mathbf{k}} = \pm c_0 \mathbf{n}\,, \tag{1.27d}$$

is independent of frequency and has constant magnitude. This has important consequences, since it results in sound waves being *non-dispersive* (see later).

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### 1.5.3 Energies of harmonic plane waves

Energies are **quadratic**; hence you **must take the real part before multiplication**. However, there is an exception, and an energy saving device, when you are interested in the average over a period. For instance, assume that $\theta$ and $\psi$ are harmonic with period $\frac{2\pi}{\omega}$, and that they have the complex representations (recall, with real part assumed)

$$\theta = \Theta \exp(i\mathbf{k}.\,\mathbf{x} - i\omega t)\ , \quad \psi = \Psi \exp(i\mathbf{k}.\,\mathbf{x} - i\omega t)\ .$$

Then

$$\begin{aligned} < \theta\psi > \equiv \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \theta(\mathbf{x},t)\psi(\mathbf{x},t)dt &= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} (\Theta_r c - \Theta_i s)(\Psi_r c - \Psi_i s)dt \\ &= \tfrac{1}{2}(\Theta_r\Psi_r + \Theta_i\Psi_i) \\ &= \tfrac{1}{2}\mathrm{Re}\,(\Theta\Psi^*) = \tfrac{1}{2}\mathrm{Re}\,(\theta\psi^*)\,, \end{aligned} \tag{1.28}$$

03/16 where $c = \cos(\mathbf{k}.\,\mathbf{x} - \omega t)$, $s = \sin(\mathbf{k}.\,\mathbf{x} - \omega t)$, $\Theta_r = \mathrm{Re}\,(\Theta)$, $\Theta_i = \mathrm{Im}\,(\Theta)$, $\Psi_r = \mathrm{Re}\,(\Psi)$, $\Psi_i = \mathrm{Im}\,(\Psi)$, and $^*$ indicates a complex conjugate.

As an example, we note from (1.17c), (1.20a), etc., that for $\varphi$ given by (1.27b):

$$\begin{aligned} < E_k > &= \tfrac{1}{2}\mathrm{Re}\,(\tfrac{1}{2}\rho_0\mathbf{u}.\mathbf{u}^*) \\ &= \tfrac{1}{4}\rho_0\mathrm{Re}\,(\boldsymbol{\nabla}\varphi.\boldsymbol{\nabla}\varphi^*) \\ &= \tfrac{1}{4}\rho_0\mathrm{Re}\,(i\mathbf{k}A\exp(i\mathbf{k}.\,\mathbf{x} - i\omega t).(-i\mathbf{k})A^*\exp(-(i\mathbf{k}.\,\mathbf{x} - i\omega t))) \\ &= \tfrac{1}{4}\rho_0 k^2|A|^2\ . \end{aligned} \tag{1.29a}$$

Similarly

$$< E_p > = \tfrac{1}{4}\rho_0 k^2|A|^2\ , \tag{1.29b}$$

and

$$< \mathbf{I} > = \pm\tfrac{1}{2}\rho_0 k^2|A|^2 c_0\widehat{\mathbf{k}}\ . \tag{1.29c}$$

Since this is a special case, it is consistent with (1.26a) and (1.26b).

## 1.6 Reflection and Transmission (or Why Bass Notes Ignore Walls)

We will do a relatively simple example here, but return to similar problems when we get to elastic waves. So, consider sound bouncing off a thin ($\ll \lambda$) wall between two rooms assuming normal incidence. Express all quantities in terms of $e^{i\omega t}$, or alternatively $e^{-i\omega t}$, implicitly taking the real part. Assume that the wall

- has a mass $m$ per unit area;
- only moves a small amount so that it can be assumed to be at $x = X(t) \approx 0$, i.e. linearise the evaluation of quantities at the wall to $x = 0$ so that

$$\tilde{p}(X(t),t) = \tilde{p}(0,t) + X(t)\tilde{p}_x(0,t) + \ldots \approx \tilde{p}(0,t)\,.$$

$$x < 0 \quad \tilde{p} = \underbrace{Ae^{i\omega(x-c_0t)/c_0}}_{\text{given incident wave}} + \underbrace{Re^{-i\omega(x+c_0t)/c_0}}_{\text{reflected wave}}\,.$$

Hence from (1.25c)

$$u = \frac{1}{\rho_0 c_0}\left(Ae^{i\omega(x-c_0t)/c_0} - Re^{-i\omega(x+c_0t)/c_0}\right)\,.$$

$$\begin{aligned} x > 0 \quad \tilde{p} &= Te^{i\omega(x-c_0t)/c_0}\,, \\ u &= \frac{1}{\rho_0 c_0}Te^{i\omega(x-c_0t)/c_0}\,. \end{aligned}$$

*Remark.* We assume that there are no waves of the form $e^{-i\omega(x+c_0t)/c_0}$ in $x > 0$, i.e. there are no waves bringing energy in from $+\infty$ (e.g. there is only one person playing their sound system in $x < 0$); this is a *radiation condition.*

*Boundary conditions.* These are linearised onto $x = 0$:

Kinematic:

$$\dot{X}(t) = u(0-, t) = u(0+, t)\,,$$

Dynamic:

$$m\ddot{X}(t) = \tilde{p}(0-, t) - \tilde{p}(0+, t) = -\left[\tilde{p}\right]_-^+\,.$$

*Solution.* Put $X = X_0e^{-i\omega t}$ and solve, then

$$\begin{aligned}
-i\omega X_0 &= \frac{1}{\rho_0c_0}(A - R) = \frac{1}{\rho_0c_0}T\,,\\
-m\omega^2X_0 &= A + R - T\,.
\end{aligned}$$

Introduce the dimensionless group

$$\alpha = \frac{\rho_0c_0}{\omega m} = \frac{\rho_0/k}{m} = \frac{\text{mass of gas in wavelength}}{\text{mass of wall}}\,.$$

Then

$$R = \frac{1}{1 + 2i\alpha}A\,, \qquad T = \frac{2i\alpha}{1 + 2i\alpha}A\,.$$

*Checks*

- $m = 0 \implies \alpha = \infty,\ R = 0,\ T = A$ (perfect transmission).
- $m = \infty \implies \alpha = 0,\ R = A,\ T = 0$ (total reflection).

*Remarks*

- The wall blocks high frequencies ($\alpha \ll 1$) better than low frequencies ($\alpha \gg 1$).
- After a lot of work (at least for $x < 0$) one can show that

$$< I_x > = < \tilde{p}u > = \begin{cases} \frac{1}{2\rho_0c_0}\left(|A|^2 - |R|^2\right) & \text{in } x < 0\,,\\ \frac{1}{2\rho_0c_0}|T|^2 & \text{in } x > 0\,. \end{cases}$$

Note that the energy flux is independent of $x$.

## 1.7 Waves in Three Dimensions

### 1.7.1 Acoustic waves in a rectangular duct

Consider the model problem of acoustic waves in a duct (a crude model of a musical instrument). For simplicity we take the duct to be rectangular (a very crude model). The wave equation, and the boundary conditions of zero normal velocity at the walls, are then

$$\Box^2\varphi = \left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\varphi = 0\,,$$

$$\left.\begin{array}{lll} \varphi_y = 0 & \text{on} & y = 0, h\\ \varphi_z = 0 & \text{on} & z = 0, b \end{array}\right\}\ \forall\ x\,.$$

Assume the waves propagate in the $x-$direction, and seek a separable harmonic wave solution of the form

$$\varphi = e^{ikx-i\omega t} f(y,z) \ . \tag{1.30a}$$

This satisfies the wave equation and the boundary conditions if

$$f_{yy}+f_{zz}+\left(\frac{\omega^2}{c_0^2}-k^2\right)f=0 \ , \tag{1.30b}$$

$$f_y = 0 \qquad \text{on} \quad y=0,h \ , \tag{1.30c}$$

$$f_z = 0 \qquad \text{on} \quad z=0,b \ . \tag{1.30d}$$

Equations (1.30a)-(1.30d) define an *eigenvalue* problem (cf. Schrödinger bound states), i.e. for given $k$ seek eigenvalues $\omega_*$, and associated eigenfunctions (modes) $f_*$.

Try a separable harmonic solution (in terms of sin and cos) that satisfies the boundary conditions:

$$f_{mn} = A_{mn}\cos\frac{m\pi y}{h}\cos\frac{n\pi z}{b} \ , \quad m,n\in\mathbb{Z} \ , \tag{1.31}$$

where, wlog, $m,n\in\mathbb{N}$. Then the wave equation is satisfied when $\omega$ is given by the *dispersion relation*

$$\omega^2 = \omega_{mn}^2(k) \equiv c_0^2\left(k^2+\left(\frac{m\pi}{h}\right)^2+\left(\frac{n\pi}{b}\right)^2\right) \ . \tag{1.32}$$

*The phase velocity.* For a given mode $(m,n)$, the phase velocity is given by

$$c=\frac{\omega}{k}=\pm\,c_0\left(1+\left(\frac{m\pi}{kh}\right)^2+\left(\frac{n\pi}{kb}\right)^2\right)^{\frac12} \ . \tag{1.33}$$

Hence, unless the waves are plane waves (i.e. $m=n=0$), the phase speed, $c$, varies with wavenumber; the waves are then said to be *dispersive.*

*The 'cut-off' frequency.* A mode of a given frequency $\omega$ *propagates* only if $k$ is real. Hence to excite the $(m,n)$ mode we require

$$|\omega| > c_0\left(\left(\frac{m\pi}{h}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^{\frac12} \ . \tag{1.34}$$

This minimum frequency is called the *cut-off frequency.* Attempts to excite the $(m,n)$ mode ($m\neq 0$, $n\neq 0$) below this frequency yield exponentially decaying evanescent modes with imaginary $k$.

Hence for given $\omega$, there exist only a finite number of propagating waves: the plane wave and a limited number of 'cross-modes'. The other modes are 'cut-off' and do not propagate. 04/17

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*The velocity of energy propagation.* From (1.30a) and (1.31)

$$\mathbf{u}=\nabla\varphi = A_{mn}\left(ik\cos\frac{m\pi y}{h}\cos\frac{n\pi z}{b}, -\frac{m\pi}{h}\sin\frac{m\pi y}{h}\cos\frac{n\pi z}{b}, -\frac{n\pi}{b}\cos\frac{m\pi y}{h}\sin\frac{n\pi z}{b}\right)e^{ikx-i\omega t} \ .$$

Averaging both in time and over a cross-section, and using (1.28) and the dispersion relation (1.32), yields

$$\begin{aligned}< E_k > &= \int_0^b dz\int_0^h dy\,\tfrac12 Re[\tfrac12\rho_0\mathbf{u}\,\boldsymbol{.}\,\mathbf{u}^*] \\ &= \tfrac14\rho_0|A_{mn}|^2\left(k^2+\frac{m^2\pi^2}{h^2}+\frac{n^2\pi^2}{b^2}\right)\frac{bh}{\chi_{mn}} \\ &= \frac{\rho_0|A_{mn}|^2 bh\omega^2}{4\chi_{mn}c_0^2} \ ,\end{aligned}$$

where $\chi_{mn} = (2-\delta_{m0})(2-\delta_{n0})$ accounts for the difference between averaging $(\cos 0)^2$ compared with $(\cos\frac{\pi y}{h})^2$, etc. Similarly, using definition (1.20a), and $\tilde{p} = c_0^2\tilde{\rho} = i\omega\rho_0\varphi$ from (1.17c),

$$\begin{aligned} < E_p >=& \int_0^b dz \int_0^h dy\, \tfrac{1}{2} Re\left[\tfrac{1}{2}c_0^2\frac{\tilde{\rho}\tilde{\rho}^*}{\rho_0}\right] \\ =&\frac{\rho_0|A_{mn}|^2 bh\omega^2}{4\chi_{mn}c_0^2} =< E_k > \ . \end{aligned}$$

04/19 *Remark.* We again have equi-partition of energy.

Further, from (1.17c) and (1.20b),

$$\begin{aligned} < I_x >=& \int_0^b dz \int_0^h dy\, \tfrac{1}{2} Re\left[\tilde{p}\varphi_x^*\right] \\ =& \int_0^b dz \int_0^h dy\, \tfrac{1}{2}\omega\rho_0 k|A_{mn}|^2 \cos^2\frac{m\pi y}{h}\cos^2\frac{n\pi z}{b} \\ =&\frac{\omega\rho_0 k|A_{mn}|^2 bh}{2\chi_{mn}}\,. \end{aligned}$$

Define the *mean energy propagation velocity* to be

$$U(k) = \frac{< I_x >}{< E_k + E_p >} = \frac{kc_o^2}{\omega}\,, \tag{1.35a}$$

and define the *group velocity* to be

$$c_g(k) = \frac{\partial\omega}{\partial k}\,. \tag{1.35b}$$

Then from (1.32)

$$c_g(k) = \pm\frac{c_0}{\left(1+\left(\frac{m\pi}{kh}\right)^2+\left(\frac{n\pi}{kb}\right)^2\right)^{\frac{1}{2}}} = \frac{kc_0^2}{\omega}\,. \tag{1.35c}$$

Hence for this example

$$U(k) = c_g(k) = \frac{\partial\omega}{\partial k}\,. \tag{1.35d}$$

This is an important result, and holds much more generally (e.g. there exists a vector equivalent).

*Remarks*

(i) Note that

$$c_g(k) \neq c(k)$$

i.e.

Energy propagation velocity $\neq$ Wave crest/trough velocity

(ii) Other than for plane waves ($m = n = 0$), it follows from (1.33) and (1.35c) that

(a) $|c_g| < c_0$, while $|c| > c_0$.

(b) Both $|c_g| \to c_0$ and $|c| \to c_0$ as $k \to \infty$ (short waves).

(c) $|c_g| \to 0$ while $|c| \to \infty$ as $k \to 0$ (long waves), so the phase speed can, in principle, be larger than the speed of light.

(iii) It is possible to interpret the waveguide as a superposition of reflecting plane waves.

For instance with $m \neq 0$, $n = 0$, let $\sin\psi = m\pi c_0/\omega h$. Then

$$\varphi \propto \left[ e^{\frac{i\omega}{c_0}(x\cos\psi + y\sin\psi - c_0 t)} + e^{\frac{i\omega}{c_0}(x\cos\psi - y\sin\psi - c_0 t)} \right] .$$

The *apparent* phase speed in the $x$-direction is

$$c = \frac{c_0}{\cos\psi} > c_0 \,,$$

while the component of the plane wave speed in the $x$-direction is $c_0\cos\psi = c_g < c_0$.

### 1.7.2 Spherically symmetric waves

In spherical polar co-ordinates

$$\begin{aligned} \nabla^2 \tilde{p} &= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\tilde{p})\,, \\ &= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\tilde{p}\,. \end{aligned}$$

So for spherically symmetric waves the wave equation (1.16), $\Box^2 \tilde{p} = 0$, becomes (after multiplying by $r$)

$$\frac{\partial^2}{\partial t^2}(r\tilde{p}) - c_0^2 \frac{\partial^2}{\partial r^2}(r\tilde{p}) = 0\,.$$

This is the 1D wave equation for $r\tilde{p}$, so

$$\tilde{p}(r,t) = \underbrace{\frac{f(r - c_0 t)}{r}}_{\text{outgoing wave}} + \underbrace{\frac{g(r + c_0 t)}{r}}_{\text{incoming wave}}\,. \tag{1.36}$$

*Remark.* Often one has a radiation condition of no incoming waves from $\infty$: then $g \equiv 0$.

It is conventional to write the outgoing solution in the form

$$\tilde{p}(r,t) = \frac{\dot{q}(t - r/c_0)}{4\pi r}\,. \tag{1.37a}$$

Then the density perturbation is given by

$$\tilde{\rho} = \frac{1}{c_0^2}\tilde{p} = \frac{\dot{q}(t - r/c_0)}{4\pi c_0^2 r}\,, \tag{1.37b}$$

and from (1.17c), i.e. $\tilde{p} = -\rho_0 \frac{\partial \varphi}{\partial t}$, the potential is given by

$$\varphi = -\frac{q(t - r/c_0)}{4\pi\rho_0 r}\,. \tag{1.37c}$$

The velocity is given by

$$u = \nabla\varphi = \frac{\partial\varphi}{\partial r}\widehat{\mathbf{r}} = \frac{1}{4\pi\rho_0}\left[\underbrace{\frac{\dot q(t-r/c_0)}{c_0 r}}_{\substack{\text{dominates}\\ \text{for } r\gg 1}} + \underbrace{\frac{q(t-r/c_0)}{r^2}}_{\substack{\text{dominates}\\ \text{for } r\ll 1}}\right]\widehat{\mathbf{r}}\,. \tag{1.37d}$$

Hence

$$\begin{aligned}\text{Mass flux out of a sphere of radius } r &= 4\pi r^2\,\rho_0\,u(r,t)\\ &= \frac{r}{c_0}\,\dot q(t-r/c_0) + q(t-r/c_0)\\ &\to q(t) \quad \text{as} \quad r\to 0\,.\end{aligned}$$

$q(t)$ is the mass flux from a point source at the origin.

### 1.7.3 Solution for a pulsating sphere

Consider a sphere performing small radial pulsations

$$r = a + \varepsilon e^{i\omega t} \quad \text{with} \quad \varepsilon \ll a\,, \tag{1.38a}$$

where is it assumed that the real part of the equation is taken. Then on the surface of the sphere, $r = a + \varepsilon e^{i\omega t}$, the velocity of the sphere is

$$u = \frac{\partial\varphi}{\partial r} = i\omega\varepsilon e^{i\omega t}\,. \tag{1.38b}$$

But

$$u(a+\varepsilon e^{i\omega t}, t) \approx u(a,t) + \underbrace{\varepsilon e^{i\omega t} u_r(a,t)}_{\text{doubly small}} + \dots\,.$$

Hence apply the boundary condition (b.c.) on $r = a$. Based on the form of the forcing, seek a solution with $q(t) \propto e^{i\omega t}$, e.g. from (1.37c)

$$\varphi = A\frac{e^{i\omega(t-\frac{r-a}{c_0})}}{4\pi\rho_0 r}\,.$$

Then from (1.37d)

$$u = -\frac{A}{4\pi\rho_0}\left(\frac{1}{r^2} + \frac{i\omega}{c_0 r}\right)\,e^{i\omega(t-\frac{r-a}{c_0})}\,.$$

Evaluate $u$ on $r = a$ to deduce from (1.38b) that

$$-\frac{A}{4\pi\rho_0 a^2}\left(1 + \frac{i\omega a}{c_0}\right) = i\omega\varepsilon\,,$$

i.e.

$$A = -\frac{4\pi i\rho_0 a^2\omega\varepsilon}{(1+i\omega a/c_0)}\,,$$

and hence

$$\tilde p = -i\omega\rho_0\varphi = -\frac{\varepsilon a^2\omega^2\rho_0}{(1+i\omega a/c_0)}\frac{e^{i\omega(t-(r-a)/c_0)}}{r}\,.$$

*Remarks*

(i) The sound has wavelength $\lambda = \frac{2\pi c_0}{\omega}$, hence we define

$$\begin{aligned}&\text{a 'large' sphere to be one such that} \quad a \gg \frac{2\pi c_0}{\omega}, \quad \text{i.e.} \quad \frac{\omega a}{c_0} = ka \gg 1\,,\\ &\text{a 'small' sphere to be one such that} \quad a \ll \frac{2\pi c_0}{\omega}, \quad \text{i.e.} \quad \frac{\omega a}{c_0} = ka \ll 1\,,\end{aligned}$$

(ii) A small sphere is said to be 'compact'. One can show that compact sources are inefficient sound sources because $\tilde{p}$ and $u$ are out of phase (see question 5 on Example Sheet 1).

### 1.7.4 Loud speakers: why tall and thin is good

A *small/compact* vibrating disk is an inefficient sound source, because the flow is locally incompressible; it acts as a dipole (the strength of which can be shown to be proportional to the force exerted to move the disk).

One way forward to improve efficiency is to use an exponential horn (megaphone) longer than a wavelength, so that a quasi-2D flow is set up (see Example Sheet 1).

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Slower next year

Alternatively, effectively change the 'dipole' to a monopole by inhibiting the reverse flow, e.g. with a baffle. The boundary conditions then become, say at $x = 0+$,

$$u_n = \begin{cases} U(y,z,t) & \text{on moving part,} \\ 0 & \text{on stationary baffle.} \end{cases}$$

Using the *method of images*, this is equivalent to having no baffle, and an additional image boundary condition of $u_n = -U$ at $x = 0-$. For a disk of area $dy\,dz$, this is equivalent to a source of strength $q = 2\rho_0 U(y,z,t) dy\,dz$.

To find the solution for a loud speaker, we add up lots of little moving parts (see also Tripos question 2009/1/38). Then, from the solution for a point source, i.e. (1.37a),

$$\tilde{p}(x,y,z,t) = \iint \frac{2\rho_0 \dot{U}(Y,Z,t-R/c_0)}{4\pi R} dY\,dZ\,,$$

where $R$ is the distance between the field point $(x,y,z)$ and the source point $(0,Y,Z)$:

$$R = \left(x^2 + (y-Y)^2 + (z-Z)^2\right)^{1/2}\,.$$

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When far from the loudspeaker

$$\begin{aligned} R = &\ (x^2+y^2+z^2)^{1/2}\left(1 - \frac{2(yY+zZ)}{x^2+y^2+z^2} + \dots\right)^{1/2} \\ \sim &\ \underset{O(r)}{r} \; - \; \underset{O(1)}{\frac{yY+zZ}{r}} + \dots\,. \end{aligned}$$

For simplicity, consider a single frequency $\omega$ so that

$$U = U(Y,Z)e^{i\omega t}\,,$$

then in far field

$$\tilde{p} \sim \frac{i\omega\rho_0}{2\pi r} e^{i\omega(t-r/c_0)} \underbrace{\iint U(Y,Z) e^{i(yY+zZ)\omega/rc_0} dY\,dZ}_{\text{Fourier transform of } U}\,.$$

Note that it is OK to use the leading-order approx to $R$ in denominator but not in exponent (since $\cos(2019) \not\approx \cos(2020)$).

Suppose further that

$$U = \begin{cases} U_0 & |Y| \leqslant a, |Z| \leqslant b \\ 0 & \text{otherwise.} \end{cases} ,$$

and that

$$\frac{\omega a}{c_0} \ll 1\,, \quad \frac{\omega b}{c_0} \gtrsim 1\,,$$

then

$$\tilde{p} \sim \frac{i\omega\rho_0 e^{i\omega(t-r/c_0)}}{2\pi r} 4abU_0 \left[\frac{\sin(\omega bz/rc_0)}{\omega bz/rc_0}\right].$$

For this solution most power is radiated in directions with

$$\frac{\omega bz}{rc_0} \lesssim \pi\,,$$

i.e. for

$$\sin\theta = \frac{z}{r} \lesssim \frac{\pi c_0}{\omega b}\,.$$

If $b$ is large enough, then it is possible to ensure that little sound hits the roof of a theatre, etc.

# Appendix

## 1.A * Thermodynamics [for Dummies] *(unlectured)* *

### 1.A.1 $0^{\text{th}}$ Law – there exists temperature

After several collisions with other molecules, the velocity of a molecule becomes randomised and the gas attains *thermal equilibrium* characterised by a *temperature* $T$; this time for air is $10^{-9}$ s and for water $10^{-12}$ s. A *reversible change* is a change to the system that is slow compared with the above times, so that the system always remains near to thermal equilibrium.

More generally we say that a fluid is in *thermodynamic equilibrium* if all past history is forgotten, and all macroscopic quantities do not change with time.

**Parameters of State.** *Parameters of state* (also sometimes called *functions of state*) are macroscopic quantities which, in any change in thermodynamic equilibrium, vary by an amount independent of the path taken between the initial and final states. Examples are the pressure, $p$, the density, $\rho$, and the [absolute] temperature, $T$.

**Equations of State.** An *equation of state* is a functional relation between the parameters of state for a system in thermodynamic equilibrium, e.g.

$$f(p,\rho,T) = 0\ . \tag{1.A.1a}$$

E.g. for a *perfect gas*

$$p = \rho R T\ . \tag{1.A.1b}$$

### 1.A.2 $1^{\text{st}}$ Law – energy is conserved if all forms are counted

Consider a unit mass of gas, then the *specific volume* (volume per unit mass) is $V = 1/\rho$. The *First Law of Thermodynamics* states that this volume has a (specific) *internal energy* $e(\rho, T)$ that arises from the kinetic energy of thermal motion of molecules and possibly from the potential energy of excited vibrational modes. The internal energy depends strongly on the current temperature and density, but not on past values.

More generally the *First Law of Thermodynamics* states that every thermodynamic system possesses a parameter of state, the *internal energy*, $e$. This parameter can be increased by adding heat energy, $\delta Q$, to the system, or by performing mechanical work, $\delta W$, on the system:

$$de = \delta Q + \delta W\ . \tag{1.A.2}$$

$Q$ and $W$ are *not* parameters of state.

### 1.A.3 Adiabatic processes, isolated systems and reversible changes

An *adiabatic process*[2] is a process in which there is no heat change, i.e.

$$\delta Q = 0\ . \tag{1.A.3a}$$

- A system is *isolated* when it does not interact with its surroundings, i.e. when no heat can be transferred to/from it, and no work can be performed on it.

[2] Some authors, e.g. Pippard, call this an *adiathermal process*.

- A *reversible change*[3] from one equilibrium state to another is an 'infinitely' slow quasi-static process during which the thermodynamic system remains infinitesimally close to thermodynamic equilibrium. At each stage of a reversible process, the parameters of state have a well-defined meaning, hence we can write
$$de = dQ - pdV \ , \tag{1.A.3b}$$
since the mechanical work done on a simple gas against pressure is $-p\,dV$ (note that positive work is done under compression, i.e. $dV < 0$).

**Specific Heats.** The specific heat of a material is defined as $dQ/dt$ in a reversible change (i.e. it is effectively the heat input to increase the temperature of a unit mass by 1 K in a reversible change). The specific heats at constant volume and constant pressure are therefore respectively equal to
$$c_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial e}{\partial T}\right)_{V \text{ or } \rho} , \tag{1.A.3c}$$
$$c_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial e}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p . \tag{1.A.3d}$$

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### 1.A.4 2nd Law – chaos increases

The *Second Law of Thermodynamics* states that all thermodynamic systems possess a parameter of state called the *entropy*, $S$, such that in a *reversible* change
$$TdS = dQ \ . \tag{1.A.4}$$
Further, the entropy of an isolated system can only *increase*, i.e. for an *irreversible* adiabatic change $dS > 0$.

*Remark.* Roughly, $S$ is proportional to the number of arrangements of the molecules at a given volume $V$ and temperature $T$. Thence increasing entropy means increasing chaos, e.g. heat flows from hot to cold.

**An Exact Differential For Entropy.** For a reversible process, (1.A.3b) and (1.A.4) imply that
$$TdS = de + pdV \ . \tag{1.A.5a}$$
This equation involves only parameters of state, and hence must be *valid for any infinitesimal transition*, whether reversible or not; thus $S \equiv S(e, V)$. However if the transition is irreversible (e.g. because of work done against friction) then
$$\delta Q \neq TdS \ , \quad \delta W \neq -pdV \ . \tag{1.A.5b}$$

**The Maxwell Relations.** It is very important to display what is a function of what, and so what is being held constant during partial differentiation. One is permitted to take any two parameters of state as independent variables, e.g. $e$ & $V$, or $\rho$ & $T$, or $\rho$ & $p$, or $S$ & $\rho$.

Consider $de = T\,dS - p\,dV$, and view all variables ($e$, $T$, $\rho$) as functions of $S$ & $V$. Then
$$T = \left(\frac{\partial e}{\partial S}\right)_V \quad \text{and} \quad p = -\left(\frac{\partial e}{\partial V}\right)_S .$$
But by cross differentiation
$$\left(\frac{\partial T}{\partial V}\right)_S = \frac{\partial^2 e}{\partial S \partial V} = -\left(\frac{\partial p}{\partial S}\right)_V .$$
Another Maxwell relation is obtained by the following trick $d(e + pV) = T\,dS + V\,dp$, so
$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p .$$
The combination $h = e + pV$, the *enthalpy*, is important in fluid mechanics.

[3] Beware: some authors, e.g. Pippard, call a *reversible adiathermal process* an adiabatic process!

**Ideal Fluids.** *Ideal fluids* are fluids in which all molecular diffusivities are zero. No heat can therefore be transferred between fluid particles, and the fluid must always be in thermodynamic equilibrium. Hence for a dissipationless fluid

$$\frac{DS}{Dt} = 0 \,, \tag{1.A.5c}$$

i.e. the flow is *isentropic*.

If the entropy takes a uniform constant value everywhere, the flow is *homentropic*. For future reference note that molecular diffusivities are *not* negligible in shocks.

**A Closed System.** Equation (1.A.5c), together with the equation of state relating $S$ to $p$ and $\rho$, close the system of governing equations.

## 1.B * Perfect Gases *(unlectured)* *

Consider dilute gases, e.g. air for which $\rho_{\text{air}} = 10^{-3}\rho_{\text{liquid air}}$.

### 1.B.1 Pressure

The pressure force on a surface is the momentum exchanged from collisions in unit time, i.e. $p = Nm\langle v_x^2\rangle$, where

- $N = nN_A/v$ is the number of molecules per unit volume ($n$ is the number of moles of gas, $N_A = 6.02\,10^{23}$ is *Avogadro's number*, and $v$ is volume), and
- it follows from statistical physics that $m\langle v_x^2\rangle = k_B T$ ($k_B = 1.38\,10^{-23}\,\mathrm{J\,K^{-1}}$ is *Boltzmann's constant*).

Hence

$$pv = n\bar{R}T \,,$$

where the *universal gas constant* $\bar{R} = N_A k_B$.

### 1.B.2 Internal energy and specific heats

In a perfect gas, the internal energy $e$ varies with temperature $T$ but **not** with density $\rho$ — because 'dilute' means the size of the molecules is irrelevant. If we further assume that the specific heats do not vary with temperature (OK for 100-2000 K when vibrational modes are not excited), then

$$c_V = \left(\frac{\partial e}{\partial T}\right)_V = \text{constant} \quad \text{and} \quad \left(\frac{\partial e}{\partial V}\right)_T = 0 \,.$$

So

$$e = c_V T \,, \tag{1.B.6a}$$

setting constant of integration to zero (so that there is no energy at $0^o$K).

From *kinetic theory*, each molecule has $\frac{1}{2}k_B T$ internal energy per degree of freedom of internal motion. Hence $c_V = \frac{3}{2}Nk_B = \frac{3}{2}R$ for monatomic gases He and Ar, and $c_V = \frac{5}{2}Nk_B = \frac{5}{2}R$ for diatomic gases $H_2$, $N_2$, $O_2$, CO (since the molecules can have rotational KE represented by, say, two Euler angles, but no vibrational energy).

The other specific heat is

$$c_p = \left(\frac{\partial e}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = c_V + R \,,$$

from $e = c_V T$ and $V = 1/\rho = RT/p$. Thence the *ratio of specific heats*

$$\gamma = \frac{c_p}{c_V} = \frac{c_V + R}{c_V} \,,$$

is equal to $\frac{5}{3}$ for monatomic gases and $\frac{7}{5}$ for diatomic gases. A useful alternative expression for the internal energy is, cf. (1.5e),

$$e = c_V T = \frac{c_V}{R}\, pV = \frac{1}{\gamma - 1}\,\frac{p}{\rho}\,. \tag{1.B.6b}$$

Hence the enthalpy of a perfect gas is given by, cf. (1.5g),

$$h \equiv e + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1}\,\frac{p}{\rho}. \tag{1.B.6c}$$

### 1.B.3 Entropy of a perfect gas

From (1.A.5a), (1.B.6a) and (1.A.1b), $T\,dS = de + p\,dV$, $e = c_V T$ and $p = \rho RT = RT/V$. Hence

$$dS = c_V \frac{dT}{T} + \frac{R}{V} dV\,.$$

By integrating we obtain

$$\begin{aligned}
S - S_0 =& c_V \ln \frac{T}{T_0} + R \ln \frac{V}{V_0} \\
=& c_V \ln \left[ \frac{T}{T_0} \left( \frac{V}{V_0} \right)^{R/c_V} \right] \\
=& c_V \ln \left[ \frac{pV}{p_0 V_0} \left( \frac{V}{V_0} \right)^{\gamma - 1} \right].
\end{aligned}$$

So

$$S = S_0 + c_V \ln \left[ \frac{p}{p_0} \left( \frac{\rho_0}{\rho} \right)^{\gamma} \right]. \tag{1.B.6d}$$

## 2 Finite Amplitude 1D Waves

Our aim is to examine the effects of nonlinearity on (plane) sound waves. In doing so we will learn why waves 'break' and shocks form. The theory is also applicable to why you can feel your pulse and why traffic jams develop.

