Pattern formation in lattice dynamical systems

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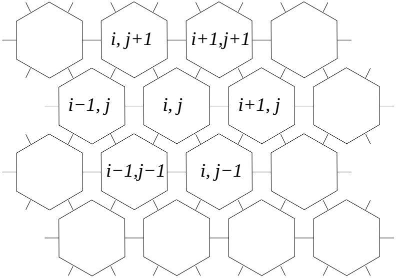
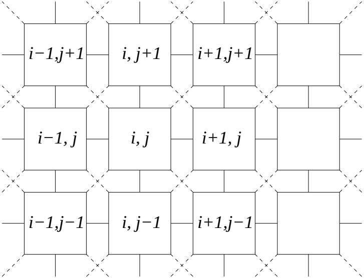
Outline

1. Introduction and motivation
2. Exactly periodic patterns
3. Long wavelength modulation
4. Localized patterns
5. Summary

What is a lattice dynamical system?

- A lattice of points, with each point having its own internal dynamics, and coupled to nearby points.
- We look at couplings with finite range and which respect the symmetries of the lattice.





Example: spatially discrete Allen-Cahn equation

The SDAC equation (on a square lattice) is

$$u_t = \beta^+ \triangle^+ u + \beta^\times \triangle^\times u + ru - u^3$$

where the operators $\triangle^+$ and $\triangle^\times$ are nearest- and next-nearest-neighbour interactions:

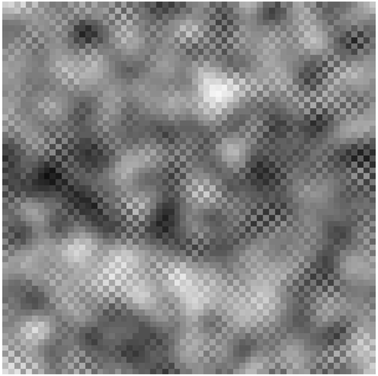
$$\begin{aligned} \triangle^+ u_{ij} &= u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} \\ \triangle^\times u_{ij} &= u_{i+1,j+1} + u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} - 4u_{ij} \end{aligned}$$

and we don't place any restrictions on the sign of β^+, $\beta^\times$, r.

Numerical results

SDAC equation on a 63x63 grid, periodic boundaries (snapshots)

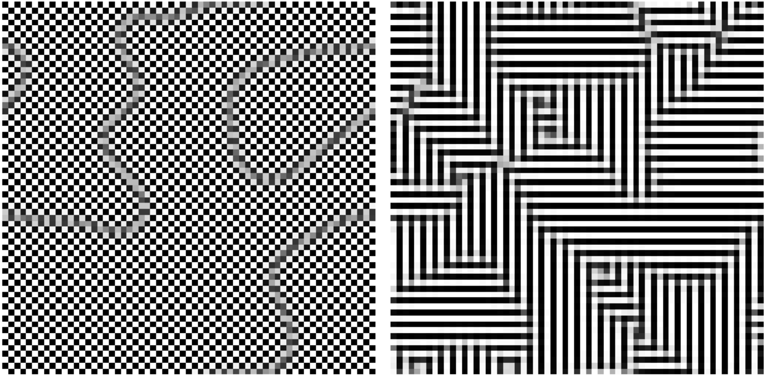
(a) $\beta^+ = 0.0625$, $\beta^\times = 0.2188$, $r = 1.0$, $T = 5$



(b) $\beta^+ = 0.025$, $\beta^\times = 0.2375$, $r = 1.0$, $T = 5$

Numerical results

SDAC equation on a 63x63 grid, periodic boundaries (snapshots)



(c) $\beta^{+} = -0.25$, $\beta^{\times} = 0.125$, $r = -1.0$, $T = 50$

(d) $\beta^{+} = 0.125$, $\beta^{\times} = -0.3125$, $r = -1.0$, $T = 50$

Example: spatially discrete Cahn-Hilliard equation

A system with longer-range coupling is the SDCH equation:

$$u_t = -\triangle(\beta^{+}\triangle^{+}u + \beta^{\times}\triangle^{\times}u + ru - u^3)$$

