

# Part II Dynamical Systems: Revision

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## 1 Fixed points

A 2-d flow  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  has a fixed point  $\mathbf{x}^*$  wherever  $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$ . The fixed points can be classified by the eigenvalues of the **linearization**:

$$\left[ \begin{array}{cc} \partial f_1 / \partial x & \partial f_1 / \partial y \\ \partial f_2 / \partial x & \partial f_2 / \partial y \end{array} \right] \Big|_{(x,y)=(x^*,y^*)} .$$

The most common possibilities are:

- If eigenvalues are both negative then  $\mathbf{x}^*$  is a **stable node**.
- If eigenvalues are both positive then  $\mathbf{x}^*$  is an **unstable node**.
- If eigenvalues are of different sign then  $\mathbf{x}^*$  is a **saddle**.
- If eigenvalues are a complex conjugate pair then  $\mathbf{x}^*$  is a **stable focus** (real parts negative) or **unstable focus** (real parts positive).
- If eigenvalues are pure imaginary then  $\mathbf{x}^*$  is a **centre**.

If the flow is **Hamiltonian** (i.e. there exists  $H$  such that  $f_1 = \partial H / \partial x$  and  $f_2 = -\partial H / \partial y$ ) then the fixed points are saddles or centres, and the flow is along level curves of  $H$ .

For a **hyperbolic** fixed point (real part of eigenvalues nonzero) the flow of the linearised fixed point is homeomorphic to the nonlinear flow (sufficiently close to the fixed point). This is not true for non-hyperbolic fixed points.

The **stable subspace**  $E^S$  is the space spanned by the eigenvectors whose eigenvalues are negative. The **unstable subspace**  $E^U$  is spanned by eigenvectors with real part positive. The stable manifold theorem says that there are stable/unstable manifolds (curves)  $W^S$  and  $W^U$  which are tangent to  $E^S$  and  $E^U$  at the origin.

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## 2 Stability

A subset  $\Sigma$  of phase space is **Lyapunov stable** if for any radius  $\epsilon$ , you can find a perturbation  $\delta$  sufficiently small that trajectories remain within  $\epsilon$  of  $\Sigma$  (starts near, stays near).

$\Sigma$  is **quasi-asymptotically stable** if trajectories eventually tend to it.

$\Sigma$  is **asymptotically stable** if it is both Lyapunov and quasi-asymptotically stable.

A **Lyapunov function** for a dynamical system on  $\mathbf{R}^n$  is a function  $V : \mathbf{R}^n \rightarrow \mathbf{R}$  which satisfies:

(i)  $V(\mathbf{0}) = 0$  and  $V(\mathbf{x}) > 0$  for  $\mathbf{x} \neq \mathbf{0}$

(ii)  $\dot{V} \leq 0$ .

Additionally,  $V$  is a **strict Lyapunov function** if the inequality in (ii) is strict.

**Theorem 1 (Lyapunov's first theorem).** *If a Lyapunov function exists then  $\mathbf{x} = \mathbf{0}$  is Lyapunov stable.*

**Theorem 2 (Lyapunov's second theorem).** *If a strict Lyapunov function exists then  $\mathbf{x} = \mathbf{0}$  is asymptotically stable.*

**Theorem 3 (LaSalle's invariance principle).** *If  $V$  is a Lyapunov function on a domain  $D$ , and  $O(\mathbf{x}) \subseteq D$ , then  $\phi_t(\mathbf{x})$  tends to an invariant subset of  $\{\dot{V} = 0\}$ .*

## 3 Periodic orbits

The Poincaré index of a sink or a source is  $+1$ . The Poincaré index of a saddle is  $-1$ .

**Theorem 4 (Poincaré index test).** *A periodic orbit must enclose fixed points whose Poincaré indexes sum to  $+1$ .*

**Theorem 5 (Dulac's criterion).** *If there exists a function  $\phi$  on a domain  $D$  such that  $\nabla \cdot (\phi \mathbf{f}) \neq 0$  then there are no periodic orbits lying entirely in  $D$ .*

**Theorem 6 (Gradient test).** *If there exist functions  $h > 0$  and  $V$  such that  $\mathbf{f} = -h\nabla V$  then the system has no periodic orbits.*

**Theorem 7 (Poincaré-Bendixson).** *If  $O^+(\mathbf{x})$  tends to a compact domain  $K$ , and  $K$  has no fixed points, then  $\omega(\mathbf{x})$  is a periodic orbit (i.e.  $K$  contains at least one periodic orbit).*

A flow is **nearly Hamiltonian** if  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(\mathbf{x})$ , where  $\mathbf{f}$  is Hamiltonian. We integrate around an orbit of the Hamiltonian flow to find

$$\Delta H = \int \dot{H} dt.$$

Periodic orbits of the near-Hamiltonian flow exist at  $\Delta H = 0$ .

The stability of a periodic orbit is given by calculating the Floquet multiplier:

$$\lambda = \exp \left\{ \int \nabla \cdot \mathbf{f}|_{H=H_0} dt \right\}$$

where the integration is around one period of the Hamiltonian system, and  $H_0$  is a zero of  $\Delta H$ . If  $\lambda > 1$  the periodic orbit is unstable, if  $\lambda < 0$  the periodic orbit is stable.

## 4 Bifurcations

### 4.1 Bifurcations of flows

A **bifurcation** is a change in qualitative behaviour of a system as a parameter  $\mu$  passes through 0.

The **centre subspace**  $E^C$  is the space spanned by the eigenvector of the linearisation with eigenvalue 0. The centre manifold theorem guarantees the existence of a manifold  $W^C$  tangent to  $E^C$  at the origin.

