Chris's guide to stationary points

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1. What are stationary points?

A function of two variables $f(x, y)$ can be thought of as giving the height of a surface above the (x, y) plane. The partial derivatices $\partial f/\partial x$ and $\partial f/\partial y$ give the slope of the surface in the x and y directions.

A *stationary point* is a point on the surface where the slope is zero in all directions. Remember that for functions of one variable, a stationary point is either a minimum or a maximum. For functions of two variables there are three possibilities:

1. A maximum
2. A minimum
3. A saddle

For example, the function $f(x, y) = (x^2 - y^2) \exp(-x^2 - y^2)$ is plotted in Figure 1.1, along with its contour plot. This function has a saddle at $(0, 0)$, two maxima at $(\pm 1, 0)$ and two minima at $(0, \pm 1)$. Its contour plot is also shown—the lines on a contour plot are lines of constant height.

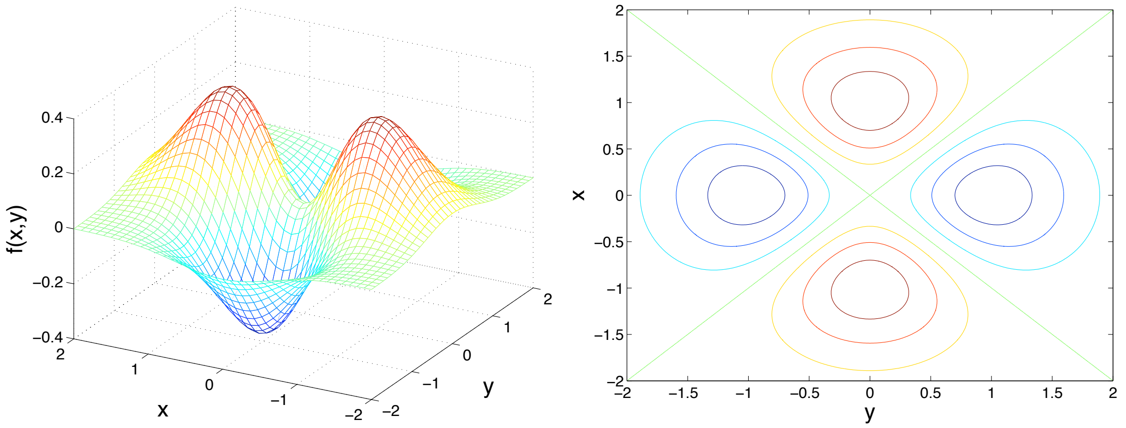

Figure 1.1: Functions of two variables. On the left we have a surface plot of the function $f(x, y) = (x^2 - y^2) \exp(-x^2 - y^2)$. On the right we have the corresponding contour plot.

2. Finding and classifying stationary points

To find and classify the stationary points of a function, follow these easy steps. To save on notation we're going to write f_x to mean $\partial f/\partial x$, f_{xx} to mean $\partial^2 f/\partial x^2$ etc.

1. Calculate the partial derivatives f_x and f_y, and simultaneously solve the equations

   $$f_x = 0\,, \quad f_y = 0\,.$$

   This gives the locations of the stationary points. Make sure you find *all* the solutions!

2. Calculate the second partial derivatives f_{xx}, f_{yy} and f_{xy} and evaluate them at each of the stationary points.

3. If $f_{xx}f_{yy} - f_{xy}^2 < 0$ then you have a *saddle*. Congratulations, you're done! If not:

4. If $f_{xx} + f_{yy} > 0$ then you have a *minimum*.
   If $f_{xx} + f_{yy} < 0$ then you have a *maximum*.

3. Worked example A

We're going to use the four-step process above to find and classify all of the stationary points of the function plotted in Figure 1.1, $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$.

1. We calculate the first partial derivatives, which are given by

   $$f_x = 2xe^{-x^2-y^2} + (x^2 - y^2)(-2x)e^{-x^2-y^2} = 2x(1 - x^2 + y^2)e^{-x^2-y^2}$$

   and

   $$f_y = -2ye^{-x^2-y^2} + (x^2 - y^2)(-2y)e^{-x^2-y^2} = -2y(1 + x^2 - y^2)e^{-x^2-y^2}$$

