4 Cosmological Perturbation Theory

So far, we have treated the universe as perfectly homogeneous. To understand the formation and evolution of large-scale structures, we have to introduce inhomogeneities. As long as these perturbations remain relatively small, we can treat them in perturbation theory. In particular, we can expand the Einstein equations order-by-order in perturbations to the metric and the stress tensor. This makes the complicated system of coupled PDEs manageable.

4.1 Newtonian Perturbation Theory

Newtonian gravity is an adequate description of general relativity on scales well inside the Hubble radius and for non-relativistic matter (e.g. cold dark matter and baryons after decoupling). We will start with Newtonian perturbation theory because it is more intuitive than the full treatment in GR.

4.1.1 Perturbed Fluid Equations

Consider a non-relativistic fluid with *mass* density ρ, pressure $P \ll \rho$ and velocity $\boldsymbol{u}$. Denote the position vector of a fluid element by $\boldsymbol{r}$ and time by t. The equations of motion are given by basic fluid dynamics.[1] Mass conservation implies the *continuity equation*

$$\partial_t \rho = -\boldsymbol{\nabla}_{\boldsymbol{r}} \cdot (\rho \boldsymbol{u}) \ , \tag{4.1.1}$$

while momentum conservation leads to the *Euler equation*

$$\left(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}}\right) \boldsymbol{u} = -\frac{\boldsymbol{\nabla}_{\boldsymbol{r}} P}{\rho} - \boldsymbol{\nabla}_{\boldsymbol{r}} \Phi \ . \tag{4.1.2}$$

The last equation is simply "$F = ma$" for a fluid element. The gravitational potential Φ is determined by the *Poisson equation*

$$\nabla^2_{\boldsymbol{r}} \Phi = 4\pi G \rho \ . \tag{4.1.3}$$

*Convective derivative.**—Notice that the acceleration in (4.1.2) is not given by $\partial_t \boldsymbol{u}$ (which measures how the velocity changes at a given position), but by the "convective time derivative" $D_t \boldsymbol{u} \equiv (\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{u}$ which follows the fluid element as it moves. Let me remind you how this comes about.

Consider a fixed volume in space. The total mass in the volume can only change if there is a flux of momentum through the surface. Locally, this is what the continuity equation describes: $\partial_t \rho + \nabla_j(\rho u_j) = 0$. Similarly, in the absence of any forces, the total momentum in the volume

[1] See Landau and Lifshitz, *Fluid Mechanics.*

can only change if there is a flux through the surface: $\partial_t(\rho u_i) + \nabla_j(\rho u_i u_j) = 0$. Expanding the derivatives, we get

$$\begin{aligned}\partial_t(\rho u_i) + \nabla_j(\rho u_i u_j) &= \rho\left[\partial_t + u_j \nabla_j\right] u_i + u_i \underbrace{\left[\partial_t \rho + \nabla_j(\rho u_j)\right]}_{=0} \\ &= \rho\left[\partial_t + u_j \nabla_j\right] u^i \ .\end{aligned}$$

In the absence of forces it is therefore the convective derivative of the velocity, $D_t\boldsymbol{u}$, that vanishes, not $\partial_t \boldsymbol{u}$. Adding forces gives the Euler equation.

We wish to see what these equation imply for the evolution of small perturbations around a homogeneous background. We therefore decompose all quantities into background values (denoted by an overbar) and perturbations—e.g. $\rho(t,\boldsymbol{r}) = \bar\rho(t) + \delta\rho(t,\boldsymbol{r})$, and similarly for the pressure, the velocity and the gravitational potential. Assuming that the fluctuations are small, we can linearise eqs. (4.1.1) and (4.1.2), i.e. we can drop products of fluctuations.

Static space without gravity

Let us first consider static space and ignore gravity ($\Phi \equiv 0$). It is easy to see that a solution for the background is $\bar\rho = const.$, $\bar P = const.$ and $\bar{\boldsymbol{u}} = 0$. The linearised evolution equations for the fluctuations are

$$\partial_t \delta\rho = -\boldsymbol{\nabla}_{\boldsymbol{r}} \cdot (\bar\rho \boldsymbol{u}) \ , \tag{4.1.4}$$

$$\bar\rho\, \partial_t \boldsymbol{u} = -\boldsymbol{\nabla}_{\boldsymbol{r}} \delta P \ . \tag{4.1.5}$$

Combining ∂_t (4.1.4) and $\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot$ (4.1.5), one finds

$$\partial_t^2 \delta\rho - \nabla_{\boldsymbol{r}}^2 \delta P = 0 \ . \tag{4.1.6}$$

For adiabatic fluctuations (see below), the pressure fluctuations are proportional to the density fluctuations, $\delta P = c_s^2 \delta\rho$, where c_s is called the *speed of sound*. Eq. (4.1.6) then takes the form of a wave equation

$$\left(\partial_t^2 - c_s^2 \nabla^2\right) \delta\rho = 0 \ . \tag{4.1.7}$$

This is solved by a plane wave, $\delta\rho = A \exp[i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})]$, where $\omega = c_s k$, with $k \equiv |\boldsymbol{k}|$. We see that in a static spacetime *fluctuations oscillate with constant amplitude* if we ignore gravity.

Fourier space.—The more formal way to solve PDEs like (4.1.7) is to expand $\delta\rho$ in terms of its Fourier components

$$\delta\rho(t,\boldsymbol{r}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \delta\rho_{\boldsymbol{k}}(t) \ . \tag{4.1.8}$$

The PDE (4.1.7) turns into an ODE for each Fourier mode

$$\left(\partial_t^2 + c_s^2 k^2\right) \delta\rho_{\boldsymbol{k}} = 0 \ , \tag{4.1.9}$$

which has the solution

$$\delta\rho_{\boldsymbol{k}} = A_{\boldsymbol{k}} e^{i\omega_k t} + B_{\boldsymbol{k}} e^{-i\omega_k t} \ , \qquad \omega_k \equiv c_s k \ . \tag{4.1.10}$$

