Symmetry in Nature: What does mathematics tell us about pattern formation in the natural world?

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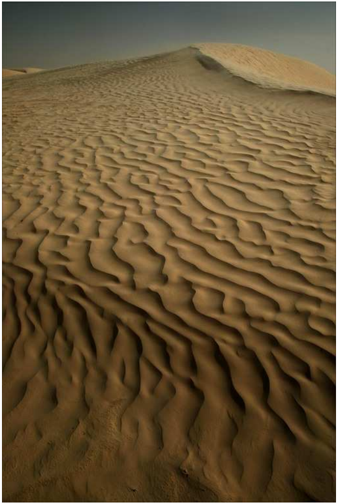
Examples of natural patterns

Patterns with (almost) periodic structure are seen all over the place in nature. Some examples are:

- Stripes in sand dunes
- Stripes on your fingertips and palms
- Animal markings:
    - Stripes on zebra, tigers, tropical fish, ...
    - Spots on leopards, cheetahs, fish, lizards, ...
    - More esoteric patterns on giraffe, shellfish, ...
- Cacti, tree bark and other plants
- Plant growth on tundra
- Minerals and precious stones

Sand dunes

Sand dunes



Things to notice:

- Definite stripey pattern
- Almost spatially periodic, but. . .
- There are defects where two stripes join together

Source: Declan McCullagh Photography

Fingerprints

Fingerprints

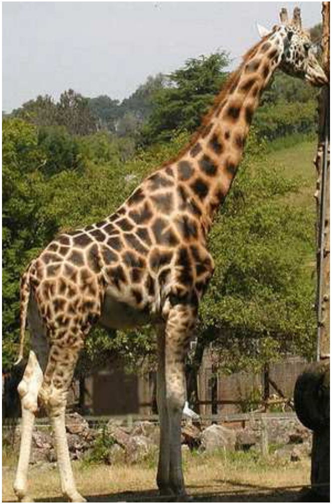
Things to notice:

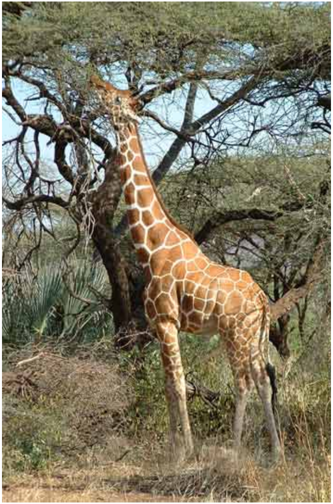
- The ridges and furrows form a pattern of stripes
- There are defects where two ridges or furrows meet
- There are also loops and spirals

Animal markings: stripes

Animal markings: spots

Giraffes





Minerals and precious stones

Examples of unnatural (laboratory!) patterns

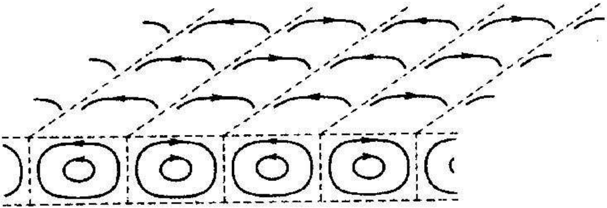
We also see plenty of patterns in the lab:

- Stripes and hexagon convection patterns in heated fluid
- Chemical reactions
- Cracking in ceramic materials
- Taylor vortices in rotating fluid
- Faraday waves in vertically shaken fluid

Rayleigh-Bénard convection

Fluid fills the space between two flat plates. The top plate is held at temperature $T = T_0$ and the bottom plate is heated to $T = T_0 + \delta T$.

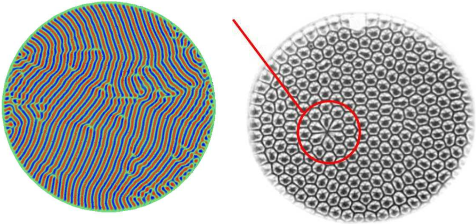
At low temperature differences, *conduction* moves heat from the bottom of the container to the top.



As δT is increased, conduction cannot transfer heat fast enough and we see the onset of *convection*.

Rayleigh-Bénard convection: patterns

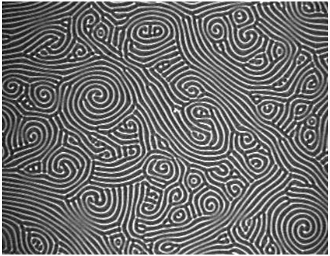
Two possible patterns of convection are *stripes* (or convection rolls) and *hexagons* (or convection cells).