   The exponential of a function is never zero, so (ignoring the constants) we have to solve the two equations

   $$x(1 - x^2 + y^2) = 0 \tag{3.1}$$
   $$y(1 + x^2 - y^2) = 0 \tag{3.2}$$

   From (3.1) we must either have $x = 0$ or $x^2 = 1 + y^2$. If we substitute $x = 0$ into (3.2) then we get

   $$y(1 - y^2) = 0$$

   and the possible solutions are $y = 0$ or $y = \pm 1$. On the other hand, if we substitute $x^2 = 1 + y^2$ then we get

   $$2y = 0$$

   and the only solution is $y = 0$, which means that $x = \pm 1$. Therefore there are five stationary points:

   $$(0,0)\,, \quad (\pm 1, 0)\,, \quad (0, \pm 1)\,.$$

Stationary point	(0,0)	(0,1)	(0,-1)	(1,0)	(-1,0)
f_{xx}	2	$4e^{-1}$	$4e^{-1}$	$-4e^{-1}$	$-4e^{-1}$
f_{yy}	-2	$4e^{-1}$	$4e^{-1}$	$-4e^{-1}$	$-4e^{-1}$
f_{xy}	0	0	0	0	0
$f_{xx}f_{yy} - f_{xy}^2$	-4	$16e^{-2}$	$16e^{-2}$	$16e^{-2}$	$16e^{-2}$
$f_{xx} + f_{yy}$	0	$8e^{-1}$	$8e^{-1}$	$-8e^{-1}$	$-8e^{-1}$

Table 3.1: Evaluating the second partial derivatives at each of the stationary points.

2. We calculate the second partial derivatives. This is usually more laborious than calculating the first derivatives, and you should take extra care. We have:

$$\begin{aligned} f_{xx} &= 2(1 - x^2 + y^2)e^{-x^2-y^2} + 2x(-2x)e^{-x^2-y^2} + 2x(1 - x^2 + y^2)(-2x)e^{-x^2-y^2} \\ &= 2(1 - 5x^2 + y^2 + 2x^4 - 2x^2y^2)e^{-x^2-y^2} \end{aligned}$$

   and

$$\begin{aligned} f_{yy} &= -2(1 + x^2 - y^2)e^{-x^2-y^2} - 2y(-2y)e^{-x^2-y^2} - 2y(1 + x^2 - y^2)(-2y)e^{-x^2-y^2} \\ &= -2(1 - 5y^2 + x^2 + 2y^4 - 2y^2x^2)e^{-x^2-y^2} \end{aligned}$$

   and finally

$$\begin{aligned} f_{xy} &= 2x(2y)e^{-x^2-y^2} + 2x(1 - x^2 + y^2)(-2y)e^{-x^2-y^2} \\ &= 4xy(x^2 - y^2)e^{-x^2-y^2} . \end{aligned}$$

   We then have to evaluate these at each of the stationary points. There are plenty of opportunities to make mistakes here, so be careful! See Table 3.1 for the results.

3. Now we need to use the results to classify the stationary points. For $(0, 0)$ we have $f_{xx}f_{yy} - f_{xy}^2 < 0$, so $(0, 0)$ is a saddle. For all of the other stationary points we have $f_{xx}f_{yy} - f_{xy}^2 > 0$, so they are either maximums or minimums, and we have to go to Step 4 to classify them.

4. For the fixed points $(0, \pm 1)$ we have $f_{xx} + f_{yy} > 0$, so they are minimums. For the stationary points $(\pm 1, 0)$ we have $f_{xx} + f_{yy} < 0$, so they are maximums.

This completes the classification of the stationary points of $f(x, y)$. Note that there are faster ways we could have done this. For example, we could have worked out that $f(x, y) = 0$ on the lines $x = \pm y$, which means that those lines are contours. If we plotted the contours on the (x, y) plane we would see that they cross at the origin. This guarantees that the origin is a saddle, since contour lines only cross at saddles. Similarly, the contour plot around a maximum or a minimum always looks like concentric circles.

However, the method presented here will always work, and is worth remembering!

4. Worked example B

Now we will apply the four-step method to find and classify all of the stationary points of the function $f(x, y) = \cos x \cos y$. This time there are infinitely many stationary points, but thankfully we can still avoid having to do too much work. Here we go:

Stationary point	$(n\pi, m\pi)$	$((n+\frac{1}{2})\pi, (m+\frac{1}{2})\pi)$
f_{xx}	$-(-1)^{m+n}$	0
f_{yy}	$-(-1)^{m+n}$	0
f_{xy}	0	$(-1)^{m+n}$
$f_{xx}f_{yy} - f_{xy}^2$	1	-1
$f_{xx} + f_{yy}$	$-2(-1)^{m+n}$	0

Table 4.1: Evaluating the second partial derivatives at each of the stationary points.