### 2.1 Nonlinear Acoustics in 1D: Governing Equations

Assume

(a) 1D motion;

(b) homentropic flow, i.e. from (1.10a)

$$p \equiv p(\rho, S_0)\,. \tag{2.1}$$

The exact 1D version of conservation of mass (1.1b) is, with $\partial_y = \partial_z = 0$, $\mathbf{u} = (u, 0, 0)$, and $Q = \mathbf{F} = 0$,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0\,. \tag{2.2a}$$

Similarly, the exact 1D version of Euler's equation, (1.2b), is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,, \tag{2.2b}$$

while from (1.10a)

$$\frac{\partial p}{\partial x} = \frac{dp}{d\rho}\frac{\partial \rho}{\partial x} = c^2\frac{\partial \rho}{\partial x}\,, \tag{2.2c}$$

where

$$c^2(\rho) = \frac{dp}{d\rho}(\rho)\,. \tag{2.2d}$$

### 2.2 Riemann Form of Equations

We can simplify the system of equations (2.2a)-(2.2d) for $u(x,t)$ and $\rho(x,t)$ by a change of coordinates.[4] First form $\lambda$.(2.2a) + (2.2b), where $\lambda(x,t)$ is to be determined, to obtain

$$\left(\frac{\partial}{\partial t} + (u + \lambda\rho)\frac{\partial}{\partial x}\right)u + \lambda\left(\frac{\partial}{\partial t} + \left(u + \frac{c^2}{\lambda\rho}\right)\frac{\partial}{\partial x}\right)\rho = 0\,.$$

We fix $\lambda$ by choosing

$$u + \lambda\rho = u + \frac{c^2}{\lambda\rho}\,, \quad \text{i.e.} \quad \lambda = \pm\frac{c}{\rho}\,,$$

so that

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)u \pm \frac{c}{\rho}\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)\rho = 0\,.$$

Define

$$Q(\rho) = \int_{\rho_0}^{\rho}\frac{c(\hat\rho)}{\hat\rho}d\hat\rho\,, \tag{2.3}$$

then

$$Q_t = \frac{c}{\rho}\rho_t\,, \quad Q_x = \frac{c}{\rho}\rho_x\,,$$

and hence

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)(u \pm Q) = 0\,. \tag{2.4}$$

[4] A transcript of Riemann's article can be downloaded from http://www.emis.de/classics/Riemann/Welle.pdf. There is also an English translation, but not for download: http://www.kendrickpress.com/Riemann.htm.

Define the *Riemann invariants*, $R_\pm(x,t)$ to be

$$R_\pm = u \pm Q\,, \tag{2.5}$$

then we obtain the *Riemann equations*

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right) R_\pm = 0\,. \tag{2.6}$$

We can integrate these equations by changing variables from $(x,t)$ to $(\xi,\eta)$ where the lines $\xi$ =const. are defined by

$$\frac{dx}{dt} = u(x,t) + c(x,t)\,, \qquad \text{(the } C_+ \text{ characteristics, } \xi \text{ =const.)}\,, \tag{2.7a}$$

where we have supposed that $u$ and $c$ are in principle known. Similarly the lines $\eta$ =const. are defined by

$$\frac{dx}{dt} = u(x,t) - c(x,t)\,, \qquad \text{(the } C_- \text{ characteristics, } \eta \text{ =const.)}\,. \tag{2.7b}$$

These two intersecting families of lines are referred to as the $C_\pm$ *characteristics* respectively.

On the $C_+$ characteristics, i.e. on lines $\xi$ =const., it follows from the first Riemann equation (2.6) and the first characteristic equation (2.7a) that

$$\frac{dR_+}{dt} = \frac{\partial R_+}{\partial t} + \frac{dx}{dt}\frac{\partial R_+}{\partial x} = \frac{\partial R_+}{\partial t} + (u+c)\frac{\partial R_+}{\partial x} = 0\,; \tag{2.8a}$$

hence

$$R_+ \equiv R_+(\xi)\,. \tag{2.8b}$$

Similarly

$$R_- \equiv R_-(\eta)\,. \tag{2.8c}$$

*Remark.* The physical interpretation of these equations is that waves carrying constant values of $R_\pm$ propagate at speeds $\pm c$ relative to the local flow speed $u$.

*Check (unlectured).* We have that

$$dt = \left.\frac{\partial t}{\partial \eta}\right|_\xi d\eta + \left.\frac{\partial t}{\partial \xi}\right|_\eta d\xi\,, \quad dx = \left.\frac{\partial x}{\partial \eta}\right|_\xi d\eta + \left.\frac{\partial x}{\partial \xi}\right|_\eta d\xi\,.$$

Hence on $\xi$ =const.,

$$d\xi = 0 \qquad \text{and} \qquad \left.\frac{\partial x}{\partial \eta}\right|_\xi = \frac{dx}{dt}\left.\frac{\partial t}{\partial \eta}\right|_\xi = (u+c)\left.\frac{\partial t}{\partial \eta}\right|_\xi\,.$$

It follows from (2.6), consistent with (2.8b), that

$$\begin{aligned}\left.\frac{\partial R_+}{\partial \eta}\right|_\xi &= \left.\frac{\partial R_+}{\partial t}\right|_x \left.\frac{\partial t}{\partial \eta}\right|_\xi + \left.\frac{\partial R_+}{\partial x}\right|_t \left.\frac{\partial x}{\partial \eta}\right|_\xi \\ &= \left(\left.\frac{\partial R_+}{\partial t}\right|_x + (u+c)\left.\frac{\partial R_+}{\partial x}\right|_t\right)\left.\frac{\partial t}{\partial \eta}\right|_\xi \\ &= 0.\end{aligned}$$

*Relationship to D'Alembert's solution.* From (2.5),

$$u = \tfrac{1}{2}(R_+(\xi) + R_-(\eta))\,, \tag{2.9a}$$

$$Q = \tfrac{1}{2}(R_+(\xi) - R_-(\eta))\,. \tag{2.9b}$$

This can be viewed as a generalisation of D'Alembert's linear solution, cf. (1.24b):

$$u = F(x - c_0 t) + G(x + c_0 t)\,,$$

as may be seen by solving (2.7a) and (2.7b) with $u = 0$ and $c = c_0$.

### 2.2.1 Example: perfect gas

From (1.5a), (1.5b) and (2.2d)

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \quad c^2 = \frac{dp}{d\rho} = \frac{\gamma p}{\rho} \quad \text{and} \quad \frac{c}{c_0} = \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}. \tag{2.10a}$$

Hence from the definitions of $Q$, see (2.3), and $R_\pm$, see (2.5),

$$Q = \frac{2}{\gamma-1}(c-c_0)\,, \qquad\qquad R_\pm = u \pm \frac{2(c-c_0)}{\gamma-1}\,, \tag{2.10b}$$

and hence

$$u = \tfrac{1}{2}(R_+(\xi) + R_-(\eta))\,, \qquad\qquad c = c_0 + \tfrac{1}{4}(\gamma-1)(R_+(\xi) - R_-(\eta))\,. \tag{2.10c}$$

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## 2.3 Solution by Characteristics

### 2.3.1 The Cauchy initial-value problem (IVP)

Suppose that initial conditions are known at $t = 0$, e.g.

$$u(x,0) = U(x), \quad \rho(x,0) = \mathrm{P}(x)\,, \tag{2.11a}$$

or since $c \equiv c(\rho)$, equivalently

$$u(x,0) = U(x), \quad c(x,0) = C(x)\,. \tag{2.11b}$$

For simplicity assume 'compact support', i.e.

$$U = 0\,,\ C = c_0 \ \text{ and } \ \mathrm{P} = \rho_0 \quad \text{for} \quad x < x_B \text{ and } x > x_F. \tag{2.11c}$$

### 2.3.2 General form of the solution

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06/19 *Regions I, II and III.* At $t = 0$ on characteristics that pass through $x < x_B$ or $x > x_F$:

$$u(x,0) = 0\,, \quad c(x,0) = c_0\,, \quad Q(x,0) = 0\,. \tag{2.12a}$$

Hence

(i) on $\xi < \xi_B$ and $\xi > \xi_F$, $R_+ = 0$;
(ii) on $\eta < \eta_B$ and $\eta > \eta_F$, $R_- = 0$.

It follows that in regions I, II and III,

$$R_+ = R_- = 0 \quad \Longrightarrow \quad u = 0\,, \quad c = c_0 \Longrightarrow \text{undisturbed flow.} \tag{2.12b}$$

*Regions IV and V.* In region IV

$$R_- = u - Q = 0 \qquad \Longrightarrow \qquad u = Q \tag{2.13}$$

On $C_+$, from (2.8b),

$$u + Q = R_+ = \text{const.}$$

Hence from (2.3) and (2.13), on $C_+$

$$u = \text{const.}, \quad Q = \text{const.}, \quad \rho = \text{const.}, \quad c = \text{const.},$$

and

$$\frac{dx}{dt} = u + c = \text{const.}.$$

Hence the $C_+$ are straight lines in region IV, corresponding to a right-running simple wave (see below).

Similarly, in region V the $C_-$ characteristics are straight, and there is a left-running simple wave.

*Region VI.* In region VI there is compound flow, where neither $C_+$ nor $C_-$ are straight. At $t = t_c$ the compound flow disentangles to give two simple waves with undisturbed regions ahead, between and behind them.

### 2.3.3 * Numerical method *

*Aim.* Find the solution at $x = X$ for $\delta t \ll 1$.

*Method.* Draw the characteristics through $(X, \delta t)$ back to $t = 0$. Since $\delta t \ll 1$, the $C_+$ characteristic can be approximated, using a Taylor expansion for $0 \leqslant t \leqslant \delta t$, as

$$\begin{aligned} x(t) &= X + \underbrace{(u(X,\delta t) + c(X,\delta t))}_{\text{slope}}(t - \delta t) + O(\delta t^2) \\ &= X + (u(X,0) + c(X,0))\,(t - \delta t) + O(\delta t^2)\,. \end{aligned} \tag{2.14a}$$

Note that $x = X$ at $t = \delta t$. Hence at $t = 0$ it follows from using the initial conditions that

$$x(0) = X - (U(X) + C(X))\,\delta t + O(\delta t^2)\,. \tag{2.14b}$$

Since $R_+$ =const. on $C_+$ this implies

$$R_+(X, \delta t) = R_+(X - (U(X) + C(X))\delta t\,, 0) + O(\delta t^2)\,. \tag{2.15a}$$

Similarly

$$R_-(X, \delta t) = R_-(X - (U(X) - C(X))\delta t\,, 0) + O(\delta t^2)\,. \tag{2.15b}$$

*Numerical Recipe.*

(a) At $t = t_0$ assume that $u$ and $c$ are known.

(b) Calculate $R_+$ and $R_-$ at $t = t_0$, e.g. using (2.10b).

(c) Calculate $R_+$ and $R_-$ at $t = t_0 + \delta t$ using (2.15a) and (2.15b).

(d) Calculate $u$ and $c$ at $t = t_0 + \delta t$, e.g. using (2.10c).

06/04 (e) Iterate after $t_0 \to t_0 + \delta t$ (remembering that interpolation is your friend).

## 2.4 Simple Waves and Shock Formation

### 2.4.1 Simple waves

*Definition.* A *simple wave* is a wave where one of $R_+$ or $R_-$ is uniformly constant (wlog zero).

As an example, consider a *right-running* simple wave (as in region IV) with

$$R_- = 0\,. \tag{2.16a}$$

Hence from (2.9a), (2.9b) and (2.5) we infer that

$$u = Q \quad \text{and} \quad R_+ = 2u = 2Q\,. \tag{2.16b}$$

However, $R_+ \equiv R_+(\xi)$ from (2.8b), i.e. *on each $C_+$ characteristic*

$$R_+ = \text{const.}\,. \tag{2.17a}$$

Hence from (2.16b), (2.3) and (2.2d)

$$u \equiv u(\xi), \quad Q \equiv Q(\xi), \quad \rho \equiv \rho(\xi), \quad c \equiv c(\xi)\,, \tag{2.17b}$$

and so these quantities are also constant on each $C_+$ characteristic. Further, from the definition (2.7a) of a $C_+$ characteristic,

$$\frac{dx}{dt} = u + c \equiv u(\xi) + c(\xi)\,, \tag{2.17c}$$

it follows that for a right-running simple wave the $C_+$ characteristics are *straight lines.*

*Solution.* Suppose that at $t = 0$ consistent initial conditions are

$$u(x,0) = U(x) \quad \text{and} \quad c(x,0) = C(x)\,. \tag{2.18a}$$

Then we deduce from (2.17b) and (2.17c) that

$$u(x,t) = U(\xi) \quad \text{and} \quad c(x,t) = C(\xi)\,, \tag{2.18b}$$

and that the equations of the $C_+$ characteristics are, from imposing $x = \xi$ at $t = 0$,

$$x(t;\,\xi) = \xi + (U(\xi) + C(\xi))t\,. \tag{2.18c}$$

Thus, for given $(x,t)$, one can in principle solve for $\xi$, and thence obtain $u(x,t)$ and $c(x,t)$ from (2.18b).

*Perfect gas.* In the case of a perfect gas it follows from (2.16b) and (2.10b) that

$$R_+ = 2u = \frac{4}{\gamma - 1}(c - c_0) = \text{const.}\,, \tag{2.19a}$$

$$c = c_0 + \tfrac{1}{2}(\gamma - 1)u\,. \tag{2.19b}$$

Hence on a $C_+$ characteristic,

$$\frac{dx}{dt} = c_0 + \tfrac{1}{2}(\gamma + 1)u\,, \tag{2.20a}$$

where, as above, the $C_+$ are straight. Hence the problem for right-running simple waves in a perfect gas can be posed either as

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = c_0 + \tfrac{1}{2}(\gamma + 1)u\,, \tag{2.20b}$$

or, from the Riemann equations (2.6) with $R_- = 0$ and $u + c = c_0 + \frac{1}{2}(\gamma + 1)u$, as

$$\left(\frac{\partial}{\partial t} + \left(c_0 + \tfrac{1}{2}(\gamma + 1)u\right)\frac{\partial}{\partial x}\right)u = 0\,. \tag{2.20c}$$

Further, from (2.18c), or from (2.20b) and imposing $x = \xi$ at $t = 0$, the equation for the $C_+$ characteristics is

$$x(t;\,\xi) = \xi + \left(c_0 + \tfrac{1}{2}(\gamma + 1)U(\xi)\right)t\,. \tag{2.20d}$$

Thus, for given $(x,t)$, one can in principle solve for $\xi$, and thence obtain

$$\begin{aligned} &u && \text{from (2.18b)}\,, \\ &c = c_0 + \tfrac{1}{2}(\gamma - 1)u && \text{from (2.19b)}\,, \\ &\frac{\rho}{\rho_0} = \left(1 + \frac{(\gamma - 1)u}{2c_0}\right)^{\frac{2}{\gamma - 1}} && \text{from (2.10a)}\,. \end{aligned}$$

### 2.4.2 Generic form

Equation (2.20c) can be written in a more suggestive form. Let

$$X = x - c_0 t \,, \tag{2.21a}$$

and introduce the *excess wavespeed*

$$v = \left(\frac{\gamma+1}{2}\right) u = u + c - c_0 \,. \tag{2.21b}$$

Then (2.20c) becomes

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial X} = 0 \,. \tag{2.22}$$

This is a special case of the *Kinematic Wave Equation*. The equation has $C_+$ characteristics

$$X(t;\,\xi) = \xi + v(\xi,0)t \,, \tag{2.23a}$$

on which $v$ is a constant, i.e.

$$v(X,t) = v(\xi,0) \,. \tag{2.23b}$$

Hence from (2.23a)

$$v(X,t) = v(X - v(\xi,0)t, 0) \,, \tag{2.23c}$$

and thence

$$\rho(X,t) = \rho(X - v(\xi,0)t, 0) \,, \tag{2.23d}$$

i.e. the density $\rho$ propagates at speed $v$ (in a frame moving with velocity $c_0$).

Each element of the wave form propagates at speed $v$. The deformation is pure distortion – i.e. there are no new values of $v$ (or $\rho$).

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### 2.4.3 Wave steepening

Calculate the slope of the wave:

$$\left.\frac{\partial v}{\partial X}\right|_t = \frac{\partial v(\xi,0)}{\partial \xi}\left.\frac{\partial \xi}{\partial X}\right|_t \ . \tag{2.24a}$$

From (2.23a)

$$1 = \left.\frac{\partial \xi}{\partial X}\right|_t \left(1 + \frac{\partial v(\xi,0)}{\partial \xi}\, t\right) \ , \tag{2.24b}$$

and hence

$$\left.\frac{\partial v}{\partial X}\right|_t = \frac{v_\xi(\xi,0)}{1 + v_\xi(\xi,0)\, t} \ . \tag{2.24c}$$

Hence parts of the profile with $v_\xi(\xi,0) < 0$ get steeper, while parts of the profile with $v_\xi(\xi,0) > 0$ get flatter. If $v_\xi(\xi,0) < 0$ somewhere, then a triple-valued waveform will form at some finite time. This is unphysical. The moment at which the slope becomes infinite is referred to as the **shock formation** time.

### 2.4.4 Shock formation

The profile is on the verge of becoming multi-valued at the first time that

$$\left.\frac{\partial v}{\partial X}\right|_t = \infty \ .$$

From (2.24c), $v_X$ is infinite when

$$t = -\frac{1}{v_\xi(\xi,0)} \ . \tag{2.25}$$

The *shock formation time* is thus

$$t_s = \min_{\xi}\left(-\frac{1}{v_\xi(\xi,0)}\right) = \left(\frac{2}{\gamma+1}\right)\frac{1}{\max_{\xi}(-u_\xi(\xi,0))} \ . \tag{2.26a}$$

Suppose that $[-u_\xi(\xi,0)]$ is maximised at $\xi = \xi_m$, then the position at which the shock forms is

$$X_s = \xi_m + v(\xi_m,0)t_s \ , \tag{2.26b}$$

i.e. the *position of shock formation* in the original co-ordinates is given by

$$\begin{aligned} x_s &= X_s + c_0 t_s \\ &= \xi_m + \left(c_0 + \tfrac{1}{2}(\gamma+1)u(\xi_m,0)\right) t_s \ . \end{aligned} \tag{2.26c}$$

### 2.4.5 Alternative estimate of shock formation time

The shock forms when neighbouring $C_+$ *first* touch, i.e. when contradicting information from different $C_+$ characteristics first propagate to the same point. From (2.20d)

$$x = \xi + (c_0 + \tfrac{1}{2}(\gamma+1)u(\xi,0))t \ .$$

The $C_+$ touch when

$$\left.\frac{\partial x}{\partial \xi}\right|_t = 0 \ ,$$

i.e. the $C_+$ touch when

$$t = -\frac{2}{(\gamma+1)u_\xi(\xi,0)} \ .$$

Hence, in agreement with (2.26a),

$$t_s = \frac{2}{(\gamma+1)}\min_{\xi}\left(-\frac{1}{u_\xi(\xi,0)}\right) \ . \tag{2.27}$$

### 2.4.6 Example

Assume

$$u = u_0 \sin kx \qquad \text{at} \qquad t = 0\ , \quad u_0 > 0\,,$$

so that

$$u_\xi(\xi, 0) = k u_0 \cos k\xi\,.$$

$\max(-u_\xi)$ occurs at $k\xi = \pi, 3\pi, \ldots$, so that it follows

$$t_s = \frac{2}{(\gamma+1)ku_0}\ ,$$
$$x_s = \frac{(2n+1)\pi}{k} + c_0 t_s\ .$$

*Remarks.*

(i) Note that, using $\omega = kc$,

$$\frac{t_s}{t_{period}} = \frac{2}{(\gamma+1)ku_0}\frac{c_0 k}{2\pi} = \frac{1}{\pi(\gamma+1)}\frac{c_0}{u_0} \propto \frac{1}{M}\ ,$$

where, as defined in (1.18c), the *Mach number*, $M$, is

$$M = \frac{u_0}{c_0}\ .$$

(ii) For an incredibly large sound of 160 dB,

$$\frac{u_0}{c_0} \approx 10^{-3}\ ,$$

which implies that a wave will propagate for 1000 periods before a shock forms.

(iii) For normal speech of about 60 dB,

$$\frac{u_0}{c_0} \approx 10^{-13}\ ,$$

which implies that a shock never forms before viscous effects dissipate the wave.

(iv) For $t \ll t_s$ and $x \ll x_s$ we can use **linear theory** for *small amplitude* waves. We conclude that for many practical applications of acoustics, linear theory is sufficient.

## 2.5 Piston Problems

- At $t = 0$ assume a perfect gas at rest in $x \geqslant 0$; so $u = 0$, $c = c_0$ in $x \geqslant 0$.
- Piston moves with path $x = X_p(t)$ where $X_p(0) = 0$.

What happens in the gas? Partly a BVP since need to take into account information from the piston.

### 2.5.1 Piston accelerates away from the gas

Assume all $C_-$ characteristics originate from $t = 0$, $x > 0$, so that

$$R_- = 0 \quad \text{everywhere.}$$

Hence the flow is a *simple wave*, and the $C_+$ are straight. Specifically, from the $C_+$ characteristic equation (2.20a),

$$C_+ : \qquad \frac{dx}{dt} = c_0 + \tfrac{1}{2}(\gamma + 1)u, \quad u = \text{const.}$$

*Region I.* If a $C_+$ meets $t = 0$ at $x > 0$, then

07/04 $$R_+ = 0\,, \quad u = 0\,, \quad c = c_0\,.$$

*Region II.* Assume that the piston is moving slow enough that the $C_+$ intersect the piston (see below), and that the $C_+$ meet the piston path at

$$t = \tau\,, \quad x = X_p(\tau) \quad \text{where} \quad \underbrace{u = \dot{X}_p(\tau)}_{\text{boundary condition}} \quad . \tag{2.28a}$$

The flow is then a simple wave ($u$=$Q$), with the $C_+$ satisfying the equation (cf. (2.20d))

$$x - X_p(\tau) = \Big(c_0 + \tfrac{1}{2}(\gamma + 1)\dot{X}_p(\tau)\Big)\ (t - \tau)\,, \tag{2.28b}$$

where $\tau$ plays the role of $\xi$ here. Given $(x, t)$ in region II, this is an implicit equation for $\tau(x, t)$; solve for $\tau$, then from (2.28b)

$$u(x, t) = \dot{X}_p(\tau)\,, \tag{2.28c}$$

and from (2.19b) and (2.10a)

$$\rho = \rho_0 \left(1 + \frac{\gamma - 1}{2}\frac{\dot{X}_p(\tau)}{c_0}\right)^{\frac{2}{\gamma-1}} . \tag{2.28d}$$

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*Characteristics 'diverge'.* While the value of $\dot{X}_p$ decreases with $\tau$, the slope $\frac{dx}{dt}$ decreases with $\tau$ (and the diagram slope $\frac{dt}{dx}$ increases); in this case the characteristics 'diverge'. A characteristic with $\dot{X}_p = -\frac{2c_0}{\gamma+1}$ is 'vertical'.

*Escape speed.* Note that

$$\rho = 0 \qquad \text{if} \qquad \dot{X}_p = -\frac{2c_0}{\gamma - 1}\,.$$

$\frac{2c_0}{\gamma-1}$ is the maximum speed of the piston for contact with gas to be maintained. For speeds greater than this, the $C_+$ characteristics no longer reach the piston, and there is a vacuum between the limiting $C_-$ characteristic and the piston. Hence, $\frac{2c_0}{\gamma-1}$ is the speed of expansion of a gas into a vacuum: the 'escape' speed ($M \sim 5$; $M \gtrsim 5$ sometimes referred to as *hypersonic*).

*Expansion fan.* Consider rapid acceleration up to a constant speed $-V$. The slopes of $C_+$ are uniform in each of regions I and IIB, and range over intermediate values in region IIA.

Now take the limit of an impulsive start. Region IIA reduces to a wedge, called an *expansion fan* with many $C_+$ coming out of the origin. The slopes of $C_+$ in region IIA are constant and must be $t/x$ in order to go through the origin.

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### 2.5.2 Piston accelerates towards the gas (at some point)

A shock will form if the piston moves towards the gas whatever the form of $\dot{X}(t) > 0$, since the $C_+$ characteristics have slope $c_0$ in region I, and slope $c_0 + \frac{1}{2}(\gamma+1)\dot{X}(\tau) > c_0$ in region II, and hence the characteristics must eventually cross. A similar argument for successive $C_+$ characteristics shows that a shock also forms if $\ddot{X}(t) > 0$ for any $t$.

As for a receding piston, the equations of the $C_+$ characteristics are, see (2.28b),

$$x(t;\,\tau) = X_p(\tau) + \Big(c_0 + \tfrac{1}{2}(\gamma+1)\dot{X}_p(\tau)\Big)\,(t-\tau)\,.$$

The characteristics touch when

$$\left.\frac{\partial x}{\partial \tau}\right|_t = 0\,,$$

i.e. when

$$\tfrac{1}{2}(\gamma+1)\ddot{X}_p t = \tfrac{1}{2}(\gamma+1)\ddot{X}_p \tau + c_0 + \tfrac{1}{2}(\gamma-1)\dot{X}_p\,.$$

Hence a shock forms at

$$t = t_s \equiv \min_{\tau>0}\left(\tau + \frac{2c_0 + (\gamma-1)\dot{X}_p}{(\gamma+1)\ddot{X}_p}\right)\,,$$

on the characteristic corresponding to the minimising $\tau = \tau_m$, i.e. at $x = x_s \equiv x(t_s;\,\tau_m)$, subject to the requirement that $x_s \geqslant X_p(t_s)$ and no vacuum has formed.

*Example.* Uniform positive acceleration: $X_p(\tau) = \frac{1}{2}f\tau^2$, with $f > 0$. Two adjacent characteristics touch when

$$t = \widehat{t}(\tau) \equiv \frac{\frac{1}{2}(\gamma+1)f\tau + c_0 + \frac{1}{2}(\gamma-1)f\tau}{\frac{1}{2}(\gamma+1)f} = \frac{2(\gamma f\tau + c_0)}{(\gamma+1)f} \quad \text{for} \quad \tau \geqslant 0\,.$$

The minimum overlap time, i.e. the shock formation time, is thus:

$$t_s = \min_{\tau>0} \widehat{t}(\tau) = \frac{2c_0}{(\gamma+1)f}\,.$$

This corresponds to the characteristic $\tau = 0$ through $x = t = 0$, and hence (since $\gamma + 1 > 0$)

$$x_s = c_0 t_s = \frac{2c_0^2}{(\gamma+1)f} > X_p(t_s)\,.$$

## 2.6 Shock Waves

A triple–valued region is unphysical. In practice it is prevented by mechanisms/physics so far neglected, e.g. dissipation. When the wave form is steep, viscous and heat conduction terms which were small become large.

For instance, the simple wave equation (2.20c) can be modified to

$$u_t + (c_0 + \tfrac{1}{2}(\gamma+1)u)u_x = \nu u_{xx}\,, \tag{2.29}$$

where the extra term is proportional to the kinematic viscosity, $\nu$ ($10^{-5}\mathrm{m}^2\mathrm{s}^{-1}$ for air). An analytic solution to (2.29) exists (e.g. see Part III). The triple–valued region is avoided by a region of rapid change, the thickness of which is proportional to $\nu$.

Outside the so–called 'shock' region, we can use the characteristic solution of the simple wave equation.

For 'real' problems we cannot find exact solutions to equations like (2.29). However, if $\nu$ is very small, shocks are very thin, and across them $p$, $\rho$, $u$ and $T$ jump almost discontinuously in value. Hence, in such circumstances, might we treat the **shock** as a **discontinuity**. Further, without knowing the internal details of the shock:

- Might we be able to impose 'jump conditions' across the shock?
- Might we be able to identify where the shock is, and how fast it moves?

For example, we know that mass is conserved, i.e.

$$\int_{-\infty}^{\infty} \rho\, dx = \text{const.}\,.$$

Hence, in a plot of $\rho$ against $x$, a shock should be inserted to ensure that it cuts off lobes of equal area either side of the shock.

### 2.6.1 Rankine-Hugoniot relations

Consider a **steady** shock separating two uniform regions (where by uniform we mean uniform on the scale of the shock width). Take coordinates in shock:

*Mass conservation.* Equate the mass entering and leaving the control volume:

$$(\delta t \delta y \delta z)\, \rho_1 u_1 = \rho_2 u_2\, (\delta t \delta y \delta z)\,,$$

i.e.

$$\rho_1 u_1 = \rho_2 u_2 \tag{2.30}$$

*Momentum conservation.* Similarly for momentum and the rate of forcing:

$$p_1 + (\rho_1 u_1) u_1 = p_2 + (\rho_2 u_2) u_2\,. \tag{2.31}$$

*Energy conservation.* For the energy budget we need to include $e$, the internal energy per unit mass, cf. (1.5e); thence

$$p_1 u_1 + \left(\tfrac{1}{2}\rho_1 u_1^2 + \rho_1 e_1\right) u_1 = p_2 u_2 + \left(\tfrac{1}{2}\rho_2 u_2^2 + \rho_2 e_2\right) u_2\,. \tag{2.32a}$$

It follows from dividing through by $\rho_1 u_1$ and, using (2.30), that

$$\tfrac{1}{2}u_1^2 + e_1 + \frac{p_1}{\rho_1} = \tfrac{1}{2}u_2^2 + e_2 + \frac{p_2}{\rho_2}\,, \tag{2.32b}$$

which is a statement of Bernoulli's principle. From (1.5g), $e + p/\rho$ is the enthalpy of the gas.

*Rankine-Hugoniot relations.* (2.30), (2.31) and (2.32b) are known as the **Rankine-Hugoniot relations**. There are three equations relating the six variables $\rho_1$, $p_1$, $u_1$, $\rho_2$, $p_2$ and $u_2$.

*Oblique shocks.* In the case of oblique shocks, the velocity parallel to the shock is unchanged across the shock.

### 2.6.2 The Hugoniot adiabatic

From (2.30) and (2.31)

$$u_1^2 = \frac{p_2 - p_1}{\rho_1} + \frac{(\rho_2 u_2)^2}{\rho_1 \rho_2} = \frac{p_2 - p_1}{\rho_1} + \frac{\rho_1 u_1^2}{\rho_2}\,,$$

so

$$u_1^2 = \frac{\rho_2 (p_2 - p_1)}{\rho_1 (\rho_2 - \rho_1)}\,. \tag{2.33}$$

Hence

$$\begin{array}{lll} \text{either} & p_2 > p_1 \;\;\&\;\; \rho_2 > \rho_1\,, \\ \text{or} & p_2 < p_1 \;\;\&\;\; \rho_2 < \rho_1. \end{array}$$

Substitute for $u_1^2$ and $u_2^2$ into (2.32b) to obtain

$$e_2 - e_1 = \tfrac{1}{2}(p_1 + p_2)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\,. \tag{2.34}$$

This is the *Hugoniot adiabatic*: it relates thermodynamic quantities across the shock.