Static space with gravity

Now we turn on gravity. Eq. (4.1.7) then gets a source term

$$\left(\partial_t^2 - c_s^2 \nabla_{\boldsymbol{r}}^2\right) \delta\rho = 4\pi G \bar{\rho}\, \delta\rho \ , \tag{4.1.11}$$

where we have used the perturbed Poisson equation, $\nabla^2 \delta\Phi = 4\pi G \delta\rho$. This is still solved by $\delta\rho = A \exp[i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})]$, but now with

$$\omega^2 = c_s^2 k^2 - 4\pi G \bar{\rho} \ . \tag{4.1.12}$$

We see that there is a critical wavenumber for which the frequency of oscillations is zero:

$$k_{\rm J} \equiv \frac{\sqrt{4\pi G \bar{\rho}}}{c_s} \ . \tag{4.1.13}$$

For small scales (i.e. large wavenumber), $k > k_{\rm J}$, the pressure dominates and we find the same oscillations as before. However, on large scales, $k < k_{\rm J}$, gravity dominates, the frequency ω becomes imaginary and the *fluctuations grow exponentially*. The crossover happens at the *Jeans' length*

$$\lambda_{\rm J} = \frac{2\pi}{k_{\rm J}} = c_s \sqrt{\frac{\pi}{G\bar{\rho}}} \ . \tag{4.1.14}$$

Expanding space

In an expanding space, we have the usual relationship between physical coordinates $\boldsymbol{r}$ and comoving coordinates $\boldsymbol{x}$,

$$\boldsymbol{r}(t) = a(t)\boldsymbol{x} \ . \tag{4.1.15}$$

The velocity field is then given by

$$\boldsymbol{u}(t) = \dot{\boldsymbol{r}} = H\boldsymbol{r} + \boldsymbol{v} \ , \tag{4.1.16}$$

where $H\boldsymbol{r}$ is the Hubble flow and $\boldsymbol{v} = a\dot{\boldsymbol{x}}$ is the proper velocity. In a static spacetime, the time and space derivates defined from t and $\boldsymbol{r}$ were independent. In an expanding spacetime this is not the case anymore. It is then convenient to use space derivatives defined with respect to the comoving coordinates $\boldsymbol{x}$, which we denote by $\boldsymbol{\nabla}_{\boldsymbol{x}}$. Using (4.1.15), we have

$$\boldsymbol{\nabla}_{\boldsymbol{r}} = a^{-1} \boldsymbol{\nabla}_{\boldsymbol{x}} \ . \tag{4.1.17}$$

The relationship between time derivatives at fixed $\boldsymbol{r}$ and at fixed $\boldsymbol{x}$ is

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)_{\boldsymbol{r}} = \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} &= \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial a^{-1}(t)\boldsymbol{r}}{\partial t}\right)_{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \\ &= \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}} - H\boldsymbol{x}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} \ . \end{aligned} \tag{4.1.18}$$

From now on, we will drop the subscripts $\boldsymbol{x}$.

With this in mind, let us look at the fluid equations in an expanding universe:

- *Continuity equation*

  Substituting (4.1.17) and (4.1.18) for $\boldsymbol{\nabla}_{\boldsymbol{r}}$ and ∂_t in the continuity equation (4.1.1), we get

$$\left[\frac{\partial}{\partial t} - H\boldsymbol{x}\cdot\boldsymbol{\nabla}\right]\left[\bar{\rho}(1+\delta)\right] + \frac{1}{a}\boldsymbol{\nabla}\cdot\left[\bar{\rho}(1+\delta)(Ha\boldsymbol{x}+\boldsymbol{v})\right] = 0\ , \tag{4.1.19}$$

  Here, I have introduced the *fractional density perturbation*

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}}\ . \tag{4.1.20}$$

  Sometimes δ is called the *density contrast*.

  Let us analyse this order-by-order in perturbation theory:

  - At zeroth order in fluctuations (i.e. dropping the perturbations δ and $\boldsymbol{v}$), we have

$$\frac{\partial\bar{\rho}}{\partial t} + 3H\bar{\rho} = 0\ , \tag{4.1.21}$$

    where I have used $\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{x} = 3$. We recognise this as the continuity equation for the homogeneous *mass* density, $\bar{\rho} \propto a^{-3}$.

  - At first order in fluctuations (i.e. dropping products of δ and $\boldsymbol{v}$), we get

$$\left[\frac{\partial}{\partial t} - H\boldsymbol{x}\cdot\boldsymbol{\nabla}\right]\left[\bar{\rho}\delta\right] + \frac{1}{a}\boldsymbol{\nabla}\cdot\left[\bar{\rho}Ha\boldsymbol{x}\delta + \bar{\rho}\boldsymbol{v}\right] = 0\ , \tag{4.1.22}$$

    which we can write as

$$\left[\frac{\partial\bar{\rho}}{\partial t} + 3H\bar{\rho}\right]\delta + \bar{\rho}\,\frac{\partial\delta}{\partial t} + \frac{\bar{\rho}}{a}\,\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\ . \tag{4.1.23}$$

    The first term vanishes by (4.1.21), so we find

$$\boxed{\dot{\delta} = -\frac{1}{a}\boldsymbol{\nabla}\cdot\boldsymbol{v}}\ , \tag{4.1.24}$$

    where we have used an overdot to denote the derivative with respect to time.

- *Euler equation*

  Similar manipulations of the Euler equation (4.1.2) lead to

$$\boxed{\dot{\boldsymbol{v}} + H\boldsymbol{v} = -\frac{1}{a\bar{\rho}}\boldsymbol{\nabla}\delta P - \frac{1}{a}\boldsymbol{\nabla}\delta\Phi}\ . \tag{4.1.25}$$

  In the absence of pressure and gravitational perturbations, this equation simply says that $\boldsymbol{v} \propto a^{-1}$, which is something we already discovered in Chapter 1.

- *Poisson equation*

  It takes hardly any work to show that the Poisson equation (4.1.3) becomes

$$\boxed{\boldsymbol{\nabla}^2\delta\Phi = 4\pi G a^2\bar{\rho}\delta}\ . \tag{4.1.26}$$

Exercise.—Derive eq. (4.1.25).

4.1.2 Jeans' Instability

Combining ∂_t(4.1.24) with $\boldsymbol{\nabla}\cdot$(4.1.25) and (4.1.26), we find

$$\boxed{\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2}\nabla^2\delta = 4\pi G\bar{\rho}\,\delta} \ . \tag{4.1.27}$$

This implies the same Jeans' length as in (4.1.14), but unlike the case of a static spacetime, it now depends on time via $\bar{\rho}(t)$ and $c_s(t)$. Compared to (4.1.11), the equation of motion in the expanding spacetime includes a friction term, $2H\dot{\delta}$. This has two effects: Below the Jeans' length, the fluctuations oscillate with decreasing amplitude. Above the Jeans' length, the *fluctuations experience power-law growth*, rather than the exponential growth we found for static space.

