Mathematical Methods IA Natsci: Michaelmas Term Revision

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1 Vectors

1.1 Points and vectors

It is important to distinguish between a **point** and a **vector**. A point P is a set of coordinates, e.g. $(3, 1, 1)$. A vector is an arrow between two points.

A point has an associated **position vector** $\overrightarrow{OP}$, which is a vector from the origin O to the point P. This is important because, for example, if a plane contains the points $P = (3, 1, 2)$ and $Q = (5, 1, 3)$ it **doesn't** necessarily contain the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, but it **does** contain the vector $\overrightarrow{PQ} = (2, 0, 1)$.

1.2 Magnitude and distance

The **magnitude** of a 3-d vector $\mathbf{r} = (r_1, r_2, r_3)$ is given by

$$|\mathbf{r}| = \sqrt{r_1^2 + r_2^2 + r_3^2}\,. \tag{1.1}$$

If two points have position vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ then the vector from $\mathbf{a}$ to $\mathbf{b}$ is $\mathbf{b} - \mathbf{a}$, and the distance between the points is

$$|\mathbf{b} - \mathbf{a}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}\,. \tag{1.2}$$

1.3 Scalar product

The **scalar product** (or dot product) between two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \tag{1.3}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Another useful form is

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3\,. \tag{1.4}$$

Some facts about the scalar product:

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(a) It is commutative: $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$.

(b) It is distributive: $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$.

(c) The scalar product of a vector with itself is its magnitude squared: $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$.

(d) Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (i.e. perpendicular) if and only if $\mathbf{a}\cdot\mathbf{b}=0$.

(e) If $\mathbf{n}$ is a unit vector, the component of $\mathbf{a}$ in the direction $\mathbf{n}$ is given by $\mathbf{a}\cdot\mathbf{n}$.

1.4 Vector product

The **vector product** (or cross product) of two vectors $\mathbf{a}=(a_1,a_2,a_3)$ and $\mathbf{b}=(b_1,b_2,b_3)$ is defined by

$$\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\mathbf{n} \tag{1.5}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{n}$ is the unit vector perpendicular to both, in a way which makes $\mathbf{a},\mathbf{b},\mathbf{n}$ into a right-handed triad. Another useful form is

$$\mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}\,. \tag{1.6}$$

You can remember this using the determinant rule:

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\,. \tag{1.7}$$

Some properties of the vector product:

(a) It is anticommutative: $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.

(b) It is distributive: $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$.

(c) It is not associative: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

(d) If $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ then either one of $\mathbf{a}$ or $\mathbf{b}$ is zero, or $\mathbf{a}$ and $\mathbf{b}$ are parallel.

(e) The vector product of a vector with itself is always zero: $\mathbf{a}\times\mathbf{a}=\mathbf{0}$.

1.5 Lines

The equation of a line through the points $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})\,. \tag{1.8}$$

The equation of a line through the point with position vector $\mathbf{a}$ and parallel to the unit vector $\mathbf{t}$ is

$$\mathbf{r}=\mathbf{a}+\lambda\mathbf{t}\,. \tag{1.9}$$

1.6 Planes

The equation of a plane with unit normal $\mathbf{n}$ passing through the point $\mathbf{a}$ is

$$(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0\,. \tag{1.10}$$

An alternative expression, where d is the perpendicular distance of the plane from the origin and $\mathbf{n}$ is the unit normal, is

$$\mathbf{r}\cdot\mathbf{n}=d\,. \tag{1.11}$$

Combining (1.10) and (1.11) we see that the perpendicular distance of a plane from the origin is given by $d=\mathbf{a}\cdot\mathbf{n}$.

To find the equation of a plane through the points with position vectors $\mathbf{a},\mathbf{b},\mathbf{c}$: calculate two vectors in the plane, say $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$. The normal to the plane is $\mathbf{n}=(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$ (normalised if necessary), and you can write the equation of the plane as $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$.