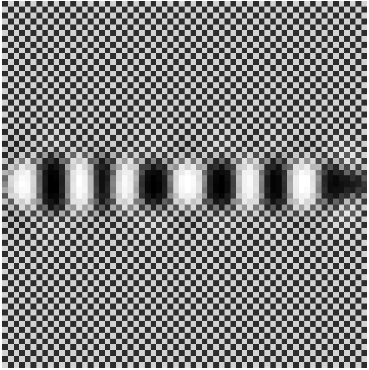
where $\triangle = \frac{1}{2}\triangle^{+} + \frac{1}{4}\triangle^{\times}$. There is a conserved quantity:

$$M = \sum_{i,j} u_{ij}$$

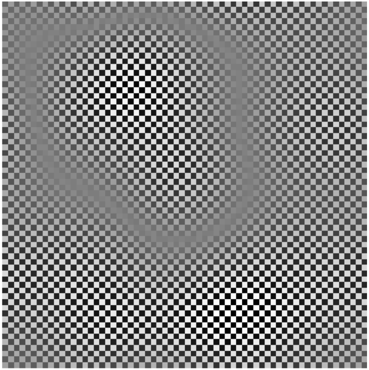
which leads to the existence of a large-scale mode that we have to take into account (c.f. Matthews and Cox 2000).

Numerical results

SDCH equation on a 64x64 grid, periodic boundaries (snapshots)



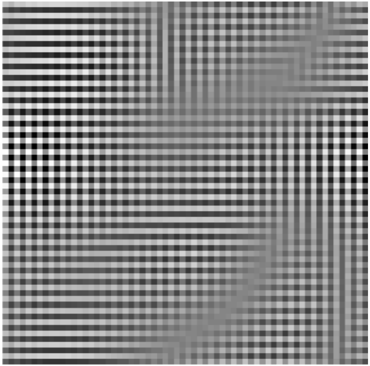
(c) $\beta^+ = 0.0125$, $\beta^\times = 0.0437$, $r = 0.15$, $T = 250$



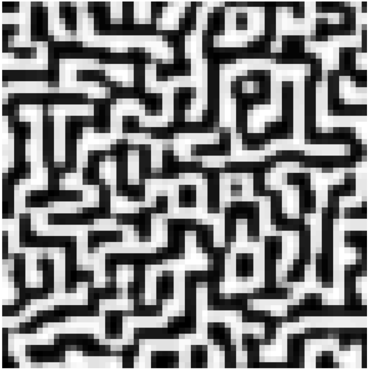
(d) $\beta^+ = -0.1313$, $\beta^\times = -0.0281$, $r = -1.0$, $T = 100$

Numerical results

SDCH equation on a 64x64 grid, periodic boundaries (snapshots)



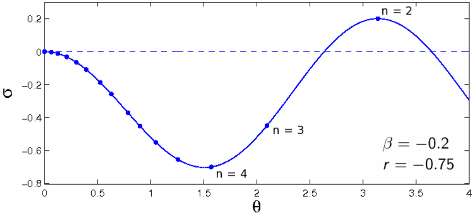
(c) $\beta^{+} = -0.0938$, $\beta^{\times} = -0.0844$, $r = -1.0$, $T = 100$



(d) $\beta^{+} = 0.0563$, $\beta^{\times} = -0.0094$, $r = 0.2$, $T = 100$

Exactly periodic patterns

- In 1D the state variable is $u = \{u_j\}_{j \in \mathbf{Z}}$.
- Perturb around the zero state, $u_j = 0$ for all j, and examine stability to perturbations of period $n = 1, 2, \cdots$.
- Let $u_j = \hat{u}(\theta) e^{\sigma t + \mathrm{i}\theta j}$, giving a dispersion relation $\sigma = f(\theta)$.
- Example: for $u_t = -\triangle(\beta \triangle u + ru - u^3)$:



- The $n = 2$ mode is excited, and $\theta = 0$ is neutrally stable.

Exactly periodic patterns

- It is often the case that the shortest scales are excited first (cf. numerical instability) so we look for solutions that are spatially periodic with small period.
- Algorithm to follow:

1. Classify patterns by their symmetry using rep theory.
2. Apply the equivariant branching lemma to get primary branches.
3. Use isotypic decomposition to analyse their stability and draw bifurcation diagrams.