08/04 09/17 09/18 *Remark.* Physical intuition might suggest that it is 'obvious' that $\rho_1 < \rho_2$ if $p_1 < p_2$; however, this intuition is based on constant entropy (or constant temperature) thinking.

### 2.6.3 The entropy jump

Let
$$V_1 = \frac{1}{\rho_1} \quad \text{and} \quad V_2 = \frac{1}{\rho_2}\,.$$
Then for a given mass of gas, plot $p$ versus $V$ at constant entropy, $S$, assuming that
$$\left.\frac{d^2p}{dV^2}\right|_S > 0\,.$$
Suppose that $\rho_2 > \rho_1$; hence $V_2 < V_1$, and from (2.33) $p_2 > p_1$.

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- **If** entropy were to be conserved, say $S = S_0$, then $p_2$ and $p_1$ would be given by A and B in the diagram (where we do not [yet] assume that $\rho_2$, etc. are necessarily 'downstream'). Since we are assuming that $dS = 0$, it follows from (1.5c), or (1.A.5a), that
$$e(V_2, S_0) - e(V_1, S_0) = -\int_{V_1}^{V_2} p dV = \text{area under } \textit{curve} \text{ AB}.$$
- **However**, from the Hugoniot adiabatic, (2.34), we know that we require
$$\begin{aligned} e(V_2, p_2) - e(V_1, p_1) &= \tfrac{1}{2}(p_1 + p_2)(V_1 - V_2) \\ &= \text{area under } \textit{chord} \text{ AB} \\ &> e(V_2, S_0) - e(V_1, S_0)\,. \end{aligned} \tag{2.35}$$
Hence if $\left.\frac{d^2p}{dV^2}\right|_s > 0$ (or equivalently $\left.\frac{d^2V}{dp^2}\right|_s > 0$), a constant entropy shock is impossible. This is not surprising given that inside the shock there is viscous dissipation, heat generation, thermal conduction, etc., and hence the flow is not adiabatic.
- If we keep $V_1$ and $V_2$ fixed, then the value of the RHS of (2.35), i.e. the area under the chord, is unchanged if we 'rotate' AB about the mid-point to, say, CD. Moreover, if $\left.\frac{\partial e}{\partial p}\right|_V > 0$, then as we rotate clockwise, $e(V_2, S_2) - e(V_1, S_1)$ increases, until (2.35) is satisfied.
- The *Second Law of Thermodynamics* implies, e.g. see (1.A.5a),
$$de = TdS - pdV\,.$$
Hence
$$\left.\frac{\partial e}{\partial p}\right|_V = T \left.\frac{\partial S}{\partial p}\right|_V\,,$$
and consequently, since $T > 0$,

  the entropy at C is greater than at A,
  the entropy at D is less than at B.

  So if $\rho_2 > \rho_1$, then
$$S_2 > S_1\,. \tag{2.36}$$
But the entropy of a fixed mass of gas cannot decrease. Hence

$$S_2 \text{ is downstream of } S_1.$$

Indeed, the full equations for the shock's internal structure have no solution unless this is the case. We conclude that the transition is from a low density to a high density; we say that we have a *compressive shock*.

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### 2.6.4 Upstream/downstream flow is supersonic/subsonic

Plot $p$ versus $V$ for $S = S_1$ and $S = S_2$.

Then from (2.33), with $V = 1/\rho$:

$$\begin{aligned} u_1^2 = V_1^2 \frac{(p_2 - p_1)}{V_1 - V_2} &= -V_1^2 \times \text{slope of CD} \\ &> -V_1^2 \times \text{slope of tangent at D} \\ &= -V_1^2 \left. \frac{\partial p}{\partial V} \right|_S (V_1) = c_1^2 \,, \end{aligned}$$

since $c^2 = \left.\frac{\partial p}{\partial \rho}\right|_S = -V^2 \left.\frac{\partial p}{\partial V}\right|_S$. So the flow is *supersonic upstream* since

$$M_1 = \frac{u_1}{c_1} > 1 \,. \tag{2.37a}$$

It is also possible to show (but it is messy, e.g. see Landau and Lifschitz), that the flow is *subsonic downstream*, i.e.

$$M_2 = \frac{u_2}{c_2} < 1 \,. \tag{2.37b}$$

*Exercise:* find an easy proof.

### 2.6.5 Upstream/downstream flow is supersonic/subsonic – perfect gas *(unlectured)*

For a perfect gas (see (1.5f), (1.5e) and (1.5b))

$$c^2 = \frac{\gamma p}{\rho} \,, \qquad e = c_V T = \frac{p}{(\gamma - 1)\rho} \,. \tag{2.38}$$

Define

$$\textit{Shock Strength } = \beta = \frac{p_2 - p_1}{p_1} \,, \tag{2.39a}$$

so that $p_2 = (1 + \beta) p_1$. Then from (2.34) and (2.38)

$$\frac{\rho_2}{\rho_1} = \frac{2\gamma + (\gamma + 1)\beta}{2\gamma + (\gamma - 1)\beta} \,. \tag{2.39b}$$

Then from (2.33), (2.38), (2.39a) and (2.39b), and assuming that $\beta > 0$ and $\gamma > 0$ (in practice $\gamma > 1$),

$$\left(\frac{u_1}{c_1}\right)^2 = 1 + \left(\frac{\gamma+1}{2\gamma}\right)\beta > 1 \,, \tag{2.39c}$$

$$\left(\frac{u_2}{c_2}\right)^2 = \frac{1}{1+\beta}\left(1 + \left(\frac{\gamma-1}{2\gamma}\right)\beta\right) < 1 \,. \tag{2.39d}$$

This confirms that, at least for a perfect gas,

*shocks are transitions from supersonic to subsonic flow.*

Further, from (1.5h)

$$S = c_V \log\left(\frac{p}{\rho^\gamma}\right) + \text{ const} \,.$$

It follows from (2.39a) and (2.39b) that, if $\beta > 0$ and $\gamma > 1$,

$$\begin{aligned} \frac{S_2 - S_1}{c_V} &= \log(1+\beta) - \gamma \log\left(\frac{2\gamma + (\gamma+1)\beta}{2\gamma + (\gamma-1)\beta}\right) \\ &> 0\,. \end{aligned} \tag{2.39e}$$

Further, if the shock is *weak*, i.e. $\beta \ll 1$, then

$$\frac{S_2 - S_1}{c_V} = \frac{(\gamma^2 - 1)}{12\gamma^2}\beta^3 + \dots \,. \tag{2.39f}$$

Hence for weak shocks, the entropy jump is much smaller than the jumps in pressure, density, velocity, etc.

*Remark.* For a general gas, the coefficient in (2.39f) is proportional to $\left.\frac{\partial^2 V}{\partial p^2}\right|_S$, consistent with our earlier assumption that $\left.\frac{\partial^2 V}{\partial p^2}\right|_S > 0$ .

### 2.6.6 Moving shocks

Consider a shock moving at the velocity $V$, with velocities $v_1$ and $v_2$ on either side of the shock. Switch to the frame in which the shock is at rest, corresponding to velocities

$$u_1 = v_1 - V, \quad \text{and} \quad u_2 = v_2 - V \quad \text{respectively.}$$

Then from (2.30), (2.31) and (2.34)

$$\rho_1(v_1 - V) = \rho_2(v_2 - V)\,, \tag{2.40a}$$

$$p_1 + \rho_1(v_1 - V)^2 = p_2 + \rho_2(v_2 - V)^2\,, \tag{2.40b}$$

$$e_2 - e_1 = \tfrac{1}{2}(p_1 + p_2)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right). \tag{2.40c}$$

Note that if $V > v_2$, then $p_2, \rho_2, u_2$ become 'upstream'; it is then necessary to be careful with signs of square roots in (2.39c), (2.39d) etc.

### 2.6.7 Shock propagating into a perfect gas at rest.

For example, as generated by a steadily moving piston or an explosion.

There are three independent equations, and six variables: $\rho_0$, $p_0$, $\rho_2$, $p_2$, $v_2$ and $V$. However, we know $\rho_0$ and $p_0$, and one of $\rho_2$, $p_2$ and $v_2$. For a piston problem it might be natural to specify $v_2$, but we will specify $p_2$. It is then convenient to define

$$\textit{Shock Strength} \ = \beta = \frac{p_2 - p_0}{p_0}\,, \tag{2.41}$$

so that $p_2 = (1+\beta)p_0$.

*Remark.* A word of warning: since the algebra can be 'detailed', consider what a question asks, before embarking on messy algebra to eliminate variables. In this case we first aim for $\rho_2(\rho_0, p_0, p_2)$.

For a perfect gas (see (1.5f), (1.5e) and (1.5b))

$$c^2 = \frac{\gamma p}{\rho}\,, \qquad e = c_V T = \frac{p}{(\gamma-1)\rho}\,. \tag{2.42}$$

Then from (2.40c), (2.41) and (2.42)

$$\frac{\rho_2}{\rho_0} = \frac{(\gamma-1)p_0 + (\gamma+1)p_2}{(\gamma+1)p_0 + (\gamma-1)p_2} = \frac{2\gamma + (\gamma+1)\beta}{2\gamma + (\gamma-1)\beta}\,. \tag{2.43a}$$

As before, from conservation of mass and momentum,

$$\rho_0(-V) = \rho_2(v_2 - V)\,, \tag{2.43b}$$

$$p_0 + \rho_0 V^2 = p_2 + \rho_2(v_2 - V)^2\,. \tag{2.43c}$$

Thence from (2.41) and (2.43a),

$$V^2 = \frac{\rho_2(p_2 - p_0)}{\rho_0(\rho_2 - \rho_0)} = \left(1 + \frac{\gamma+1}{2\gamma}\,\beta\right) c_0^2\,. \tag{2.43d}$$

Moreover, from (2.43a), (2.43b), (2.43d) and some algebra,

$$v_2 = \frac{\beta c_0^2}{\gamma V}\,. \tag{2.43e}$$

09/04 Hence, if $V > v_2 > 0$, then $\beta > 0$, and from (2.43d) the shock propagates supersonically into the gas at rest.

## 2.7 Nonlinear Shallow Water Waves in 1D

Shallow water means that the wavelength $\gg$ water depth. This means that we can assume that the flow is almost horizontal and independent of depth (cf. hydraulic flows in IB).

### 2.7.1 Governing Equations

Consider a thin control volume of width $\delta x$. Then from conservation of mass

$$\frac{\partial}{\partial t}(\rho_{\mathrm{w}} h \delta x) = (\rho_{\mathrm{w}} h u)(x) - (\rho_{\mathrm{w}} h u)(x+\delta x)\,,$$

thence by Taylor series, and assuming that the density of water, $\rho_{\mathrm{w}}$, is a constant,

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0\,. \tag{2.44a}$$

Similarly, from conservation of momentum

$$\frac{\partial}{\partial t}(\rho_{\mathrm{w}} h u \delta x) = (\rho_{\mathrm{w}} h u^2)(x) + P(x) - (\rho_{\mathrm{w}} h u^2)(x+\delta x) - P(x+\delta x)\,,$$

where $P$ is the pressure force integrated over the depth; thence by Taylor series

$$\frac{\partial (hu)}{\partial t} = -\frac{\partial (hu^2)}{\partial x} - \frac{1}{\rho_{\mathrm{w}}}\frac{\partial P}{\partial x}\,.$$

Since we are assuming the the flow is primarily in the $x$-direction, it is consistent to assume that the vertical accelerations are negligible, and that the pressure is hydro-static to leading order, i.e.

$$p = \rho_{\mathrm{w}} g (h - z)\,,$$

where $p = 0$ has been assumed at the free surface. Hence

$$P = \int_0^h p\,dz = \tfrac{1}{2}\rho_{\mathrm{w}} g h^2\,, \tag{2.44b}$$

and so the momentum equation becomes

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2)}{\partial x} = -gh\frac{\partial h}{\partial x}\,. \tag{2.44c}$$

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The *shallow water equations* are thus, after simplification of (2.44c) using (2.44a),

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0\,, \tag{2.45a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0\,. \tag{2.45b}$$

These are the same as the 1D gas equations (2.2a), (2.2b) and (2.2c) if we identify the gas density, $\rho$, with $h$, and the wavespeed $c$ with $\sqrt{gh}$, i.e.

$$\rho = h \quad \text{and} \quad c = \sqrt{g\rho} = \sqrt{gh}\,. \tag{2.45c}$$

For a perfect gas, $c = c_0\,(\rho/\rho_0)^{\frac{\gamma-1}{2}}$; hence the shallow water equations are equivalent to 1D gas dynamics with $\gamma = 2$. As above, these equations can therefore be solved using $C_\pm$ characteristics, with the solutions
10/17 portraying wave steepening, shock formation, etc.
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### 2.7.2 * Hydraulic jumps *

The 'shocks' that form from the steepening of shallow water waves are termed *hydraulic jumps* or *bores*.

The extra physics that controls a hydraulic jump is different from that in a gas shock. In a gas shock, $p$, $\rho$ and $u$ jump so that mass. momentum and energy are conserved across the shock; however, across a
09/16 hydraulic jump only $h$ and $u$ can jump, and energy is not conserved.

Work in the frame of the jump.

*Mass conservation.* Divide through by $\rho_\mathrm{w}$ to obtain

$$h_1 u_1 = h_2 u_2 \,. \tag{2.46a}$$

*Momentum conservation.*

$$P_1 + \rho_\mathrm{w} h_1 u_1^2 = P_2 + \rho_\mathrm{w} h_2 u_2^2 \,,$$

where from (2.44b) the force due to the integrated hydrostatic pressure is given by

$$P_1 = \tfrac{1}{2}\rho_\mathrm{w} g h_1^2 \,, \quad P_2 = \tfrac{1}{2}\rho_\mathrm{w} g h_2^2 \,.$$

Hence

$$\tfrac{1}{2} g h_1^2 + h_1 u_1^2 = \tfrac{1}{2} g h_2^2 + h_2 u_2^2 \,. \tag{2.46b}$$

Analogously to a gas shock, these can be manipulated to obtain

$$F_1^2 \equiv \frac{u_1^2}{c_1^2} = \frac{u_1^2}{g h_1} = \frac{(h_1+h_2)h_2}{2h_1^2} = (1+\tfrac{1}{2}\beta)(1+\beta) \,, \tag{2.47a}$$

$$F_2^2 \equiv \frac{u_2^2}{c_2^2} = \frac{u_2^2}{g h_2} = \frac{(h_1+h_2)h_1}{2h_2^2} = \frac{(1+\tfrac{1}{2}\beta)}{(1+\beta)^2} \,, \tag{2.47b}$$

where $F_1$ and $F_2$ are *Froude* numbers (the ratio of fluid speed to wavespeed, cf. Mach number), and $h_2 = (1+\beta)h_1$, where $\beta$ is the shock strength.

*Energy loss.* The energy flux is

$$\begin{aligned} I &= \underbrace{\left(\tfrac{1}{2}\rho_\mathrm{w} g h^2\right) u}_{\text{Work done by pressure}} + \underbrace{\left(\tfrac{1}{2}\rho_\mathrm{w} u^2 + \tfrac{1}{2}\rho_\mathrm{w} g h\right) uh}_{\text{Volume flux of (KE+GPE)}} \\ &= \rho_\mathrm{w} uh \left(\tfrac{1}{2}u^2 + gh\right) \,. \end{aligned}$$

While $\rho_\mathrm{w} uh$ is continuous across the jump, $\left(\tfrac{1}{2}u^2 + gh\right)$ is not, because 'Bernoulli' does not apply. The change in energy flux is given by

$$I_1 - I_2 = \rho_\mathrm{w} u_1 h_1 \left(\tfrac{1}{2}(u_1^2 - u_2^2) + g(h_1 - h_2)\right) \tag{2.48a}$$

$$= \frac{\rho_\mathrm{w} g u_1 h_1^2 \beta^3}{4(1+\beta)} \,. \tag{2.48b}$$

The energy loss has to be positive, from which we conclude that if the height upstream is $h_1$, then

$$\beta > 0 \,, \quad h_2 > h_1 \,, \quad F_1 > 1 \quad \text{and} \quad F_2 < 1 \,.$$

Hence the flow upstream is 'supercritical', i.e. the fluid moves faster than the long-wave speed (cf. supersonic), and the flow downstream is 'subcritical', i.e. the fluid moves slower than the long-wave speed (cf. subsonic).

There are two possibilities for energy loss.

**$\beta \gtrsim 0.5$:** this generally corresponds to a turbulent bore, where energy is lost to small scale motion and viscous dissipation.

**$\beta \lesssim 0.5$:** this generally corresponds to an undular bore where [semi-long] waves behind the jump carry energy away.

**Counting characteristics* (unlectured).* In the frame in which the jump is at rest, both the $C_+$ and $C_-$ characteristics point towards the shock in the supercritical region, while only the $C_+$ characteristic points away from the shock in the subsonic region. There is thus too much information at the jump (two routes in and one route out), and something has to 'give' (in this case energy, while in the case of shocks it was entropy).

The number of characteristics approaching the shock corresponds to the total number of parameters that can be specified on both sides of the shock.

*Example.* Consider a jump propagating with speed $V$ into quiescent fluid of depth $h_0$. On the assumption that the 'upstream' height $h_2 = (1+\beta)h_0$ is known, find $V$. Note: there are three characteristics approaching the jump, and we specify $u_0 = 0$, $h_0$ and $h_2$.

In the frame in which the jump is at rest, 'upstream' is 'downstream'. It follows from conservation of mass and momentum, cf. (2.46a) and (2.46b),

$$h_0(-V) = h_2(u_2 - V)\,, \tag{2.49a}$$

$$\tfrac{1}{2}gh_0^2 + h_0V^2 = \tfrac{1}{2}gh_2^2 + h_2(u_2 - V)^2\,, \tag{2.49b}$$

and thence, cf. (2.47a),

$$\begin{aligned}\frac{V^2}{gh_0} &= \frac{(h_0+h_2)h_2}{2h_0^2}\\ &= (1+\tfrac{1}{2}\beta)(1+\beta) > 1\,.\end{aligned} \tag{2.49c}$$

Hence the jump propagates at a supercritical velocity.

# Appendix

## 2.A * Shock Formation in Trombones *

Model the trombone as a semi-infinite straight tube $x > 0$ with fluid at rest for $t < 0$. For $t > 0$ consider pressure perturbations at $x = 0$. In the quiescent fluid (from which the $C_-$ characteristics emanate), $u = 0$ and $c = c_0$ , so $R_- = 0$. Let $u(t)$, $c(t)$ and $p(t)$ denote the velocity, sound speed and pressure at $x = 0$ when $t \geqslant 0$, and assume that $p(t)$ is known.

Proceeding as in § 2.5.1, the flow is again a *simple wave*. Hence the $C_+$ are straight with characteristic equation, see (2.20a),

$$C_+ : \qquad \frac{dx}{dt} = c_0 + \tfrac{1}{2}(\gamma+1)u, \quad u = \text{const.} \tag{2.A.1a}$$

For characteristics emanating from $x = 0$ for $t \geqslant 0$, the $C_+$ satisfy the equation, cf. (2.28b),

$$x = \left(c_0 + \tfrac{1}{2}(\gamma+1)u(\tau)\right)(t-\tau), \tag{2.A.1b}$$

where $\tau$ is used to label the characteristics. From (2.10a), (2.10b) and (2.10c), $u(\tau)$ is given in terms of the pressure by

$$u(\tau) = \left(\frac{2c_0}{\gamma-1}\right)\left(\left(\frac{p(\tau)}{p_0}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right). \tag{2.A.1c}$$

Hence, given $(x,t)$, (2.A.1b) and (2.A.1c) are an implicit system of equations for $\tau(x,t)$ in terms of the known pressure at $x = 0$.

Shock waves form when the characteristics intersect, i.e. when $\left.\frac{dx}{d\tau}\right|_t = 0$. Hence the time of shock formation is given by

$$t_s = \min_{\tau \geqslant 0}\left(\tau + \frac{c_0 + \frac{1}{2}(\gamma+1)u(\tau)}{\frac{1}{2}(\gamma+1)\dot{u}(\tau)}\right). \tag{2.A.2a}$$

If the RHS of (2.A.2a) is minimised for $\tau = \tau_m$, then the position of shock formation is given by

$$x_s = \frac{\left(c_0 + \frac{1}{2}(\gamma+1)u(\tau_m)\right)^2}{\frac{1}{2}(\gamma+1)\dot{u}(\tau_m)}. \tag{2.A.2b}$$

For simplicity, assume that the pressure perturbations are small and that $|u| \ll c_0$, then the above expression can be linearised to obtain

$$t_s \sim \min_{\tau \geqslant 0}\left(\tau + \frac{2\gamma p_0}{(\gamma+1)\dot{p}(\tau)}\right) \quad \text{and} \quad x_s \sim \frac{2\gamma c_0 p_0}{(\gamma+1)\dot{p}(\tau_m)}. \tag{2.A.2c}$$

Thus for a shock to form downstream of the mouthpiece, we require $\dot{p}(\tau_m) > 0$. Moreover, the shock will first form on the characteristic, or at least close to the characteristic, where $\dot{p}(\tau)$ is largest.

For instance, if for $\tau \geqslant 0$

$$p(\tau) = A(1 - e^{-\alpha\tau}), \quad \text{so} \quad \dot{p}(\tau) = A\alpha e^{-\alpha\tau}, \tag{2.A.3a}$$

then

$$t_s \sim \min_{\tau \geqslant 0}\left(\tau + \frac{2\gamma p_0\, e^{\alpha\tau}}{(\gamma+1)A\alpha}\right) = \frac{2\gamma p_0}{(\gamma+1)A\alpha} \qquad (\tau_m = 0), \tag{2.A.3b}$$

$$x_s \sim \frac{2\gamma c_0 p_0}{(\gamma+1)A\alpha}. \tag{2.A.3c}$$

Typical values of $\max(\dot{p})$ at the mouthpiece of a trombone range from around $5,000$ kPa s$^{-1}$ when played at piano (soft) level to around $20,000$ kPa s$^{-1}$ when played at fortissimo (loud) for low frequencies. Using values of $\gamma \approx 1.4$, $p_0 = 101$ kPa and $c_0 = 340$ m s$^{-1}$, this gives a shock formation distance of around

6.8 m (piano) to around 1.7 m (fortissimo). At higher frequencies this distance is approximately in the order of tens of metres (piano) to around 1 m (fortissimo). With the typical length of a trombone's tubing around 2-3 metres, one would expect shocks to form at higher dynamic playing levels. Indeed, this is what is observed by Hirschberg *et al.* (1996)[5] when measuring the horn exit pressure.

For a video see

http://www.bbc.co.uk/news/science-environment-13574197
and/or https://www.youtube.com/watch?v=TOhxr643YuA.



[5] Hirschberg, A., Gilbert, J., Msallam, R. & Wijnands, A.P.J, *Shock Waves in Trombones*. Journal of the Acoustical Society of America, Acoustical Society of America, 1995, 99(3), 1754-1758.

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# 3 Linear Elastic Waves

## 3.1 Governing Equations of Linear Elasticity

### 3.1.1 Deformation of a solid

As in fluid dynamics, we make the *continuum approximation* (averaging over volumes large enough to contain many molecules but much smaller than the scales in interest) in order to define fields of density $\rho(\mathbf{x}, t)$, etc.

When a body is deformed, a particle moves from its original 'reference' position $\boldsymbol{\xi}$ to a new position $\mathbf{x}(\boldsymbol{\xi}, t)$; since the position must be invertible, $\boldsymbol{\xi} \equiv \boldsymbol{\xi}(\mathbf{x}, t)$. A key feature of an elastic solid is that it remembers the original reference configuration $\boldsymbol{\xi}$. It is conventional in solid mechanics to denote the

$$\text{particle displacement by} \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{x}(\boldsymbol{\xi}, t) - \boldsymbol{\xi}\,, \tag{3.1a}$$

$$\text{particle velocity by} \quad \mathbf{v}(\mathbf{x}, t) = \frac{d\mathbf{u}}{dt}\,, \tag{3.1b}$$

$$\text{particle acceleration by} \quad \mathbf{a}(\mathbf{x}, t) = \frac{d\mathbf{v}}{dt}\,, \tag{3.1c}$$

where $\dfrac{d}{dt} = \left.\dfrac{\partial}{\partial t}\right|_{\boldsymbol{\xi}}$ is the rate of change moving with the particle, written more usually in fluid dynamics as

$$\frac{D}{Dt} \equiv \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \mathbf{v}\cdot\boldsymbol{\nabla}\,.$$

*Warning:* in solid mechanics $\mathbf{u}$ denotes *displacement* **not** *velocity*.

It is convenient to write all fields as functions of the current (Eulerian) position $\mathbf{x}$, rather than the original (Lagrangian) position $\boldsymbol{\xi}$.

### 3.1.2 The stress tensor

As in fluid dynamics, the forces acting the material can be divided into

(a) *surface forces*, or tractions/stresses, acting on and proportional to the surface area, like pressure;

(b) *body forces* acting on and proportional to the volume, like gravity, where we denote the body force by $\mathbf{F}$, or $\rho\mathbf{f}$.

Consider a small area element $\mathbf{n}\,dS$ at position $\mathbf{x}$ within a solid body. The force exerted **by the outside on the solid inside** of the element (with 'outside' defined by the **outward normal n**) is assumed to be a surface force $\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n})\,dS$; by Newton three, $\boldsymbol{\tau}(\mathbf{x}, t; -\mathbf{n}) = -\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n})$.

The *traction* or *stress*, $\boldsymbol{\tau}$, acting on the area element $\mathbf{n}\,dS$ depends on the orientation $\mathbf{n}$. (The units of traction are force/area, and forces are obtained by integrating tractions over an area.) In an inviscid fluid $\boldsymbol{\tau} = -p\mathbf{n}$. In general, $\boldsymbol{\tau}(\mathbf{x}, t; \mathbf{n})$ is linearly related to $\mathbf{n}$ by a second-rank tensor $\boldsymbol{\sigma}$. To see this we proceed as follows.

Consider a small material tetrahedron with three faces aligned parallel to the Cartesian coordinate planes and a fourth sloping face with area $\varepsilon^2$ and normal $\mathbf{n}$. By geometry, the areas of the other faces are $\varepsilon^2 n_i$ and the normals are $-\mathbf{e}_i$, where $\mathbf{e}_i$ is the $i^{\text{th}}$ basis vector.

The surface forces on the tetrahedron are $\propto \varepsilon^2$, whereas both the body forces and the inertia of the tetrahedron are $\propto \varepsilon^3$.It follows that the surface forces must balance each other as $\varepsilon \to 0$, i.e.

$$\varepsilon^2\big\{\boldsymbol{\tau}(\mathbf{x},t;\mathbf{n}) - \boldsymbol{\tau}(\mathbf{x},t;\mathbf{e}_1)n_1 - \boldsymbol{\tau}(\mathbf{x},t;\mathbf{e}_2)n_2 - \boldsymbol{\tau}(\mathbf{x},t;\mathbf{e}_3)n_3\big\} = O(\varepsilon^3)\,,$$

or equivalently in the limit $\varepsilon \to 0$,

$$\tau_i(\mathbf{x},t;\mathbf{n}) = \tau_i(\mathbf{x},t;\mathbf{e}_j)n_j\,. \tag{3.2a}$$

Hence the traction $\boldsymbol{\tau}$ exerted by the outside on the inside of a surface with outward normal $\mathbf{n}$ is given by

$$\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathbf{n}\,, \tag{3.2b}$$

where $\boldsymbol{\sigma}$ is the second-rank tensor, the (Cauchy) *stress tensor*, given by

$$\sigma_{ij}(\mathbf{x},t) = \tau_i(\mathbf{x},t;\mathbf{e}_j)\,. \tag{3.2c}$$

*Warning:* 'stress' is also often used to refer to the vector $\boldsymbol{\tau}$ as well as the tensor $\boldsymbol{\sigma}$.

### 3.1.3 Momentum, angular momentum and energy

*Momentum.* Consider an arbitrary *material* control volume $\mathcal{V}(t)$ with surface $\mathcal{S}(t)$ and normal $\mathbf{n}$. Integrate '$\mathbf{F} = m\mathbf{a}$' over all the particles in the control volume, to yield

$$\int_{\mathcal{V}} \rho(\mathbf{x},t)\mathbf{a}(\mathbf{x},t)\,dV = \int_{\mathcal{V}} \mathbf{F}(\mathbf{x},t)\,dV + \int_{\mathcal{S}} \boldsymbol{\sigma}(\mathbf{x},t)\cdot\mathbf{n}\,dS\,. \tag{3.3a}$$

Using the divergence theorem and the arbitrariness of $\mathcal{V}$, we obtain the *momentum equation*

$$\rho a_i = F_i + \frac{\partial \sigma_{ij}}{\partial x_j}\,. \tag{3.3b}$$

*Angular Momentum.* Provided that long-range forces exert no body couple on the material (a counter-example is provided by a solid containing magnetic particles in an external magnetic field), we can show that the stress tensor is symmetric from the angular momentum balance. Taking moments about $\mathbf{0}$,

$$\int_{\mathcal{V}} \rho\mathbf{x}\times\mathbf{a}\,dV = \int_{\mathcal{V}} \mathbf{x}\times\mathbf{F}\,dV + \int_{\mathcal{S}} \mathbf{x}\times\boldsymbol{\sigma}\cdot\mathbf{n}\,dS\,. \tag{3.4a}$$

and hence

$$\epsilon_{pqi}x_q(\rho a_i - F_i) = \frac{\partial}{\partial x_j}(\epsilon_{pqi}x_q\sigma_{ij}) = \epsilon_{pqi}x_q\frac{\partial \sigma_{ij}}{\partial x_j} + \epsilon_{pji}\sigma_{ij}\,. \tag{3.4b}$$

Subtracting $\epsilon_{pqi}x_q \times$ (3.3b) gives

$$\epsilon_{pji}\sigma_{ij} = 0\,.$$

Hence $\boldsymbol{\sigma}$ is symmetric:

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$$\sigma_{ij} \;= \sigma_{ji}. \tag{3.4c}$$

*Energy.* We obtain the energy equation by multiplying the momentum equation (3.3b) by the velocity $v_i$. As a preliminary note that from the definition of acceleration, conservation of mass, and the product rule,

$$v_i a_i = \frac{d}{dt}(\tfrac{1}{2}v^2)\,, \tag{3.5a}$$

$$\frac{d}{dt}(\rho\, dV) = 0 \qquad \text{for a material volume element}\,, \tag{3.5b}$$

$$v_i \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial (v_i \sigma_{ij})}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\,\sigma_{ij}\,. \tag{3.5c}$$

Hence, using the symmetry of the stress tension, (3.4c), we obtain by integrating over a *material* volume

$$\underbrace{\frac{d}{dt}\int_{\mathcal{V}(t)} \tfrac{1}{2}\rho v^2\, dV}_{\text{Change in KE}} = \underbrace{\int_{\mathcal{V}} \mathbf{v}\cdot\mathbf{F}\, dV}_{\substack{\text{Rate of work}\\ \text{by body forces}}} + \underbrace{\int_{\mathcal{S}} \mathbf{v}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\, dS}_{\substack{\text{Rate of work}\\ \text{by surface forces}}} - \underbrace{\int_{\mathcal{V}} \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\sigma_{ij}\, dV}_{\substack{\text{Rate of release}\\ \text{of internal energy}}}\,. \tag{3.5d}$$

We will return to the final term; in the meantime note that in the simple case $\sigma_{ij} = -p\delta_{ij}$ the final
11/18 term reduces to $+\int p\,\boldsymbol{\nabla}\cdot\mathbf{v}\, dV$, the result for the reversible adiabatic compression of a gas.
11/19

### 3.1.4 Fluid and solid continua

All of the arguments above apply to any continuum – fluid or solid, whether viscous, inviscid, elastic, plastic, viscoelastic or whatever. They are presented for a viscous fluid in Part II *Fluid Dynamics*. The main difference between these various sorts of continua lies in the *constitutive relationship* between the stress $\boldsymbol{\sigma}$ and the deformation or rate of deformation. (The notation also varies from one application to another e.g. $\mathbf{u}$ or $\mathbf{v}$ for velocity, $D/Dt$ or $d/dt$ for material derivative, $\mathbf{e}$ or $\dot{\mathbf{e}}$ for strain rate.)