1.7 Other surfaces

The equation of a sphere with radius ρ centred at $\mathbf{a}$ is

$$|\mathbf{r}-\mathbf{a}|=\rho\,. \tag{1.12}$$

The equation of a cylinder with radius R and axis parallel to the unit vector $\mathbf{n}$ is

$$|\mathbf{r}-(\mathbf{r}\cdot\mathbf{n})\mathbf{n}|=R\,. \tag{1.13}$$

The equation of a cone whose axis is parallel to the unit vector $\mathbf{n}$ and with semi-angle α is

$$\mathbf{r}\cdot\mathbf{n}=|\mathbf{r}|\cos\alpha\,. \tag{1.14}$$

1.8 Calculating distances

Let L be a line through points $\mathbf{a}$ and $\mathbf{b}$ (as in (1.8)) and let Q be a point with position vector $\mathbf{q}$. The shortest distance between L and Q is

$$d=\frac{|(\mathbf{q}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})|}{|\mathbf{b}-\mathbf{a}|}\,. \tag{1.15}$$

If L is the line through $\mathbf{a}$ and perpendicular to the unit normal $\mathbf{t}$ (as in (1.9)) then the shortest distance between L and Q is

$$d=|(\mathbf{q}-\mathbf{a})\times\mathbf{t}|\,. \tag{1.16}$$

Let P be the plane through $\mathbf{a}$ with unit normal $\mathbf{n}$ (as in (1.10)) and Q be a point with position vector $\mathbf{q}$. The shortest distance between P and Q is

$$d=(\mathbf{q}-\mathbf{a})\cdot\mathbf{n}\,. \tag{1.17}$$

1.9 Scalar triple product

The **scalar triple product** of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is defined by

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\,. \tag{1.18}$$

You can remember the component form easily using the determinant rule:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\,. \tag{1.19}$$

The scalar triple product is unchanged if we make a cyclic permutation of $\mathbf{a}, \mathbf{b}, \mathbf{c}$, but we introduce a minus sign if we swap any two of $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]\,. \tag{1.20}$$

The scalar triple product gives the volume of a parallelipiped whose sides are the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

1.10 Vector triple product

The **vector triple product** of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. The vector product is not associative, so it matters where you put the brackets! A useful formula for the vector triple product is

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,. \tag{1.21}$$

1.11 Vector area

The **vector area** of a planar (i.e. flat) shape with area A and unit normal $\mathbf{n}$ is defined by be

$$\mathbf{S} = A\mathbf{n}\,. \tag{1.22}$$

The area A' of the projection of the shape onto the plane with unit normal $\mathbf{n}'$ is

$$A' = \mathbf{S} \cdot \mathbf{n}'\,. \tag{1.23}$$

For a more complicated shape made up of planar faces, the total vector area is the sum of the vector areas of each of the faces (remembering to make consistent choices for the normals, i.e. all pointing in or all pointing out).

For any closed surface (i.e. one which has a separate inside and outside) the total vector area is zero.

1.12 Basis vectors

Consider n-dimensional vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. A set of N vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ is **linearly dependendent** if you can find constants $\lambda_1, \lambda_2, \ldots, \lambda_N$ (not all equal to zero) such that

$$\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \cdots + \lambda_N\mathbf{a}_N = \mathbf{0}\,. \tag{1.24}$$

The vectors are **linearly independent** if the only way to satisfy this equation is to have $\lambda_1 = \cdots = \lambda_N = 0$. In three dimensions, three vectors are linearly independent if and only if their triple scalar product is nonzero.

A **basis** is a set of vectors which satisfy the following two properties:

(a) They are linearly independent.

(b) The number of vectors is equal to the dimension of the space, i.e. $N = n$.

So you need three vectors to form a basis in three dimensions.

The scalar product in n dimensions is

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i \,. \tag{1.25}$$

Two n-dimensional vectors are orthogonal if their scalar product is zero.

1.13 Other coordinate systems

Cartesian coordinates $\mathbf{r} = (x, y)$ (in two dimensions) or $\mathbf{r} = (x, y, z)$ (in three dimensions) are only one possible coordinate system.

