Natural Sciences Part IA — Professor M.J. Perry
Mathematics II (course A)
Examples Sheet 3 — Lent Term 2008

Please communicate any errors in this sheet to M.J.Perry@damtp.cam.ac.uk. Note that this examples sheet is for course A.

Skills Questions Questions in this section are intended to give practice in routine calculation.

Scalar and vector fields

S1 For each of the following functions $f(x,y,z)$ or $f(\mathbf{x})$, evaluate ∇f.

(a) $f = xyz$, (b) $f = \exp(-1/(x+y+z))$, (c) $f = \exp(-\alpha^2 x^2 - \beta^2 y^2 - \gamma^2 z^2)$, (d) $f = r$, (e) $f = r^{-1}$, (f) $f = G(r)$. [In (d), (e) and (f) $r = |\mathbf{x}| = (x^2+y^2+z^2)^{1/2}$.]

S2 Write down parametrizations of the following curves:

(a) circle in xy plane, radius 1, centre at $(1,1,0)$.

(b) straight line from $(-1,0,0)$ to $(1,1,1)$.

(c) triangle with corners $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

S3 A vector field $\mathbf{F}(\mathbf{x})$ has components $(x^3+3y+z^2, y^3, x^2+y^2+3z^2)$. Evaluate
 (i) the divergence of $\mathbf{F}(\mathbf{x})$ and
 (ii) the curl of $\mathbf{F}(\mathbf{x})$.

S4 Let $\mathbf{a}$ and $\mathbf{b}$ be fixed vectors. Show that $\boldsymbol{\nabla}(\mathbf{a}.\mathbf{x}) = \mathbf{a}$. The following are vector functions of $\mathbf{x}$:

$$\mathbf{x}; \qquad \mathbf{a}(\mathbf{x}.\mathbf{b}); \qquad \mathbf{a}\times\mathbf{x}; \qquad \mathbf{x}/r^3, \text{ where } \; r = |\mathbf{x}| \;.$$

Evaluate the divergence and the curl of each of these functions.

S5 Determine whether the following are solutions to Laplace's equation in two dimensions

$$\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = 0$$

or the wave equation in two dimensions

$$\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right) = 0$$

or neither or both:

$$x, x^2, x^3, x^4-y^4, x^5+y^5, \sin(y/x), \sin^{-1}(x/y), \tan^{-1}(y/x),$$

$$\tan^{-1}(x/y), \sin x \sinh y, \ln(x^2+y^2), \ln(x^2-y^2).$$

Standard questions

Scalar and Vector Fields

6 For the function $f(x, y, z) = \ln(x^2 + y^2) + z$, find ∇f.
Consider a cylinder of radius 5 whose axis is along the z-axis.
  (i) What is the rate of change of $f(x, y, z)$ in the direction normal to the cylinder at the point $(3, -4, 4)$?
  (ii) What is the rate of change of $f(x, y, z)$ in the direction $\mathbf{m} = \mathbf{i} + 2\mathbf{j}$ at the same point, where $\mathbf{i}, \mathbf{j}$ are unit vectors parallel to the x, y axes respectively.

7 Find the normal to the surface $xz + z^2 - xy^2 = 5$ at the point (1,1,2). Deduce the equation of the tangent plane at this point.

8 Obtain the equation of the plane that is tangent to the surface $z = 3x^2 y \sin(\pi x/2)$ at the point $x = y = 1$. Take North to be the direction $(0, 1, 0)$ and East to be the direction $(1, 0, 0)$. In which direction will a marble roll if placed on the surface at $x = 1, y = \frac{1}{2}$?

9 Evaluate

$$\int_\Gamma \{P(x, y)dx + Q(x, y)dy\}$$

where $P = -x^2 y, \quad Q = y^2 x$ and Γ is the closed curve consisting of the semi-circle $x^2 + y^2 = a^2 (y > 0)$, and the segment $(-a, a)$ of the x-axis, described anti-clockwise. Verify that this is equal to

$$\int\int_D \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dxdy$$

where D is the plane surface enclosed by Γ. (This illustrates Stokes's theorem in the plane – see later.)

10 Determine a function $f(y)$ such that

$$\int_\Gamma [f(y)\, dx + x \cos y\, dy] = 0$$

for all closed contours Γ in the (x, y) plane.
[Hint: when is the integrand an exact differential?]

11 The work done by a force $\mathbf{F}$ acting on a particle which moves along a curve C is defined as the line integral

$$W = \int_C \mathbf{F} \cdot \mathbf{dx} \quad .$$

  (i) When $\mathbf{F} = \mathbf{c} \times \mathbf{v}$, where $\mathbf{c}$ is a constant vector, and $\mathbf{v} = \dfrac{d\mathbf{x}}{dt}$ is the velocity of the particle, show that the work done is zero.
  (ii) A particle moves along the helical path given by

$$x = \cos t, \quad y = \sin t, \quad z = t \quad .$$

Calculate the work W done in the time interval $0 \le t \le \pi$ by each of the forces $\mathbf{F}$ given by

(a) $y\mathbf{i} - x\mathbf{j} - \mathbf{k}$,

(b) $x\mathbf{i} + y\mathbf{j}$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors parallel to the x, y, z axes respectively.