10/16 In a linear elastic solid the *stress* is linearly related to the *strain*.

### 3.1.5 Strain

Consider the elastic stresses that result from the *change* in separation of two neighbouring material elements:

$$\begin{aligned} dx_i - d\xi_i &= u_i(\mathbf{x}+\mathbf{dx},t) - u_i(\mathbf{x},t) \\ &= \frac{\partial u_i}{\partial x_j}\, dx_j + \text{h.o.t.}\,. \end{aligned} \tag{3.6a}$$

Assume further that the deformation is small, i.e.

$$|\mathbf{dx} - \mathbf{d}\boldsymbol{\xi}| \ll |\mathbf{dx}|\,, \tag{3.6b}$$

so that

$$\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1\,, \tag{3.6c}$$

as is appropriate for hard solids like metal and rock (but *not* rubber), where a small deformation produces a large restoring force.

Write

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{3.7a}$$

$$= \qquad e_{ij} \qquad + \qquad \omega_{ij}\,, \tag{3.7b}$$

where $e_{ij}$ and $\omega_{ij}$ are the symmetric and antisymmetric parts of the tensor $\frac{\partial u_i}{\partial x_j}$. Also write

$$\omega_{ij} = -\tfrac{1}{2}\epsilon_{ijk}\,\omega_k, \quad \text{where} \quad \boldsymbol{\omega} = \text{curl}\ \mathbf{u}\ ,$$

then

$$\omega_{ij}\,\delta x_j \;=\; \tfrac{1}{2}\left(\boldsymbol{\omega}\times\boldsymbol{\delta}\mathbf{x}\right)_i \;.$$

With small deformations this term gives only a rigid rotation, and hence induces no elastic stresses. curl $\mathbf{u}$ is referred to as the *rotation*; so there is plenty of potential for confusion between solids mechanics and fluid mechanics (in the latter case $\mathbf{u}$, $\boldsymbol{\omega}$ and $\mathbf{e}$ are the velocity, vorticity and the strain rate).

If $\mathbf{e}$ is zero then to first order there is no deformation. $\mathbf{e}$ is called the (Cauchy, or infinitesimal) *strain tensor.*

*Exercise.* Show that with small deformations, that $\mathbf{e}$ is related to changes in length by

$$|\boldsymbol{\delta}\mathbf{x}|^2 - |\boldsymbol{\delta\xi}|^2 = 2\,\boldsymbol{\delta\xi}\cdot\mathbf{e}\cdot\boldsymbol{\delta\xi}\,.$$

### 3.1.6 Constitutive equation for a linear elastic solid

The constitutive equation is the relationship between stress and strain. For a elastic body subject to small strains about the reference state $\mathbf{e}=0$:

(i) We will assume that the constitutive equation is instantaneous and local so that $\boldsymbol{\sigma}\equiv\boldsymbol{\sigma}(\mathbf{e})$, i.e. there is no dependence on

$$\frac{\partial e_{ij}}{\partial t} \quad\text{or}\quad \frac{\partial e_{ij}}{\partial x_k} \quad\text{or}\quad \ldots\;.$$

(ii) We will assume that the constitutive equation is linear, i.e.

$$\sigma_{ij}(\mathbf{x},t) \;=\; c_{ijk\ell}\,e_{k\ell}(\mathbf{x},t)\,, \tag{3.8a}$$

where $\mathbf{c}$ is a fourth-order tensor that is a property of the material (cf. Hooke's law for which force $\propto$ extension, with $\boldsymbol{\sigma}$ =force/area $\propto$ $\mathbf{e}$ =extension/original-length).

Since $\sigma_{ij}=\sigma_{ji}$, and $e_{k\ell} \;=\; e_{\ell k}$,

$$c_{ijk\ell} \;=\; c_{jik\ell} \;=\; c_{ij\ell k}. \tag{3.8b}$$

Hence there are up to 36 parameters in a general anisotropic material.

(iii) We will also assume that the material is *isotropic*; then $\mathbf{c}$ is an isotropic tensor

$$c_{ijk\ell} \;=\; \lambda\,\delta_{ij}\,\delta_{k\ell} + \mu\,\delta_{ik}\,\delta_{j\ell} + \bar{\mu}\,\delta_{i\ell}\,\delta_{jk}. \tag{3.8c}$$

Moreover, (3.8b), or equivalently the symmetry of $\boldsymbol{\sigma}$, implies that $\bar{\mu}=\mu$; hence

$$\sigma_{ij} \;=\; \lambda\,\delta_{ij}\,e_{kk} + 2\mu\,e_{ij} \;=\; \lambda\,\delta_{ij}\,\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\,. \tag{3.8d}$$

$\lambda$ and $\mu$ are the *Lamé constants/moduli.*

Since $\sigma_{kk}=(3\lambda+2\mu)e_{kk}$, it follows from (3.8d) that

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}\sigma_{kk}\right)\,. \tag{3.8e}$$

### 3.1.7 Simple deformations

*Dilatation.*

Consider a small change of material volume

$$\Delta V = \int \mathbf{u}\cdot\mathbf{n}\,dS = \int \nabla\cdot\mathbf{u}\,dV\,.$$

We define the [local] dilatation to be

$$\theta \equiv \nabla\cdot\mathbf{u} = \frac{\partial u_k}{\partial x_k} = e_{kk}\;. \tag{3.9}$$

*Hydrostatic pressure.* In the case of *hydrostatic pressure*, i.e. when

$$\sigma_{ij} \;=\; -p\,\delta_{ij} \quad \text{and} \quad \boldsymbol{\tau}(\mathbf{x},t;\mathbf{n}) \;=\; -p\mathbf{n}\,, \tag{3.10a}$$

then from (3.8d),

$$p = -\tfrac{1}{3}\sigma_{\ell\ell} = -\left(\lambda + \frac{2\mu}{3}\right) e_{kk} = -\kappa\theta\,, \tag{3.10b}$$

where $\kappa = (\lambda + \frac{2}{3}\mu)$ is the *bulk modulus* or *modulus of incompressibility.* On physical grounds we expect that

$$\kappa > 0\,. \tag{3.10c}$$

*Otherwise.* When *not* in a state of hydrostatic pressure, we define

$$p = -\tfrac{1}{3}\sigma_{kk}\,, \tag{3.11a}$$

and write

$$\sigma_{ij} = -p\,\delta_{ij} + \overline{\sigma}_{ij}\,, \tag{3.11b}$$

where $\overline{\sigma}_{ij}$ is the *deviatoric stress.*

*Simple shear.*

Consider the *simple shear* $\mathbf{u} = (\gamma y, 0, 0)$. Then

$$\mathbf{e} = \gamma \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad \boldsymbol{\sigma} = \gamma\mu \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

$\mu$ is referred to as the *shear modulus* or *modulus of rigidity.* In a fluid $\mu = 0$; more generally we expect on physical grounds that

$$\mu \geqslant 0\,. \tag{3.12}$$

*Uniaxial extension.*

Consider a uniaxial extension induced by a stress

$$\boldsymbol{\sigma} = \frac{F}{A} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

Then from (3.8e)

$$\mathbf{e} = \frac{F}{2A(3\lambda + 2\mu)\mu} \begin{pmatrix} 2(\lambda+\mu) & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = \frac{F}{EA} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{pmatrix}\,,$$

where

$$E = \frac{(3\lambda + 2\mu)\mu}{\lambda + \mu} = \frac{3\kappa\mu}{\lambda+\mu} \geqslant 0\,, \qquad \textit{Young's modulus}\,, \tag{3.13a}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)}\,, \qquad \textit{Poisson's ratio}\,. \tag{3.13b}$$

10/04 *Remark.* $\nu$ is *not* the kinematic viscosity, and can be negative (in *auxetic* materials).

### 3.1.8 Equations of motion

From substituting the isotropic constitutive relation (3.8d) into the momentum equation (3.3b),

$$\rho a_j \;=\; \rho F_j + \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\,. \tag{3.14}$$

Next, recall that the velocity $\mathbf{v}(\mathbf{x}, t)$ of a material point is given by

$$v_i = \left.\frac{\partial}{\partial t}\right|_{\xi} x_i(\boldsymbol{\xi}, t) = \frac{d}{dt}\left(u_i(\mathbf{x}(\boldsymbol{\xi}, t), t)\right) = \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_j}\frac{dx_j}{dt} = \frac{\partial u_i}{\partial t} + v_j \frac{\partial u_i}{\partial x_j}.$$

However, since we are assuming that the deformation is small, i.e. $\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$ from (3.6c), it is consistent to neglect the convective derivative in the material derivative, so that

$$v_i \approx \frac{\partial u_i}{\partial t}\,. \tag{3.15a}$$

Similarly

$$a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \approx \frac{\partial v_i}{\partial t} \approx \frac{\partial^2 u_i}{\partial t^2}\,. \tag{3.15b}$$

Further, for small deformations the dilatation is small, and hence the density can be taken to be that in the undeformed state (relieving the need to consider the conservation of mass equation henceforth):

$$\rho(\mathbf{x}, t) = \rho(\boldsymbol{\xi} + \mathbf{u}, t) \approx \rho(\boldsymbol{\xi}, 0)\,. \tag{3.15c}$$

From substituting the above linearisations into the momentum equation (3.3b), and using (3.8d), it follows that

$$\rho\frac{\partial^2 u_j}{\partial t^2} = \rho F_j + \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left(\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)\,. \tag{3.16a}$$

Henceforth we will assume that the material is *homogeneous*, i.e. that $\rho$, $\lambda$ and $\mu$ are constants throughout the material. Then

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho\mathbf{F} + (\lambda + \mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{u}) + \mu\nabla^2\mathbf{u} \tag{3.16b}$$

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$$= \rho\mathbf{F} + (\lambda + 2\mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{u}) - \mu\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u})\,. \tag{3.16c}$$

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**Boundary conditions**

*Rigid/clamped boundary.*

$$\mathbf{u} = 0\,. \tag{3.17a}$$

*Free surface.* At a constant pressure free surface

$$\boldsymbol{\tau} = \boldsymbol{\sigma}{\cdot}\mathbf{n} = -p_0\mathbf{n} \quad (\implies \mathbf{n}\times\boldsymbol{\sigma}{\cdot}\mathbf{n} = 0)\,. \tag{3.17b}$$

*Solid/solid interface.* No relative motion and Newton three imply the six conditions

$$[\mathbf{u}]_-^+ = [\boldsymbol{\sigma}{\cdot}\mathbf{n}]_-^+ = 0\,. \tag{3.17c}$$

*Solid/inviscid-fluid interface.* No relative normal motion, and continutity of stress (an inviscid fluid only has a pressure traction/stress), imply the four conditions

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$$[\mathbf{u}{\cdot}\mathbf{n}]_-^+ = 0\,, \quad \boldsymbol{\sigma}{\cdot}\mathbf{n} = -p_0\mathbf{n}\,. \tag{3.17d}$$

### 3.1.9 Energy equation

Recall the energy equation (3.5d)

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\,dV = \int_V \mathbf{v}\cdot\mathbf{F}\,dV + \int_S \mathbf{v}{\cdot}\boldsymbol{\sigma}{\cdot}\mathbf{n}\,dS - \int_V \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\sigma_{ij}\,dV\,.$$

Assuming small deformations, so that $\mathbf{v}\approx\dot{\mathbf{u}}$, this becomes

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,dV = \int_V \dot{\mathbf{u}}\cdot\mathbf{F}\,dV + \int_S \dot{\mathbf{u}}{\cdot}\boldsymbol{\sigma}{\cdot}\mathbf{n}\,dS - \int_V \dot{e}_{ij}\sigma_{ij}\,dV\,. \tag{3.18a}$$

Further, for small displacements and an isotropic material

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\tfrac{1}{2}\sigma_{ij}e_{ij}\right) =& \tfrac{1}{2}c_{ijk\ell}\left(e_{k\ell}\dot{e}_{ij}+\dot{e}_{k\ell}e_{ij}\right) && \text{from(3.8a)}\\
=& c_{ijk\ell}e_{k\ell}\dot{e}_{ij} && c_{ijk\ell}=c_{k\ell ij}\ \text{from(3.8b)}\\
=& \sigma_{ij}\dot{e}_{ij}\,. && 
\end{aligned} \tag{3.18b}$$

Hence, using the fact that the density is approximately constant,

$$\frac{d}{dt}\int_V \tfrac{1}{2}\left(\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}+\sigma_{ij}e_{ij}\right)\,dV-\int_S \dot{\mathbf{u}}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\,dS=\int_V \dot{\mathbf{u}}\cdot\mathbf{F}\,dV\,, \tag{3.18c}$$

i.e.

$$\frac{\partial}{\partial t}(E_k+E_p)+\boldsymbol{\nabla}\cdot\mathbf{I}=\dot{\mathbf{u}}\cdot\mathbf{F}\,, \tag{3.18d}$$

where, assuming no strain energy when the material is undeformed,

$$\text{Kinetic Energy} = E_k = \tfrac{1}{2}\rho\dot{\mathbf{u}}^2\,, \tag{3.18e}$$

$$\text{Elastic Potential Energy} = E_p = \rho\mathcal{E} = \tfrac{1}{2}\sigma_{ij}e_{ij}\,, \tag{3.18f}$$

$$\text{Energy Flux Vector} = \mathbf{I} = -\dot{\mathbf{u}}\cdot\boldsymbol{\sigma}\,. \tag{3.18g}$$

Here $\mathcal{E}$ is the strain energy density per unit mass. Further, from (3.8d),

$$\begin{aligned}
\rho\mathcal{E} =& \tfrac{1}{2}\left(\lambda e_{kk}e_{jj}+2\mu e_{ij}e_{ij}\right)\\
=& \tfrac{1}{2}\left(\lambda+\tfrac{2}{3}\mu\right)\left(e_{kk}e_{jj}\right)+\mu\left(e_{ij}-\tfrac{1}{3}e_{kk}\delta_{ij}\right)\left(e_{ij}-\tfrac{1}{3}e_{\ell\ell}\delta_{ij}\right)\,.
\end{aligned}$$

Hence $\mathcal{E}$ positive definite iff

$$\kappa=\lambda+\tfrac{2}{3}\mu>0\,,\quad \mu>0\,. \tag{3.19}$$

11/16 These are the same restrictions as obtained in (3.10c) and (3.12) by physical reasoning.

## 3.2 Compressional Waves and Shear Waves

### 3.2.1 Waves of dilatation and rotation

Recall from (3.16b) that linear disturbances to an isotropic homogeneous material are governed by, assuming no body force,

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=(\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{u})+\mu\boldsymbol{\nabla}^2\mathbf{u}\,. \tag{3.20a}$$

$$=(\lambda+2\mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{u})-\mu\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u}) \tag{3.20b}$$

*Primary/compressional/pressure waves.* Take the divergence of (3.20a) to obtain

$$\frac{\partial^2\theta}{\partial t^2}-c_P^2\boldsymbol{\nabla}^2\theta=0\,, \tag{3.21a}$$

where $\theta=\boldsymbol{\nabla}\cdot\mathbf{u}$ and

$$c_P^2=\frac{\lambda+2\mu}{\rho}=\left(\frac{\kappa+\frac{4}{3}\mu}{\rho}\right)>0\,. \tag{3.21b}$$

$c_P$ is the dilatational (or compressional) wave speed.

*Secondary/shear waves.* Next take the curl of (3.20a) to obtain

$$\frac{\partial^2\boldsymbol{\omega}}{\partial t^2}-c_S^2\boldsymbol{\nabla}^2\boldsymbol{\omega}=0 \tag{3.22a}$$

where $\boldsymbol{\omega}=\operatorname{curl}\mathbf{u}$ and

$$0<\frac{\mu}{\rho}=c_S^2<c_P^2\,. \tag{3.22b}$$

$c_S$ is the shear wave speed.

Hence a general disturbance, e.g. as generated by an earthquake, propagates [at least] at two speeds. The first to arrive, the primary $P$ wave, is the dilatational part. The second to arrive, the secondary $S$ wave, is the shear wave.

*Elastic fluids.* If $\mu = 0$ (i.e. an elastic fluid) then $c_S = 0$ and only compressional, i.e. acoustic/sound, waves are supported with

$$c_P^2 = \frac{\kappa}{\rho} = \frac{dp}{d\rho}\,, \tag{3.23}$$

since for an elastic fluid

$$p = -\,\tfrac{1}{3}\sigma_{kk} = -(\lambda + \tfrac{2}{\cancel{3}}\cancel{\mu})e_{kk} = -\kappa\theta\,,$$

and (with some slight abiguity in the use of $\rho = \rho_0 + \tilde{\rho}$)

$$\frac{\partial\tilde{\rho}}{\partial t} \approx -\,\rho_0\boldsymbol{\nabla}.\dot{\mathbf{u}} \approx -\rho_0\frac{\partial\theta}{\partial t} \quad\Longrightarrow\quad \tilde{\rho} \approx -\rho_0\theta \quad\Longrightarrow\quad p \approx \frac{\kappa}{\rho_0}\tilde{\rho} \quad\Longrightarrow\quad \frac{dp}{d\rho} \approx \frac{\kappa}{\rho}\,.$$

11/04 *Example.* For steel, $c_P \sim 6\,.\,10^3\,\mathrm{ms}^{-1}$, $c_S \sim 3\,.\,10^3\,\mathrm{ms}^{-1}$.

### 3.2.2 Plane waves

Look for solutions to (3.20b) of the form

$$\mathbf{u} = \mathbf{p}\left(\widehat{\mathbf{k}}\cdot\mathbf{x} - ct\right)\,, \tag{3.24a}$$

where $\mathbf{p}$ is the *polarisation* and $\widehat{\mathbf{k}}$ is a unit vector. Then we require

$$c^2\mathbf{p}'' = c_P^2\,\widehat{\mathbf{k}}(\widehat{\mathbf{k}}\cdot\mathbf{p}'') - c_S^2\,\widehat{\mathbf{k}}\times\left(\widehat{\mathbf{k}}\times\mathbf{p}''\right)\,. \tag{3.24b}$$

It follows that

$$(c^2 - c_P^2)\,\widehat{\mathbf{k}}\cdot\mathbf{p}'' = 0\,, \tag{3.24c}$$

$$(c^2 - c_S^2)\,\widehat{\mathbf{k}}\times\mathbf{p}'' = 0\,. \tag{3.24d}$$

Since $c_P \neq c_S$ there are two possibilities.

*Longitudinal plane waves.* If $c = c_P$, then $\widehat{\mathbf{k}}\times\mathbf{p}'' = 0$, and we let [6]

$$\mathbf{u} = \widehat{\mathbf{k}}\,f\left(\widehat{\mathbf{k}}\cdot\mathbf{x} - c_Pt\right)\,,\quad \theta = \boldsymbol{\nabla}\cdot\mathbf{u} = f'\left(\widehat{\mathbf{k}}\cdot\mathbf{x} - c_Pt\right)\quad\text{and}\quad \boldsymbol{\omega} = \operatorname{curl}\mathbf{u} = 0\,. \tag{3.25}$$

This is a plane wave of arbitrary shape with $\mathbf{u}$ parallel to $\widehat{\mathbf{k}}$, i.e. the displacement is parallel to the direction of travel of the wave. This is a *longitudinal* wave.

*Transverse plane waves.* If $c = c_S$, then $\widehat{\mathbf{k}}\cdot\mathbf{p}'' = 0$, , and we let

$$\mathbf{u} = \widehat{\mathbf{k}}\times\mathbf{g}\left(\widehat{\mathbf{k}}\cdot\mathbf{x} - c_St\right)\,,\quad \theta = \boldsymbol{\nabla}\cdot\mathbf{u} = 0\quad\text{and}\quad \boldsymbol{\omega} = \operatorname{curl}\mathbf{u} = -\mathbf{g}'\left(\widehat{\mathbf{k}}\cdot\mathbf{x} - c_St\right)\,, \tag{3.26}$$

where, without loss of generality, we can assume that $\widehat{\mathbf{k}}\cdot\mathbf{g} = 0$. This is a plane wave of arbitrary shape with $\mathbf{u}\cdot\widehat{\mathbf{k}} = 0$, i.e. the displacement is perpendicular to the direction of travel of the wave. This is a *transverse* wave. If the direction of $\widehat{\mathbf{k}}\times\mathbf{g}$ is fixed, then the wave is said to be *polarised.*

*Plane waves: stress tensor.* From (3.8d)

$$\sigma_{ij} = \begin{cases} \left(\lambda\delta_{ij} + 2\mu\widehat{k}_i\widehat{k}_j\right)f' & : \text{P wave,}\\ \mu\left(\epsilon_{i\ell m}\widehat{k}_j + \epsilon_{j\ell m}\widehat{k}_i\right)\widehat{k}_\ell g_m' & : \text{S wave.}\end{cases}$$

[6] We do not consider the cases of constant uniform displacement or uniform translation of the solid.

*Plane waves: energy density.* From (3.18e) and (3.18f)

$$E_k = \tfrac{1}{2}\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} = \begin{cases} \tfrac{1}{2}\rho c_P^2 f'^2 & : \text{P wave,} \\ \tfrac{1}{2}\rho c_S^2 (\widehat{\mathbf{k}}\times\mathbf{g}')\cdot(\widehat{\mathbf{k}}\times\mathbf{g}') & : \text{S wave,} \end{cases}$$

$$E_p = \tfrac{1}{2}\lambda e_{kk}e_{jj} + \mu e_{ij}e_{ij} = \begin{cases} \tfrac{1}{2}(\lambda+2\mu) f'^2 & : \text{P wave,} \\ \tfrac{1}{2}\mu(\widehat{\mathbf{k}}\times\mathbf{g}')\cdot(\widehat{\mathbf{k}}\times\mathbf{g}') & : \text{S wave,} \end{cases}$$

Hence, since $\rho c_P^2 = \lambda + 2\mu$ from (3.21b), and $\rho c_S^2 = \mu$ from (3.22b), for both P and S waves there is equi-partition of energy:

$$E_k = E_p.$$

*Plane waves: energy flux.* From (3.18g)

$$\mathbf{I} = -\boldsymbol{\sigma}\cdot\dot{\mathbf{u}} = \begin{cases} (\lambda+2\mu)c_P\, f'^2\,\widehat{\mathbf{k}} & : \text{P wave,} \\ \mu c_S\,(\widehat{\mathbf{k}}\times\mathbf{g}')\cdot(\widehat{\mathbf{k}}\times\mathbf{g}')\,\widehat{\mathbf{k}} & : \text{S wave.} \end{cases}$$

*Plane waves: energy propagation.* Hence, for either type of plane wave *separately*,

$$\mathbf{I} = (E_k + E_p)\, c\,\widehat{\mathbf{k}},$$

where $c$ is the corresponding phase speed (i.e. $c_P$ or $c_S$). Hence energy is propagated at the wavespeed in the direction of travel of the waves (because the waves are non-dispersive). However, remember that while linear displacements can be added, quadratic energies cannot in general.

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### 3.2.3 Harmonic waves

*P waves.* Proceeding as in §1.5.2, it follows from (3.25) that a harmonic P-wave solution is

$$\mathbf{u} = A\mathbf{k}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{3.27}$$

where $k = |\mathbf{k}|$, $\omega = kc_P$ and $A$ is a constant.

*S waves and polarisation.* From (3.26) a harmonic S-wave solution is

$$\mathbf{u} = \mathbf{k}\times\mathbf{B}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{3.28a}$$

where $\omega = kc_S$ and $\mathbf{B}$ is a constant. For an S wave, the displacement is perpendicular to $\mathbf{k}$, with
12/16 $\mathbf{k}\times\mathbf{B}$ being the direction of polarisation.

To fix ideas, suppose that we have a plane boundary, say $y = 0$, defining the 'horizontal' (as in the earth). W.l.o.g. we need only consider two-dimensional waves, e.g. by rotating axes so that $\mathbf{k}$ lies in the $(x, y)$ plane, say $\mathbf{k} = k(\sin\theta, \cos\theta, 0)$. Recalling that $\mathbf{k}\cdot\widehat{\mathbf{z}} = 0$, we can then decompose $\mathbf{k}\times\mathbf{B}$ into

(i) a part that is horizontally polarised, i.e. parallel to $\widehat{\mathbf{z}}$, called a SH-wave

$$\mathbf{u} = (\mathbf{k}\times\mathbf{B}\cdot\widehat{\mathbf{z}})\,\widehat{\mathbf{z}}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}; \tag{3.28b}$$

(ii) a part that is 'vertically' polarised, i.e. which lies in the same plane as $\mathbf{k}$ and the vertical $\widehat{\mathbf{y}}$ and is parallel to $\mathbf{k}\times\widehat{\mathbf{z}}$, called a SV-wave

$$\mathbf{u} = (\mathbf{B}\cdot\widehat{\mathbf{z}})\,\mathbf{k}\times\widehat{\mathbf{z}}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{3.28c}$$

Note that despite the name, displacements are not purely "vertical" in SV waves.

13/18 *Remark.* A general disturbance contains P, SV and SH waves.

*Attenuated/evanescent waves (near a boundary).* If $\mathbf{k}$ is allowed to be complex, say $\mathbf{k} = \mathbf{k}_r + i\mathbf{k}_i$, then, taking P waves as an example,

$$\mathbf{u} = A\mathbf{k}\exp(i\mathbf{k}_r.\,\mathbf{x} - \mathbf{k}_i.\,\mathbf{x} - i\omega t)\,,$$

where

$$\begin{aligned}\omega^2 &= c_P^2(\mathbf{k}_r + i\mathbf{k}_i)\cdot(\mathbf{k}_r + i\mathbf{k}_i)\\ &= c_P^2(k_r^2 - k_i^2) + 2ic_P^2\,\mathbf{k}_r\cdot\mathbf{k}_i\,.\end{aligned} \tag{3.29a}$$

Hence

- there is propagation in the $\mathbf{k}_r$ direction;
- there is attenuation/evanescence in the $\mathbf{k}_i$ direction;
- if $\omega$ is real, then we need $k_r^2 > k_i^2$ and $\mathbf{k}_r\cdot\mathbf{k}_i = 0$, in which case

$$\text{frequency:}\quad \omega = \pm c_P(k_r^2 - k_i^2)^{\frac{1}{2}}\,, \tag{3.29b}$$

$$\text{phase speed:}\quad \frac{\omega}{k_r} = \pm c_P(1 - k_i^2/k_r^2)^{\frac{1}{2}}\,. \tag{3.29c}$$

Hence attenuated waves move slower than unattenuated waves, i.e. propagation along a boundary is slower than through the interior (evanescent waves do not occur in the interior of a solid, but may occur near an interface).

### 3.2.4 Interface boundary conditions

Suppose there is a plane interface at $y = 0$ when undisturbed. Let the disturbed position be

$$y \;=\; \eta(x, z, t). \tag{3.30}$$

If the solid material is not to fracture, then we require the displacement to be continuous at the interface, i.e., as in (3.17c),

$$[\mathbf{u}(x, \eta, z, t)]_-^+ \;=\; 0. \tag{3.31a}$$

If the displacements are small, i.e. $|\eta| \ll 1$, we can linearise this to

$$[\mathbf{u}]_-^+ \;=\; 0, \quad \text{on} \quad y \;=\; 0. \tag{3.31b}$$

Throughout our study of solid dynamics we have built into our analysis an assumption that, by Newton three, the traction on the material on one side of an element of surface, due to the material on the other side of the element of surface, is equal and opposite. Hence we also have the boundary condition (again see (3.17c))

$$[\boldsymbol{\sigma}\cdot\mathbf{n}]_-^+ \;=\; 0 \quad \text{on} \quad y \;=\; \eta, \quad \text{where} \quad \mathbf{n} \;=\; (-\eta_x, 1, -\eta_z)\,. \tag{3.32a}$$

On linearising this becomes

$$[\sigma_{xy}]_-^+ \;=\; [\sigma_{yy}]_-^+ \;=\; [\sigma_{zy}]_-^+ \;=\; 0 \quad \text{on} \quad y \;=\; 0\,, \quad \text{where} \quad \mathbf{n} \;=\; (0, 1, 0)\,. \tag{3.32b}$$

### 3.2.5 Rayleigh waves

It is not possible to have a self-sustained SH wave at a boundary, but is is possible to have a self-contained combination of P and SV waves, called a Rayleigh wave.

Suppose that an elastic material is bounded by a stress-free boundary, initially at $y = 0$. Suppose that the surface is slightly displaced to $y = \eta(x, z, t)$. If $|\eta| \ll 1$, then the linearised stress-free boundary condition becomes

$$\boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on} \quad y = 0\,, \quad \text{where} \quad \mathbf{n} = (0, 1, 0)\,. \tag{3.33a}$$

The is no restriction on $\mathbf{u}$ since the surface is free to move. Seek 2D solutions; then we have *two* boundary conditions,

$$\sigma_{xy} = 0 = \sigma_{yy} \quad \text{on} \quad y = 0\,, \tag{3.33b}$$

and hence we need both P and SV waves.

Seek attenuated wave solutions

$$\begin{aligned}\mathbf{u} &= A\,\mathbf{k}_P\,e^{i(kx-\omega t)+k\alpha y} + B\,\mathbf{k}_S \times \widehat{\mathbf{z}}\,e^{i(kx-\omega t)+k\beta y}\,, &(3.34a)\\ &= \left(A\,e^{k\alpha y} - i\beta B\,e^{k\beta y}, -i\alpha A\,e^{k\alpha y} - B\,e^{k\beta y}\right) k\,e^{i(kx-\omega t)}\,, &(3.34b)\end{aligned}$$

where

$$\mathbf{k}_P = k\widehat{\mathbf{x}} - ik\alpha\widehat{\mathbf{y}}, \quad \mathbf{k}_S = k\widehat{\mathbf{x}} - ik\beta\widehat{\mathbf{y}}\,, \tag{3.34c}$$

and we have taken both the P wave and the SV wave to have the *same* frequency and $x$-wavenumber so that the boundary conditions can be satisfied (also see below). Further, the trick in many of these calculations is to keep the algebra under control; to this end we have chosen a notation such that $k$ is the $x$-wavenumber, and $-k\alpha$ and $-k\beta$ are the complex $y$-wavenumbers (and $k \neq |\mathbf{k}_j|$).

If $k > \omega/c_S$ ($> \omega/c_P$), decaying solutions exist as $y \to -\infty$ if we choose (see the dispersion relation (3.29b) and the analogous result for a shear wave)

$$\alpha = +\left(1 - c^2/c_P^2\right)^{\frac{1}{2}}\,, \quad \beta = +\left(1 - c^2/c_S^2\right)^{\frac{1}{2}}\,, \tag{3.35}$$

12/04 where $c = \omega/k$ is the phase velocity of the wave.

