Example Sheet 4

1. By considering the graph of $F(x)$ (with $x \in \mathbb{R}$), for each of the maps

$$F(x) = -x, \qquad F(x) = x + x^2, \qquad F(x) = x - x^3 \qquad \text{and} \qquad F(x) = x + x^3,$$

state whether the (non-hyperbolic) fixed point at the origin is Lyapunov stable, asymptotically stable or neither.

2. Find and analyse the successive bifurcations in the map $F : \mathbb{R} \to \mathbb{R}$ given by $x_{n+1} = \mu - \frac{1}{4}x_n^2$ as μ increases from $-\infty$ to 5.

3. By considering the binary representation, show that the sawtooth map has exactly two 3-cycles and express them as fractions.

*Use a similar technique to find the two 3-cycles of the tent map for $\mu = 2$.

4*. Show that the sawtooth map has an orbit that is dense in $[0, 1)$ i.e. an orbit that comes arbitrarily close to every point in the interval. (There are many such orbits – construct one!)

5. Let F be a continuous map on $\mathbb{R}$ and let $x_0 < x_1 < x_2 < x_3$ be the members of a 4-cycle with $F(x_n) = x_{n+1[\text{mod } 4]}$. Prove (formally, using the IVT etc.) that F has a fixed point, a 2-cycle and a 3-cycle. Explain informally why F has at least one 4-cycle in addition to $(x_0, \ldots, x_3)$.

6. (1996/2/11) Let F be a continuous map on $\mathbb{R}$ and let $x_4 < x_2 < x_0 < x_6 < x_1 < x_3 < x_5$ be the members of a 7-cycle with $F(x_n) = x_{n+1[\text{mod } 7]}$. Show (less formally, using a directed graph) that F has N-cycles for all $N > 8$ and for all even N.

*Sketch the graph of such an F that does not have a 3-cycle or 5-cycle, and explain why.

7*. (1997/2/12) Find a continuous map of $[0, 1]$ onto itself that has a fixed point and cycles of all even periods, but no cycles of odd periods ≥ 3.

8. Consider the skewed tent map

$$F(x) = \mu x, \quad 0 < x < a$$
$$F(x) = \mu a(1 - x)/(1 - a), \quad a < x < 1$$

where $0 < a < 1$. When is F a map of $[0, 1]$ into itself? When is there a nontrivial fixed point? Show that this fixed point is at $x_0 = \mu/[\mu + \mu_s(a)]$ (with $\mu_s(\frac{1}{2}) = 1$). When is x_0 stable? Show that there is a value $\mu_c(a)$ (with $\mu_c(\frac{1}{2}) = \sqrt{2}$) such that F^2 has a horseshoe for $\mu > \mu_c$. When is the map chaotic?

9. Consider the (usual) tent map for $2^{1/8} \leq \mu < 2^{1/4}$. Explain why F^8 has a horseshoe and describe the extent of the chaotic attractor. For what values of N can you now be sure that F has an N-cycle? *Show further that F has a 12-cycle if $\mu > [(1 + \sqrt{5})/2]^{1/4}$ but no 6-cycle.

Questions 10–12 refer to the logistic map $F(x) = \mu x(1 - x)$ on $[0, 1]$.

10. Sketch the graphs of x, $F(x)$ and $F^2(x)$ using one diagram for each of the cases $i < \mu \leq i + 1$, $i = 0, 1, 2, 3$. Find all the fixed points of F and F^2, determining which of them are attractors.

[Note that F^2 has at most three extrema, one of which is $\frac{1}{2}$ by symmetry, and some cases split on whether $\frac{1}{2}$ is a maximum or a minimum.]

11*. Use a computer to examine the limiting behaviour of x_n as $n \to \infty$ for each of the following values of μ and various initial values of x (including $x = \frac{1}{2}$):

$$\mu = 2.6,\ 3.1,\ 3.5,\ 3.55,\ 3.566,\ 3.829,\ 3.845,\ 3.9222.$$

In each case decide whether or not the ω-limit set appears to be a cycle of finite order.

12. (i) Show that for $2 < \mu < 3.678\ldots$ there are exactly two points for each $n > 1$ such that

$$F^{n+1}(x) = 1 - \mu^{-1}, \qquad F^n(x) \neq 1 - \mu^{-1},$$

where $\mu_c = 3.678\ldots$ is a root of $\mu^4 - 4\mu^3 + 16 = 0$. Show further that the set of all such points as n varies has 0 and 1 as its only points of accumulation.

(ii) Hence identify the domain of attraction of the 2-cycle in $3 < \mu < 1 + \sqrt{6}$.

Please email corrections/comments to lister@damtp.cam.ac.uk