Exactly periodic patterns

- For example, the 1D lattice has translation and reflection symmetry:

$$T(u)_j = u_{j+1} \qquad R(u)_j = u_{-j}$$

- This natural representation on $\{u_j\}_{j=1,\dots,N}$ is reducible, and the decomposition into irreps is:

$$\begin{array}{ll} \langle \mathbf{x}_0 \rangle, \langle \mathbf{x}_1, \mathbf{y}_1 \rangle, \langle \mathbf{x}_2, \mathbf{y}_2 \rangle, \dots, \langle \mathbf{x}_{\frac{N}{2}} \rangle & (N \text{ even}) \\ \langle \mathbf{x}_0 \rangle, \langle \mathbf{x}_1, \mathbf{y}_1 \rangle, \langle \mathbf{x}_2, \mathbf{y}_2 \rangle, \dots, \langle \mathbf{x}_{\frac{N-1}{2}}, \mathbf{y}_{\frac{N-1}{2}} \rangle & (N \text{ odd}) \end{array}$$

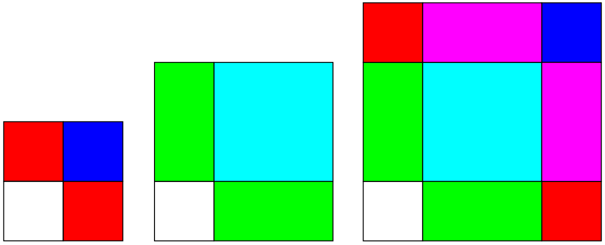
- where

$$\mathbf{x}_j = \mathfrak{Re}(1, \omega^j, \omega^{2j}, \dots, \omega^{(N-1)j})$$
$$\mathbf{y}_j = \mathfrak{Im}(1, \omega^j, \omega^{2j}, \dots, \omega^{(N-1)j})$$

and ω is a primitive Nth root of unity.

Exactly periodic patterns

- To get irreps for higher-dimensional lattices we simply take tensor products of the $\mathbf{x}_i$ and $\mathbf{y}_i$.
- In 2D there is a nice pictorial representation of these:



$N = 2$ $N = 3$ $N = 4$

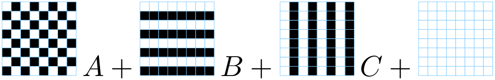
- We can do the same thing for lattices in 3 or more dimensions.

Exactly periodic patterns: Example

- For a square lattice the state variable is $u = \{u_{ij}\}_{i,j\in\mathbf{Z}}$.
- For $N = 2$ the problem is four dimensional, and there are three invariant subspaces:

$$u_{ij}(t) = (-1)^{i+j}A + (-1)^{j}B + (-1)^{i}C + D$$

- Or more explicitly:

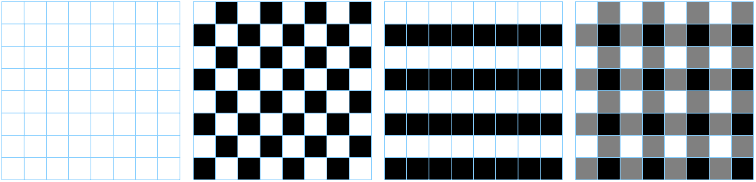


$$u_{ij}(t) = \quad A + \quad B + \quad C + \quad D$$

- The lattice symmetry group acts on the (A, B, C, D) in a natural way.

Exactly periodic patterns: Example

- Use the equivariant branching lemma to deduce which solution branches emerge in a primary bifurcation. We find:



(a) Flat (b) Checks (c) Stripes (d) Gingham

- The first three arise naturally in simple lattice systems with linear coupling.
- To see the gingham pattern (aka squares) we need something a bit more special (either nonlinear coupling or conservation laws).

Exactly periodic patterns: Example

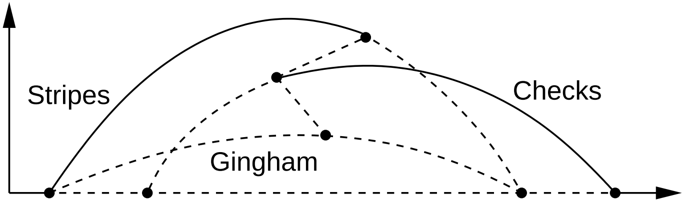
- We can also get equations for the evolution of the amplitudes A, B, C, D. To cubic order:

$$\begin{aligned}
A_t &= A(\mu_1 - a_1 A^2 - b_1(B^2 + C^2) - d_1 D^2) - e_1 BCD \\
B_t &= B(\mu_2 - a_2 B^2 - b_2 C^2 - c_2 A^2 - d_2 D^2) - e_2 ACD \\
C_t &= C(\mu_2 - a_2 C^2 - b_2 B^2 - c_2 A^2 - d_2 D^2) - e_2 ABD \\
D_t &= D(\mu_3 - a_3 D^2 - b_3(B^2 + C^2) - c_3 A^2) - e_3 ABC
\end{aligned}$$

- The parameters μ_1, μ_2, μ_3 are the growth rates of the check, stripe and flat modes respectively.
- Bifurcation structure depends on the form of the nonlinear terms in the amplitude equations.