Now apply the boundary conditions at $y = 0$:

$$\begin{aligned}0 = \sigma_{xy} &= \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\\ &= \mu\left(2\alpha A - i\left(1+\beta^2\right)B\right)k^2\,e^{i(kx-\omega t)}\\ &= \mu\left(2\alpha A - i\Big(2 - \frac{c^2}{c_S^2}\Big)B\right)k^2\,e^{i(kx-\omega t)},\\ 0 = \sigma_{yy} &= \lambda\,\frac{\partial u_1}{\partial x} + (\lambda + 2\mu)\,\frac{\partial u_2}{\partial y}\\ &= \left(-i((\lambda+2\mu)\alpha^2 - \lambda)\,A - 2\mu\beta B\right)k^2\,e^{i(kx-\omega t)}\\ &= \mu\left(-i\left(2 - \frac{c^2}{c_S^2}\right)A - 2\beta B\right)k^2\,e^{i(kx-\omega t)}\,,\end{aligned}$$

using $(\lambda + 2\mu)c_S^2 = \mu c_P^2$. For a non-trivial solution for $A$, $B$ we require that the *dispersion relation*

$$\left(2 - \frac{c^2}{c_S^2}\right)^2 = 4\alpha\beta = 4\left(1 - \frac{c^2}{c_P^2}\right)^{\frac{1}{2}}\left(1 - \frac{c^2}{c_S^2}\right)^{\frac{1}{2}}\,, \tag{3.36a}$$

is satisfied. This equation for the unknown wave speed $c$ is *Rayleigh's equation* (which is one of many). It can be rewritten, given $c \neq 0$, as

$$f(\xi) = \xi^3 - 8\xi^2 + 8\left(3 - \frac{2c_S^2}{c_P^2}\right)\xi - 16\left(1 - \frac{c_S^2}{c_P^2}\right) = 0\,, \tag{3.36b}$$

where $\xi = \frac{c^2}{c_S^2}$. Note that

$$f(0) \;=\; -16\left(1-\frac{c_S^2}{c_P^2}\right) \;<\; 0\;, \quad \text{and} \quad f(1) \;=\; 1 \;>\; 0\;;$$

hence $\exists$ at least one real root with $0 < c^2 < c_S^2$ (in fact $\exists$ only one root). This is the **Rayleigh wave**.

*Remarks.*

(i) Rayleigh waves are non-dispersive, i.e. $c$ is independent of $k$.

(ii) The displacements are given by

$$\begin{aligned} u_1 = \;& kA\left(e^{k\alpha y} - \left(1-\frac{c^2}{2c_S^2}\right) e^{k\beta y}\right) e^{i(kx-\omega t)}\,, \\ u_2 = \;& -ik\alpha A\left(e^{k\alpha y} - \left(1-\frac{c^2}{2c_S^2}\right)^{-1} e^{k\beta y}\right) e^{i(kx-\omega t)}\,. \end{aligned}$$

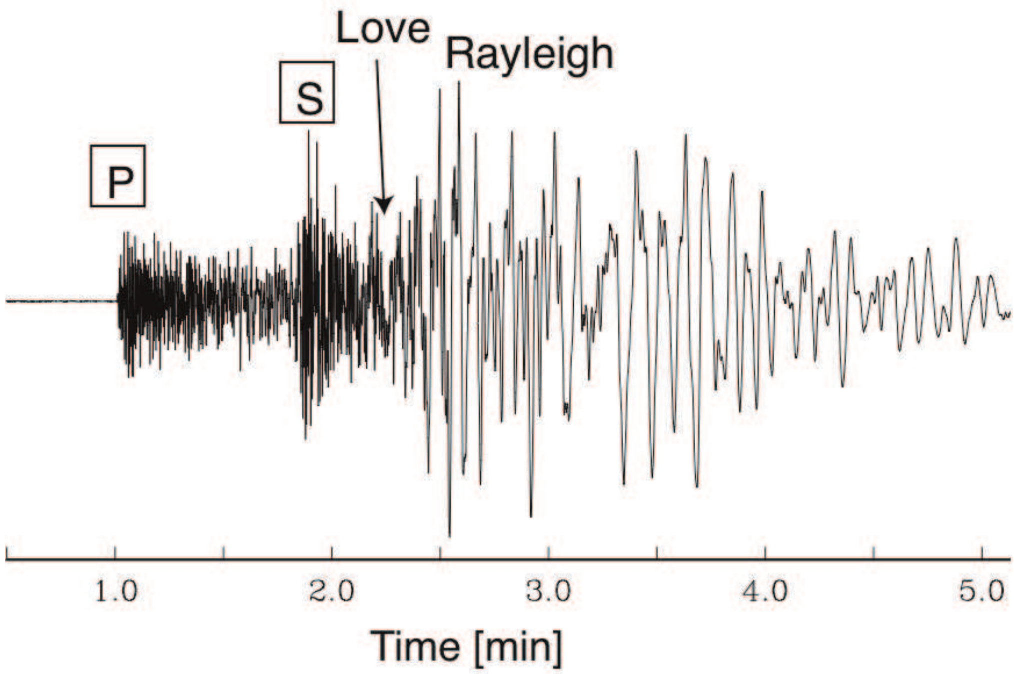
- Hence at $y=0$, $u_1$ is $-\frac{\pi}{2}$ out of phase with $u_2$.
- Also, since $\alpha > \beta$, $u_1$ changes sign for some $y$, while $u_2$ is one signed for all $y$.

(iii) Rayleigh waves form the principal part of the seismic signal since P and S waves spread through a 3D volume, while Rayleigh waves spread out in a 2D region.

(iv) In earthquakes:

- P waves arrive first, then
- S waves arrive, and then
- Rayleigh waves arrive (and knock your buildings down).






Vertical-component seismogram showing the arrival of the P(rimary), (S)econdary, Love and Rayleigh waves.[7] The Rayleigh waves have the largest amplitude; for Love waves see § 4.1.2.

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[7] Gabi Laske, http://igppweb.ucsd.edu/˜gabi/sio15/sio15-13/lectures/Lecture07.html.

## 3.3 Reflection and Refraction of Plane Waves

If an elastic wave is incident on an interface where the properties of the medium change discontinuously, reflection and refraction can occur.

In general, an incident wave of one type generates reflected and refracted waves of other types.

- For instance, SV and P waves can excite each other at an interface (cf. Rayleigh waves).
- However, SH waves can only excite other SH waves (since they are the only types of waves with displacements in the spanwise $z$-direction).

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### 3.3.1 Reflection and refraction of SH waves at an interface

The waves on either side of the interface must satisfy the relevant dispersion relation. Hence

*Incident SH wave:*

$$\mathbf{u}_I = (0,0,1)\, e^{i\mathbf{k}_I\cdot\mathbf{x}-i\omega_I t} \quad \text{where} \quad \mathbf{k}_I = \frac{\omega_I}{c_S}\,(\sin\theta,\cos\theta,0)\ , \tag{3.37a}$$

*Reflected SH wave:*

$$\mathbf{u}_R = (0,0,R)\, e^{i\mathbf{k}_R\cdot\mathbf{x}-i\omega_R t} \quad \text{where} \quad \mathbf{k}_R = \frac{\omega_R}{c_S}\,(\sin\Theta,-\cos\Theta,0)\ , \tag{3.37b}$$

*Transmitted SH wave:*

$$\mathbf{u}_T = (0,0,T)\, e^{i\mathbf{k}_T\cdot\mathbf{x}-i\omega_T t} \quad \text{where} \quad \mathbf{k}_T = \frac{\omega_T}{\bar{c}_S}\left(\sin\bar{\theta},\cos\bar{\theta},0\right)\ . \tag{3.37c}$$

The appropriate boundary conditions on $y=0$ are (since $\sigma_{xy}=\sigma_{yy}=0$)

$$\mathbf{u}_I+\mathbf{u}_R = \mathbf{u}_T\ , \tag{3.38a}$$

$$(\sigma_{yz})_I+(\sigma_{yz})_R = (\sigma_{yz})_T\ . \tag{3.38b}$$

These can only be satisfied $\forall x$ and $t$ if the frequencies are the same, i.e. if

$$\omega_I = \omega_R = \omega_T \equiv \omega\,, \tag{3.39a}$$

and the wavelengths in the $\widehat{\mathbf{x}}$ direction are equal, i.e. if

$$\frac{\sin\theta}{c_S} = \frac{\sin\Theta}{c_S} = \frac{\sin\bar{\theta}}{\bar{c}_S}\,. \tag{3.39b}$$

Hence

$$\underbrace{\theta = \Theta}_{\substack{\text{Angles of incidence}\\ \text{and reflection equal}}}\,, \qquad \underbrace{\frac{\sin\theta}{c_S} = \frac{\sin\bar{\theta}}{\bar{c}_S}}_{\substack{\text{Snell's law}\\ \text{of refraction}}}\,. \tag{3.39c}$$

Now

$$(\sigma_{yz})_I = \mu\frac{\partial(\mathbf{u}_I)_z}{\partial y} = i\omega\,\frac{\mu\cos\theta}{c_S}\,e^{i\mathbf{k}_I\cdot\mathbf{x}-i\omega t}\,, \quad \text{etc.,}$$

hence from applying the boundary conditions (3.38a) and (3.38b)

$$1+R = T\,, \tag{3.40a}$$

$$(1-R)\,\frac{\mu\,\cos\theta}{c_S} = T\,\frac{\bar{\mu}\,\cos\bar{\theta}}{\bar{c}_S}\,. \tag{3.40b}$$

Solve for $R$ and $T$ to obtain

$$R = \frac{\overline{Z}-Z}{\overline{Z}+Z}\,, \quad T = \frac{2\overline{Z}}{\overline{Z}+Z}\,, \tag{3.41a}$$

where $Z$ and $\overline{Z}$ are impedances defined by

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$$Z = \frac{c_S}{\mu\cos\theta}\,, \quad \overline{Z} = \frac{\bar{c}_S}{\bar{\mu}\cos\bar{\theta}}\,. \tag{3.41b}$$

*Energy flux.* From (3.18g), the flux of energy is given by

$$\mathbf{I} = -\boldsymbol{\sigma}.\frac{\partial\mathbf{u}}{\partial t}\,, \tag{3.42}$$

and so

$$\langle\mathbf{I}_y\rangle_I = -\tfrac{1}{2}\mathrm{Re}\,\left[\sigma_{yz}\,(\dot{\mathbf{u}})_z^*\right]_I = \frac{\mu\omega^2}{2c_S}\,\cos\theta\,,$$

$$\langle\mathbf{I}_y\rangle_R = -\frac{\mu\omega^2|R|^2}{2c_S}\,\cos\theta\,,$$

$$\langle\mathbf{I}_y\rangle_T = \frac{\bar{\mu}\omega^2|T|^2}{2\bar{c}_S}\,\cos\bar{\theta}\,.$$

Thus

$$\begin{aligned}\langle\mathbf{I}_y\rangle_I + \langle\mathbf{I}_y\rangle_R &= \frac{\mu\omega^2}{2c_S}\,\cos\theta\,\left(1-|R|^2\right)\\ &= \frac{2\omega^2\overline{Z}}{\left(\overline{Z}+Z\right)^2} = \frac{\omega^2}{2\overline{Z}}\,|T|^2 = \langle\mathbf{I}_y\rangle_T\,.\end{aligned} \tag{3.43}$$

It follows that the mean incident energy flux is either reflected or transmitted.

*Remarks*

(i) There is *total transmission* if the impedances match, i.e. if $Z = \overline{Z}$.

(ii) If $\bar{c}_S < c_S$ then from (3.39c)

$$\sin\bar{\theta} = \frac{\bar{c}_S}{c_S}\sin\theta < \sin\theta\,.$$

Hence, whatever the value of $\theta$, there exists a maximum value of $\bar{\theta}$ given by

$$\bar{\theta}_{max} = \sin^{-1}\left(\frac{\bar{c}_S}{c_S}\right)\,. \tag{3.44}$$

There can be no transmitted wave with $\bar{\theta} > \bar{\theta}_{max}$.

(iii) If $\bar{c}_S > c_S$ then $\bar{\theta} > \theta$, and there exists an incident angle for which $\bar{\theta} = \frac{\pi}{2}$, namely

$$\theta = \theta_{max} = \sin^{-1}\left(\frac{c_S}{\bar{c}_S}\right)\,. \tag{3.45a}$$

For incident angles greater than this critical angle, i.e. for $\theta > \theta_{max}$, the solutions in $y > 0$ are evanescent waves of the form (from using (3.37c) and (3.39c) )

$$\mathbf{u}_T = (0,0,T)\exp\left(i\frac{\omega}{c_S}\sin\theta\,x - \frac{\omega}{c_S}\beta y - i\omega t\right)\,, \tag{3.45b}$$

where (cf. (3.29b))

$$\beta = \left(\sin^2\theta - \frac{c_S^2}{\bar{c}_S^2}\right)^{\frac{1}{2}}\,. \tag{3.45c}$$

$\mathbf{u}_T$ is an **evanescent** wave, and there is no energy flux away from the interface.

The only modification necessary to (3.41a), (3.41b), etc. is

$$i\cos\bar{\theta} \to -\frac{\bar{c}_S}{c_S}\beta\,.$$

It follows from (3.41a) that

$$R = \frac{i\mu\cos\theta + \bar{\mu}\left(\sin^2\theta - \frac{c_S^2}{\bar{c}_S^2}\right)^{\frac{1}{2}}}{i\mu\cos\theta - \bar{\mu}\left(\sin^2\theta - \frac{c_S^2}{\bar{c}_S^2}\right)^{\frac{1}{2}}}\,, \tag{3.46}$$

and hence $|R| = 1$. So for $\theta > \theta_{max}$ all incident energy is reflected and there is total internal reflection.

### 3.3.2 Reflection of P waves at a rigid interface

Consider reflection at a rigid barrier so that $\mathbf{u} = 0$ on $y = 0$ (alternatively one might consider a free surface). As before

(a) all waves must have the same frequency, $\omega$;

(b) the angles of incidence and reflection of the P wave must be equal, i.e. $\Theta = \theta$.

*Incident P wave:*

$$\mathbf{u}_I = (\sin\theta,\cos\theta,0)e^{i\mathbf{k_I}\cdot\mathbf{x}-i\omega t} \quad \text{where} \quad \mathbf{k}_I = \frac{\omega}{c_P}(\sin\theta,\cos\theta,0)\,, \tag{3.47a}$$

*Reflected P wave:*

$$\mathbf{u}_R = R(\sin\theta, -\cos\theta, 0)e^{i\mathbf{k}_R\cdot\mathbf{x}-i\omega t} \quad \text{where} \quad \mathbf{k}_R = \frac{\omega}{c_P}(\sin\theta, -\cos\theta, 0)\,, \tag{3.47b}$$

*Reflected SV wave:*

$$\mathbf{u}_{\overline{R}} = \overline{R}(\cos\bar{\theta}, \sin\bar{\theta}, 0)e^{i\bar{k}_R\cdot\mathbf{x}-i\omega t} \quad \text{where} \quad \bar{k}_R = \frac{\omega}{c_S}(\sin\bar{\theta}, -\cos\bar{\theta}, 0)\,. \tag{3.47c}$$

From matching the phases

$$\frac{\sin\theta}{c_P} = \frac{\sin\bar{\theta}}{c_S}\,, \quad \text{i.e.} \quad \bar{\theta} = \sin^{-1}\left(\frac{c_S}{c_P}\sin\theta\right)\,, \tag{3.48a}$$

and hence

$$\bar{\theta} < \theta\,. \tag{3.48b}$$

Applying the rigid boundary condition, $\mathbf{u} = 0$ on $y = 0$:

$$\begin{aligned}(1+R)\sin\theta + \overline{R}\cos\bar{\theta} &= 0\,,\\ (1-R)\cos\theta + \overline{R}\sin\bar{\theta} &= 0\,.\end{aligned}$$

Hence

$$R = \frac{\cos(\theta+\bar{\theta})}{\cos(\theta-\bar{\theta})}\,, \qquad \overline{R} = -\frac{\sin 2\theta}{\cos(\theta-\bar{\theta})}\,. \tag{3.49}$$

*Remarks*

(i) The SV waves reflect at a smaller angle than the P waves.

(ii) Both $R$ and $\overline{R}$ are real, so the waves are in phase at $y = 0$.

(iii) For normal incidence, $\theta = 0$, there is no reflected $SV$ wave (since $\overline{R} = 0$).

14/16 (iv) If $\theta + \bar{\theta} = \frac{\pi}{2}$, i.e. $\tan\theta = \frac{c_P}{c_S}$, then a pure $SV$ wave is reflected ('mode conversion').

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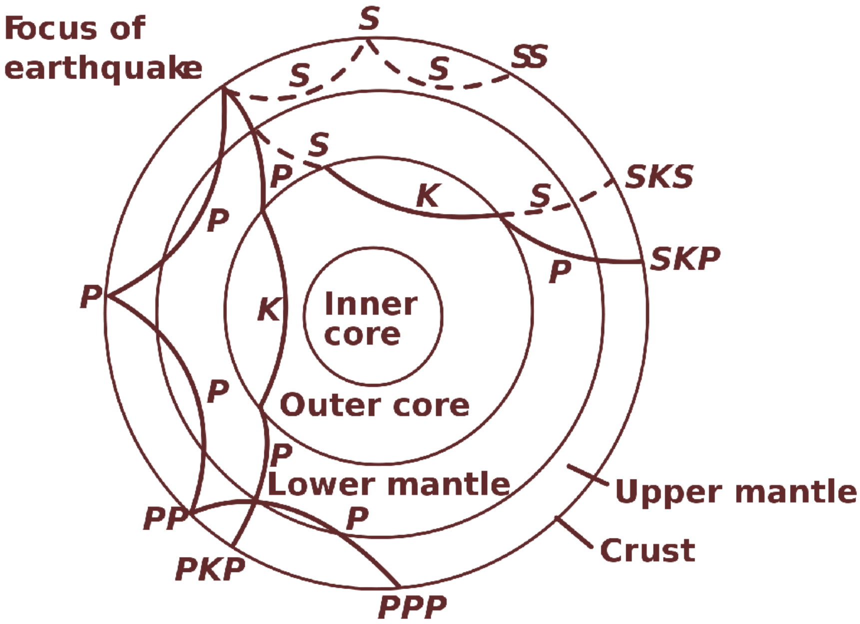
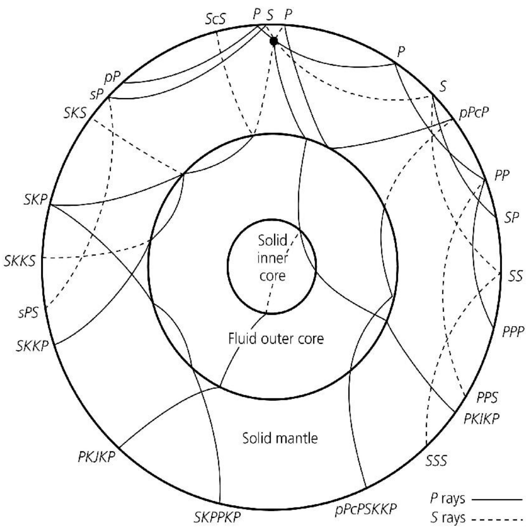
### 3.3.3 Seismic tomography

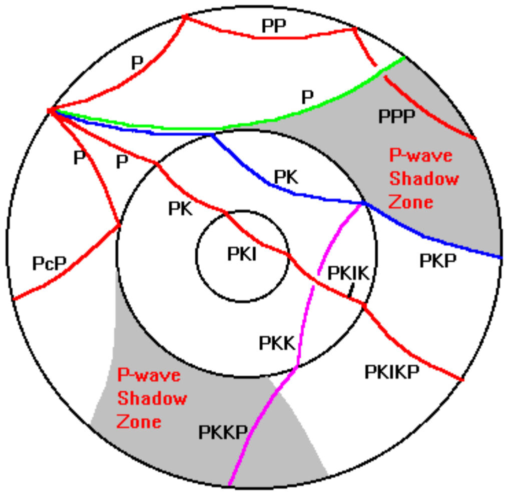
Travel-time measurements of P and S waves generated by earthquakes, or man-made explosions, can be used to make inferences about the material the waves have passed through, since the wavespeed, etc. varies with material properties. This is an example of an *inverse problem* where a set of observations are used to deduce the causal factors that produced them; in inverse problems you start with the results and then calculate the causes (while in forward problems you start with the causes and then calculate the results). There are many similar problems, e.g. PET/brain scans.

An important early discovery using such methods was that the outer core of the earth is liquid (since $\mu = 0$ in fluids, S waves cannot propagate through the liquid core).




Typical ray paths of reflected and refracted P and S waves in the interior of the earth.[8] For the reason why changes in material properties mean that the paths are curved, see *Ray Theory* in §5.

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15/19 Ray paths of reflected and refracted P and S waves.[9]

P wave paths illustrating shadow zones.[10]

[8] United States Geological Survey, http://commons.wikimedia.org/wiki/File:Earthquake_wave_paths.svg.
[9] An Introduction to Seismology, Earthquakes, and Earth Structure, Stein, S. & Wysession, W., Blackwell Publishing (2003); see also http://levee.wustl.edu/seismology/book/.
[10] Steven Dutch, http://www.uwgb.edu/dutchs/earthsc202notes/quakes.htm.

 

# 4 Linear Dispersive Waves

In the plane-wave, $e^{ikx-i\omega t}$, systems studied so far (with the exception of the waveguide in § 1.7.1), the [signed] phase speed,

$$c = \frac{\omega}{|\mathbf{k}|}\,, \tag{4.1}$$

has been independent of $\mathbf{k}$ (because there has been nothing in the geometry or material parameters to give a lengthscale, or direction, to measure $\mathbf{k}$ against). However, for most systems the phase speed $c$ depends on $\mathbf{k}$ (e.g. water waves). If the phase speed depends on the wavenumber, $\mathbf{k}$, then the different Fourier components of an initial local disturbance travel at different speeds and 'disperse'. This has consequences for the speed of energy propagation, which now differs from the velocity of propagation of the crests/troughs, $\mathbf{c} = c\,\widehat{\mathbf{k}}$.[11]

## 4.1 Geometric Dispersion in Wave Guides

### 4.1.1 Acoustic waves in a rectangular duct

In § 1.7.1, we have already seen that for acoustic waves in a rectangular waveguide, the phase speed
15/16 depends on the wavenumber, $\mathbf{k}$.

### 4.1.2 Love waves

A Rayleigh wave is a mixed interfacial $P/SV$ wave. The question might be posed whether it is possible to have an interfacial $SH$ waves. The answer is yes, but only within a *layered* half space.

Love waves are trapped in a finite layer (cf. bound states in a finite well in $QM$) by

(i) reflection at a free surface;

(ii) total internal reflection at the lower interface ... hence we require $\bar{c}_S > c_S$ (i.e. the 'crust' is a low velocity layer).

For $SH$ waves, where $\mathbf{u}$ is the displacement,

$$\mathbf{u} = (0, 0, u)\,, \quad \begin{cases} c_S^2 \boldsymbol{\nabla}^2 u - u_{tt} = 0\,, & 0 < y < h\,, \\ \bar{c}_S^2 \boldsymbol{\nabla}^2 u - u_{tt} = 0\,, & y < 0\,. \end{cases}$$

Again insist that waves in both layers have the same $x$-wavenumber and frequency. Seek solutions of the form (that satisfy the dispersion relation)

$$0 < y < h: \qquad u = (A_1 \sin \ell y + A_2 \cos \ell y) e^{i(kx-\omega t)}\,,$$

$$\ell = k\left(\frac{\omega^2}{k^2 c_S^2} - 1\right)^{\frac{1}{2}} = k\left(\frac{c^2}{c_S^2} - 1\right)^{\frac{1}{2}} > 0\,,$$

where we have assumed $c_S < c \equiv \frac{\omega}{k}$ for a propagating mode,[12] and

[11] The solution to $\mathbf{c}_g \equiv \nabla_{\mathbf{k}}\omega = \frac{\omega}{k}\,\widehat{\mathbf{k}} \equiv \mathbf{c}$, i.e. group velocity equals phase velocity, is $\frac{\omega}{k} = \text{const}$.

[12] Some prefer the solution $(\mathcal{A}_1 \sin \ell(h-y) + \mathcal{A}_2 \cos \ell(h-y))e^{i(kx-\omega t)}$, but it is swings and roundabouts.

$$y < 0: \qquad u = \alpha e^{\beta y} e^{i(kx-\omega t)}\,,$$
$$\beta = k\left(1 - \frac{c^2}{\bar{c}_S^2}\right)^{\frac{1}{2}} > 0\,,$$

15/04 est. where we have assumed $c < \bar{c}_S$ for an evanescent mode.

The boundary conditions are

$$\begin{array}{lll} y = h: & \sigma_{yz} = 0 & A_1 \cos \ell h - A_2 \sin \ell h = 0\,, \\ y = 0: & [\sigma_{yz}]_-^+ = 0 & \mu A_1 \ell = \bar{\mu} \alpha \beta\,, \\ y = 0: & [u]_-^+ = 0 & A_2 = \alpha\,. \end{array}$$

Hence

$$\bar{\mu}\beta = \mu \ell \tan \ell h\,,$$

which, on substitution, yields the dispersion relation relating $c(k)$ to $k$:

$$\tan\left[\left(\frac{c^2}{c_S^2} - 1\right)^{\frac{1}{2}} kh\right] = \frac{\bar{\mu}}{\mu}\left(\frac{1 - \frac{c^2}{\bar{c}_S^2}}{\frac{c^2}{c_S^2} - 1}\right)^{\frac{1}{2}}. \tag{4.2}$$

This equation can be solved graphically by plotting the LHS and RHS for $c_S < c < \bar{c}_S$, noting that over this range, $\phi = \left(\frac{c^2}{c_S^2} - 1\right)^{\frac{1}{2}} kh$ increases from 0 to $\left(\frac{\bar{c}_S^2}{c_S^2} - 1\right)^{\frac{1}{2}} kh$.

*Remarks*

(i) From the graph we see that there is at least one solution with $c_S < c < \bar{c}_S$ for all $k$. Alternatively this can also be deduced by considering

$$f(c,k) = \tan\left[\left(\frac{c^2}{c_S^2} - 1\right)^{\frac{1}{2}} kh\right] - \frac{\bar{\mu}}{\mu}\left(\frac{1 - \frac{c^2}{\bar{c}_S^2}}{\frac{c^2}{c_S^2} - 1}\right)^{\frac{1}{2}}. \tag{4.3}$$

The dispersion relation is $f(c,k) = 0$, while

$$f(\bar{c}_S, k) > 0\,, \quad \text{and} \quad f(c,k) \to -\infty \quad \text{as} \quad c \to c_S+\,.$$

Thus there exists at least one root for all $k$.

(ii) There are $n$ solutions if

$$(n-1)\pi \leqslant \left(\frac{\bar{c}_S^2}{c_S^2} - 1\right)^{\frac{1}{2}} kh < n\pi\,.$$

(iii) As $k$ and $\phi$ increase, the tan curves 'squash' to the left, which implies that each new mode appears with $c = \bar{c}_S$ and the phase speed then decreases towards $c_S$. The $n^{\text{th}}$ mode has a minimum 'cut-off' frequency (using the fact that $\omega \equiv kc = k\bar{c}_S$ at cut-off)

$$\omega = \frac{n\pi \bar{c}_S}{h}\left(\frac{\bar{c}_S^2}{c_S^2} - 1\right)^{-\frac{1}{2}}.$$

(iv) Since $c$ depends on $k$, i.e. $c \equiv c(k)$, the waves are *dispersive.*

(v) If the lower layer is not present, i.e. $\bar{\mu} = 0$, then the dispersion relation (4.2) reduces to

$$\tan\left[\left(\frac{c^2}{c_S^2} - 1\right)^{\frac{1}{2}} kh\right] = 0\,, \tag{4.4a}$$

i.e.

$$\omega^2 = c_S^2\left(k^2 + \left(\frac{m\pi}{h}\right)^2\right)\,. \tag{4.4b}$$

This is the dispersion relation for the 'classical' 2D waveguide (cf. (1.32)).

(vi) *Mean energy propagation velocity (unlectured).* If sufficiently determined/perverse, one can confirm that the mean energy propagation velocity is equal to the group velocity.

$$\begin{aligned}
< E_k > &= \int_{-\infty}^{0} \tfrac{1}{2}\mathrm{Re}\left[\tfrac{1}{2}\bar{\rho}\dot{u}\dot{u}^*\right] dy + \int_0^h \tfrac{1}{2}\mathrm{Re}\left[\tfrac{1}{2}\rho\dot{u}\dot{u}^*\right] dy \\
&= \frac{\bar{\mu}A_2^2\omega^2}{8\beta c_S^2}\left[\frac{1 - \frac{c_S^2}{\bar{c}_S^2}}{\frac{c^2}{c_S^2} - 1} + \frac{2\beta^2 h}{\ell\sin 2\ell h}\right]. \\
< E_p > &= \int_{-\infty}^{h} \tfrac{1}{4}\mathrm{Re}\,[\sigma_{ij}e_{ij}^*] dy \\
&= \int_{-\infty}^{0} \tfrac{1}{2}\bar{\mu}\mathrm{Re}\left[e_{xz}e_{xz}^* + e_{zx}e_{zx}^* + e_{yz}e_{yz}^* + e_{zy}e_{zy}^*\right] dy + \int_0^h \mu\mathrm{Re}\left[e_{xz}e_{xz}^* + e_{yz}e_{yz}^*\right] dy \\
&= \frac{\bar{\mu}A_2^2\omega^2}{8\beta c_S^2}\left[\frac{1 - \frac{c_S^2}{\bar{c}_S^2}}{\frac{c^2}{c_S^2} - 1} + \frac{2\beta^2 h}{\ell\sin 2\ell h}\right] \\
&= < E_k > \\
< I_x > &= \int_{-\infty}^{h} -\tfrac{1}{2}\mathrm{Re}\left[\sigma_{xz}\dot{u}^*\right] \\
&= \frac{\bar{\mu}A_2^2\omega k}{4\beta}\left[\frac{c^2\left(1 - \frac{c_S^2}{\bar{c}_S^2}\right)}{c_S^2\left(\frac{c^2}{c_S^2} - 1\right)} + \frac{2\beta^2 h}{\ell\sin 2\ell h}\right] \\
c_g \equiv \frac{\partial\omega}{\partial k} &= \frac{c_S^2}{c}\left[\frac{c^2\left(1 - \frac{c_S^2}{\bar{c}_S^2}\right)}{c_S^2\left(\frac{c^2}{c_S^2} - 1\right)} + \frac{2\beta^2 h}{\ell\sin 2\ell h}\right]\left[\frac{1 - \frac{c_S^2}{\bar{c}_S^2}}{\frac{c^2}{c_S^2} - 1} + \frac{2\beta^2 h}{\ell\sin 2\ell h}\right]^{-1}.
\end{aligned}$$

Hence

$$\text{Mean energy propagation velocity} = U(k) = \frac{< I_x >}{< E_k + E_p >} = c_g \neq c\,.$$

## 4.2 Initial Value (Cauchy) Problems

### 4.2.1 General solution of the beam equation

For clarity reduce from 2D/3D to 1D and consider the simpler beam equation (imagine a twanged ruler):

$$\varphi_{tt} + \gamma^2\varphi_{xxxx} = 0\,, \tag{4.5a}$$

where $\gamma^2 = \frac{B}{m} > 0$, and $m$ and $B$ are the mass and bending moment per unit length of the beam. Since the equation is second order in time, two initial conditions are required; suppose that at $t = 0$

$$\varphi(x, 0) = \varphi_0(x) \quad \text{and} \quad \varphi_t(x, 0) = v_0(x)\,. \tag{4.5b}$$

Seek a solution by Fourier transform:

$$\varphi(x, t) = \int_{-\infty}^{\infty} \widehat{\varphi}(k, t)\, e^{ikx} dk\,. \tag{4.6a}$$

Then we require that

$$\widehat{\varphi}_{tt} + \gamma^2 k^4 \widehat{\varphi} = 0\,, \tag{4.6b}$$

with solution

$$\widehat{\varphi} = \mathcal{A}(k)\, e^{-i\omega t}\,, \quad \text{where} \quad -\omega^2 + \gamma^2 k^4 = 0\,. \tag{4.6c}$$

Hence $\omega$ satisfies the dispersion relation for plane waves, $e^{i(kx-\omega t)}$,

$$\omega = \pm\gamma k^2 = \pm\Omega(k) \quad \text{where} \quad \Omega(k) = \gamma k^2\,. \tag{4.6d}$$

The general solution to (4.5a) is thus a linear superposition of plane waves $e^{i(kx\pm\Omega t)}$:

$$\varphi(x,t) = \int_{-\infty}^{\infty} A(k) e^{ikx - i\Omega(k)t} dk + \int_{-\infty}^{\infty} B(k) e^{ikx + i\Omega(k)t} dk\,. \tag{4.7}$$

If the initial conditions (4.5b) are to be satisifed then

$$\varphi_0(x) = \int_{-\infty}^{\infty} [A(k) + B(k)] e^{ikx} dk\,, \tag{4.8a}$$

$$v_0(x) = -\,i \int_{-\infty}^{\infty} \Omega(k)[A(k) - B(k)] e^{ikx} dk\,. \tag{4.8b}$$

Invert these Fourier transforms to obtain

$$A + B = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_0(x) e^{-ikx} dx\,, \tag{4.9a}$$

$$-i\gamma k^2 (A - B) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v_0(x) e^{-ikx} dx\,. \tag{4.9b}$$

For definiteness, suppose further that $v_0 = 0$, so that $A(k) = B(k)$. Then to obtain a solution one needs to be able to evaluate

$$\varphi_{\pm}(x,t) = \int_{-\infty}^{\infty} A(k) e^{ikx \pm i\Omega(k)t} dk\,. \tag{4.10}$$

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*Remarks*

(i) All information that (4.7) is a solution to the beam equation (4.5a), rather than another equation, is contained in the solutions to the dispersion relation, i.e. from (4.6d), $\omega = \pm\Omega(k)$.