4.1.3 Dark Matter inside Hubble

The Newtonian framework describes the evolution of matter fluctuations. We can apply it to the evolution dark matter on sub-Hubble scales. (We will ignore small effects due to baryons.)

- During the *matter-dominated era*, eq. (4.1.27) reads

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0 \ , \tag{4.1.28}$$

  where we have dropped the pressure term, since $c_s = 0$ for linearised CDM fluctuations. (Non-linear effect produce a finite, but small, sound speed.) Since $a \propto t^{2/3}$, we have $H = 2/3t$ and hence

$$\ddot{\delta}_m + \frac{4}{3t}\dot{\delta}_m - \frac{2}{3t^2}\delta_m = 0 \ , \tag{4.1.29}$$

  where we have used $4\pi G\bar{\rho}_m = \frac{3}{2}H^2$. Trying $\delta_m \propto t^p$ gives the following two solutions:

$$\delta_m \propto \begin{cases} t^{-1} & \propto \ a^{-3/2} \\ t^{2/3} & \propto \ a \end{cases} \ . \tag{4.1.30}$$

  Hence, the *growing mode* of dark matter fluctuations grows like the scale factor during the MD era. This is a famous result that is worth remembering.

- During the *radiation-dominated era*, eq. (4.1.27) gets modified to

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\sum_I \bar{\rho}_I\,\delta_I = 0 \ , \tag{4.1.31}$$

  where the sum is over matter and radiation. (It is the *total* density fluctuation $\delta\rho = \delta\rho_m + \delta\rho_r$ which sources $\delta\Phi$!) Radiation fluctuations on scales smaller than the Hubble radius oscillate as sound waves (supported by large radiation pressure) and their time-averaged density contrast vanishes. To prove this rigorously requires relativistic perturbation theory (see below). It follows that the CDM is essentially the only clustered component during the acoustic oscillations of the radiation, and so

$$\ddot{\delta}_m + \frac{1}{t}\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m \approx 0 \ . \tag{4.1.32}$$

Since δ_m evolves only on cosmological timescales (it has no pressure support for it to do otherwise), we have

$$\ddot{\delta}_m \sim H^2\delta_m \sim \frac{8\pi G}{3}\bar{\rho}_r\delta_m \gg 4\pi G\bar{\rho}_m\delta_m \ , \tag{4.1.33}$$

where we have used that $\bar{\rho}_r \gg \bar{\rho}_m$. We can therefore ignore the last term in (4.1.32) compared to the others. We then find

$$\delta_m \propto \begin{cases} const. \\ \ln t \ \propto \ \ln a \end{cases} . \tag{4.1.34}$$

We see that the rapid expansion due to the effectively unclustered radiation reduces the growth of δ_m to only logarithmic. This is another fact worth remembering: we need to wait until the universe becomes matter dominated in order for the dark matter density fluctuations to grow significantly.

- During the Λ-*dominated era*, eq. (4.1.27) reads

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\sum_i \bar{\rho}_I\,\delta_I = 0 \ , \tag{4.1.35}$$

where $I = m, \Lambda$. As far as we can tell, dark energy doesn't cluster (almost by definition), so we can write

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\,\delta_m = 0 \ , \tag{4.1.36}$$

Notice that this is *not* the same as (4.1.28), because H is different. Indeed, in the Λ-dominated regime $H^2 \approx const. \gg 4\pi G\bar{\rho}_m$. Dropping the last term in (4.1.36), we get

$$\ddot{\delta}_m + 2H\dot{\delta}_m \approx 0 \ , \tag{4.1.37}$$

which has the following solutions

$$\delta_m \propto \begin{cases} const. \\ e^{-2Ht} \ \propto \ a^{-2} \end{cases} . \tag{4.1.38}$$

We see that the matter fluctuations stop growing once dark energy comes to dominate.

4.2 Relativistic Perturbation Theory

The Newtonian treatment of cosmological perturbations is inadequate on scales larger than the Hubble radius, and for relativistic fluids (like photons and neutrinos). The correct description requires a full general-relativistic treatment which we will now develop.

4.2.1 Perturbed Spacetime

The basic idea is to consider small perturbations $\delta g_{\mu\nu}$ around the FRW metric $\bar{g}_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \ . \tag{4.2.39}$$

Through the Einstein equations, the metric perturbations will be coupled to perturbations in the matter distribution.

Perturbations of the Metric

To avoid unnecessary technical distractions, we will only present the case of a flat FRW background spacetime,

$$\mathrm{d}s^2 = a^2(\tau)\Big[\mathrm{d}\tau^2 - \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\Big] . \tag{4.2.40}$$

The perturbed metric can then be written as

$$\mathrm{d}s^2 = a^2(\tau)\Big[(1+2A)\mathrm{d}\tau^2 - 2B_i\,\mathrm{d}x^i\mathrm{d}\tau - (\delta_{ij}+h_{ij})\mathrm{d}x^i\mathrm{d}x^j\Big] , \tag{4.2.41}$$

where A, B_i and h_{ij} are functions of space and time. We shall adopt the useful convention that Latin indices on spatial vectors and tensors are raised and lowered with δ_{ij}, e.g. $h^i{}_i = \delta^{ij}h_{ij}$.