Exactly periodic patterns: Example

- For the equation $u_t = \beta^+ \triangle^+ u + \beta^\times \triangle^\times u + ru - u^3$ with $\mu_3 = r < 0$ we can consider a path around the origin in the μ_1-μ_2 plane.
- Bifurcation diagram:



- Bistability between checks and stripes for some parameter values (complicated dynamics & spatial chaos, c.f. Chow 2003).

Exactly periodic patterns: Summary

- Growing modes are typically periodic with short wavelength.
- The symmetries of the system give us good coordinate systems to work with.
- Bifurcation problems are mode interactions.
- Can use equivariant bifurcation theory to classify stable patterns near onset.
- Lots of interesting dynamics in the amplitude equations—heteroclinic cycles, periodic orbits etc.

Long wavelength modulation (in 1D)

- The equation $u_t = \beta \triangle u + ru - u^3$ has many solutions which are approximately periodic, but modulated over longer length scales.
- We can allow for this modulation by invoking a 'multiple scales expansion'
$$u_j(t) = \varepsilon A(X,T) e^{\mathrm{i}\theta j} + \text{c.c.} + \text{h.o.t.}$$
where $X = \varepsilon j$ and $T = \varepsilon^2 t$ are (continuous) long space and time scales.
- We are approximating a discrete, finite-dimensional system with an infinite-dimensional one. But with good reason!

Long wavelength modulation (in 1D)

- Recall that

$$\triangle u_j = u_{j+1} - 2u_j + u_{j-1}$$

- So for $u_j = \varepsilon A(X,T)e^{\mathrm{i}\theta j}$ we have

$$\begin{aligned}\triangle u_j &= \left[A(X+\varepsilon)e^{\mathrm{i}\theta} - 2A(X) + A(X-\varepsilon)e^{-\mathrm{i}\theta}\right]e^{\mathrm{i}\theta j}\\ &= \Big[(A + \varepsilon A_X + \tfrac{1}{2}\varepsilon^2 A_{XX} + \cdots)e^{\mathrm{i}\theta} - 2A +\\ &\qquad\qquad + (A - \varepsilon A_X + \tfrac{1}{2}\varepsilon^2 A_{XX})e^{-\mathrm{i}\theta}\Big]e^{\mathrm{i}\theta j}\\ &= \left[2(\cos\theta - 1)A + 2\mathrm{i}\varepsilon A_X \sin\theta + \varepsilon^2 A_{XX}\cos\theta + \cdots\right]e^{\mathrm{i}\theta j}\end{aligned}$$

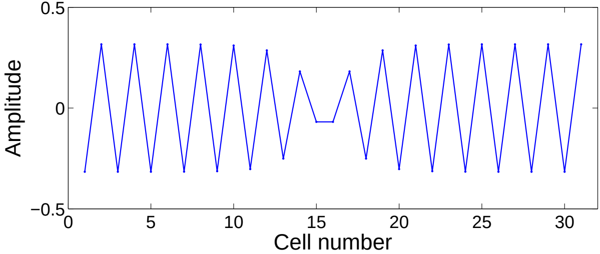
- Often interested in $\theta = 0$ or $\theta = \pi$.

Long wavelength modulation (in 1D)

- Two approaches: direct substitution into the lattice equation, or arguments from symmetry.
- We end up with Ginzburg-Landau type amplitude equations. Using $u_j = \varepsilon A(X,T)(-1)^j +$ c.c. in the SDAC equation gives

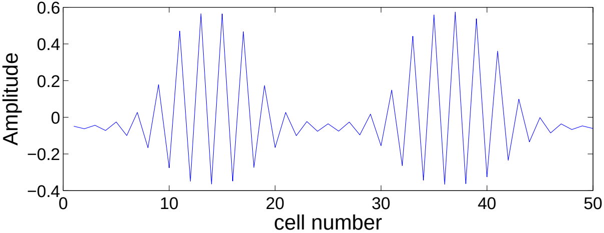
$$A_T = \mu A + A_{XX} - A^3$$

- Well-known tanh solution, representing a front:



Long wavelength modulation (in 1D)

- When the lattice system has a conserved quantity we also need to take a large-scale mode into account, which leads to modulational instabilities (as seen in Cox and Matthews 2000).
- Typical solution profile:



- In 2D more interesting instabilities are possible, such as growth of a large-scale mode from a front between two patterned modes. We are still working on understanding this.