1. We calculate the first partial derivatives:

$$f_x = -\sin x \cos y\,, \quad f_y = -\cos x \sin y$$

to find the stationary points. For $f_x = 0$ we either have $x = n\pi$ or $y = (n+\frac{1}{2})\pi$, where n is an integer. Substituting $x = n\pi$ into the second equation gives

$$(-1)^n \sin y = 0$$

since $\cos n\pi = (-1)^n$, which means that $y = m\pi$ for some integer m. Alternatively, substituting $y = (n+\frac{1}{2})\pi$ gives

$$(-1)^n \cos x = 0$$

since $\sin(n+\frac{1}{2})\pi = (-1)^n$, which means that $x = (m+\frac{1}{2})\pi$. Therefore the fixed points are

$$(n\pi, m\pi)\,, \quad ((n+\tfrac{1}{2})\pi, (m+\tfrac{1}{2})\pi)$$

where n and m are integers.

2. We calculate the second partial derivatives:

$$f_{xx} = -\cos x \cos y\,, \quad f_{yy} = -\cos x \cos y\,, \quad f_{xy} = \sin x \sin y\,.$$

We now evaluate these at the stationary points. Notice that we get different results depending on whether n and m are even or odd. The results are in Table 4.1.

3. The stationary points at $((n+\frac{1}{2})\pi, (m+\frac{1}{2})\pi)$ have $f_{xx}f_{yy} - f_{xy}^2 < 0$, so they're all saddles. The stationary points at $(n\pi, m\pi)$ have $f_{xx}f_{yy} - f_{xy}^2 > 0$, so we need to go to Step 4 to classify them.

4. If $m+n$ is even then $f_{xx} + f_{yy} < 0$, and the stationary point at $(n\pi, m\pi)$ is a maximum. In particular, the stationary point at the origin is a maximum. If $m+n$ is odd then the stationary point at $(n\pi, m\pi)$ is a minimum.

This completes the classification of the fixed points of $f(x, y) = \cos x \cos y$. Again, notice that we could have accomplished this more quickly by looking for the contours where $f(x, y) = 0$, this time finding that those contours are $x = (n+\frac{1}{2})\pi$ and $y = (m+\frac{1}{2})\pi$ for integers n and m. Plotting these contours, we see that they cross at the points $((n+\frac{1}{2})\pi, (m+\frac{1}{2})\pi)$, which are therefore saddles.

The surface plot and contour plot for $f(x, y)$ with $-2\pi \leq x, y \leq 2\pi$ are shown in Figure 4.1. Notice how the saddle points appear as intersecting lines on the contour plot and the maxima and minima appear as concentric circles.

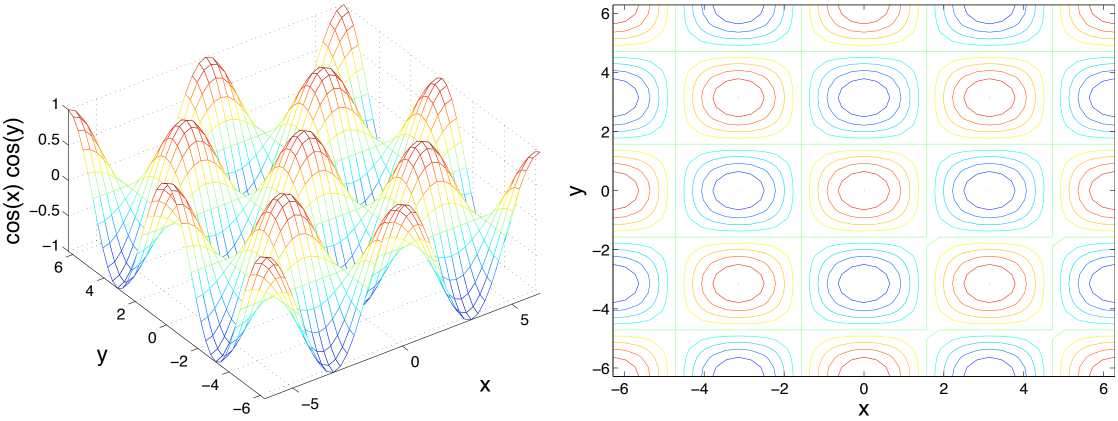

Figure 4.1: On the left we have a surface plot of the function $f(x, y) = \cos x \cos y$. On the right we have the corresponding contour plot.

5. Exercises

Find and classify the stationary points of each of these functions, and draw their contour plots:

1. $f(x, y) = x^2 + y^2 + 8x + 8$
2. $f(x, y) = x^2 + 4xy + 2y^2$
3. $f(x, y) = 2ax^2 + 2ay^2 - x^4 - y^4 - 6x^2y^2$ for $a > 0$
4. $f(x, y) = 2ax^2 + 2ay^2 - x^4 - y^4 - 6x^2y^2$ for $a < 0$
5. $f(x, y) = \frac{1}{2}y^2 + 1 - \cos x$