Scalar, Vectors and Tensors

It will be extremely useful to perform a scalar-vector-tensor (SVT) decomposition of the perturbations. For 3-vectors, this should be familiar. It simply means that we can split any 3-vector into the gradient of a scalar and a divergenceless vector

$$B_i = \underbrace{\partial_i B}_{\text{scalar}} + \underbrace{\hat{B}_i}_{\text{vector}} , \tag{4.2.42}$$

with $\partial^i \hat{B}_i = 0$. Similarly, any rank-2 symmetric tensor can be written

$$h_{ij} = \underbrace{2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E}_{\text{scalar}} + \underbrace{2\partial_{(i}\hat{E}_{j)}}_{\text{vector}} + \underbrace{2\hat{E}_{ij}}_{\text{tensor}} , \tag{4.2.43}$$

where

$$\partial_{\langle i}\partial_{j\rangle}E \equiv \Big(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\Big)E , \tag{4.2.44}$$

$$\partial_{(i}\hat{E}_{j)} \equiv \frac{1}{2}\Big(\partial_i\hat{E}_j + \partial_j\hat{E}_i\Big) . \tag{4.2.45}$$

As before, the hatted quantities are divergenceless, i.e. $\partial^i\hat{E}_i = 0$ and $\partial^i\hat{E}_{ij} = 0$. The tensor perturbation is traceless, $\hat{E}^i{}_i = 0$. The 10 degrees of freedom of the metric have thus been decomposed into $4+4+2$ SVT degrees of freedom:

- *scalars:* A, B, C, E
- *vectors:* $\hat{B}_i$, $\hat{E}_i$
- *tensors:* $\hat{E}_{ij}$

What makes the SVT-decomposition so powerful is the fact that the Einstein equations for scalars, vectors and tensors don't mix at linear order and can therefore be treated separately. In these lectures, we will mostly be interested in scalar fluctuations and the associated density perturbations. Vector perturbations aren't produced by inflation and even if they were, they would decay quickly with the expansion of the universe. Tensor perturbations are an important prediction of inflation and we will discuss them briefly in Chapter 6.

The Gauge Problem

Before we continue, we have to address an important subtlety. The metric perturbations in (4.2.41) aren't uniquely defined, but depend on our choice of coordinates or the *gauge choice*. In particular, when we wrote down the perturbed metric, we implicitly chose a specific time slicing of the spacetime and defined specfic spatial coordinates on these time slices. Making a different choice of coordinates, can change the values of the perturbation variables. It may even introduce fictitious perturbations. These are fake perturbations that can arise by an inconvenient choice of coordinates even if the background is perfectly homogeneous.

For example, consider the homogeneous FRW spacetime (4.2.40) and make the following change of the spatial coordinates, $x^i \mapsto \tilde{x}^i = x^i + \xi^i(\tau, \boldsymbol{x})$. We assume that ξ^i is small, so that it can also be treated as a perturbation. Using $\mathrm{d}x^i = \mathrm{d}\tilde{x}^i - \partial_\tau \xi^i \mathrm{d}\tau - \partial_k \xi^i \mathrm{d}\tilde{x}^k$, eq. (4.2.40) becomes

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - 2\xi_i' \mathrm{d}\tilde{x}^i \mathrm{d}\tau - \left(\delta_{ij} + 2\partial_{(i}\xi_{j)}\right)\mathrm{d}\tilde{x}^i \mathrm{d}\tilde{x}^j\right] \ , \tag{4.2.46}$$

where we have dropped terms that are quadratic in ξ^i and defined $\xi_i' \equiv \partial_\tau \xi_i$. We apparently have introduced the metric perturbations $B_i = \xi_i'$ and $\hat{E}_i = \xi_i$. But these are just fictitious *gauge modes* that can be removed by going back to the old coordinates.

Similar, we can change our time slicing, $\tau \mapsto \tau + \xi^0(\tau, \boldsymbol{x})$. The homogeneous density of the universe then gets perturbed, $\rho(\tau) \mapsto \rho(\tau + \xi^0(\tau, \boldsymbol{x})) = \bar{\rho}(\tau) + \bar{\rho}'\xi^0$. So even in an unperturbed universe, a change of the time coordinate can introduce a fictitious density perturbation

$$\delta\rho = \bar{\rho}'\xi^0 \ . \tag{4.2.47}$$

Similarly, we can remove a real perturbation in the energy density by choosing the hypersurface of constant time to coincide with the hypersurface of constant energy density. Then $\delta\rho = 0$ although there are real inhomogeneities.

These examples illustrate that we need a more physical way to identify true perturbations. One way to do this is to define perturbations in such a way that they don't change under a change of coordinates.

Gauge Transformations

Consider the coordinate transformation

$$X^\mu \ \mapsto \ \tilde{X}^\mu \equiv X^\mu + \xi^\mu(\tau, \boldsymbol{x}) \ , \qquad \text{where} \qquad \xi^0 \equiv T \ , \quad \xi^i \equiv L^i = \partial^i L + \hat{L}^i \ . \tag{4.2.48}$$

We have split the spatial shift L^i into a scalar, L, and a divergenceless vector, $\hat{L}^i$. We wish to know how the metric transforms under this change of coordinates. The trick is to exploit the invariance of the spacetime interval,

$$\mathrm{d}s^2 = g_{\mu\nu}(X)\mathrm{d}X^\mu \mathrm{d}X^\nu = \tilde{g}_{\alpha\beta}(\tilde{X})\mathrm{d}\tilde{X}^\alpha \mathrm{d}\tilde{X}^\beta \ , \tag{4.2.49}$$

where I have used a different set of dummy indices on both sides to make the next few lines clearer. Writing $\mathrm{d}\tilde{X}^\alpha = (\partial \tilde{X}^\alpha / \partial X^\mu)\mathrm{d}X^\mu$ (and similarly for $\mathrm{d}X^\beta$), we find

$$g_{\mu\nu}(X) = \frac{\partial \tilde{X}^\alpha}{\partial X^\mu}\frac{\partial \tilde{X}^\beta}{\partial X^\nu}\, \tilde{g}_{\alpha\beta}(\tilde{X}) \ . \tag{4.2.50}$$

This relates the metric in the old coordinates, $g_{\mu\nu}$, to the metric in the new coordinates, $\tilde{g}_{\alpha\beta}$.