1.13.1 Plane polar coordinates

Plane polar coordinates (r, θ) are related to 2-d cartesian coordinates by

$$x = r\cos\theta \,, \quad y = r\sin\theta \tag{1.26}$$

or

$$r^2 = x^2 + y^2 \,, \quad \theta = \arctan(y/x) \,. \tag{1.27}$$

1.13.2 Cylindrical polar coordinates

Cylindrical polar coordinates (r, θ, z') are related to 3-d cartesian coordinates by

$$x = r\cos\theta \,, \quad y = r\sin\theta \,, \quad z = z' \tag{1.28}$$

or

$$r^2 = x^2 + y^2 \,, \quad \theta = \arctan(y/x) \,, \quad z' = z \,. \tag{1.29}$$

1.13.3 Spherical polar coordinates

Spherical polar coordinates (r, θ, ϕ) are related to 3-d cartesian coordinates by

$$x = r\sin\theta\cos\phi \,, \quad y = r\sin\theta\sin\phi \,, \quad z = r\cos\theta \tag{1.30}$$

or

$$r = x^2 + y^2 + z^2 \,, \quad \theta = \arccos(z/r) \,, \quad \phi = \arccos(x/r\sin\theta) \,. \tag{1.31}$$

Remember that r, θ in spherical coordinates are different to r, θ in cylindrical coordinates!

2 Complex Numbers

2.1 Properties

A complex number is written

$$z = x + iy \tag{2.1}$$

where $i^2 = -1$ and x and y are the real and imaginary parts, sometimes written:

$$\Re(z) = x \tag{2.2}$$

$$\Im(z) = y\,. \tag{2.3}$$

2.2 Modulus and argument

The **modulus** (or magnitude) of a complex number is given by

$$r = |z| = \sqrt{x^2 + y^2} \tag{2.4}$$

and the **argument** (i.e. the angle that it makes with the real axis) is

$$\theta = \arg z = \arctan(y/x)\,. \tag{2.5}$$

We often restrict the argument so that $-\pi < \theta \leq \pi$. This is called the **principal argument**. Using the modulus and argument, we can write a complex number as

$$z = r(\cos\theta + i\sin\theta)\,. \tag{2.6}$$

2.3 Complex conjugate

The **complex conjugate** of $z = x + iy$ is

$$z^* = x - iy\,. \tag{2.7}$$

Some properties of the complex conjugate:

(a) $(z_1 + z_2)^* = z_1^* + z_2^*$.

(b) $(z_1 z_2)^* = z_1^* z_2^*$.

(c) $|z^*| = |z|$.

(d) $zz^* = |z|^2$.

2.4 Complex exponential

The exponential of a complex number can be found using Euler's formula

$$\exp(i\theta) = \cos\theta + i\sin\theta \tag{2.8}$$

and the fact that $\exp(x+iy) = \exp(x)\exp(iy)$. We can therefore write complex numbers in modulus-argument form as

$$z = r\exp(i\theta)\,. \tag{2.9}$$

If $z_1 = r_1\exp(i\theta_1)$ and $z_2 = r_2\exp(i\theta_2)$ are two complex numbers, then their product is

$$z_1 z_2 = r_1 r_2 \exp i(\theta_1 + \theta_2) \tag{2.10}$$

so we multiply moduli and add arguments (note the alliteration!) Their quotient is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\exp i(\theta_1 - \theta_2) \tag{2.11}$$

so we divide moduli and subtract arguments. The complex conjugate of $z = r\exp(i\theta)$ is

$$z^* = r\exp(-i\theta)\,. \tag{2.12}$$

We can use (2.8) to show that

$$\cos\theta = \frac{\exp(i\theta) + \exp(-i\theta)}{2}\,, \quad \sin\theta = \frac{\exp(i\theta) - \exp(-i\theta)}{2i}\,. \tag{2.13}$$

2.5 Roots of unity

The n**th roots of unity** (i.e. the numbers z which satisfy $z^n = 1$) are

$$z = \exp(i2\pi m/n) \tag{2.14}$$

for $m = 0, 1, 2, \ldots, n-1$. The nth roots of unity always add up to zero.