Localized patterns

- A lattice system can exhibit bistability, i.e. two different modes are linearly stable.
- For example, in the bistable SDAC equation

  $$u_t = \beta \triangle u + ru + su^3 - u^5$$

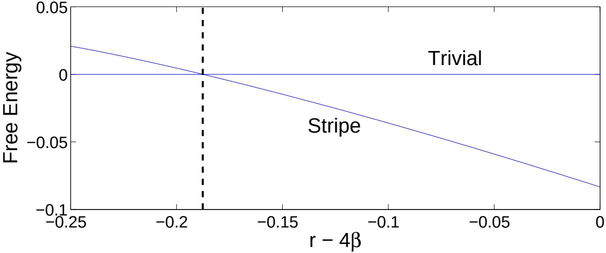
  both the trivial equilibrium and a stripe solution can be stable.
- The dynamics are variational, with free energy

  $$F[u(t)] = \sum_j \tfrac{1}{2}\beta|\nabla u|^2 - \tfrac{1}{2}ru^2 - \tfrac{1}{4}su^4 + \tfrac{1}{6}u^6$$

- The system acts to minimize this free energy, so the most energetically favourable pattern will invade the other.

Localized patterns

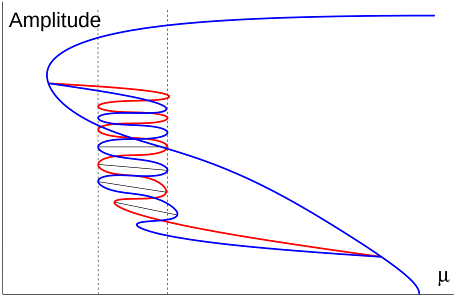
- The trivial solution has zero free energy, and the free energy of the stripe solution varies with the parameters of the system.

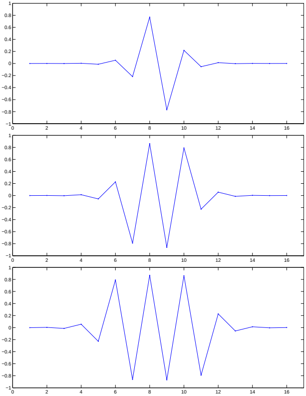


- At the 'Maxwell point' $r - 4\beta = -3s^2/16$ the two solutions have identical free energy so neither is preferred, and we expect to see localized structures.
- Due to pinning effects there is an interval around the Maxwell point in which localized structures can exist.

Localized patterns

Localized structures in the 1D bistable SDAC equation:



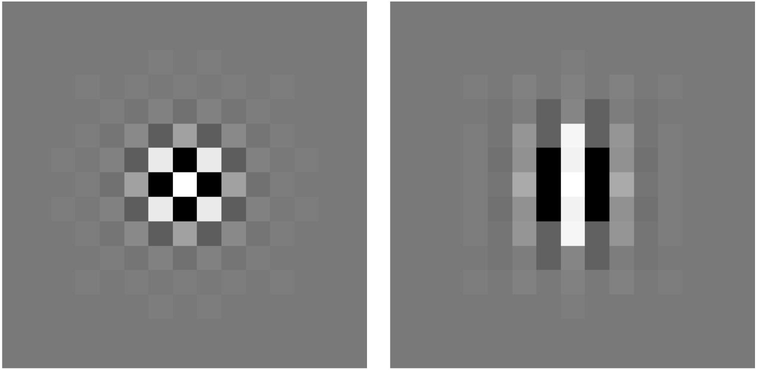


$\beta = -0.0788$, $r = -0.5$, $s = 1$, $T = 10000$

- Snaking in an interval around the Maxwell point.

Localized patterns

Localized structures in the 2D bistable SDAC equation:



$\beta^+ = -0.0436,\ \beta^\times = 0.0219,$
$r = -0.5,\ T = 10000$

$\beta^+ = 0.0125,\ \beta^\times = -0.05,$
$r = -0.5,\ T = 10000$

Summary

- Although lattice systems are spatially discrete, many of the techniques developed for continuous systems can be applied, including equivariant bifurcation theory and Ginzburg-Landau asymptotics.
- But the discreteness allows richer behaviour—many more normal modes.
- Localized structures and snaking are present, and the effects of discreteness still need to be understood.
- Lots more directions to explore!