Let us see what (4.2.50) implies for the transformation of the metric perturbations in (4.2.41). I will work out the 00-component as an example and leave the rest as an exercise. Consider $\mu = \nu = 0$ in (4.2.50):

$$g_{00}(X) = \frac{\partial \tilde{X}^\alpha}{\partial \tau}\frac{\partial \tilde{X}^\beta}{\partial \tau}\,\tilde{g}_{\alpha\beta}(\tilde{X}) \ . \tag{4.2.51}$$

The only term that contributes to the l.h.s. is the one with $\alpha = \beta = 0$. Consider for example $\alpha = 0$ and $\beta = i$. The off-diagonal component of the metric $\tilde{g}_{0i}$ is proportional to $\tilde{B}_i$, so it is a first-order perturbation. But $\partial \tilde{X}^i/\partial\tau$ is proportional to the first-order variable ξ^i, so the product is second order and can be neglected. A similar argument holds for $\alpha = i$ and $\beta = j$. Eq. (4.2.51) therefore reduces to

$$g_{00}(X) = \left(\frac{\partial\tilde{\tau}}{\partial\tau}\right)^2 \tilde{g}_{00}(\tilde{X}) \ . \tag{4.2.52}$$

Substituting (4.2.48) and (4.2.41), we get

$$\begin{aligned} a^2(\tau)\big(1+2A\big) &= \big(1+T'\big)^2 a^2(\tau+T)\big(1+2\tilde{A}\big) \\ &= \big(1+2T'+\cdots\big)\big(a(\tau)+a'T+\cdots\big)^2\big(1+2\tilde{A}\big) \\ &= a^2(\tau)\big(1+2\mathcal{H}T+2T'+2\tilde{A}+\cdots\big) \ , \end{aligned} \tag{4.2.53}$$

where $\mathcal{H} \equiv a'/a$ is the Hubble parameter in conformal time. Hence, we find that at first order, the metric perturbation A transforms as

$$A \mapsto \tilde{A} = A - T' - \mathcal{H}T \ . \tag{4.2.54}$$

I leave it to you to repeat the argument for the other metric components and show that

$$B_i \mapsto \tilde{B}_i = B_i + \partial_i T - L_i' \ , \tag{4.2.55}$$

$$h_{ij} \mapsto \tilde{h}_{ij} = h_{ij} - 2\partial_{(i}L_{j)} - 2\mathcal{H}T\delta_{ij} \ . \tag{4.2.56}$$

Exercise.—Derive eqs. (4.2.55) and (4.2.56).

In terms of the SVT-decomposition, we get

$$A \mapsto A - T' - \mathcal{H}T \ , \tag{4.2.57}$$

$$B \mapsto B + T - L' \ , \qquad \hat{B}_i \mapsto \hat{B}_i - \hat{L}_i' \ , \tag{4.2.58}$$

$$C \mapsto C - \mathcal{H}T - \frac{1}{3}\nabla^2 L \ , \tag{4.2.59}$$

$$E \mapsto E - L \ , \qquad \hat{E}_i \mapsto \hat{E}_i - \hat{L}_i \ , \qquad \hat{E}_{ij} \mapsto \hat{E}_{ij} \ . \tag{4.2.60}$$

Gauge-Invariant Perturbations

One way to avoid the gauge problems is to define special combinations of metric perturbations that do not transform under a change of coordinates. These are the *Bardeen variables*:

$$\Psi \equiv A + \mathcal{H}(B - E') + (B - E')' \ , \qquad \hat{\Phi}_i \equiv \hat{E}_i' - \hat{B}_i \ , \qquad \hat{E}_{ij} \ , \tag{4.2.61}$$

$$\Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E \ . \tag{4.2.62}$$

Exercise.—Show that Ψ, Φ and $\hat{\Phi}_i$ don't change under a coordinate transformation.

These gauge-invariant variables can be considered as the 'real' spacetime perturbations since they cannot be removed by a gauge transformation.

Gauge Fixing

An alternative (but related) solution to the gauge problem is to *fix the gauge* and keep track of *all* perturbations (metric and matter). For example, we can use the freedom in the gauge functions T and L in (4.2.48) to set two of the four scalar metric perturbations to zero:

- *Newtonian gauge.*—The choice
$$B = E = 0 \,, \tag{4.2.63}$$
gives the metric
$$\mathrm{d}s^2 = a^2(\tau)\left[(1+2\Psi)\mathrm{d}\tau^2 - (1-2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] \,. \tag{4.2.64}$$
Here, we have renamed the remaining two metric perturbations, $A \equiv \Psi$ and $C \equiv -\Phi$, in order to make contact with the Bardeen potentials in (4.2.61) and (4.2.62). For perturbations that decay at spatial infinity, the Newtonian gauge is unique (i.e. the gauge is fixed completely).[2] In this gauge, the physics appears rather simple since the hypersurfaces of constant time are orthogonal to the worldlines of observers at rest in the coordinates (since $B = 0$) and the induced geometry of the constant-time hypersurfaces is isotropic (since $E = 0$). In the absence of anisotropic stress, $\Psi = \Phi$. Note the similarity of the metric to the usual weak-field limit of GR about Minkowski space; we shall see that Ψ plays the role of the gravitational potential. Newtonian gauge will be our preferred gauge for studying the formation of large-scale structures (Chapter 5) and CMB anisotropies (Chapter **??**).

- *Spatially-flat qauge.*—A convenient gauge for computing inflationary perturbations is
$$C = E = 0 \,. \tag{4.2.65}$$
In this gauge, we will be able to focus most directly on the fluctuations in the inflaton field $\delta\phi$ (see Chapter 6) .

4.2.2 Perturbed Matter

In Chapter 1, we showed that the matter in a homogeneous and isotropic universe has to take the form of a perfect fluid
$$\bar{T}^\mu{}_\nu = (\bar{\rho} + \bar{P})\bar{U}^\mu\bar{U}_\nu - \bar{P}\delta^\mu_\nu \,, \tag{4.2.66}$$
where $\bar{U}_\mu = a\delta^0_\mu$, $\bar{U}^\mu = a^{-1}\delta^\mu_0$ for a comoving observer. Now, we consider small perturbations of the stress-energy tensor
$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu + \delta T^\mu{}_\nu \,. \tag{4.2.67}$$

[2] More generally, a gauge transformation that corresponds to a small, time-dependent but spatially constant boost – i.e. $L^i(\tau)$ and a compensating time translation with $\partial_i T = L_i(\tau)$ to keep the constant-time hypersurfaces orthogonal – will preserve $E_{ij} = 0$ and $B_i = 0$ and hence the form of the metric in eq. (4.4.168). However, such a transformation would not preserve the decay of the perturbations at infinity.