2.6 De Moivre's theorem

De Moivre's theorem is

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta\,. \tag{2.15}$$

Note that n does not have to be an integer.

2.7 Complex logarithm

The logarithm of $z = r\exp(i\theta)$ is

$$\ln z = \ln r + i\theta\,. \tag{2.16}$$

where θ is the principal argument. We can add multiples of 2π to the argument without changing its meaning, so sometimes the complex logarithm is written as

$$\ln z = \ln r + i(\theta + 2\pi n) \tag{2.17}$$

for $n = 0, \pm1, \pm2, \ldots$.

2.8 Oscillations

The position of a particle is given as a function of time by

$$x(t) = a\cos\omega t + b\sin\omega t\,. \tag{2.18}$$

You can write this in modulus-argument form as

$$x(t) = \Re[A\exp(i\omega t)] \tag{2.19}$$

where $A = a - ib$ is the complex amplitude of the oscillation. The real amplitude (i.e. the distance of the endpoints of the oscillation from the centre) is given by $|A|$.

To find the velocity v of the particle as a function of time you multiply by $i\omega$:

$$v(t) = \Re[i\omega A\exp(i\omega t)]\,. \tag{2.20}$$

3 Hyperbolic Functions

3.1 Definitions

The hyperbolic functions are defined by

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2} \tag{3.1}$$

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2} \tag{3.2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)}\,. \tag{3.3}$$

We also have

$$\operatorname{sech} x = \frac{1}{\cosh x}\,, \quad \operatorname{cosech} x = \frac{1}{\sinh x}\,, \quad \coth x = \frac{1}{\tanh x}\,. \tag{3.4}$$

3.2 Identities

The relationships with the trigonometric functions are:

$$\cos(ix) = \cosh x \tag{3.5}$$

$$\sin(ix) = i \sinh x \tag{3.6}$$

$$\tan(ix) = i \tanh x\,. \tag{3.7}$$

Another useful identity is

$$\cosh^2 x - \sinh^2 x = 1\,. \tag{3.8}$$

By dividing this identity by either $\cosh^2 x$ or $\sinh^2 x$ you can get other useful identities.

3.3 Addition formulae

The addition formulae for cosh and sinh are

$$\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B \tag{3.9}$$

$$\sinh(A + B) = \sinh A \cosh B + \sinh B \cosh A\,. \tag{3.10}$$

You can find the addition formula for tanh by dividing (3.10) by (3.9).

The rule for finding hyperbolic identities is to take the corresponding trigonometric identity, replace ‘cos’ by ‘cosh’ etc, and change the sign anywhere there is a product of two ‘sin’ functions.

3.4 Inverses

The inverses for hyperbolic functions are given by

$$\operatorname{arccosh} x = \pm \ln\left(x + \sqrt{x^2 - 1}\right) \tag{3.11}$$

$$\operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right) \tag{3.12}$$

$$\operatorname{arctanh} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)\,. \tag{3.13}$$

4 Differentiation

4.1 First principles

The derivative of the curve $y = f(x)$ is defined to be

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h \to 0} \left[\frac{y(x+h) - y(x)}{h} \right] . \tag{4.1}$$

4.2 Product rule

The derivative of the product $u(x)v(x)$ is

$$\frac{\mathrm{d}(uv)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} v + u \frac{\mathrm{d}v}{\mathrm{d}x} . \tag{4.2}$$

4.3 Quotient rule

The derivative of the quotient $u(x)/v(x)$ is

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{u}{v} \right) = \frac{v(\mathrm{d}u/\mathrm{d}x) - u(\mathrm{d}v/\mathrm{d}x)}{v^2} . \tag{4.3}$$

4.4 Chain rule

The derivative of a function of a function $f(u(x))$ is

$$\frac{\mathrm{d}f(u(x))}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u} \frac{\mathrm{d}u}{\mathrm{d}x} . \tag{4.4}$$

You can think of this as ‘cancelling the $\mathrm{d}u$’s’.