12 Write down a condition obeyed by a conservative vector field $\mathbf{V} = \{P(x,y),\ \ Q(x,y)\}$. Do the following choices for $P,\ Q$ yield conservative fields ?

(i) $P = x^2y + y, \qquad Q = xy^2 + x$;

(ii) $P = ye^{xy} + 2x + y, \quad Q = xe^{xy} + x$.

In the case that $\mathbf{V}$ is conservative, find a function $f(x,y)$ such that $\mathbf{V} = \nabla f$.

Consider the following curves, each joining $(0,0)$ to $(1,1)$:

(a) $C_1 : x = t, \quad y = t \quad (0 \le t \le 1)$

(b) $C_2 : x = 0, \quad y = t \quad (0 \le t \le 1); \quad x = t, \quad y = 1 \quad (0 \le t \le 1)$.

Evaluate the integrals $\int_{C_1} Pdx + Qdy$ and $\int_{C_2} Pdx + Qdy$ for each of the fields (i) and (ii) above.

13 Consider the vector field

$$\mathbf{V} = (4x^3z + 2x, \quad z^2 - 2y, \quad x^4 + 2yz).$$

Evaluate the line integral $\int_c \mathbf{V} \cdot \mathbf{ds}$ along

(i) the sequence of straight-line paths joining $(0,0,0)$ to $(0,0,1)$ to $(0,1,1)$ to $(1,1,1)$.

(ii) the straight line joining $(0,0,0)$ to $(1,1,1)$, given parametrically by

$$x = y = z = t \qquad (0 \le t \le 1).$$

Show that $\mathbf{V}$ is conservative by finding a function $f(x,y,z)$ such that $\mathbf{V} = \nabla f$.

14 Let $\mathbf{E} = (-ye^{-2t}, \quad xe^{-2t}, \quad 0)$ and $\mathbf{B} = (0, 0, e^{-2t})$. Evaluate $\int_S \mathbf{B} \cdot \mathbf{dS}$, $\int_C \mathbf{E} \cdot \mathbf{dx}$, where S is the surface of the circular disc $x^2 + y^2 < 1, \quad z = 0$ and C is the curve bounding S. Show that

$$\int_C \mathbf{E} \cdot \mathbf{dx} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dS}.$$

15 Calculate $\int\int \mathbf{F} \cdot \mathbf{n} dS$, where $\mathbf{F} = \alpha x^3 \mathbf{i} + \beta y^3 \mathbf{j} + \gamma z^3 \mathbf{k}$ (with α, β and γ constants):

(i) over the surface of a sphere of radius a, centred at the origin.

(ii) over the surface of a cylinder of radius a and height $2h$, centred at the origin with its axis in the z-direction.

[Hint: You can reduce the amount of algebra by exploiting the symmetry of the sphere and the cylinder.]

16 A cube is defined by $0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le z \le 1$. Evaluate the surface integral

$$\int\int \mathbf{F} \cdot \mathbf{n} dS$$

over the surface of the cube where $\mathbf{F} = (x^2 + ay^2,\ 3xy,\ 6z)$ and a is a constant. [$\mathbf{n}$ is a unit vector in the direction of the outward normal from the volume across a surface element dS; e.g. on the $x = 0$ face, $dS = dydz$ and $\mathbf{n} = (-1, 0, 0)$.] Evaluate also

$$\int\int\int f dxdydz$$

over the volume of the same cube, where $f = bx + 6$ and b is a constant. For what values of a and b do these integrals have the same value?

17 Let $\mathbf{u}$ be the vector field $\mathbf{u} = Q\mathbf{x}/4\pi\epsilon_0 r^3$ in three dimensions, where $\mathbf{x}$ is the position vector and $r = |\mathbf{x}|$ (Q, ϵ_0 constants). Show that

$$\int_S \mathbf{u} \cdot \mathbf{n} dS = Q/\epsilon_0 \quad ,$$

where S is a sphere of radius a centred on the origin, and $\mathbf{n}$ is a unit normal vector pointing radially outward from the sphere. [This is Gauss's law for a point charge.]

18

(i) Find all values of a and b for which the integrals in Question 16 are equal by using the divergence theorem.

(ii) For $\mathbf{E}$ and $\mathbf{B}$ as in Question 14 show show that

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and apply Stokes theorem to deduce the equality of the integrals.

19 State the divergence theorem.

Let $\mathbf{F}(r) = (x^3 + 3y + z^2,\ y^3,\ x^2 + y^2 + 3z^2)$ and let S be the (open) surface

$$1 - z = x^2 + y^2,\ 0 \leq z \leq 1\ .$$

Evaluate $\int_S \mathbf{F} \cdot \mathbf{n} dS$, where $\mathbf{n}$ is the unit normal on S having a positive component in the z-direction. [*Hint: construct a closed surface including S and use the divergence theorem.*]

20 State Stokes theorem, and verify it for the hemispherical surface $r = 1,\ z \geq 0$, and the vector field $\mathbf{A}(\mathbf{x}) = (y, -x, z)$.