If one eigenvalue passes through 0 at  $\mu = 0$  and the others are negative, then  $W^C$  is one-dimensional and the evolution on  $W^C$  is  $\dot{x} = f(x, \mu)$ .

A **steady state** bifurcation occurs when an eigenvalue passes through 0. The three most common steady-state bifurcations are:

- A **saddle-node** bifurcation (Fig 1(a)) has normal form  $\dot{x} = \mu - x^2$ .
- A **transcritical** bifurcation (Fig 1(b)) has normal form  $\dot{x} = \mu x - x^2$ .
- A **pitchfork** bifurcation has normal form  $\dot{x} = \mu x - ax^3$ , where  $a > 0$  is a **supercritical** bifurcation (Fig 1(c)) and  $a < 0$  is a **subcritical** bifurcation (Fig 1(d)).

The **Hopf bifurcation** is an oscillatory bifurcation (eigenvalues pass through  $\pm i\omega$ ) with the normal form (in polar coordinates):

$$\begin{aligned} \dot{r} &= \mu r - ar^3 \\ \dot{\theta} &= \omega + \mu c - br^2 \end{aligned}$$

where  $a > 0$  is a **supercritical** bifurcation and  $a < 0$  is a **subcritical** bifurcation.

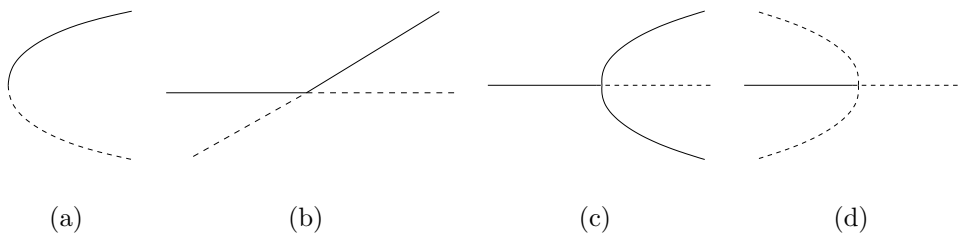


Figure 1: Bifurcation diagrams for the three most common one-dimensional bifurcations.

## 4.2 Bifurcations of maps

A map takes the form  $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$  for  $n = 0, 1, 2, \dots$ . A fixed point  $\mathbf{x}^*$  satisfies  $\mathbf{x}^* = \mathbf{F}(\mathbf{x}^*)$ . Linearising about the fixed point gives the map  $\mathbf{y}_{n+1} = A\mathbf{y}_n$ , where  $A$  is a matrix with eigenvalues  $\lambda_i$ . Then:

- $\mathbf{x}^*$  is **asymptotically stable** if  $|\lambda_i| < 1$  for all  $i$ .
- $\mathbf{x}^*$  is **unstable** if  $|\lambda_i| > 1$  for any  $i$ .
- $\mathbf{x}^*$  is **non-hyperbolic** if  $|\lambda_i| = 1$  for any  $i$ .

A bifurcation of a map occurs when an eigenvalue of  $A$  crosses the unit circle.

If  $\lambda = 1$  then you see one of the following **steady state** bifurcations:

- A **saddle-node** bifurcation has normal form  $x_{n+1} = x_n + \mu - x_n^2$ .
- A **transcritical** bifurcation has normal form  $x_{n+1} = x_n + x_n(\mu - x_n)$ .
- A **pitchfork** bifurcation has normal form  $x_{n+1} = x_n + \mu x_n - ax_n^3$ , where  $a > 0$  is supercritical and  $a < 0$  is subcritical.

If  $\lambda = -1$  we have a **period-doubling** bifurcation, which creates a 2-cycle. You can find the 2-cycle by looking for fixed points of  $\mathbf{F}^2$  (the period-doubling bifurcation is a pitchfork bifurcation for  $\mathbf{F}^2$ ).

## 5 Chaos

Consider the 1-d map  $x_{n+1} = F(x_n)$  for  $F : I \rightarrow I$ .

$F$  has a **horseshoe** if there exists an interval  $J \subseteq I$  and open, disjoint intervals  $K_0, K_1 \subseteq I$  such that  $F(K_0) = F(K_1) = J$ .  $F$  is **chaotic** if  $F^n$  has a horseshoe for some  $n \geq 1$ .

The **Bernoulli shift map** is  $F(x) = 2x \pmod{1}$ . If  $x = 0.a_1a_2a_3a_4\dots$  in binary, then  $F(x) = 0.a_2a_3a_4\dots$ . You can use this to show that the shift map is chaotic.

For any map  $F$  with a horseshoe, you should be able to define an invariant subset  $\Lambda$  and show that  $F$  acts on  $\Lambda$  in the same way that the shift map acts on sequences  $0.a_1a_2a_3a_4\dots$  with  $a_i \in \{0, 1\}$ .

You should know how to prove that any map with a 3-cycle is chaotic, and that any map with a 3-cycle has periods of all orders.

You should be able to use directed graphs to find cycles of a given order.

**Sharkovsky's theorem** says that if  $F$  has an  $N$ -cycle, then  $F$  has an  $n$ -cycle for all  $n \prec N$  in the following ordering:

$$3 \succ 5 \succ 7 \succ 9 \succ \dots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ \dots \succ 2^2 \cdot 3 \succ 2^2 \cdot 5 \succ \dots \succ 2^j \succ \dots 8 \succ 4 \succ 2 \succ 1.$$