4.5 Implicit differentiation

If you have an equation of the form $g(y) = f(x)$ you can find $\mathrm{d}y/\mathrm{d}x$ by differentiating implicitly:

$$\frac{\mathrm{d}g}{\mathrm{d}y} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} \tag{4.5}$$

or, rearranging:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} \left(\frac{\mathrm{d}g}{\mathrm{d}y} \right)^{-1} . \tag{4.6}$$

4.6 Stationary points

The curve $y = f(x)$ has a **stationary point** (or turning point) when $\mathrm{d}y/\mathrm{d}x = 0$. You can find out what kind of stationary point it is by looking at the second derivative:

- If $\mathrm{d}^2y/\mathrm{d}x^2 > 0$ at the stationary point then it is a **minimum**.
- If $\mathrm{d}^2y/\mathrm{d}x^2 < 0$ at the stationary point the it is a **maximum**.
- If $\mathrm{d}^2y/\mathrm{d}x^2 = 0$ at the turning point then you need to look at higher derivatives. If $\mathrm{d}^3y/\mathrm{d}x^3 \neq 0$ then the stationary point is a **point of inflexion**.

4.7 Curve sketching

When trying to sketch a curve of the form $y = f(x)$, ask yourself the following questions:

- Where does the curve cross the axes, i.e. what happens when $x = 0$ or $y = 0$?
- What happens as $x \to \pm\infty$? Does the curve have any asymptotes?
- Are there any singularites, i.e. values of x where y is infinite?
- Are there any stationary points?

It is worth knowing the equation for an ellipse by heart:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{4.7}$$

This is an ellipse intercepting the x-axis at $x = \pm a$ and the y-axis at $y = \pm b$. The special case where $a = b$ is a circle of radius a.

4.8 Common derivatives

Powers:

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1} \tag{4.8}$$

Exponential and logarithm:

$$\frac{\mathrm{d}}{\mathrm{d}x} \exp x = \exp x \tag{4.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x} \tag{4.10}$$

Trigonometric and hyperbolic functions:

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \sinh x = \cosh x \tag{4.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cosh x = \sinh x \tag{4.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \tanh x \tag{4.13}$$

5 Integration

5.1 Reverse of differentiation

Integration is the reverse of differentiation. If $\mathrm{d}F/\mathrm{d}x = f(x)$, then

$$\int f(x)\,\mathrm{d}x = F(x) + c \tag{5.1}$$

where c is an arbitrary constant. This leads to the list of commonly encountered integrals at the end of this section(compare with the common derivatives).

5.2 Total derivatives

Integrands of the form $f'(x)[f(x)]^n$ can be integrated easily:

$$\int f'(x)[f(x)]^n \mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}\,. \tag{5.2}$$

5.3 Powers of trigonometric functions

To integrate an even power of a trigonometric function use one of the following identities:

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x\,, \quad \cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x \tag{5.3}$$

to remove all powers of trignometric functions. To integrate an odd power of a trigonometric function, use the identity

$$\sin^2 x + \cos^2 x = 1 \tag{5.4}$$

to remove all but one of the powers.

5.4 Partial fractions

To integrate the reciprocal of a polynomial, first split the function into its partial fractions and then integrate as usual.

5.5 Substitution

Often a clever substitution can make a hard integral easy. A good guess is to substitute the hardest-looking part of the integral for another variable; for example in the integral

$$\int \tan x \sqrt{\sec x}\,\mathrm{d}x \tag{5.5}$$

you should make the substitution $y = \sqrt{\sec x}$. If you make the substitution $y = f(x)$ then

$$\mathrm{d}x = \frac{\mathrm{d}y}{f'(x)}\,. \tag{5.6}$$

5.6 Integration by parts

The integration by parts formula is

$$\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = [uv]_a^b - \int_a^b \frac{\mathrm{d}u}{\mathrm{d}x}v\,\mathrm{d}x\,. \tag{5.7}$$

The trick is knowing how to choose u and $\mathrm{d}v/\mathrm{d}x$. A good rule is to choose $\mathrm{d}v/\mathrm{d}x$ to be something that gets simpler when integrated, and u to be a function that simplifies when differentiated. Don't forget that taking $\mathrm{d}v/\mathrm{d}x = 1$ is often a good choice. You might like to look up the 'ILATE rule' on Wikipedia.