(ii) Solutions to many equations have the generic form (4.7), where $\omega = \pm\Omega(k)$, or similar, are solutions to the dispersion relation. For instance, suppose that

$$\mathcal{L}(\partial_t, \partial_x)\varphi = 0\,, \tag{4.11a}$$

where $\mathcal{L}$ is another linear, constant coefficient, operator. Then the dispersion relation is

$$\mathcal{L}(-i\omega, ik) = 0\,. \tag{4.11b}$$

Solve this equation for $\omega = \omega_j(k)$, where the number of roots equals the order, say $n$, of the equation in time. The general solution then consists of the sum of $n$ Fourier integrals.

*Example.* Suppose

$$\frac{\partial\varphi}{\partial t} + \sqrt{gh}\,\frac{\partial\varphi}{\partial x} + h^2\sqrt{gh}\,\frac{\partial^3\varphi}{\partial x^3} = 0\,,$$

then

$$\mathcal{L}(\partial_t, \partial_x) = \partial_t + \sqrt{gh}\,\partial_x + h^2\sqrt{gh}\,\partial_x^3\,,$$

and the dispersion relation is given by

$$\mathcal{L}(-i\omega, ik) = -i\omega + \sqrt{gh}\,ik - ih^2\sqrt{gh}\,k^3 = 0\,,$$

i.e.

$$\omega = \sqrt{gh}\,k - h^2\sqrt{gh}\,k^3\,.$$

16/04 This is a first-order equation in time, so one initial condition is needed, there is one root of the
16/16 dispersion relation, and there will be one Fourier integral in the solution.
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### 4.2.2 Exact solution of the beam equation for Gaussian modulation

Consider the initial conditions

$$\varphi_0(x) = \exp\left(-\left(\frac{x}{\lambda}\right)^2\right)\exp(i\alpha x)\ , \qquad v_0 = 0\,, \tag{4.12a}$$

where $\alpha$ and $\lambda$ are positive constants, and the real part is understood. This represents a 'carrier' harmonic mode with wavenumber $\alpha$, modulated by a Gaussian of width $\lambda$. Then

$$\begin{aligned} A(k) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-(x/\lambda)^2} e^{-i(k-\alpha)x}dx \\ &= \frac{\lambda}{4\sqrt{\pi}}\exp\left(-\frac{\lambda^2(k-\alpha)^2}{4}\right)\ . \end{aligned} \tag{4.12b}$$

From (4.6d) and (4.10)

$$\begin{aligned} \varphi_\pm &= \int_{-\infty}^{\infty}\frac{\lambda}{4\sqrt{\pi}}\exp\left(-\frac{\lambda^2(k-\alpha)^2}{4} + ikx \pm i\gamma k^2 t\right)dk \\ &= \frac{\lambda}{2\left(\lambda^2 \mp 4i\gamma t\right)^{\frac{1}{2}}}\exp\left(i\alpha x \pm i\gamma\alpha^2 t - \frac{(x\pm 2\gamma\alpha t)^2}{\lambda^2\mp 4i\gamma t}\right)\ . \end{aligned} \tag{4.13}$$

*Slow modulation.* Suppose that the carrier wave is slowly modulated, i.e. $\alpha\lambda \gg 1$, and consider times $t \ll \lambda^2/\gamma$. Since $\lambda$ is large, the latter restriction still includes 'moderate' times when $t \sim \lambda/(\alpha\gamma)$. Then by Taylor expansion of (4.13)

$$\begin{aligned} \varphi_\pm &\sim \tfrac{1}{2}\exp\left(i\alpha(x\pm\gamma\alpha t)\right)\exp\left(-\left(\frac{x\pm 2\gamma\alpha t}{\lambda}\right)^2\right) \\ &\sim \tfrac{1}{2}\underbrace{\exp\left(i\alpha(x\pm ct)\right)}_{\text{carrier wave}}\underbrace{\exp\left(-\left(\frac{x\pm c_g t}{\lambda}\right)^2\right)}_{\text{modulation envelope}}, \end{aligned} \tag{4.14}$$

where, in agreement with (4.6d), the phase speed is given by $c(\alpha) = \Omega(\alpha)/\alpha = \gamma\alpha$, and the group velocity by $c_g(\alpha) = \frac{\partial\Omega}{\partial k}(\alpha) = 2\gamma\alpha$.

*Remarks*

(i) The factor of $\frac{1}{2}$ arises because half of the initial disturbance propagates to the right and half to the left.

(ii) The slowly varying modulation envelope, i.e. the 'group' of waves occupying many wavelengths, propagates with the group velocity, $c_g$, of the carrier wave.

(iii) Within the group, individual phase fronts of the carrier wave propagate with the phase speed, $c$, which maybe faster or slower than the group velocity; here $c = \frac{1}{2}c_g$.

(iv) Waves must be continually created/destroyed at the front/back of the packet, or vice versa.

(v) To make the two scales explicit, once can introduce 'slow' variables

$$X = \frac{x}{\lambda} \quad \text{and} \quad \tau = \frac{t}{\lambda}\,. \tag{4.15a}$$

Then (4.14) becomes

$$\varphi_\pm \sim \tfrac{1}{2}\exp\left(i\alpha(x \pm ct)\right)\exp\Big(-\left(X \pm c_g\tau\right)^2\Big)\,. \tag{4.15b}$$

(vi) We consider larger times below, i.e. times when the wavepacket has travelled many times its initial width (plus a bit).

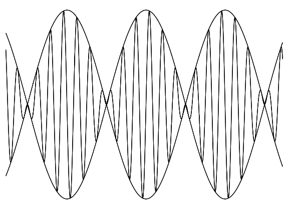
### 4.2.3 Modulation of periodic wavetrains

Consider the superposition of two waves of equal amplitude and nearly equal wavenumbers and frequencies, say

$$\begin{aligned}\varphi &= \cos(k_1x - \omega_1 t) + \cos(k_2x - \omega_2 t)\\ &= 2\cos\left(\tfrac{1}{2}(k_1+k_2)x - \tfrac{1}{2}(\omega_1+\omega_2)t\right)\cos\left(\tfrac{1}{2}(k_1-k_2)x - \tfrac{1}{2}(\omega_1-\omega_2)t\right)\\ &= 2\cos\left(kx-\omega t\right)\cos\left(\varepsilon(\kappa x - \Omega t)\right)\end{aligned} \tag{4.16a}$$

where

$$k = \tfrac{1}{2}(k_1+k_2)\,,\quad \omega = \tfrac{1}{2}(\omega_1+\omega_2)\,,\quad \varepsilon\kappa = \tfrac{1}{2}(k_1-k_2)\,,\quad \varepsilon\Omega = \tfrac{1}{2}(\omega_1-\omega_2)\,. \tag{4.16b}$$



This is a modulated wave where the individual wavecrests move at the phase speed

$$c = \frac{\omega}{k} = \frac{\omega_1+\omega_2}{k_1+k_2}\,, \tag{4.17a}$$

but the envelope moves at the group velocity

$$c_g = \frac{\varepsilon\Omega}{\varepsilon\kappa} = \frac{\omega_1-\omega_2}{k_1-k_2} \approx \frac{d\omega}{dk}\,. \tag{4.17b}$$

### 4.2.4 Large time solution — Kelvin's method of stationary phase

Dispersive waves disperse!

For an initially compact disturbance, at large times we might expect to find disturbances with wavenumber $\alpha$ a distance $c_g(\alpha)t$ away. Hence if $c_g$ is not constant, the initial disturbance will spread out, and we expect the wavenumber and amplitude to be slowly varying functions of $x$ at large times.

Seek the large time behaviour of

$$\Phi(x,t) = \int_{-\infty}^{\infty} A(k) e^{ikx - i\omega(k)t}\, dk \ . \tag{4.18a}$$

For large times $t \gg 1$ we expect $x$ to be large (since $x = O(c_g t)$); hence write

$$x = Vt \ . \tag{4.18b}$$

The integral (4.18a) can then be rewritten as

$$\Phi = \int_{-\infty}^{\infty} A(k) e^{it\psi(k)}\, dk\,, \tag{4.18c}$$

where the *phase* $\psi$ is given by

$$\psi(k) = kV - \omega(k) \ . \tag{4.18d}$$

In order to find a large-time approximation of (4.18c) we first note that 'rapid oscillations lead to cancellation'.

*The 'tails' of Fourier transforms.* For example, consider

$$J(t) = \int_{-\infty}^{\infty} A(k) e^{itk}\, dk \ ,$$

where $A$ is $C^\infty$, and $A^{(n)}(k) \to 0$ as $|k| \to \infty$ (e.g. $A(k) = e^{-k^2}$). Then, from integration by parts,

$$\begin{aligned} J(t) &= -\frac{1}{it} \int_{-\infty}^{\infty} A'(k) e^{itk}\, dk \\ &= \left(\frac{i}{t}\right)^2 \int_{-\infty}^{\infty} A''(k) e^{itk}\, dk \ , \\ &= \left(\frac{i}{t}\right)^n \int_{-\infty}^{\infty} A^{(n)}(k) e^{itk}\, dk \ . \end{aligned}$$

If $\int_{-\infty}^{\infty} \left|A^{(n)}(k)\right| dk$ is well behaved, it follows that as $t \to \infty$, $J(t) \to 0$ faster than any algebraic power of $t$. Colloquially $J(t)$ is said to be exponentially small as $t \to \infty$.

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17/17 It is possible to prove similar exponential decay for $\Phi(x,t)$ as $t \to \infty$ ***if*** $\psi(k)$ is a *monotonic* function of $k$. However, if this is not the case, i.e. if there exists $k = \alpha$ such that

$$\psi' = V - \omega'(\alpha) = V - c_g(\alpha) = 0\,, \tag{4.19a}$$

then the cancellation is mitigated for wavenumbers $k$ close to $\alpha$. To see this, Taylor expand $\psi(k)$ and $e^{it\psi(k)}$ for $|k - \alpha| \ll 1$ (or, to be more precise, for $(k-\alpha) = O(\,|\omega''(\alpha)|^{-\frac{1}{2}}\, t^{-\frac{1}{2}})$ to obtain

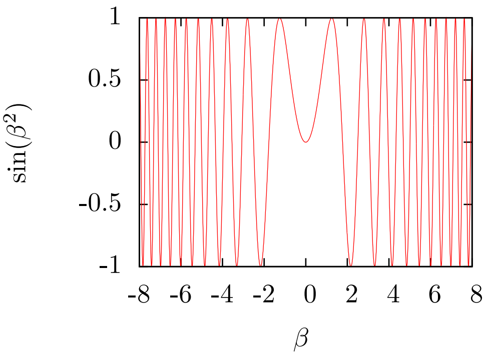
$$\begin{aligned} \psi(k) &\sim \psi(\alpha) + \tfrac{1}{2}\psi''(\alpha)\,(k-\alpha)^2 + \dots \ , \\ e^{it\psi(k)} &\sim e^{it\psi(\alpha)} \left(\cos\beta^2 + i\sigma \sin\beta^2\right) + \dots \ , \end{aligned}$$

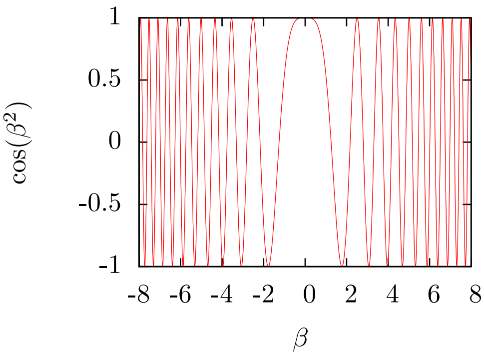
where

$$\beta = \left[\frac{t\,|\psi''(\alpha)|}{2}\right]^{\frac{1}{2}} (k-\alpha) \ , \tag{4.19b}$$

$$\sigma = \operatorname{sgn}\left[\psi''(\alpha)\right] = -\operatorname{sgn}\left[\omega''(\alpha)\right] \ . \tag{4.19c}$$

The cancellation due to the rapid change in phase is least, and hence the contributions to $\Phi$ greatest, in
17/16 the neighbourhood of wavenumber**s**, $\alpha$, where the phase $\psi$ is stationary (Kelvin).




*Rescale for $t \gg 1$.* Suppose there exists a single stationary point, and let

$$\epsilon = \left[ \frac{2}{t\,|\psi''(\alpha)|} \right]^{\frac{1}{2}} \ll 1 \, . \tag{4.20}$$

Then using (4.19b), i.e. $k = \alpha + \epsilon\beta$, rewrite (4.10) in terms of the scaled wavenumber $\beta$, and Taylor expand:

$$\begin{aligned} \Phi &= \int_{-\infty}^{\infty} A(\alpha + \epsilon\beta) \, \exp\left( it\psi(\alpha) + i\sigma\beta^2 + O(\epsilon\beta^3) \right) \epsilon \, d\beta \\ &= \epsilon A(\alpha) \, e^{it\psi(\alpha)} \int_{-\infty}^{\infty} e^{i\sigma\beta^2} \, (1 + O(\epsilon)) \, d\beta \, . \end{aligned}$$

From contour integration or otherwise,

$$\int_{-\infty}^{\infty} e^{\pm iu^2} \, du = \pi^{\frac{1}{2}} e^{\pm i\pi/4} \, , \tag{4.21}$$

thus from (4.19b)

$$\Phi = \left( \frac{2\pi}{|\psi''(\alpha)|\, t} \right)^{\frac{1}{2}} A(\alpha) \, e^{i\psi(\alpha)t + i\sigma\pi/4} \left( 1 + O(t^{-\frac{1}{2}}) \right) \, , \tag{4.22}$$

where we have assumed that

(i) $A(\alpha) \neq 0$;

(ii) $\psi''(\alpha) = -\omega''(\alpha) \neq 0$.

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*Summary.* For $t \gg 1$,

$$\Phi \sim \left( \frac{2\pi}{|\omega''(\alpha)|\, t} \right)^{\frac{1}{2}} A(\alpha) \, e^{i\alpha x - i\omega(\alpha)t + i\sigma\pi/4} \, , \tag{4.23a}$$

where $\sigma = -\text{sgn}[\omega''(\alpha)]$, and the wavenumber $\alpha$ satisfies, see (4.18d) and (4.19a),

$$\psi'(\alpha) = 0 \, ,$$

i.e.

$$x = \omega'(\alpha)\, t = c_g(\alpha)\, t \, . \tag{4.23b}$$

Hence to find the solution for given large $x$ and $t$, solve (4.23b) for the wavenumber $\alpha$, and substitute back into (4.23a).

*Interpretation.* Harmonic waves of a given wavenumber $\alpha$, and corresponding frequency $\omega(\alpha)$, are observed in the neighbourhood of points propagating with the *group velocity*. The amplitude of the waves decays like $t^{-\frac{1}{2}}$.

*Radiation condition.* Energy must travel outwards from a source, i.e. $\mathbf{c}_g$ must be directed away from a source. The direction of the phase velocity is immaterial.

Further, suppose that $t = 0$ oscillations of frequency $\omega_0$ and wavenumber $k_0$, where $\omega(k_0) = \omega_0$, are generated at $x = 0$. Then there will be no disturbance at $x = x_0$ for

$$t < \frac{x_0}{c_g(k_0)} \,.$$

*Additional remarks.*

(i) Only waves with $k$'s for which $A(k)$ is significant will be observed.

(ii) Originally localised waves are dispersed, and hence the amplitude decreases. The decay like $t^{-1/2}$ follows from conservation of energy; the extent of the disturbance is proportional to $t$, while energy $\propto \int |\varphi|^2 dx$, hence $\varphi$ must decay like $t^{-\frac{1}{2}}$.

(iii) There may be more than one point of stationary phase, i.e. more than one $\alpha$ may satisfy (4.23b); e.g. if $\omega(k)$ is odd, then $c_g(k)$ is even, and so $\pm\alpha$ satisfy (4.23b). The contributions from each point of stationary phase need to be added.

(iv) If $\omega(k)$ is even, then waves with wavenumber $(-\alpha)$ will be found where

$$\frac{x}{t} = -\omega'(\alpha) \,.$$

(v) If $\omega''(\alpha) = 0$, then it is necessary to include higher-order terms in the Taylor expansion.

### 4.2.5 Example: beam equation

Consider, from (4.7) and (4.10),

$$\varphi(x,t) = \varphi_-(x,t) + \varphi_+(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - \Omega(k)t)} dk + \int_{-\infty}^{\infty} A(k) e^{i(kx + \Omega(k)t)} dk \,. \tag{4.24a}$$

where $\Omega(k) = \gamma k^2$. From (4.18d)

$$\psi_\pm(k) = Vk - \omega(k) = Vk \mp \Omega(k) \,.$$

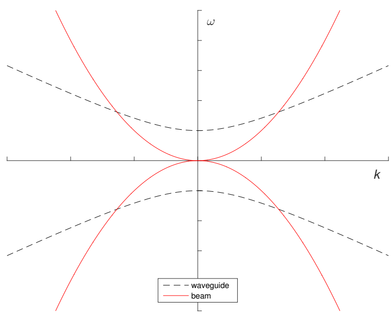
Hence the stationary points satisfy

$$V = \frac{x}{t} = \pm\Omega'(k) \,,$$

i.e. the stationary points are at

$$k = \pm\frac{x}{2\gamma t} = \pm\alpha \,,$$

where $\alpha = x/2\gamma t$. Note that, compared with our earlier notation, the stationary points are at $+\alpha$ *and* $-\alpha$, so there is a need to keep careful track of signs.




All branches of the dispersion relations for the waveguide, (1.32) & (4.25a), and the beam equation, (4.6d); that for the waveguide has a bounded slope ($\Rightarrow$ bounded $c_g$).

Next, substitute into (4.23a), first with $\omega = \Omega(k) = \gamma k^2$ and then with $\omega = -\Omega(k) = -\gamma k^2$, noting that at a stationary point $\omega''(k) = \pm 2\gamma$, and $\sigma = -\operatorname{sgn}[\omega''(k)] = \mp 1$. It follows that

$$\varphi \sim \left(\frac{\pi}{\gamma t}\right)^{\frac{1}{2}} \left( A(\alpha) e^{i\alpha x - i\gamma\alpha^2 t - i\pi/4} + A(-\alpha) e^{-i\alpha x + i\gamma\alpha^2 t + i\pi/4} \right) . \tag{4.24b}$$

Since $\varphi_0$ is real and $A(k) = B(k)$ (because we have assumed $v_0 = 0$), we conclude from (4.9a) that

$$A(-k) = A^*(k) \,,$$

where $*$ indicates the complex conjugate (c.c). Hence $\varphi$ is reassuringly real:

$$\varphi \sim \left(\frac{\pi}{\gamma t}\right)^{\frac{1}{2}} \left( A(\alpha) e^{i\alpha x - i\gamma\alpha^2 t - i\pi/4} + \text{c.c.} \right) \tag{4.24c}$$

$$= \left(\frac{\pi}{\gamma t}\right)^{\frac{1}{2}} \left( A(\alpha) e^{i(x^2/\gamma t - \pi)/4} + \text{c.c.} \right) . \tag{4.24d}$$

### 4.2.6 Example: dispersion in a waveguide *(unlectured)*

From the dispersion relation for the waveguide, (1.32),

$$\omega = \pm\left(\omega_0^2 + k^2 c_0^2\right)^{\frac{1}{2}} = \pm\Omega(k) \tag{4.25a}$$

where $\omega_0^2 = \frac{m^2\pi^2}{h^2} + \frac{n^2\pi^2}{b^2}$ for some fixed choice of $m$ and $n$. Then

$$\psi_\pm = Vk \mp \left(\omega_0^2 + k^2 c_0^2\right)^{\frac{1}{2}} \;, \tag{4.25b}$$

and $\psi_\pm' = 0$, when

$$V = \pm\frac{kc_0^2}{\left(\omega_0^2 + k^2 c_0^2\right)^{\frac{1}{2}}}\;.$$

For each $-c_0 < V = x/t < c_0$ there is one root on each branch; specifically $\psi_\pm' = 0$ at

$$k = \pm\frac{\omega_0 V}{c_0\left(c_0^2 - V^2\right)^{\frac{1}{2}}} = \pm\alpha\;. \tag{4.25c}$$

From (4.5a), (4.6d) and $A(k) = B(k)$,

$$\varphi(x,t) = \int_{-\infty}^{\infty} A(k)\left(e^{i\psi_+ t} + e^{i\psi_- t}\right)\,dk\,,$$

and

$$\psi_\pm''(k) = \mp\frac{\omega_0^2 c_0^2}{\left(\omega_0^2 + k^2 c_0^2\right)^{\frac{3}{2}}} = \mp\frac{\omega_0^2 c_0^2}{\Omega^3(k)}\;.$$

So from (4.22), (4.23a) and (4.23b)

$$\varphi(x,t) \approx \left[\frac{2\pi\Omega^3(\alpha)}{\omega_0^2 c_0^2 t}\right]^{\frac{1}{2}}\left(A(\alpha)\,e^{i\alpha x - i\Omega(\alpha)t - \frac{i\pi}{4}} + A(-\alpha)\,e^{-i\alpha x + i\Omega(-\alpha)t + \frac{i\pi}{4}}\right)\;.$$

As in the previous example, if $\varphi_0$ is real, then it follows from $A(k) = B(k)$ and (4.8a) that

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$$A(-k) = A^*(k)\,.$$

Hence, since $\Omega(-k) = \Omega(k)$,

$$\varphi(x,t) \approx \underbrace{\left[\frac{2\pi\Omega(\alpha)^3}{\omega_0^2 c_0^2 t}\right]^{\frac{1}{2}}\left(A(\alpha)\,e^{i\alpha x - i\Omega(\alpha)t - \frac{i\pi}{4}} + \text{c.c.}\right)}_{\text{real}}\;, \tag{4.26a}$$

where

$$\alpha = \frac{\omega_0 V}{c_0\left(c_0^2 - V^2\right)^{\frac{1}{2}}} = \frac{\omega_0 x}{c_0\left(c_0^2 t^2 - x^2\right)^{\frac{1}{2}}}\;. \tag{4.26b}$$

*Remark.* There is no stationary phase point if $|V| \geqslant c_0$, i.e. $|x| \geqslant c_0 t$. $\psi$ is then monotonic, and the solution is exponentially small.

## 4.3 Capillary-Gravity Waves

Assume inviscid, irrotational, incompressible flow. Hence

$$\mathbf{u} = \boldsymbol{\nabla}\varphi\,, \quad \boldsymbol{\nabla}.\mathbf{u} = 0 \quad \text{and} \quad \nabla^2\varphi = 0\,. \tag{4.27}$$

*Bottom boundary condition.* The boundary condition at the rigid bottom is

$$\varphi_y = 0 \qquad \text{on} \qquad y = -h \ . \tag{4.28}$$

*Kinematic boundary condition at surface.*

$$\frac{D}{Dt}(y - \eta(x, z, t)) = 0 \qquad \text{on} \qquad y = \eta \ ,$$

i.e.

$$v = \eta_t + u\eta_x + w\eta_z \qquad \text{on} \qquad y = \eta \ . \tag{4.29a}$$

Linearise for small amplitude waves such that

$$|\eta| \ll \min(h, \lambda) \quad \text{and} \quad |u| \ll \omega\lambda \, , \tag{4.29b}$$

where $\lambda$ and $\omega$ are the wavelength and wave frequency, respectively. Then, since

$$v(x, \eta, z, t) \approx v(x, 0, z, t) + \eta v_y(x, 0, z, t) + \dots \ ,$$

we have

$$\varphi_y = v = \eta_t \qquad \text{on} \qquad y = 0 \ . \tag{4.29c}$$

*Dynamic boundary condition at surface.* Assume that the free surface is at constant pressure and, unlike Part IB, allow for surface tension. Then

$$p - p_{atm} = -T(\kappa_1 + \kappa_2) = T\,\nabla \cdot \mathbf{n} \quad \text{on} \quad y = \eta \ , \tag{4.30a}$$

where $T$ is a the coefficient of surface tension, $\kappa_1$ and $\kappa_2$ are the principal curvatures (positive if concave upward), and $\mathbf{n}$ is the unit normal.[13] From Bernoulli's equation for unsteady potential flow with gravity:

$$\rho(\varphi_t + \tfrac{1}{2}|\nabla\varphi|^2 + gy) + p - p_{atm} = F(t) \ . \tag{4.30b}$$

W.l.o.g we can assume $F(t) = 0$, by $\varphi \to \varphi + \int F dt$. Substitute into (4.30a) to obtain

$$\rho(\varphi_t + \tfrac{1}{2}|\nabla\varphi|^2 + g\eta) + T\,\nabla \cdot \mathbf{n} = 0 \qquad \text{on} \qquad y = \eta \ . \tag{4.30c}$$

Linearise to obtain

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$$\rho\varphi_t + \rho g\eta - T(\eta_{xx} + \eta_{zz}) = 0 \qquad \text{on} \qquad y = 0 \ . \tag{4.30d}$$

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[13] $(\kappa_1 + \kappa_2)$ is twice the *mean curvature*, and is given by (see *Vector Calculus* for 2D),

$$\begin{aligned}\kappa_1 + \kappa_2 &= -\nabla \cdot \mathbf{n} = -\nabla \cdot \left( \frac{\boldsymbol{\nabla}(y - \eta)}{|\boldsymbol{\nabla}(y - \eta)|} \right) \\ &= \left[ \left( 1 + \left( \frac{\partial \eta}{\partial z} \right)^2 \right) \frac{\partial^2 \eta}{\partial x^2} - 2 \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial z} \frac{\partial^2 \eta}{\partial x \partial z} + \left( 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 \right) \frac{\partial^2 \eta}{\partial z^2} \right] \left( 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 + \left( \frac{\partial \eta}{\partial z} \right)^2 \right)^{-3/2} .\end{aligned}$$

*Seek Fourier-mode solution.* Try

$$\varphi = f(y)e^{ik_1x+ik_3z-i\omega t}\,, \tag{4.31a}$$

$$\eta = A\; e^{ik_1x+ik_3z-i\omega t}\,. \tag{4.31b}$$

From (4.27), $\nabla^2\varphi = 0$, which implies

$$f'' - k^2 f = 0\,, \quad \text{where} \quad k = \pm(k_1^2 + k_3^2)^{\frac{1}{2}}\,.$$

The bottom boundary condition, (4.28), implies that $f'(-h) = 0$, and hence

$$f = B\cosh k(y+h)\ . \tag{4.32a}$$

Then from the kinematic boundary condition, (4.29c), it follows that the amplitude, $A$, of the wave is given by

$$A = \frac{ikB}{\omega}\sinh kh\,. \tag{4.32b}$$

Finally, the dynamic boundary condition, (4.30d), yields the *dispersion relation* for gravity waves with surface tension, i.e. capillary-gravity waves:

$$\omega^2 = gk\left(1 + \frac{Tk^2}{\rho g}\right)\tanh kh\,. \tag{4.33a}$$

18/18
18/19 *Group velocity.* The *group velocity* is given by

$$c_g = \frac{\partial\omega}{\partial k} = \frac{\omega}{2k}\left(\frac{1+3\beta k^2}{1+\beta k^2} + \frac{2kh}{\sinh 2kh}\right)\,, \tag{4.33b}$$

where the surface tension parameter, $\beta$, is defined by

$$\beta = \frac{T}{\rho g}\,. \tag{4.33c}$$

As in previous examples, it is possible to show that the group velocity, $c_g$, is again the mean wave-energy propagation speed.

### 4.3.1 Limiting Cases

(i) *Gravity waves*: $T = 0$, i.e. $\beta = 0$.

$$\omega^2 = gk\,\tanh kh\ , \tag{4.34a}$$

$$c_g = \frac{\omega}{2k}\left(1 + \frac{2kh}{\sinh 2kh}\right) < \frac{\omega}{k} = c_p\,, \tag{4.34b}$$

where, to fix signs, we assume $\omega k > 0$. 'New' waves appear at the *back* of the group.

(ii) *Shallow-water waves*: $kh \to 0$ by letting the depth, $h$, become small.

$$\omega^2 \approx ghk^2\left(1+\beta k^2\right)\ , \tag{4.35a}$$

$$c_g \approx \frac{\omega}{k}\left(\frac{1+2\beta k^2}{1+\beta k^2}\right) > \frac{\omega}{k} = c_p\,, \tag{4.35b}$$

where again we assume $\omega k > 0$. 'New' waves appear at the *front* of the group. For surface tension to be important need $h \approx 1\,\text{mm}$, at which point it is debatable as to whether viscous effects are negligible.

(iii) *Long gravity waves (or ocean waves)*: $kh \to 0$, $\beta k^2 \to 0$ by letting the wavenumber, $k$, become small.

$$\omega^2 \approx ghk^2\ , \tag{4.36a}$$

i.e.

$$\omega \approx \pm\, c\, k \quad \text{where} \quad c = \sqrt{gh}\,. \tag{4.36b}$$

Non-dispersive waves: $c_g \approx c_p$. Relevant, for instance, to tides, tsunamis ($\lambda = 100$km, $h = 1$km, $\Rightarrow$ $c = 100\,\text{ms}^{-1}$).

(iv) *Deep-water, capillary-gravity waves*: $|k|h \to \infty$ by letting the depth, $h$, become large.

$$\omega^2 = g|k|\left(1 + \frac{T}{\rho g}k^2\right) = g|k|\left(1+\beta k^2\right) , \tag{4.37a}$$

$$c_g = \frac{\omega}{2k}\left(\frac{1+3\beta k^2}{1+\beta k^2}\right) \to \begin{cases} \frac{1}{2}c_p & \text{as } k \to 0, \\ \frac{3}{2}c_p & \text{as } k \to \infty. \end{cases} \tag{4.37b}$$

*Remarks (for definiteness, concentrate on the first quadrant of the dispersion relation).*

(a) *Minimum phase speed.* There is a wavenumber,

$$k_2 = \beta^{-\frac{1}{2}} , \tag{4.37c}$$

at which the phase velocity has a minimum. Since

$$\frac{\partial c_p}{\partial k} = \frac{\partial}{\partial k}\left(\frac{\omega}{k}\right) = \frac{1}{k}\left(c_g - c_p\right) , \tag{4.37d}$$

it follows that at the minimum, $c_p(k_2) = c_g(k_2)$. Further

$$c_p \gtrless c_g \quad \text{according as} \quad k \lessgtr k_2 . \tag{4.37e}$$

(b) *Ocean waves.* If $\beta k^2 \ll 1$ (wavelength $\gtrsim 2\,\mathrm{cm}$), then we have 'ocean' waves, with $c_p > c_g$.

(c) *Capillary waves.* If $\beta k^2 \gg 1$ (wavelength $\lesssim 2\,\mathrm{cm}$), then we have capillary waves/ripples, with $c_p < c_g$.