Perturbations of the Stress-Energy Tensor

In a perturbed universe, the energy density ρ, the pressure P and the four-velocity U^μ can be functions of position. Moreover, the stress-energy tensor can now have a contribution from *anisotropic stress*, $\Pi^\mu{}_\nu$. The perturbation of the stress-energy tensor is

$$\delta T^\mu{}_\nu = (\delta\rho + \delta P)\bar{U}^\mu \bar{U}_\nu + (\bar\rho + \bar P)(\delta U^\mu \bar{U}_\nu + \bar{U}^\mu \delta U_\nu) - \delta P\, \delta^\mu_\nu - \Pi^\mu{}_\nu \ . \tag{4.2.68}$$

The spatial part of the anisotropic stress tensor can be chosen to be traceless, $\Pi^i{}_i = 0$, since its trace can always be absorbed into a redefinition of the isotropic pressure, P. The anisotropic stress tensor can also be chosen to be orthogonal to U^μ, i.e. $U^\mu \Pi_{\mu\nu} = 0$. Without loss of generality, we can then set $\Pi^0{}_0 = \Pi^0{}_i = 0$. In practice, the anisotropic stress will always be negligible in these lectures. We will keep it for now, but at some point we will drop it.

Perturbations in the four-velocity can induce non-vanishing *energy flux*, $T^0{}_j$, and *momentum density*, $T^i{}_0$. To find these, let us compute the perturbed four-velocity in the perturbed metric (4.2.41). Since $g_{\mu\nu}U^\mu U^\nu = 1$ and $\bar g_{\mu\nu}\bar U^\mu \bar U^\nu = 1$, we have, at linear order,

$$\delta g_{\mu\nu} \bar U^\mu \bar U^\nu + 2\bar U_\mu \delta U^\mu = 0 \ . \tag{4.2.69}$$

Using $\bar U^\mu = a^{-1}\delta^0_\mu$ and $\delta g_{00} = 2a^2 A$, we find $\delta U^0 = -Aa^{-1}$. We then write $\delta U^i \equiv v^i/a$, where $v^i \equiv dx^i/d\tau$ is the *coordinate velocity*, so that

$$U^\mu = a^{-1}[1 - A, v^i] \ . \tag{4.2.70}$$

From this, we derive

$$U_0 = g_{00}U^0 + \overbrace{g_{0i}U^i}^{\mathcal{O}(2)} = a^2(1+2A)a^{-1}(1-A) = a(1+A) \ , \tag{4.2.71}$$

$$U_i = g_{i0}U^0 + g_{ij}U^j = -a^2 B_i a^{-1} - a^2 \delta_{ij} a^{-1} v^j = -a(B_i + v_i) \ , \tag{4.2.72}$$

i.e.

$$U_\mu = a[1 + A, -(v_i + B_i)] \ . \tag{4.2.73}$$

Using (4.2.70) and (4.2.73) in (4.2.68), we find

$$\delta T^0{}_0 = \delta\rho \ , \tag{4.2.74}$$

$$\delta T^i{}_0 = (\bar\rho + \bar P) v^i \ , \tag{4.2.75}$$

$$\delta T^0{}_j = -(\bar\rho + \bar P)(v_j + B_j) \ , \tag{4.2.76}$$

$$\delta T^i{}_j = -\delta P \delta^i_j - \Pi^i{}_j \ . \tag{4.2.77}$$

We will use q^i for the *momentum density* $(\bar\rho + \bar P)v^i$. If there are several contributions to the stress-energy tensor (e.g. photons, baryons, dark matter, etc.), they are added: $T_{\mu\nu} = \sum_I T^I_{\mu\nu}$. This implies

$$\delta\rho = \sum_I \delta\rho_I \ , \quad \delta P = \sum_I \delta P_I \ , \quad q^i = \sum_I q^i_I \ , \quad \Pi^{ij} = \sum_I \Pi^{ij}_I \ . \tag{4.2.78}$$

We see that the perturbations in the density, pressure and anisotropic stress simply add. The velocities do *not* add, but the momentum densities do.

Finally, we note that the SVT decomposition can also be applied to the perturbations of the stress-energy tensor: $\delta\rho$ and δP have scalar parts only, q_i has scalar and vector parts,

$$q_i = \partial_i q + \hat{q}_i \ , \tag{4.2.79}$$

and Π_{ij} has scalar, vector and tensor parts,

$$\Pi_{ij} = \partial_{\langle i}\partial_{j\rangle}\Pi + \partial_{(i}\hat{\Pi}_{j)} + \hat{\Pi}_{ij} \ . \tag{4.2.80}$$

Gauge Transformations

Under the coordinate transformation (4.2.48), the stress-energy tensor transform as

$$T^\mu{}_\nu(X) = \frac{\partial X^\mu}{\partial \tilde{X}^\alpha}\frac{\partial \tilde{X}^\beta}{\partial X^\nu}\,\tilde{T}^\alpha{}_\beta(\tilde{X}) \ . \tag{4.2.81}$$

Evaluating this for the different components, we find

$$\delta\rho \mapsto \delta\rho - T\bar{\rho}' \ , \tag{4.2.82}$$

$$\delta P \mapsto \delta P - T\bar{P}' \ , \tag{4.2.83}$$

$$q_i \mapsto q_i + (\bar{\rho} + \bar{P})L_i' \ , \tag{4.2.84}$$

$$v_i \mapsto v_i + L_i' \ , \tag{4.2.85}$$

$$\Pi_{ij} \mapsto \Pi_{ij} \ . \tag{4.2.86}$$

Exercise.—Confirm eqs. (4.2.82)–(4.2.86).

Gauge-Invariant Perturbations

There are various gauge-invariant quantities that can be formed from metric and matter variables. One useful combination is

$$\bar{\rho}\Delta \equiv \delta\rho + \bar{\rho}'(v + B) \ , \tag{4.2.87}$$

where $v_i = \partial_i v$. The quantity Δ is called the *comoving-gauge density perturbation.*

Exercise.—Show that Δ is gauge-invariant.

Gauge Fixing

Above we used our gauge freedom to set two of the metric perturbations to zero. Alternatively, we can define the gauge in the matter sector:

- *Uniform density gauge.*—We can use the freedom in the time-slicing to set the total density perturbation to zero

$$\delta\rho = 0 \ . \tag{4.2.88}$$

- *Comoving gauge.*—Similarly, we can ask for the scalar momentum density to vanish,

$$q = 0 \ . \tag{4.2.89}$$

Fluctuations in comoving gauge are most naturally connected to the inflationary initial conditions. This will be explained in §4.3.1 and Chapter 6.

There are different versions of uniform density and comoving gauge depending on which of the metric fluctuations is set to zero. In these lectures, we will choose $B = 0$.