5.7 Reduction formulae

Finding a reduction formula can often enable you to do a whole family of integrals easily. Reduction formula work best for definite integrals. Given a family of integrals I_n which depend on n, the idea is to find I_n in terms of I_{n-1} or I_{n-2}, and then successively apply the formula to the integration. This is often more of an art than a science, and the best way to learn it is to do lots of examples.

5.8 Common integrals

Powers:

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c \text{ (for } n \neq -1) \tag{5.8}$$

$$\int x^{-1}\mathrm{d}x = \ln x + c \tag{5.9}$$

Exponential:

$$\int \exp(ax)\,\mathrm{d}x = \frac{1}{a}\exp(ax) + c \tag{5.10}$$

Trigonometric functions:

$$\int \sin x\,\mathrm{d}x = -\cos x + c \qquad \int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \arcsin(x/a) + c \tag{5.11}$$

$$\int \cos x\,\mathrm{d}x = \sin x + x \qquad \int \frac{-\mathrm{d}x}{\sqrt{a^2 - x^2}} = \arccos(x/a) + c \tag{5.12}$$

$$\int \sec^2 x\,\mathrm{d}x = \tan x + c \qquad \int \frac{\mathrm{d}x}{a^2 + x^2} = \frac{1}{a}\arctan(x/a) + c \tag{5.13}$$

6 Power Series

6.1 Taylor series

The **Taylor series** for $f(x)$ about $x = a$ is given by

$$\boxed{f(a+h) = f(a) + f'(a)h + f''(a)\frac{h^2}{2!} + f'''(a)\frac{h^3}{3!} + \cdots + f^{(n)}(a)\frac{h^n}{n!} + R_{n+1}\,.} \tag{6.1}$$

where R_{n+1} is the remainder, or error term. Taylor's theorem says that there is a point $x = \zeta$, in the range $a < \zeta < a + h$, such that

$$R_{n+1} = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\zeta)\,. \tag{6.2}$$

The worst-case error can be found by asking what the largest value of $f^{(n+1)}(\zeta)$ is, for $a < \zeta < a + h$.

6.2 Maclaurin series

A **Maclaurin series** is a special case of a Taylor series, where $a = 0$:

$$\boxed{f(h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + \cdots + f^{(n)}(0)\frac{h^n}{n!} + R_{n+1}\,.} \tag{6.3}$$

6.3 Power series

If a function is infinitely differentiable then you can carry on the Taylor series to an infinite number of terms. This is called the **power series** for f. The power series about $x = 0$ is:

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots \tag{6.4}$$

$$= \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}\,. \tag{6.5}$$

6.4 Common power series

Exponentials:

$$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{(for all } x\text{)} \tag{6.6}$$

$$\exp(-x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \qquad \text{(for all } x\text{)} \tag{6.7}$$

Trigonometric functions:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \text{(for all } x\text{)} \tag{6.8}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{(for all } x\text{)} \tag{6.9}$$

6.5 The binomial expansion

The power series (or binomial expansion) for $(1+x)^n$ is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots . \tag{6.10}$$

Remember that n does not have to be an integer.

In the special case where n is an integer the series is finite, because all terms after the nth are multiplied by $(n-n)$, which is zero. We write it as

$$(1+x)^n = \sum_{r=1}^{n} {}^nC_r x^r \tag{6.11}$$

where

$${}^nC_r = \frac{n!}{r!(n-r)!} . \tag{6.12}$$

You can find binomial expansions for functions like $(a+bx)^n$ by taking a factor of a outside the bracket:

$$(a+bx)^n = a^n(1+bx/a)^n \tag{6.13}$$

and finding the binomial expansion for $(1+bx/a)^n$ as normal.