(d) *Degenerate stationary point.* There is a wavenumber, say $k_1$, at which the group velocity has a minimum, and the stationary point is degenerate, i.e.

$$c_g'(k_1) = \omega''(k_1) = 0 \quad \text{for} \quad k_1^2 = \frac{1}{\beta}\left(\frac{2}{\sqrt{3}} - 1\right) . \tag{4.37f}$$

Since

$$\text{amplitude} \propto \left(\frac{1}{\psi''(k)}\right)^{\frac{1}{2}} \propto \left(\frac{1}{\omega''(k)}\right)^{\frac{1}{2}} ,$$

the amplitude increases without bound as $k \to k_1$. In fact the stationary phase approximation fails for $x \approx c_g(k_1)\,t$, with the 'infinite' amplitude problem being resolved by Airy functions.

(e) *Example.* Suppose

$$A(k) = 0 \qquad \text{for} \quad k < k_0 \text{ and } k > k_3 ,$$

where

$$0 < k_0 < k_1 < k_2 < k_3 \qquad \text{and} \qquad c_g(k_0) > c_g(k_3) .$$

Then at large times and $x \gg 1$,

- for $x < c_g(k_1)\,t$, $x > c_g(k_0)\,t$, there will be 'no waves',
- for $c_g(k_3)\,t < x < c_g(k_0)\,t$, there will be one wave,
- for $c_g(k_1)\,t < x < c_g(k_3)\,t$, there will be two waves.

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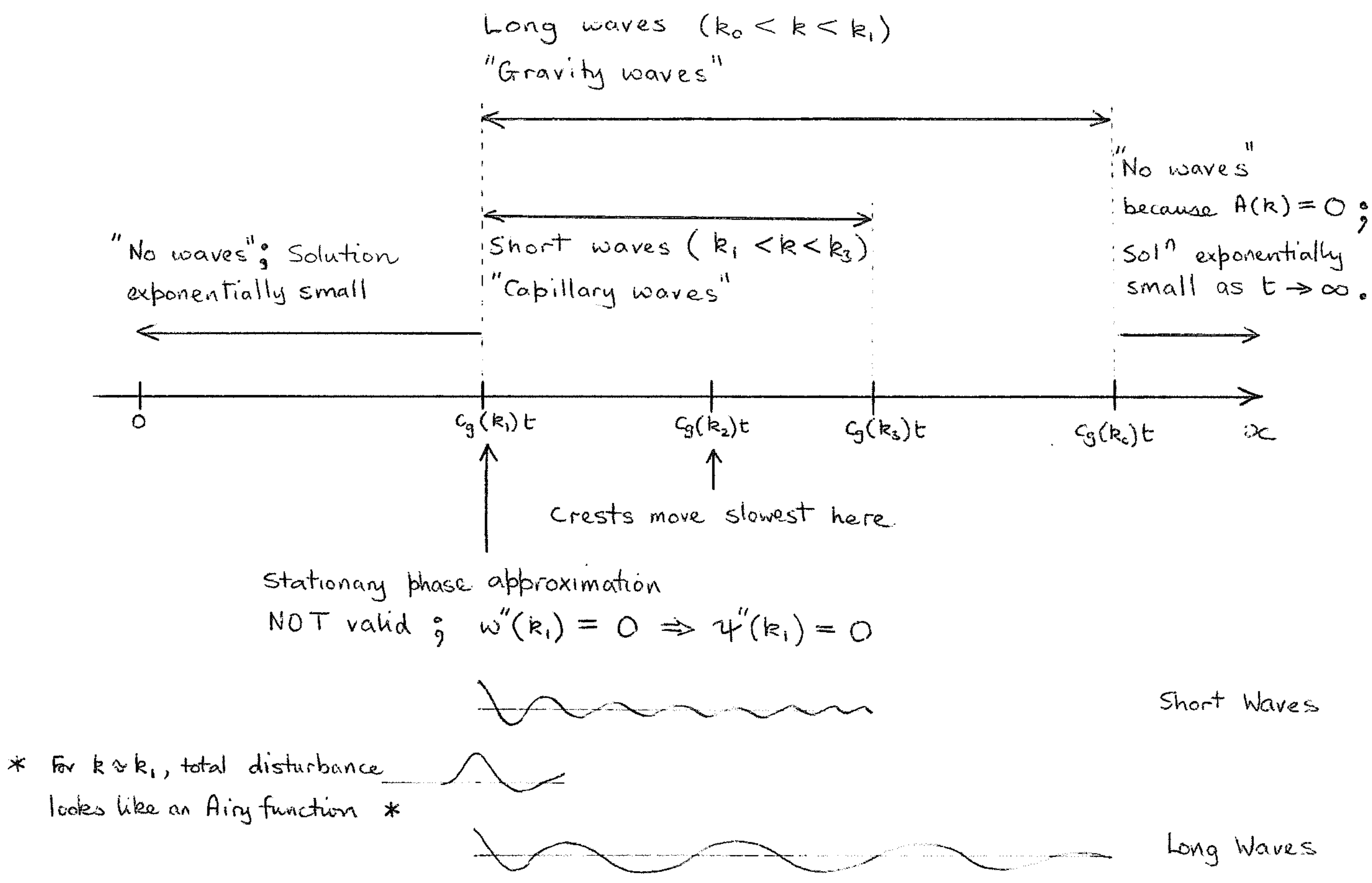

Long waves ($k_0 < k < k_1$)
"Gravity waves"
"No waves"; Solution exponentially small
Short waves ($k_1 < k < k_3$)
"Capillary waves"
"No waves" because $A(k) = 0$; Sol$^n$ exponentially small as $t \to \infty$.
0
$c_g(k_1)t$
$c_g(k_2)t$
$c_g(k_3)t$
$c_g(k_c)t$
$x$
Crests move slowest here
Stationary phase approximation
NOT valid ; $\omega''(k_1) = 0 \Rightarrow \psi''(k_1) = 0$
Short Waves
* For $k \approx k_1$, total disturbance looks like an Airy function *
Long Waves

## 4.4 Internal Gravity Waves in a Stratified Incompressible Fluid

Change convention and put gravity in the $z$ direction. Then, consider a fluid medium in which the mean pressure, $p_0(z)$, and the density, $\rho_0(z)$, are in hydrostatic balance when there is no motion, i.e.

$$\frac{dp_0}{dz} = -\rho_0 g\,. \tag{4.38}$$

Assume that the vertical lengthscale for $O(1)$ changes in $\rho_0$ is $L$, e.g. $L$ is the lengthscale over which the density halves.

19/17 Assume also that the fluid is incompressible, i.e.

$$\frac{D\rho}{Dt} = \rho_t + u\rho_x + v\rho_y + w\rho_z = 0\,, \tag{4.39a}$$

*and*

$$\boldsymbol{\nabla}.\mathbf{u} = u_x + v_y + w_z = 0\,, \tag{4.39b}$$

where $\mathbf{u} = (u, v, w)$. For an inviscid fluid, the motion is governed by momentum equation

$$\rho\frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla}p - \rho g\widehat{\mathbf{z}}\ . \tag{4.39c}$$

Finally, assume that there are small perturbations to the mean state, i.e.

$$p = p_0(z) + \tilde{p}(\mathbf{x}, t)\ , \quad \rho = \rho_0(z) + \tilde{\rho}(\mathbf{x}, t)\ , \quad \text{etc.}$$

Linearise (4.39a), (4.39b) and (4.39c), and use the hydrostatic balance equation (4.38), to obtain

$$\tilde{\rho}_t + w\rho_0'(z) = 0\ , \tag{4.40a}$$
$$u_x + v_y + w_z = 0\ , \tag{4.40b}$$
$$\rho_0 u_t = -\tilde{p}_x\ , \tag{4.40c}$$
$$\rho_0 v_t = -\tilde{p}_y\ , \tag{4.40d}$$
$$\rho_0 w_t = -\tilde{p}_z - \tilde{\rho}g\ . \tag{4.40e}$$

19/16

19/18 Then

$$\text{(4.40a) and (4.40e)} \Rightarrow \qquad \rho_0 w_{tt} = -\tilde{p}_{zt} + \rho_0' g w\,, \tag{4.41a}$$
$$\text{(4.40b), (4.40c) and (4.40d)} \Rightarrow \qquad \rho_0 w_{zt} = \tilde{p}_{xx} + \tilde{p}_{yy}\ . \tag{4.41b}$$

Hence

$$(\rho_0 w_{zz} + \rho_0' w_z)_{tt} - (\partial_{xx}^2 + \partial_{yy}^2)(g\rho_0' - \rho_0\partial_{tt}^2)w = 0\,, \tag{4.42a}$$

19/19 or equivalently

$$\partial_{tt}^2\left(\partial_{xx}^2 w + \partial_{yy}^2 w + \partial_{zz}^2 w + \frac{\rho_0'}{\rho_0}\partial_z w\right) = -N^2(z)(\partial_{xx}^2 w + \partial_{yy}^2 w)\ , \tag{4.42b}$$

where the *Brunt–Väisälä frequency*, $N$, is given by

$$N^2(z) = -\frac{g\rho_0'(z)}{\rho_0(z)} > 0\,, \tag{4.42c}$$

and it has been assumed that the stratification is stable with the density decreasing with increasing height. Hence $w$ satisfies a linear equation of the form

$$\mathcal{L}(\partial_t, \partial_x, \partial_y, \partial_z; z)w = 0\ . \tag{4.42d}$$

*Slowly varying assumption.* We now make a slowly varying assumption, in that we assume that a plane harmonic wave has a relatively short vertical lengthscale, i.e.

$$\frac{2\pi}{k_3} \ll L \quad \text{where} \quad \frac{\rho_0}{\rho_0'} = O(L)\ . \tag{4.43a}$$

Then

$$\frac{\rho_0'}{\rho_0} w_z = O\left(\frac{wk_3}{L}\right), \quad w_{zz} = O\left(wk_3^2\right)\ , \tag{4.43b}$$

and so

$$\rho_0' w_z \ll \rho_0 w_{zz}\,. \tag{4.43c}$$

Under this assumption, (4.42b) simplifies to

$$w_{zztt} + (\partial_{xx}^2 + \partial_{yy}^2)\left(N^2(z) + \partial_{tt}^2\right) w = 0\,. \tag{4.44}$$

Suppose, for the time being, that $N^2$ is a constant. From (4.42c), this is the case if

$$\rho_0 = \bar{\rho}\exp(-z/L)\ , \tag{4.45}$$

where $\bar{\rho}$ is a constant and $L = g/N^2$. Then a harmonic plane wave is an exact solution of (4.44), and from (4.11b) the dispersion relation is given by

$$(-k_3^2)(-\omega^2) - (k_1^2 + k_2^2)\left(N^2 - \omega^2\right) = 0\,,$$

or equivalently

$$\omega^2 = \Omega^2(\mathbf{k}) \equiv \frac{N^2(k_1^2 + k_2^2)}{k_1^2 + k_2^2 + k_3^2}\,. \tag{4.46a}$$

*Remarks.*

(i) If as assumed $N^2 > 0$, wave solutions to (4.46a) are possible for frequencies $\omega^2 \leqslant N^2$.

(ii) If $N(z)$ is a slowly-varying function of $z$, i.e. if the vertical wavelength is much shorter than the length over which $N$ varies by an order one amount, then for a given value of $z$, 'local' harmonic-wave solutions to (4.44) can be sought (as we shall see in § 5 is the case in *ray theory*). Under this assumption the [leading-order] dispersion relation becomes a 'slow' function of the vertical co-ordinate, i.e.

$$\omega^2 = \Omega^2(\mathbf{k}; z) \equiv \frac{N^2(z)(k_1^2 + k_2^2)}{k_1^2 + k_2^2 + k_3^2}\,. \tag{4.46b}$$

(iii) (4.46a), or more generally (4.46b), are *anisotropic* dispersion relations.

(iv) The phase and group velocities are given by

$$\mathbf{c} = \frac{\omega}{k}\,\widehat{\mathbf{k}}\ , \tag{4.46c}$$

and

$$\mathbf{c}_g \equiv \left(\frac{\partial\omega}{\partial k_1}, \frac{\partial\omega}{\partial k_2}, \frac{\partial\omega}{\partial k_3}\right) = \frac{N^2 k_3}{\omega k^4}\left(k_1k_3, k_2k_3, -(k_1^2 + k_2^2)\right)\ , \tag{4.46d}$$

respectively. Hence

$$\mathbf{c}\cdot\mathbf{c}_g = \frac{N^2}{k^6}\left((k_1^2 + k_2^2)k_3^2 - (k_1^2 + k_2^2)k_3^2\right) = 0\ , \tag{4.46e}$$

i.e. the phase velocity and the group velocity are perpendicular.

*Experiment.* Consider an experiment where waves are generated by a two–dimensional oscillatory cylinder with $\omega < N$. So that the waves travel in straight lines,[14] assume that $N$ is constant. Then with

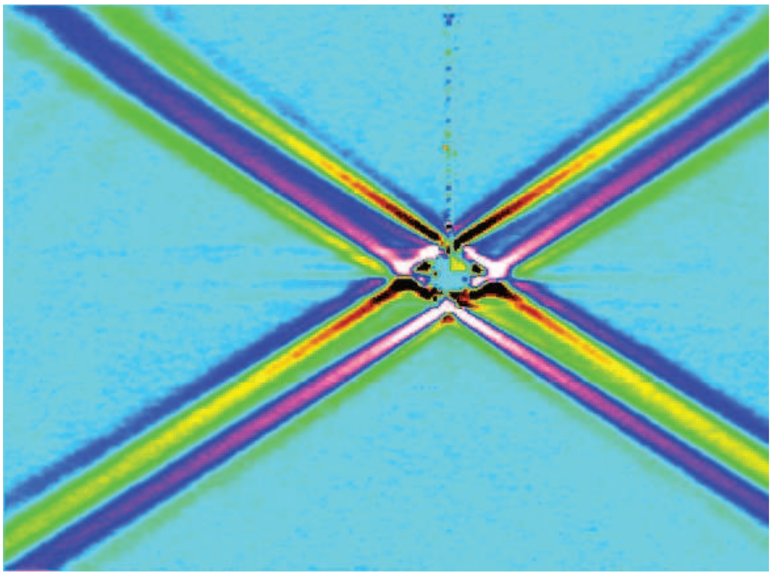
$$\mathbf{k} = k\,(\sin\phi, 0, -\cos\phi)\,, \quad k > 0\,, \quad -\pi < \phi \leqslant \pi\,,$$

(4.46b) and (4.46d) imply that

$$\frac{\omega}{N} = \pm\sin\phi\,, \quad \mathbf{c}_g = \pm\frac{N\cos\phi}{k}(\cos\phi, 0, \sin\phi) \quad \text{and} \quad \mathbf{c} = \pm N\sin\phi\,(\sin\phi, 0, -\cos\phi)\,.$$

Here the signs need to be chosen so that, according to the radiation condition, the group velocity, $\mathbf{c}_g$, is directed *away* from the cylinder, i.e. chosen so that $\pm\cos\phi > 0$.

*Downward propagating crests* imply *upward propagating energy.*



[14] Once *ray theory* is at our disposal, we can deduce, from the ray tracing equations (5.10b) and (5.10a), that $\omega$, $k_1$ and $k_2$ are constant on rays since

$$\frac{\partial\Omega}{\partial t} = \frac{\partial\Omega}{\partial x} = \frac{\partial\Omega}{\partial y} = 0\,. \tag{4.47a}$$

Also, from (5.10b),

$$\frac{dk_3}{dt} = -\frac{\partial\Omega}{\partial z} = -\frac{\Omega}{N}N'\,. \tag{4.47b}$$

Hence if (4.45) is satisfied, $k_3$ is also constant on rays, with the consequence that all rays are straight.

# 5 Ray Theory

Ray theory (geometric optics) is the generalisation to PDEs of the WKB[JLG] method for ODEs (but do not worry if you have not done *Asymptotic Methods*; we will be starting from scratch). It is an asymptotic approximation for long-distance propagation through 'slowly varying' (s.v.) media, where by slowly varying we mean that the variations must be slow on the scale of the wavelength or period (as in (4.43a) for internal waves), i.e.

$$\text{media lengthscales} \gg k^{-1} \quad \text{and/or} \quad \text{media timescales} \gg \omega^{-1}\,. \tag{5.1}$$

Ray theory is motivated by the fact that waves often transport energy, etc. over distances $\gg k^{-1}$ and times $\gg \omega^{-1}$ into regions with different properties, e.g.

- seismic waves through the earth;
- sound waves, or internal gravity waves, propagating though the lower/upper atmosphere;
- water waves propagating towards a [gently] sloping beach.

The aim is to use local solutions (on the scale of the waves) to solve on the much larger scale of the medium. In what follows we will neglect

(i) dissipation,

(ii) nonlinear effects,

(iii) rapid variations (e.g. effects near boundaries, at interfaces or at focusing points),

(iv) diffraction (i.e. spreading effects).

## 5.1 Ray Tracing

### 5.1.1 Multiple scales

We will assume that the medium varies on an $O(1)$ lengthscale/timescale, while the wavelength/period of the waves is $O(\varepsilon)$ where $\varepsilon \ll 1$. Hence seek slowly-varying solutions to

$$\mathcal{L}(\partial_t, \partial_x, \partial_y, \partial_z; \mathbf{x}, t)\,\varphi = 0\,, \tag{5.2a}$$

where the inhomogeneity in the medium is represented by the $\mathbf{x}$ dependence, and we have allowed the medium to be time dependent (e.g. arising from diurnal variations). The dispersion relation for internal gravity waves is of this form (see (4.42b) and (4.42d)), as is the equation for sound waves in a stratified atmosphere:

$$\mathcal{L}\tilde{p} \equiv \frac{\partial^2 \tilde{p}}{\partial t^2} - \rho_0(z)c_0^2(z)\frac{\partial}{\partial z}\left(\frac{1}{\rho_0(z)}\frac{\partial \tilde{p}}{\partial z}\right) = 0\,. \tag{5.2b}$$

Because the wavevector/frequency may vary across the medium, a simple local harmonic wave, $e^{i(\mathbf{k}.\mathbf{x}-\omega t)}$, with phase $(\mathbf{k}.\mathbf{x} - \omega t)$, is unlikely to work on the *medium's* scale. The key idea is to focus on the *phase* of the wave by seeking a solution of the form

$$\varphi = A(\mathbf{x}, t; \varepsilon)e^{\frac{i}{\varepsilon}\theta(\mathbf{x},t)}\,. \tag{5.3}$$

The factor of $\frac{1}{\varepsilon}$ ensures that variations in phase, $\theta/\varepsilon$, are rapid, while variations in amplitude, $A$, are slow[er]. This has consequences for derivatives, e.g.

$$\frac{\partial \varphi}{\partial t} = \underbrace{\frac{i}{\varepsilon}\frac{\partial \theta}{\partial t}}_{\text{large}}\varphi + \underbrace{\frac{1}{A}\frac{\partial A}{\partial t}}_{O(1)}\varphi \sim \frac{i}{\varepsilon}\frac{\partial \theta}{\partial t}\varphi\,. \tag{5.4a}$$

Similarly

$$\frac{\partial \varphi}{\partial x} \sim \frac{i}{\varepsilon}\frac{\partial \theta}{\partial x}\varphi\,, \quad \text{etc.} \tag{5.4b}$$

For a harmonic wave, $\varphi = e^{i(\mathbf{k}.\mathbf{x}-\omega t)}$,

$$\frac{\partial \varphi}{\partial t} = -i\omega\varphi\,, \quad \boldsymbol{\nabla}\varphi = i\mathbf{k}\varphi\,. \tag{5.4c}$$

This is somewhat suggestive, so *define* the local frequency, $\omega$, and local wavevector, $\mathbf{k}$, over $O(\varepsilon)$ scales, by

$$\omega = -\frac{1}{\varepsilon}\,\theta_t\,, \quad \mathbf{k} = \frac{1}{\varepsilon}\boldsymbol{\nabla}\theta\,. \tag{5.4d}$$

Substitute into (5.2a), then we obtain, at *leading order*, the *local* dispersion relation

$$\mathcal{L}\left(\frac{i\theta_t}{\varepsilon}, \frac{i\theta_x}{\varepsilon}, \frac{i\theta_y}{\varepsilon}, \frac{i\theta_z}{\varepsilon}; \mathbf{x}, t\right) = \mathcal{L}(-i\omega, ik_1, ik_2, ik_3; \mathbf{x}, t) = 0\,. \tag{5.5a}$$

or, on solving,

$$\omega = \Omega(\mathbf{k}; \mathbf{x}, t)\,, \tag{5.5b}$$

where the dependence on

- $\mathbf{k}$ represents dispersion;
- $\mathbf{x}$ represents the [slow] spatial inhomogeneity of the medium;
- 20/16 $t$ represents the [slow] time dependence of the medium.

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### 5.1.2 Phase and wavecrests

Assuming that partial derivatives of the phase commute, we note using (5.4d) that

$$\frac{\partial^2\theta}{\partial x_j \partial x_i} = \frac{\partial^2\theta}{\partial x_i \partial x_j} \Rightarrow \qquad \frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i}\,, \tag{5.6a}$$

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\theta) = \boldsymbol{\nabla}\left(\frac{\partial \theta}{\partial t}\right) \Rightarrow \qquad \frac{\partial \mathbf{k}}{\partial t} = -\boldsymbol{\nabla}_{\mathbf{x}}\omega\,. \tag{5.6b}$$

(5.6a) is equivalent to $\boldsymbol{\nabla} \times \mathbf{k} = 0$. Hence, by Stokes' theorem,

$$\theta(\mathbf{x}_2) - \theta(\mathbf{x}_1) = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \boldsymbol{\nabla}\theta \cdot \mathbf{dx} = \varepsilon \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{k} \cdot \mathbf{dx}\,, \tag{5.6c}$$

is independent of path; it follows that the number of crests between $\mathbf{x}_1$ and $\mathbf{x}_2$ does not vary by path..

(5.6b) is the *continuity of phase* equation. It is a conservation equation stating that the rate of change of *phase density*, $\mathbf{k}$, is balanced by the flux, or *phase flow rate*, $\omega$. In integral form

$$\frac{d}{dt}\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{k} \cdot \mathbf{dx} = \omega(\mathbf{x}_1) - \omega(\mathbf{x}_2)\,. \tag{5.6d}$$

Since wavecrests are points of constant $\theta$, we can interpret this as the rate of change in the number of wavecrests in $[\mathbf{x}_1, \mathbf{x}_2]$ equals the wavecrests entering minus those leaving.

### 5.1.3 Ray-tracing equations

As a wavepacket moves through the medium, the wavevector, $\mathbf{k}$, and frequency, $\omega$, must vary in order to satisfy the local dispersion relation (5.5b):

$$\omega(\mathbf{x},t)=\Omega(\mathbf{k}(\mathbf{x},t);\mathbf{x},t)\,. \tag{5.7}$$

From the chain rule

$$\left.\frac{\partial\omega}{\partial t}\right|_{\mathbf{x}}=\left.\frac{\partial\Omega}{\partial k_i}\right|_{\mathbf{x},t}\left.\frac{\partial k_i}{\partial t}\right|_{\mathbf{x}}+\left.\frac{\partial\Omega}{\partial t}\right|_{\mathbf{k},\mathbf{x}}, \tag{5.8a}$$

and hence from (5.6b)

$$\frac{\partial\omega}{\partial t}+(\mathbf{c}_g\cdot\boldsymbol{\nabla}_{\mathbf{x}})\,\omega=\frac{\partial\Omega}{\partial t}\,, \tag{5.8b}$$

where the local group velocity is given by

$$\mathbf{c}_g=\boldsymbol{\nabla}_{\mathbf{k}}\Omega\,. \tag{5.8c}$$

Similarly,

$$\left.\frac{\partial\omega}{\partial x_j}\right|_{t}=\left.\frac{\partial\Omega}{\partial k_i}\right|_{\mathbf{x},t}\left.\frac{\partial k_i}{\partial x_j}\right|_{t}+\left.\frac{\partial\Omega}{\partial x_j}\right|_{\mathbf{k},t}, \tag{5.9a}$$

and hence from (5.6a) and (5.6b)

$$\frac{\partial\mathbf{k}}{\partial t}+(\mathbf{c}_g\cdot\boldsymbol{\nabla}_{\mathbf{x}})\,\mathbf{k}=-\boldsymbol{\nabla}_{\mathbf{x}}\Omega\,. \tag{5.9b}$$

Equations (5.8b) and (5.9b) can be written in characteristic form as

$$\left.\frac{d\omega}{dt}\right|_{\mathbf{c}_g}=\frac{\partial\Omega}{\partial t}\,, \tag{5.10a}$$

$$\left.\frac{d\mathbf{k}}{dt}\right|_{\mathbf{c}_g}=-\boldsymbol{\nabla}_{\mathbf{x}}\Omega\,, \tag{5.10b}$$

where

$$\left.\frac{d}{dt}\right|_{\mathbf{c}_g}=\frac{\partial}{\partial t}+\mathbf{c}_g\cdot\boldsymbol{\nabla}_{\mathbf{x}}\,, \tag{5.10c}$$

and the characteristics are the paths, $\mathbf{x}=\mathbf{x}(t)$, satisfying

$$\frac{d\mathbf{x}}{dt}=\mathbf{c}_g(\mathbf{k}(\mathbf{x},t);\mathbf{x},t)\,. \tag{5.10d}$$

*Definition.* A *ray* is a path satisfying (5.10d).

*Remarks.*

(i) In general rays are *curved.*

(ii) Equation (5.10c) represents the time derivative of an observer moving with velocity $\mathbf{c}_g$ along a ray.

(iii) From (5.4d), the evolution of the phase is given by

$$\frac{1}{\varepsilon}\left.\frac{d\theta}{dt}\right|_{\mathbf{c}_g}=\frac{1}{\varepsilon}\left(\frac{\partial\theta}{\partial t}+\mathbf{c}_g\cdot\boldsymbol{\nabla}_{\mathbf{x}}\theta\right)=-\omega+\mathbf{c}_g.\mathbf{k}\,. \tag{5.10e}$$

*Solution.* Given $\Omega$, and initial conditions $\mathbf{k}_0$, $\omega_0$ and $\theta_0$ at a position $\mathbf{x}_0$, it is possible to integrate the *ray tracing equations* (5.10a), (5.10b), (5.10d) and (5.10e) to solve for $\mathbf{k}$, $\omega$ and $\theta$ at $\mathbf{x}$. It is necessary to do this for each ray (with lots of initial positions and wavevectors) to find $\mathbf{k}$ and $\omega$ everywhere.

*Evolution of frequency.* From (5.10a), the frequency, $\omega$, of a wavepacket moving with velocity $\mathbf{c}_g$ changes iff the medium varies with $t$. If the medium is time independent, i.e.

$$\frac{\partial \Omega}{\partial t} = 0\,, \tag{5.11a}$$

then from (5.10a)

$$\frac{d\omega}{dt} = 0\,, \tag{5.11b}$$

i.e. the frequency, $\omega$, is constant along rays.

*Evolution of wavenumber.* Analogously, from (5.10b), the wavevector, $\mathbf{k}$, of a wavepacket moving with velocity $\mathbf{c}_g$ changes iff there is spatial inhomogeneity. This is referred to as *refraction.* If the medium is homogeneous, i.e.

$$\boldsymbol{\nabla}_{\mathbf{x}}\Omega = 0\,, \tag{5.11c}$$

then from (5.10b)

$$\frac{d\mathbf{k}}{dt} = 0\,, \tag{5.11d}$$

i.e. the wavevector, $\mathbf{k}$, is constant along rays.

*Homogeneous time-independent media.* If both (5.11a) and (5.11c) are satisfied, i.e. if $\Omega \equiv \Omega(\mathbf{k})$, then $\mathbf{c}_g \equiv \mathbf{c}_g(\mathbf{k})$, and hence, from the fact that $\mathbf{k}$ is constant along a ray, the rays/characteristics are straight.

### 5.1.4 Hamilton's equations of classical mechanics

From (5.10b) and (5.10d)

$$\frac{dk_j}{dt} = -\frac{\partial \Omega}{\partial x_j}\,. \quad \frac{dx_j}{dt} = \frac{\partial \Omega}{\partial k_j}\,. \tag{5.12a}$$

These are Hamilton's equations with

$$\mathbf{x} = \mathbf{q}\,, \quad \mathbf{k} = \mathbf{p}\,, \quad \Omega(\mathbf{k};\mathbf{x},t) = H(\mathbf{q},\mathbf{p},t)\,. \tag{5.12b}$$

Hence waves are like particles travelling with the group velocity. Further, let $S = \theta/\varepsilon$, then from (5.4d) and (5.5b)

$$\frac{\partial S}{\partial t} + \Omega\left(\boldsymbol{\nabla} S;\mathbf{x},t\right) = 0\,. \tag{5.12c}$$

This is the **Hamilton-Jacobi equation**, with the phase as the action.

*Remark.* The Hamiltonian is a first integral and, thus, so is the dispersion relation. Since a first integral of the ray equations exists, the number of ray equations that need to be solved is reduced by one.

### 5.1.5 * Wave action *

The local *energy density*, $E$, is *not* necessarily conserved for a slowly-varying wavetrain, but the *wave action*, $I$, is (statement with no proof):

$$\frac{\partial I}{\partial t} + \boldsymbol{\nabla}.(\mathbf{c}_g I) = 0\,, \tag{5.13a}$$

where

$$I = \frac{E}{\omega}\,. \tag{5.13b}$$

*Remarks.*

(i) This is the equivalent of an adiabatic invariant in classical mechanics.

(ii) If the medium is time independent, so that the frequency is constant along a ray, then energy is conserved.

21/04 (iii) Equations (5.13a) and (5.13b) can be used to calculate the amplitude of waves in slowly-varying media (e.g. see Question 6 on Example Sheet 4).

### 5.1.6 Example: gravity waves approaching a beach (surfing)

Suppose that there is a sloping beach with a shore line at $x = 0$. Let the depth of the water be

$$h \equiv h(x, z)\ . \tag{5.14}$$

Assume

(i) that $h \to \infty$ as $x \to -\infty$, and

(ii) that $\mathbf{k} = (k_1, k_3) \to k_\infty(\cos\phi_\infty, \sin\phi_\infty)$ as $x \to -\infty$ , i.e. that far from the beach we have uniform plane waves.

From (4.34a) the dispersion relation is

$$\omega^2 = \Omega^2(\mathbf{k}; \mathbf{x}) = gk \tanh kh\,, \tag{5.15a}$$

where surface tension has been neglected, and

$$k^2 = k_1^2 + k_3^2\,. \tag{5.15b}$$

From (5.11a) and (5.11b), $\omega$ is a constant on each ray. Further, all rays originate from $x = -\infty$, and hence

$$\omega^2 = gk_\infty \tag{5.16a}$$

everywhere. Thus

$$\tanh kh = \frac{k_\infty}{k}\,, \tag{5.16b}$$

and so $k \equiv k(x, z)$, and

21/16 $$\{\text{measuring}\ \ k\} \Leftrightarrow \{\text{measuring}\ \ h\}\ .$$

Further, water waves are an example of **isotropic dispersion**, i.e.

$$\Omega \equiv \Omega(k; \mathbf{x})\,, \quad \text{where} \quad k = |\mathbf{k}|\,. \tag{5.17a}$$

In such cases

$$\mathbf{c}_g = \frac{\partial\Omega}{\partial k}\widehat{\mathbf{k}}\ , \tag{5.17b}$$

21/17 i.e. the rays are parallel to $\mathbf{k}$. It follows from the characteristic equation (5.10d) that the rays are specified
21/18 by the equation

$$\frac{dx}{dz} = \frac{\frac{dx}{dt}}{\frac{dz}{dt}} = \frac{k_1}{k_3}\ . \tag{5.17c}$$

To solve for the wavevector, one more equation is required; from the ray-tracing equations (5.10b), on a ray

$$\frac{dk_3}{dk_1} = \frac{\frac{dk_3}{dt}}{\frac{dk_1}{dt}} = \frac{\Omega_z}{\Omega_x}\ . \tag{5.17d}$$

21/18

*Wavecrests.* As noted both in § 1.5.2 and above, *wavecrests* are lines (or in 3D, surfaces) of constant $\theta$. The normal to these lines (or surfaces) is in the direction of $\nabla\theta$. Hence, as concluded in § 1.5.2, or from (5.4d), the unit normal $\mathbf{n}$ is given by

$$\mathbf{n} = \widehat{\mathbf{k}}\,. \tag{5.18a}$$

It follows from (5.17b) that, for isotropic dispersion relations, the wavecrests are perpendicular to the rays (in contrast with internal gravity waves). Further, for our 2D example, the equation for the wavecrests is

$$\theta_x dx + \theta_z dz = 0\,,$$

or, from (5.4d),

$$\frac{dx}{dz} = -\frac{k_3}{k_1}\ . \tag{5.18b}$$

*Check.* Consistent with (5.17b) and (5.18a), the rays (5.17c) and wavecrests (5.18b) are orthogonal.