We can understand how these patterns exchange stability via the use of a *bifurcation diagram*, a diagram which indicates qualitative changes in the behaviour of a system.

Rayleigh-Bénard convection: spiral defect chaos

Another apparently stable state is known as *spiral defect chaos*.



This chaotic convection pattern is very poorly understood at present, and it is the subject of much research.

It is believed that the onset of spiral waves in electrical signals in your heart is responsible for ventricular fibrillation, a potentially fatal cardiac rhythm.

The Belousov-Zhabotinsky reaction

This reaction is a mix of:

- potassium bromate
- cerium(IV) sulfate
- propanedioic acid
- citric acid

. . . in dilute sulfuric acid.

The colour difference is due to the proportion of cerium(III) and cerium(IV) ions.

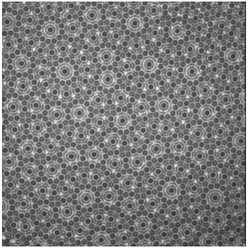
Faraday wave experiment

In the Faraday wave experiment, a container of fluid (with viscosity about 50 times that of water) is vibrated vertically at high frequency.

Initially there is no pattern. As the frequency is increase a pattern of standing waves appears, and increasing the frequency still further results in a pattern of squares or hexagons.

Faraday wave experiment: quasipatterns

If two different frequencies of vibration are used (for example, in a ratio 4:5) then something new and entirely surprising happens:



This 12-fold quasipattern has no translational symmetry, but it does have local rotational symmetries. At present there is no good theoretical basis for the formation of the quasipattern.

Commonalities of patterns

We have seen that similar patterns occur in very different situations, which suggests that it might be worth thinking about pattern formation in a model-independent, mathematical way.

Some observations about patterns:

- Spatial/spatio-temporal structures with *symmetry*
- Typically two-dimensional
- Often periodic in space
- Tesselate the plane: stripes, squares, hexagons . . .
- . . . or not: spirals, targets, quasipatterns . . .
- They arise (or *emerge*) spontaneously in both natural and laboratory setting.

Modelling pattern-forming systems

We assume that our pattern-forming system has only one interesting variable $u \in \mathbf{R}$ (for example the vertical fluid velocity, chemical concentration, amplitude of oscillation) which is a function of position and time:

$$u = u(\mathbf{x}, t)\,.$$

We further assume that there is a parameter $\lambda \in \mathbf{R}$ which is under our control (for example temperature differential or frequency of shaking).

Modelling pattern-forming systems

Finally, we assume that there is a partial differential equation for u which tells us how the system evolves:

$$\frac{\partial u}{\partial t} = F_\lambda(u, \nabla u, \nabla^2 u, \dots\,).$$

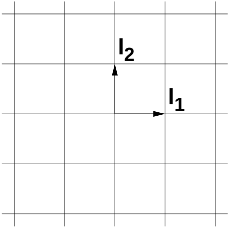
But partial differential equations are hard to solve, so we make another assumption. We know that lots of interesting patterns are approximately periodic, so we restrict our attention to functions which are spatially periodic. This involves using *lattices*.

Lattices

A planar lattice L is a subset of $\mathbf{R}^2$ generated by two vectors:

$$L = \{n_1\mathbf{l}_1 + n_2\mathbf{l}_2 : n_1, n_2 \in \mathbf{Z}\}.$$

For example, the choice $\mathbf{l}_1 = (1,0)$ and $\mathbf{l}_2 = (0,1)$ generates the square lattice:



Modelling pattern-forming systems

The functions that are periodic on a lattice can be written as combinations of sines and cosines, or more conveniently as complex exponentials:

$$u(\mathbf{x}, t) = z_1(t)e^{2\pi ix} + z_2(t)e^{2\pi iy} + c.c.$$

where $\mathbf{x} = (x, y)$ and 'c.c.' means complex conjugate.

Now all the time-dependence is in the amplitudes z_1, z_2, z_3, so we can write down a system of first-order ODEs for the amplitudes.

Modelling pattern-forming systems

Our hypothesis is going to be:

"A spatial pattern is the result of a symmetry-breaking bifurcation from a homogeneous equilibrium, which occurs as a parameter λ passes through some critical value"

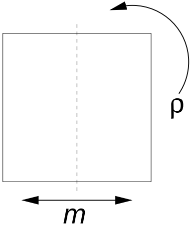
This suggests that our theory of pattern formation should have two components:

- Bifurcation theory: small changes have large effects.
- Group theory: the study of symmetry.

Equivariance

An important concept is that of the *automorphism group* of a mathematical object, which is the set of transformations on an object which leave it unchanged.