7 Probability

7.1 Basics

The possible results of a random experiment are the **outcomes** $\omega_1, \omega_2, \ldots$ and the set of possible outcomes is the **sample space** $\Omega = \{\omega_1, \omega_2, \ldots\}$. Each outcome has an associated probability $p_1, p_2, \ldots$. An **event** A is a subset of Ω. The **probability** of event A occuring is

$$P(A) = \sum_{\omega_i \in A} p_i \,. \tag{7.1}$$

Some basic properties of probability are:

- $0 \leq P(A) \leq 1$ for all events A,
- $P(\Omega) = 1$, i.e. we must get some outcome,
- If A and B are mutually exclusive then $P(A \text{ or } B) = P(A) + P(B)$.

The **complement** of A is the event that A does not occur, and is written A'. The event that A **or** B occurs is $A \cup B$, and the event that A **and** B occur is $A \cap B$. Some facts about these:

- $P(A') = 1 - P(A)$,
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If A and B are mutually exclusive then $P(A \cap B) = 0$.

7.2 Conditional probability

The probability that event B occurs given that event A has already occured is called the **conditional probability** and is denoted $P(B|A)$. The formula for conditional probability is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \,. \tag{7.2}$$

The probability that A occurs given that B has occurred is

$$P(A|B) = \frac{P(B \cap A)}{P(B)} \,. \tag{7.3}$$

If A and B are mutually exclusive then $P(B|A) = P(A|B) = 0$. If A and B are independent then $P(B|A) = P(B)$.

The most important result in conditional probability is **Bayes' theorem**:

$$\boxed{P(A|B) = \frac{P(A)}{P(B)} P(B|A) \,.} \tag{7.4}$$

7.3 Combinatorics

The number of ways of choosing r objects from a collection of n objects, where order of choosing matters, is the number of **permutations**:

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!}\,. \tag{7.5} $$

The number of ways of choosing r objects from a collection of n objects, where order does not matter, is the number of **combinations**:

$$ {}^{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}\,. \tag{7.6} $$

7.4 Discrete random variables

A **random variable** X is a real-valued variable that depends on the outcome ω. If A is the event consisting of all outcomes where X takes the value x, then the probability that X takes the value x is

$$ P(X = x) = P(A)\,. \tag{7.7} $$

The **mean** or **expectation** of a random variable is

$$ \mathrm{E}(X) = \sum xP(X = x)\,. \tag{7.8} $$

The **variance** of a random variable is

$$ \mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2\,. \tag{7.9} $$

The **binomial distribution** $B(n, p)$ has mean np and variance $np(1-p)$ and has probabilities:

$$ P(X = r) = \binom{n}{r} p^r (1-p)^{n-r} \quad \text{for } r = 0, 1, 2, \ldots, n\,. \tag{7.10} $$

The **Poisson distribution** has mean λ and variance λ, and probability distribution:

$$ P(X = r) = \frac{\lambda^r e^{-\lambda}}{r!} \quad \text{for } r = 0, 1, 2, \ldots \tag{7.11} $$

7.5 Continuous random variables

Continuous random variables X have a **probability density** $f(x)$ which is normalised, so that

$$ \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1\,. \tag{7.12} $$

The probability that X is between a and b is

$$ P(a \le X \le b) = \int_a^b f(x)\,\mathrm{d}x\,. \tag{7.13} $$

The **mean** and **variance** of X are

$$ E(X) = \int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x\,, \quad \mathrm{Var}(X) = E(X^2) - E(X)^2\,. \tag{7.14} $$

The **Gaussian distribution** (or **normal distribution**) $N(\mu, \sigma^2)$ has density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \,. \tag{7.15}$$

It has mean μ and variance σ^2. The **standard normal distribution** $N(0, 1)$ has mean 0 and variance 1, and density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \,. \tag{7.16}$$

Note that by symmetry we have $P(X \geq 0) = 1/2$ for a standard normal distribution.