Adiabatic Fluctuations

Simple inflation models predict initial fluctuations that are *adiabatic* (see Chapter 6). Adiabatic perturbations have the property that the local state of matter (determined, for example, by the energy density ρ and the pressure P) at some spacetime point $(\tau, \boldsymbol{x})$ of the perturbed universe is the same as in the *background* universe at some slightly different time $\tau + \delta\tau(\boldsymbol{x})$. (Notice that the time shift varies with location $\boldsymbol{x}$!) We can thus view adiabatic perturbations as some parts of the universe being "ahead" and others "behind" in the evolution. If the universe is filled with multiple fluids, adiabatic perturbations correspond to perturbations induced by a *common, local shift in time* of all background quantities; e.g. adiabatic density perturbations are defined as

$$\delta\rho_I(\tau, \boldsymbol{x}) \equiv \bar\rho_I(\tau + \delta\tau(\boldsymbol{x})) - \bar\rho_I(\tau) = \bar\rho_I' \delta\tau(\boldsymbol{x}) \ , \tag{4.2.90}$$

where $\delta\tau$ is the same for all species I. This implies

$$\delta\tau = \frac{\delta\rho_I}{\bar\rho_I'} = \frac{\delta\rho_J}{\bar\rho_J'} \qquad \text{for all species } I \text{ and } J \ . \tag{4.2.91}$$

Using[3] $\bar\rho_I' = -3\mathcal{H}(1 + w_I)\bar\rho_I$, we can write this as

$$\frac{\delta_I}{1 + w_I} = \frac{\delta_J}{1 + w_J} \qquad \text{for all species } I \text{ and } J \ , \tag{4.2.92}$$

where we have defined the *fractional density contrast*

$$\delta_I \equiv \frac{\delta\rho_I}{\bar\rho_I} \ . \tag{4.2.93}$$

Thus, for adiabatic perturbations, all matter components ($w_m \approx 0$) have the same fractional perturbation, while all radiation perturbations ($w_r = \frac{1}{3}$) obey

$$\delta_r = \frac{4}{3}\delta_m \ . \tag{4.2.94}$$

It follows that for adiabatic fluctuations, the total density perturbation,

$$\delta\rho_{\rm tot} = \bar\rho_{\rm tot}\delta_{\rm tot} = \sum_I \bar\rho_I \delta_I \ , \tag{4.2.95}$$

is dominated by the species that is dominant in the background since all the δ_I are comparable. We will have more to say about adiabatic initial conditions in §4.3.

[3] If there is no energy transfer between the fluid components at the background level, the energy continuity equation is satisfied by them separately.

Isocurvature Fluctuations

The complement of adiabatic perturbations are *isocurvature perturbations*. While adiabatic perturbations correspond to a change in the total energy density, isocurvature perturbations only correspond to perturbations between the different components. Eq. (4.2.92) suggests the following definition of isocurvature fluctuations

$$S_{IJ} \equiv \frac{\delta_I}{1+w_I} - \frac{\delta_J}{1+w_J} \ . \tag{4.2.96}$$

Single-field inflation predicts that the primordial perturbations are purely adiabatic, i.e. $S_{IJ} = 0$, for all species I and J. Moreover, all present observational data is consistent with this expectation. We therefore won't consider isocurvature fluctuations further in these lectures.

4.2.3 Linearised Evolution Equations

Our next task is to derive the perturbed Einstein equations, $\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}$, from the perturbed metric and the perturbed stress-energy tensor. We will work in Newtonian gauge with

$$g_{\mu\nu} = a^2 \left(\begin{array}{cc} 1+2\Psi & 0 \\ 0 & -(1-2\Phi)\delta_{ij} \end{array} \right) \ . \tag{4.2.97}$$

In these lectures, we will never encounter situations where anisotropic stress plays a significant role. From now on, we will therefore set anisotropic stress to zero, $\Pi_{ij} = 0$. As we will see, this enforces $\Phi = \Psi$.

Perturbed Connection Coefficients

To derive the field equations, we first require the perturbed connection coefficients. Recall that

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho} \right) \ . \tag{4.2.98}$$

Since the metric (4.2.97) is diagonal, it is simple to invert

$$g^{\mu\nu} = \frac{1}{a^2} \left(\begin{array}{cc} 1-2\Psi & 0 \\ 0 & -(1+2\Phi)\delta^{ij} \end{array} \right) \ . \tag{4.2.99}$$

Substituting (4.2.97) and (4.2.99) into (4.2.98), gives

$$\Gamma^0_{00} = \mathcal{H} + \Psi' \ , \tag{4.2.100}$$

$$\Gamma^0_{0i} = \partial_i \Psi \ , \tag{4.2.101}$$

$$\Gamma^i_{00} = \delta^{ij} \partial_j \Psi \ , \tag{4.2.102}$$

$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} - \left[\Phi' + 2\mathcal{H}(\Phi + \Psi) \right] \delta_{ij} \ , \tag{4.2.103}$$

$$\Gamma^i_{j0} = \mathcal{H}\delta^i_j - \Phi' \delta^i_j \ , \tag{4.2.104}$$

$$\Gamma^i_{jk} = -2\delta^i_{(j}\partial_{k)}\Phi + \delta_{jk}\delta^{il}\partial_l \Phi \ . \tag{4.2.105}$$

I will work out Γ^0_{00} as an example and leave the remaining terms as an exercise.

Example.—From the definition of the Christoffel symbol we have

$$\begin{aligned}\Gamma^0_{00} &= \frac{1}{2}g^{00}(2\partial_0 g_{00} - \partial_0 g_{00}) \\ &= \frac{1}{2}g^{00}\partial_0 g_{00} \ . \end{aligned} \tag{4.2.106}$$

Substituting the metric components, we find

$$\begin{aligned}\Gamma^0_{00} &= \frac{1}{2a^2}(1-2\Psi)\partial_0[a^2(1+2\Psi)] \\ &= \mathcal{H} + \Psi' \ , \end{aligned} \tag{4.2.107}$$

at linear order in Ψ.

Exercise.—Derive eqs. (4.2.101)–(4.2.105).