*Uniform beach.* Suppose $h \equiv h(x)$. Then

$$\Omega \equiv \Omega(k; x) \ ,$$

and so

$$\frac{dk_3}{dk_1} = 0\,, \quad \text{or equivalently} \quad \frac{dk_3}{dt} = 0\,.$$

Thus, since all characteristics originate from $-\infty$,

$$k_3 = k_\infty \sin\phi_\infty$$

everywhere. Further, from (5.16b), as $h \to 0$

$$k \sim \sqrt{\frac{k_\infty}{h}} \to \infty\,, \quad \text{and} \quad kh \sim \sqrt{k_\infty h} \to 0\,.$$

But $k_3$ is fixed, so $k_1 \to \infty$ as $h \to 0$. Hence at the shoreline, $\mathbf{k}$ is perpendicular to the beach, while the wavecrests are parallel to the beach. Good news for surfing. ☺

## 5.2 Fermat's Principle

*Claim.* The ray-tracing equations (5.10d) and (5.10b) are equivalent to the **variational principle**

$$\delta \int_{t_1}^{t_2} \Phi(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{k}, t) dt = 0, \tag{5.19a}$$

where

$$\Phi = \mathbf{k}.\dot{\mathbf{x}} - \Omega(\mathbf{k}; \mathbf{x}, t), \tag{5.19b}$$

$t_1, t_2$ are fixed, and $\boldsymbol{\delta}\mathbf{x}$ and $\boldsymbol{\delta}\mathbf{k}$ are independently varied s.t.

$$\boldsymbol{\delta}\mathbf{x}(t_1) = \boldsymbol{\delta}\mathbf{x}(t_2) = 0 \tag{5.19c}$$

*Proof.* (5.19a) can be rewritten

$$\int_{t_1}^{t_2} \left\{ (\dot{\mathbf{x}} - \boldsymbol{\nabla}_{\mathbf{k}}\Omega) \cdot \boldsymbol{\delta}\mathbf{k} + \mathbf{k} \cdot \delta\dot{\mathbf{x}} - \boldsymbol{\nabla}_{\mathbf{x}}\Omega \cdot \boldsymbol{\delta}\mathbf{x} \right\} dt = 0\,. \tag{5.20a}$$

Integrate by parts and use (5.19c). Since $\boldsymbol{\delta}\mathbf{k}$ and $\boldsymbol{\delta}\mathbf{x}$ can be independently varied, their coefficients must be zero; thus

$$\dot{\mathbf{x}} = \boldsymbol{\nabla}_{\mathbf{k}}\Omega \quad , \quad \dot{\mathbf{k}} = -\boldsymbol{\nabla}_{\mathbf{x}}\Omega\,. \tag{5.20b}$$

Hence from (5.10d) and (5.10b), the stationary path is a ray.

### 5.2.1 Time independent case

Suppose that the medium is time independent, i.e. $\frac{\partial\Omega}{\partial t} = 0$, and that the frequency, $\omega = \Omega$, is a uniform constant. We then need to to consider a restricted class of variations $\boldsymbol{\delta}\mathbf{k}$, $\boldsymbol{\delta}\mathbf{x}$ such that

$$\Omega(\mathbf{k} + \boldsymbol{\delta}\mathbf{k}; \mathbf{x} + \boldsymbol{\delta}\mathbf{x}) = \Omega(\mathbf{k}; \mathbf{x})\,, \tag{5.21a}$$

i.e., such that

$$\delta \int_{t_1}^{t_2} \Omega\, dt = 0\,. \tag{5.21b}$$

(5.19a) and (5.19b) then become

$$\delta \int_{t_1}^{t_2} \mathbf{k} \cdot \dot{\mathbf{x}}\, dt = 0\,, \tag{5.21c}$$

or equivalently

$$\delta \int_{x_1}^{x_2} \mathbf{k} \cdot \mathbf{dx} = 0\,. \tag{5.21d}$$

*Isotropic case.* When the dispersion relation is isotropic, then the wavevector, $\mathbf{k}$, and the group velocity, $\mathbf{c}_g$, are parallel, e.g. see (5.17b). Since a ray, i.e. stationary path, is parallel to $\mathbf{c}_g = \boldsymbol{\nabla}_{\mathbf{k}}\Omega$, it follows that on a ray $\mathbf{k}$ is parallel to $\mathbf{dx}$; hence

$$\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{k}.\mathbf{dx} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} kds = \omega \int_{\mathbf{x}_1}^{\mathbf{x}_2} \frac{ds}{c}\,, \tag{5.22a}$$

where

$$k^2 = \mathbf{k}\cdot\mathbf{k}\,, \quad ds^2 = \mathbf{dx}\cdot\mathbf{dx}\,, \quad c^2 = \left(\frac{\omega}{k}\widehat{\mathbf{k}}\right)\cdot\left(\frac{\omega}{k}\widehat{\mathbf{k}}\right) = \frac{\omega^2}{k^2}\,. \tag{5.22b}$$

Since $\omega$ is a constant, (5.21d) can be written in the classical form

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} \frac{ds}{c} = 0\,. \tag{5.22c}$$

### 5.2.2 Snell's law of refraction

Suppose that the dispersion relation is isotropic and depends only on $x_3$, i.e.

$$\Omega \equiv \Omega(k; x_3)\,. \tag{5.23a}$$

Then from (5.11d) and (5.11b), $k_1, k_2$ and $\omega$ are constant on rays, and the rays are parallel to $\mathbf{k}$. Suppose that $\mathbf{k}$ subtends an angle $\alpha$ with the $x_3$ direction. Then

$$\sin\alpha = \frac{(k_1^2+k_2^2)^{\frac{1}{2}}}{k} = \frac{(k_1^2+k_2^2)^{\frac{1}{2}}}{\omega}\,c\,,$$

and hence on a ray

$$\frac{\sin\alpha}{c} = \frac{\sin\alpha_0}{c_0} = \text{const}\,. \tag{5.23b}$$

This is Snell's law of refraction.

### 5.2.3 Example: sound waves in inhomogeneous media

Consider a sound ray in the $(x, z)$ plane, and suppose that

$$\frac{\partial\Omega}{\partial x} = \frac{\partial\Omega}{\partial y} = \frac{\partial\Omega}{\partial t} = 0\,. \tag{5.24a}$$

Then, since sound waves are non-dispersive,

$$\omega = \Omega(k; z) = kc(z)\,. \tag{5.24b}$$

Seek a ray joining $(\pm a, 0, 0)$; then from (5.22c)

$$\delta \int_{-a}^{a} \frac{ds}{c} = 0\,, \tag{5.25a}$$

or

$$\delta \int_{-a}^{a} \left(1 + \left(\frac{dz}{dx}\right)^2\right)^{\frac{1}{2}} \frac{dx}{c(z)} = 0\;. \tag{5.25b}$$

Standard calculus of variations, and manipulation, show that on a ray

$$\frac{dz}{dx} = \pm\left(\frac{c_0^2}{c^2(z)\,\sin^2\alpha_0} - 1\right)^{\frac{1}{2}}\,, \tag{5.26a}$$

where $c_0/\sin\alpha_0$ is a constant.[15]

*Alternatively*, from Snell's law

$$\frac{\sin\alpha}{c} = \text{constant on a ray.}$$

For an isotropic dispersion relation $\mathbf{k}$ is parallel to a ray, hence

$$\frac{dx}{(dx^2+dz^2)^{\frac{1}{2}}} = \frac{c}{c_0}\sin\alpha_0\ ,$$

or, as in (5.26a),

$$\frac{dz}{dx} = \pm\left(\frac{c_0^2}{c^2(z)\,\sin^2\alpha_0} - 1\right)^{\frac{1}{2}}\ . \tag{5.26b}$$

*Exercise.* Recover (5.26a), or (5.26b), from the dispersion relation (5.24b) and the ray-tracing equations (5.10d), (5.10b) and (5.10a).

*Remark.* Take the + sign, and suppose that $c$ decreases as $z$ increases ($\frac{\partial c}{\partial z} < 0$); then $\frac{dz}{dx}$ increases with height. Hence $\frac{dz}{dx} > 0$ for all $z$, and thus a ray cannot return to original level.

Alternatively suppose that $c$ increases as $z$ increases ($\frac{\partial c}{\partial z} > 0$), and that there exits $(x_0, z_0)$ such that

$$\frac{dz}{dx} = 0\ .$$

22/16 Then the ray has a maximum at this point.

*Example.* As an example suppose that

$$c = c_0\,(z+1)\,.$$

Then from solving (5.26a), we find that a ray passing through $(0,0)$ has the equation

$$(x - \cot\alpha_0)^2 + (z+1)^2 = \operatorname{cosec}^2\alpha_0\ ,$$

i.e. the rays are arcs of circles in $x$–$z$ plane.

*Remark.* On still summer nights there can be an *inversion* in the atmosphere so that close to the ground the temperature, and thence the sound speed, increase with height. On such nights sound can travel
22/04 much further.

22/17
22/19

## 5.3 Media with Mean Flows

So far we have assumed that the medium that the waves are propagating through has no mean flow. This is not always the case. As an example where **mean flow** effects are important, consider a *source* emitting waves with a frequency $\omega_s$ and moving with
22/18 a [uniform] velocity $\mathbf{U}$ through a fluid medium.

[15] Since the integrand of the functional is independent of $x$, the Euler-Lagrange equation

$$\frac{\partial}{\partial z}\left(\frac{(1+z'^2)^{\frac{1}{2}}}{c(z)}\right) - \frac{d}{dx}\left(\frac{\partial}{\partial z'}\left(\frac{(1+z'^2)^{\frac{1}{2}}}{c(z)}\right)\right) = 0\,.$$

has the first integral

$$\frac{(1+z'^2)^{\frac{1}{2}}}{c(z)} - z'\frac{\partial}{\partial z'}\left(\frac{(1+z'^2)^{\frac{1}{2}}}{c(z)}\right) = \frac{\sin\alpha_0}{c_0}\,.$$

On expansion and rearrangement, this yields (5.26a).

*Frame in which the fluid is at rest.* In the $(\mathbf{x}, t)$ frame in which the fluid is at rest, suppose that the dispersion relation for plane harmonic waves of frequency $\omega_r$ is known:

$$\omega_r = \Omega_r(\mathbf{k}; \mathrm{x}, t) \ . \tag{5.27}$$

*Frame in which the source is at rest.*

Transform to a $(\mathbf{X}, t)$ frame moving with the source, in which the fluid has velocity $-\mathbf{U}$. It is in this frame that the source 'emits' waves with frequency $\omega_s$. The frames are related by a Galilean transformation:

$$\mathbf{X} = \mathbf{x} - \mathbf{U}t \, . \tag{5.28a}$$

Thus, the transformation formula for the time derivative is given by

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} \to \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}} - U_j \frac{\partial}{\partial X_j} \ . \tag{5.28b}$$

It follows that for plane harmonic waves the frequencies in the 'at rest' and 'source/moving' frames are related by

$$-\imath\, \omega_{\mathrm{r}} = -\imath\, \omega_{\mathrm{s}} - \imath\, \mathbf{k} \cdot \mathbf{U} \ ,$$

i.e.

$$\omega_{\mathrm{s}} = \omega_r - \mathbf{k} \cdot \mathbf{U} \, , \tag{5.28c}$$

with corresponding group velocity, from (5.27),

$$\mathbf{c}_{gs} = \boldsymbol{\nabla}_{\mathbf{k}} \left( \Omega_r(\mathbf{k}) - \mathbf{k} \cdot \mathbf{U} \right) = \mathbf{c}_{gr} - \mathbf{U} \, . \tag{5.28d}$$

*Remarks.*

(i) In general this is an *anisotropic dispersion relation*, even if $\Omega_r$ is isotropic (i.e. $\Omega_r \equiv \Omega_r(k)$), since $\mathbf{U}$ introduces a direction.

(ii) Group velocities obey Galilean transformations, whereas phase velocities do not — because phase velocities are not 'true' velocities.

### 5.3.1 The Doppler effect

Consider a source moving with velocity $\mathbf{U}$ emitting *sound* waves with constant frequency $\omega_s$. We know that in a medium which is at rest, the dispersion relation for sound waves is

$$\omega_r = c_0 k \, , \tag{5.29a}$$

where $c_0$ is the constant sound speed. Let $\phi$ be the angle between the direction of motion and the ray, or equivalently the wavevector, $\mathbf{k}$.

Then from (5.29a)

$$\mathbf{k} \cdot \mathbf{U} = Uk \cos\phi = \frac{U\omega_r \cos\phi}{c_0} \, ,$$

and hence, from (5.28c), the frequency heard by a stationary observer is

$$\omega_r = \frac{\omega_s}{1 - \frac{U}{c_0} \cos\phi} \ . \tag{5.29b}$$

If $U < c_0$ , then

(i) if $\cos\phi > 0$, i.e. if the observer is in front of the moving source, then $\omega_r > \omega_s$ and the frequency heard is larger than generated.

(ii) if $\cos\phi < 0$, i.e. if the observer is behind the moving source, then $\omega_r < \omega_s$ and the frequency heard is smaller than generated.

*Exercise.* What happens if $U > c_0$?

### 5.3.2 Stationary capillary-gravity waves

In general waves are unsteady, because of the $e^{-\imath\omega t}$ factor. However, they can be steady in the frame of a moving source.

Suppose that there is a cm size object moving with velocity $(U, 0, 0)$ in deep water (e.g. a trailing finger from a punt). If $U > 0$ is large enough, then this steady force can generate a stationary wave pattern in the frame of the object.

Work in 1D for simplicity. Then, in the frame where the forcing is at rest, and the fluid has a uniform mean velocity $(-U, 0, 0)$, the dispersion relation is, from (5.27) and (5.28c),

$$\omega_{\mathrm{s}} = \Omega_r(k) - Uk\,, \tag{5.30a}$$

where, see (4.37a),

$$\Omega_r(k) = \pm|gk|^{\frac{1}{2}}\left(1 + \beta k^2\right)^{\frac{1}{2}}\,, \tag{5.30b}$$

is the dispersion relation for deep-water waves in the absence of a mean flow.

For a wave pattern generated by a **steadily** moving force we require that $\omega_s = 0$, i.e.

$$Uk = \pm|gk|^{\frac{1}{2}}(1 + \beta k^2)^{\frac{1}{2}}\ . \tag{5.30c}$$

- If $0 < U < (2g/k_2)^{\frac{1}{2}} = (4g^2\beta)^{\frac{1}{4}}$, which is about $0.25\,\mathrm{ms}^{-1}$ in water, then there is no non-zero (i.e. finite-wavelength) solution to (5.30c), and there can be no steady wave pattern. Here, as in (4.37c), $k_2 = \beta^{-\frac{1}{2}}$ is the [positive] wavenumber of the wave with the minimum phase speed when there is *no* mean flow
- If $U > (4g^2\beta)^{\frac{1}{4}}$, then there are four non-zero solutions to (5.30c). The radiation condition necessitates that the waves should propagate outwards, and from (5.28d), $\mathbf{c}_{gs} = \mathbf{c}_{gr} - \mathbf{U}$. Hence

$|k| < k_2, \quad c_{gr}(k) < U, \quad c_{gs}(k) < 0.$

Long waves occur downstream of, i.e. behind, the steady disturbance

$|k| > k_2, \quad c_{gr}(k) > U, \quad c_{gs}(k) > 0.$

Short waves occur upstream, i.e. ahead, of the steady disturbance

### 5.3.3 Mach cone

Consider an aircraft moving supersonically with velocity $\mathbf{U} = (Mc_0, 0, 0)$, where $c_0$ is the speed of sound and $M > 1$; suppose that linear theory has something to say.

In the frame of the aircraft, from (5.28c) and (5.29a),

$$\omega_s = c_0 k - \mathbf{U} \cdot \mathbf{k} = c_0 k(1 - M\cos\phi)\,. \tag{5.31a}$$

Steady waves are possible if $\phi = \cos^{-1}(1/M)$, which implies

$$\widehat{\mathbf{k}} = (\cos\phi, \sin\phi, 0) = \left(\frac{1}{M}, \left(1 - \frac{1}{M^2}\right)^{\frac{1}{2}}, 0\right)\,. \tag{5.31b}$$

*Remark.* Note that the angle is unique (up to a sign); hence waves of a variety of wavenumbers will propagate at this angle.

For these waves, from (5.28d),

$$\begin{aligned} \mathbf{c}_{gs} &= c_0 \widehat{\mathbf{k}} - \mathbf{U} &\quad (5.31\text{c})\\ &= c_0 (M^2 - 1)^{\frac{1}{2}} (-\sin\phi, \cos\phi)\,, &\quad (5.31\text{d}) \end{aligned}$$

where we note that $\mathbf{c}_{gs} \cdot \mathbf{U} < 0$ and $\mathbf{c}_{gs} \cdot \widehat{\mathbf{k}} = 0$. Hence steady waves develop on a Mach cone behind the aircraft at a semi-angle of $\alpha = \sin^{-1}(\cos\phi) = \sin^{-1}(1/M)$.

Further, since (5.31c) also holds for unsteady waves, the locus of unsteady sound waves is a sphere within
23/16 the Mach cone.
23/17

### 5.3.4 Ship/duck waves

Consider the wave pattern generated by a ship, or duck, travelling with a uniform velocity $\mathbf{U} = (U, 0, 0)$. In the frame in which the fluid is at rest, the dispersion relation for deep water gravity waves with no surface tension is (see (4.37a))

$$\Omega_r = \sqrt{gk}\,, \tag{5.32a}$$

where we take the wavevector, $\mathbf{k}$, to be given by

23/18 $$\mathbf{k} = k(\cos\phi, 0, \sin\phi)\,. \tag{5.32b}$$

In the frame moving with the duck, the dispersion relation for deep-water gravity waves is

$$(\omega_{\mathrm{s}} + \mathbf{k} \cdot \mathbf{U})^2 = gk\;. \tag{5.32c}$$

Henceforth drop the subscript '$s$', and solve for $\omega$ to obtain

$$\omega = \Omega(k_1, k_3) = \pm\sqrt{gk} - Uk\cos\phi\;. \tag{5.32d}$$

Again this is an anisotropic dispersion relation since $\Omega \not\equiv \Omega(k)$. As expected the group velocity is not parallel to $\mathbf{k}$:

23/19 $$\mathbf{c}_g = \pm\tfrac{1}{2}\sqrt{\frac{g}{k}}\,\widehat{\mathbf{k}} - \mathbf{U}\,. \tag{5.32e}$$

Seek a **steady** wave pattern with

$$\omega = 0\;.$$

From (5.32d) we need the $\pm$ signs according as $\cos\phi \gtrless 0$; so

$$U\cos\phi = \pm\sqrt{\frac{g}{k}}\,, \tag{5.33a}$$

and

$$\mathbf{c}_g = \tfrac{1}{2}U\cos\phi\,\widehat{\mathbf{k}} - \mathbf{U} = \tfrac{1}{2}U(\cos^2\phi - 2, 0, \sin\phi\cos\phi)\,. \tag{5.33b}$$

Thus $\mathbf{c}_g \cdot \widehat{\mathbf{x}} < 0$, and the waves appear behind the duck. Moreover, the rays are straight since

$$\Omega_t = 0 \quad \text{and} \quad \nabla_{\mathbf{x}}\,\Omega = 0\,. \tag{5.34a}$$

On such rays let

$$\frac{dz}{dx} = -\tan\psi\,. \tag{5.34b}$$

Then from (5.10d)

$$\tan\psi = -\frac{\frac{dz}{dt}}{\frac{dx}{dt}} = -\frac{\frac{\partial\omega}{\partial k_3}}{\frac{\partial\omega}{\partial k_1}} = -\frac{(\mathbf{c}_g)_z}{(\mathbf{c}_g)_x} = \frac{\tan\phi}{1+2\tan^2\phi}\,, \tag{5.34c}$$

and

$$\widehat{\mathbf{c}}_g = (-\cos\psi, 0, \sin\psi)\,. \tag{5.34d}$$

Hence for given $\phi$,

solve (5.33a) for $k$: $$k = \frac{g}{U^2\cos^2\phi}\,, \tag{5.35a}$$

solve (5.34c) for $\psi$: $$\psi = \arctan\left(\frac{\tan\phi}{1+2\tan^2\phi}\right)\,. \tag{5.35b}$$

We can restrict attention to

$$-\frac{\pi}{2} < \phi < \frac{\pi}{2}\,, \tag{5.36a}$$

or equivalently $\cos\phi > 0$, since this leads to no less choice for $\psi$ in (5.35b). However, in order to have waves behind the duck, we need

$$|\psi| < \frac{\pi}{2}\,. \tag{5.36b}$$

Further, there exists a maximum value of $|\psi|$. In particular

$$\frac{\partial\tan\psi}{\partial\tan\phi} = 0\,, \tag{5.37a}$$

when

$$\tan^2\phi = \tfrac{1}{2}\,, \quad \tan|\psi| = \frac{1}{2\sqrt{2}}\,. \tag{5.37b}$$

The maximum value of $|\psi|$ is approximately $19\frac{1}{2}^\circ$, i.e. the waves are all confined to a wedge of semi–angle $19\frac{1}{2}^\circ$ behind the duck.

*Geometric interpretation.* Rewrite (5.33b) as

$$\mathbf{c}_g = \tfrac{1}{4}U(\cos2\phi, 0, \sin2\phi) - \tfrac{3}{4}U(1,0,0)\,. \tag{5.38}$$

Hence the locus of $\mathbf{c}_g$ lies on a circle confined within a wedge of semi-angle $\psi_{\max} = \sin^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{2\sqrt{2}}$.

While the duck travels a distance $U\tau$, the distance travelled by waves with wavenumber $k = g(U\cos\phi)^{-2}$ is $c_{gr}\tau = \left(\frac{1}{2}U\cos\phi\right)\tau$.

*Wavecrests.* The phase can be calculated by writing

$$\begin{aligned}
\theta &= \int_{\text{any path}} \nabla\theta\cdot\mathbf{dx} \\
&= \mathbf{k}\cdot\mathbf{x} && \text{choose a ray, on which } \mathbf{k} \text{ is constant, as the path} \\
&= kr\cos(\pi-\psi-\phi) \\
&= -kr\cos\psi\cos\phi(1-\tan\psi\tan\phi) \\
&= -kr\frac{\cos\psi}{\cos\phi}\left(\frac{1}{1+2\tan^2\phi}\right) && \text{from (5.34c)}\,. \qquad (5.39a)
\end{aligned}$$

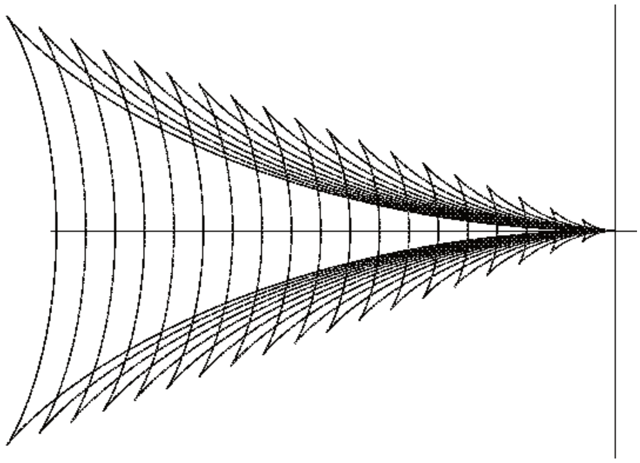
Hence

$$x = -r\cos\psi = -\theta\,\frac{\cos\phi(1+2\tan^2\phi)}{k}\,. \qquad (5.39b)$$

Thus from (5.35a) we obtain a parametric equation for the crest shape:

$$x = -\frac{U^2\theta}{g}\cos^3\phi\,(1+2\tan^2\phi)\,, \qquad (5.40a)$$

$$z = -x\tan\psi = \frac{U^2\theta}{g}\cos^2\phi\,\sin\phi\,. \qquad (5.40b)$$



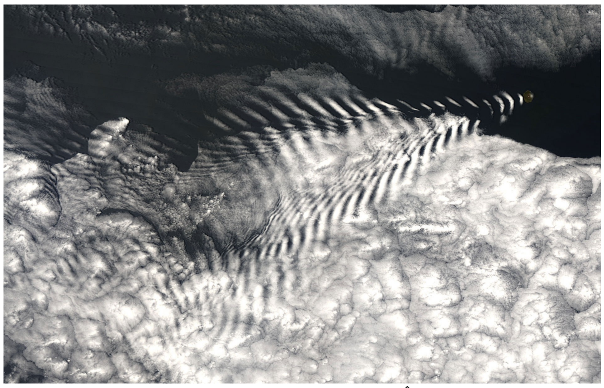
Pattern of wavecrests behind a steadily moving object.

*Remarks.*

(i) Unsteady waves overlie this pattern, especially in the wake.

(ii) Stationary phase fails in the immediate neighbourhood of the cusps.

Credit: Daderot.



Cloud pattern in the wake of the Île Amsterdam (NASA, http://tinyurl.com/Cloud-Wake).

### 5.3.5 Internal gravity waves in a horizontal wind

Assume the all motion is two-dimensional and confined to the $x$–$z$ plane. Suppose that the fluid medium has a background wind:

$$\mathbf{U} = (U(z), 0, 0)\ . \tag{5.41}$$

Assume that the lengthscale $L$ over which the wind changes velocity is long compared with the wavelength of any wave, i.e. assume that the wind is slowly varying.

An observer fixed on the ground sees a moving medium with velocity $+\mathbf{U}$ (cf. the start of §5.3 where in the frame is which the source is stationary, the fluid was assumed to have velocity $-\mathbf{U}$). Hence from (5.28c), *with a change of sign*,

$$\omega \equiv \omega_s = \omega_r + k_1 U\ , \tag{5.42a}$$

where $\omega_r$ is the frequency in a frame moving at the wind speed. Thus from (4.46b) with $k_2 = 0$:

$$\omega = \Omega(\mathbf{k}; z) \equiv \frac{N(z)k_1}{k} + k_1 U(z)\ . \tag{5.42b}$$

From the ray-tracing equations (5.10b) and (5.10a), $\omega$ and $k_1$ are constant on rays since

$$\frac{\partial\Omega}{\partial t} = \frac{\partial\Omega}{\partial x} = 0\,. \tag{5.43a}$$

In addition

$$\frac{dk_3}{dt} = -\frac{\partial\Omega}{\partial z} = -k_1\left(\frac{N'}{k} + U'\right)\ , \tag{5.43b}$$

while from (5.10d)

$$\frac{dx}{dt} = \frac{\partial\Omega}{\partial k_1} = \frac{Nk_3^2}{k^3} + U\,, \tag{5.43c}$$

$$\frac{dz}{dt} = \frac{\partial\Omega}{\partial k_3} = -\frac{Nk_1k_3}{k^3}\,. \tag{5.43d}$$

For definiteness assume that:

(i) $N = \text{constant}$; i.e. from (4.42c) assume that

$$\rho_0 = \bar\rho\exp(-zN^2/g)\,, \quad \text{where } \bar\rho \text{ is a constant.} \tag{5.44}$$

(ii) $k_1 > 0$.

(iii) $k_3 < 0$, so that the waves are propagating upwards.

Then from (5.43b) and (5.43d)

$$\frac{dk_3}{dz} = \frac{U'k^3}{Nk_3}\ , \tag{5.45a}$$

and thus since $k_1$ is a constant

$$\frac{U(z) - U(0)}{N} = \frac{1}{(k_1^2 + k_3^2(0))^{\frac{1}{2}}} - \frac{1}{(k_1^2 + k_3^2)^{\frac{1}{2}}}\,. \tag{5.45b}$$

As $z$ increases, and $U(z)$ increases from $U(0)$, $k_3$ must increase. If $U(z)$ increases sufficiently that

$$U(z_c) - U(0) = \frac{N}{(k_1^2 + k_3^2(0))^{\frac{1}{2}}} \tag{5.46a}$$

then

$$k_3(z_c) = -\infty\ . \tag{5.46b}$$

*Exercise (see also Question 9 on Example Sheet 4).* Show that as $z \to z_c-$,

$$k_3 \sim -\frac{1}{U'(z_c)(z_c - z)} \qquad \to -\infty \,, \tag{5.47a}$$

$$x \sim \frac{U(z_c)}{Nk_1\,(U'(z_c))^2}\,\frac{1}{z_c - z} \to \infty \,, \tag{5.47b}$$

$$t \sim \frac{1}{Nk_1\,(U'(z_c))^2}\,\frac{1}{z_c - z} \to \infty \,. \tag{5.47c}$$

Thus a wave packet takes an infinite time to approach the *critical level*, $z = z_c$, along a ray. Show also that $U(z_c) = \omega/k_1$, i.e. that at the critical level the fluid velocity equals the apparent phase velocity in the same direction.

** Conservation of wave action (unlectured) *.* For the current system, conservation of wave action can be expressed as

$$\frac{\partial}{\partial t}\left(\frac{E_r}{\omega_r}\right) + \boldsymbol{\nabla}\cdot\left(\frac{E_r}{\omega_r}\,\boldsymbol{\nabla}_k\Omega\right) = 0\,, \tag{5.48}$$

where $E_r$ is the wave energy density in a frame moving with the wind, and $\omega_r$ and $\Omega$ are as earlier. For the normalisation,

$$\rho_0\, w = A(\mathbf{x}, t; \epsilon)\, \exp\left(\frac{\imath}{\epsilon}\,\theta(\mathbf{x}, t)\right) \,, \tag{5.49a}$$

it follows (after some manipulation) that

$$E_r = \frac{|A|^2}{2\rho_0}\,\frac{k^2}{k_1^2} \,. \tag{5.49b}$$

*Steady wave pattern.* First consider a steady wave pattern. Then the upward component of wave-action flux is constant, i.e.

$$\frac{E_r}{\omega_r}\,\frac{\partial\Omega}{\partial k_3} = \text{const} \,. \tag{5.50a}$$

As a critical level is approached:

$$\omega_r = \omega - k_1 U = \frac{Nk_1}{k} \approx \frac{Nk_1}{k_3} \,, \tag{5.50b}$$

$$\frac{\partial\Omega}{\partial k_3} = -\frac{Nk_1k_3}{k^3} \approx -\frac{Nk_1}{k_3^2} \,. \tag{5.50c}$$

Hence as the critical level is approached

$$E_r \approx \text{const.}(-k_3) \to \infty \,,$$
$$|A| \approx \frac{\text{const.}}{(-k_3)^{\frac{1}{2}}} \to 0 \,. \tag{5.50d}$$

*Wave packet.* Alternatively, consider a wave packet. Then (5.48) implies that the total wave action of the wavepacket will be conserved. As the wavepacket approaches the critical level, then from (5.50b),

$$\omega_r \to 0 \,. \tag{5.51}$$

Hence the total wave energy of the packet must also go to zero. There is no dissipation, so where does the energy go? The answer is that is transferred to the mean flow (see the Part III
24/16 Geophysical Fluid Dynamics course or similar).

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