For example, the automorphism group of the square is $\mathcal{D}_4$, which is generated by a rotation ρ and a reflection m:



Equivariance

Take a first-order system of equations,

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$$

for $\mathbf{z} \in \mathbf{R}^n$ and $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$. A group $\mathcal{G}$ *acts* on $\mathbf{R}^n$ via a *representation*, a map $\phi : \mathcal{G} \to GL(\mathbf{R}^n)$ which associates a matrix $\phi(g)$ to each $g \in \mathcal{G}$, such that

$$\phi(gh) = \phi(g)\phi(h)$$

for all $g, h \in \mathcal{G}$.

We write $g\mathbf{z}$ as a shorthand for $\phi(g)\mathbf{z}$.

Equivariance

Take a first-order system of equations,

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$$

for $\mathbf{z} \in \mathbf{R}^n$ and $\mathbf{f} : \mathbf{R}^n \longrightarrow \mathbf{R}^n$. To say that g is a symmetry of the equations means that $g\mathbf{z}(t)$ is a solution whenever $\mathbf{z}(t)$ is a solution. This leads to the condition:

$$\mathbf{f}(g\mathbf{z}) = g\mathbf{f}(\mathbf{z}) \quad \forall g \in \mathcal{G}$$

called the *equivariance condition*.

Equivariance

The equivariance condition tells us many useful things about the system, but two of the more important are:

- If we have stationary solution $\mathbf{z}(t) = \mathbf{z_0}$, then the symmetrically related solution $g\mathbf{z}_0$ is also a stationary state.
- The linearisations about these stationary states are identical, so the two states have the same stabilities.

Symmetry and patterns

This is important for pattern formation because equations on the infinite plane have a natural symmetry group: the Euclidean group $E(2)$, of all rotations, reflections and translations in the plane.

Because we assumed that solutions were periodic on a lattice, we actually deal with a subgroup of this group.

The group has a natural action on $(x, y) \in \mathbf{R}^2$, and we extend this to an action on the amplitudes z_1, z_2.

$$u(\mathbf{x}, t) = z_1(t)e^{2\pi ix} + z_2(t)e^{2\pi iy} + c.c\,.$$

Amplitude equations

We can apply the equivariance condition to deduce the most general form for the amplitude equations. On the square lattice they are

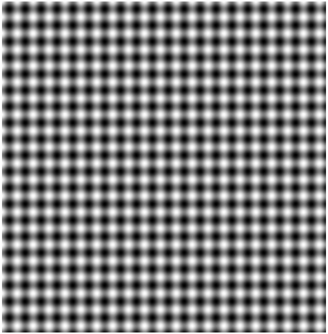
$$\begin{aligned}\dot{z}_1 &= \lambda z_1 - A z_1^3 - B z_1 z_2^2 \\ \dot{z}_2 &= \lambda z_2 - A z_2^3 - B z_2 z_1^2\end{aligned}$$

for A, B constants. Because these equations have a lot of 'symmetry', the steady-state solutions also look symmetric.

Amplitude equations

Two solutions of the abstract system on a square lattice are:

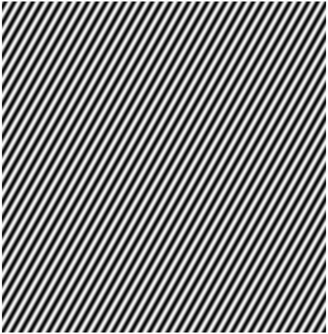
The solution $(z_1, z_2) = (a, 0)$ gives stripes.



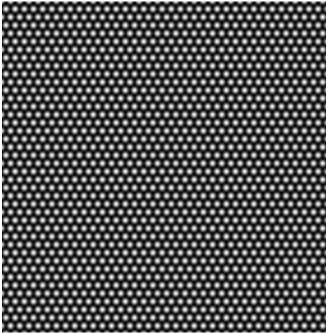
The solution $(z_1, z_2) = (a, a)$ gives squares.

Amplitude equations

Two solutions of the abstract system on a hexagonal lattice are:



The solution $(z_1, z_2, z_3) = (a, 0, 0)$ gives stripes.



The solution $(z_1, z_2, z_3) = (a, a, a)$ gives hexagons.

Conclusions

Some conclusions that can be drawn:

- Pattern forming problems can be considered as symmetry-breaking bifurcations from a spatially homogeneous equilibrium.
- There are generic equations for pattern-forming problems, which are largely independent of the physics of the system.
- Tools of group theory and bifurcation theory can help us to analyse these systems, and hence to analyse the natural world.