Perturbed Stress-Energy Conservation

Equipped with the perturbed connection, we can immediately derive the perturbed conservation equations from

$$\begin{aligned}\nabla_\mu T^\mu{}_\nu &= 0 \\ &= \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha}T^\alpha{}_\nu - \Gamma^\alpha_{\mu\nu}T^\mu{}_\alpha \ . \end{aligned} \tag{4.2.108}$$

Continuity Equation

Consider first the $\nu = 0$ component

$$\partial_0 T^0{}_0 + \partial_i T^i{}_0 + \Gamma^\mu_{\mu 0}T^0{}_0 + \underbrace{\Gamma^\mu_{\mu i}T^i{}_0}_{\mathcal{O}(2)} - \Gamma^0_{00}T^0{}_0 - \underbrace{\Gamma^0_{i0}T^i{}_0}_{\mathcal{O}(2)} - \underbrace{\Gamma^i_{00}T^0{}_i}_{\mathcal{O}(2)} - \Gamma^i_{j0}T^j{}_i = 0 \ . \tag{4.2.109}$$

Substituting the perturbed stress-energy tensor and the connection coefficients gives

$$\begin{aligned}&\partial_0(\bar\rho + \delta\rho) + \partial_i q^i + (\mathcal{H} + \Psi' + 3\mathcal{H} - 3\Phi')(\bar\rho + \delta\rho) \\ &\quad - (\mathcal{H} + \Psi')(\bar\rho + \delta\rho) - (\mathcal{H} - \Phi')\delta^i_j\big[-(\bar P + \delta P)\delta^j_i\big] = 0 \ , \end{aligned} \tag{4.2.110}$$

and hence

$$\bar\rho' + \delta\rho' + \partial_i q^i + 3\mathcal{H}(\bar\rho + \delta\rho) - 3\bar\rho\Phi' + 3\mathcal{H}(\bar P + \delta P) - 3\bar P\Phi' = 0 \ . \tag{4.2.111}$$

Writing the zeroth-order and first-order parts separately, we get

$$\bar\rho' = -3\mathcal{H}(\bar\rho + \bar P) \ , \tag{4.2.112}$$

$$\delta\rho' = -3\mathcal{H}(\delta\rho + \delta P) + 3\Phi'(\bar\rho + \bar P) - \boldsymbol{\nabla}\cdot\boldsymbol{q} \ . \tag{4.2.113}$$

The zeroth-order part (4.2.112) simply is the conservation of energy in the homogeneous background. Eq. (4.2.113) describes the evolution of the density perturbation. The first term on the right-hand side is just the dilution due to the background expansion (as in the background equation), the $\boldsymbol{\nabla}\cdot\boldsymbol{q}$ term accounts for the local fluid flow due to peculiar velocity, and the Φ' term is a purely relativistic effect corresponding to the density changes caused by perturbations to the local expansion rate [$(1-\Phi)a$ is the "local scale factor" in the spatial part of the metric in Newtonian gauge].

It is convenient to write the equation in terms of the fractional overdensity and the 3-velocity,

$$\delta \equiv \frac{\delta\rho}{\bar\rho} \qquad \text{and} \qquad \boldsymbol{v} = \frac{\boldsymbol{q}}{\bar\rho + \bar P}\ . \tag{4.2.114}$$

Eq. (4.2.113) then becomes

$$\boxed{\delta' + \left(1+\frac{\bar P}{\bar\rho}\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{v} - 3\Phi'\right) + 3\mathcal{H}\left(\frac{\delta P}{\delta\rho} - \frac{\bar P}{\bar\rho}\right)\delta = 0}\ . \tag{4.2.115}$$

This is the relativistic version of the *continuity equation*. In the limit $P \ll \rho$, we recover the Newtonian continuity equation in conformal time, $\delta' + \boldsymbol{\nabla}\cdot\boldsymbol{v} - 3\Phi' = 0$, but with a general-relativistic correction due to the perturbation to the rate of exansion of space. This correction is small on sub-horizon scales ($k \gg \mathcal{H}$) — we will prove this rigorously in Chapter 5.

Euler Equation

Next, consider the $\nu = i$ component of eq. (4.2.108),

$$\partial_\mu T^\mu{}_i + \Gamma^\mu_{\mu\rho} T^\rho{}_i - \Gamma^\rho{}_{\mu i} T^\mu{}_\rho = 0\ , \tag{4.2.116}$$

and hence

$$\partial_0 T^0{}_i + \partial_j T^j{}_i + \Gamma^\mu_{\mu 0} T^0{}_i + \Gamma^\mu_{\mu j} T^j{}_i - \Gamma^0_{0i} T^0{}_0 - \Gamma^0_{ji} T^j{}_0 - \Gamma^j_{0i} T^0{}_j - \Gamma^j_{ki} T^k{}_j = 0\ . \tag{4.2.117}$$

Using eqs. (4.2.74)–(4.2.77), with $T^0{}_i = -q_i$ in Newtonian gauge, eq. (4.2.117) becomes

$$\begin{aligned} -q_i' + \partial_j\left[-(\bar P + \delta P)\delta_i^j\right] - 4\mathcal{H}q_i - (\partial_j\Psi - 3\partial_j\Phi)\,\bar P\,\delta_i^j - \partial_i\Psi\bar\rho \\ -\mathcal{H}\delta_{ji}q^j + \mathcal{H}\delta_i^j q_j + \underbrace{\left(-2\delta^j_{(i}\partial_{k)}\Phi + \delta_{ki}\delta^{jl}\partial_l\Phi\right)\bar P\,\delta_j^k}_{-3\partial_i\Phi\,\bar P} \;=\; 0\ , \end{aligned} \tag{4.2.118}$$

or

$$-q_i' - \partial_i\delta P - 4\mathcal{H}q_i - (\bar\rho + \bar P)\partial_i\Psi = 0\,. \tag{4.2.119}$$

Using eqs. (4.2.112) and (4.2.114), we get

$$\boxed{\boldsymbol{v}' + \mathcal{H}\boldsymbol{v} - 3\mathcal{H}\frac{\bar P'}{\bar\rho'}\boldsymbol{v} = -\frac{\boldsymbol{\nabla}\delta P}{\bar\rho + \bar P} - \boldsymbol{\nabla}\Psi}\ . \tag{4.2.120}$$

This is the relativistic version of the *Euler equation* for a viscous fluid. Pressure gradients ($\boldsymbol{\nabla}\delta P$) and gravitational infall ($\boldsymbol{\nabla}\Psi$) drive $\boldsymbol{v}'$. The equation captures the redshifting of peculiar velocities ($\mathcal{H}\boldsymbol{v}$) and includes a small correction for relativistic fluids ($\bar P'/\bar\rho'$). Adiabatic fluctuations satisfy $\bar P'/\bar\rho' = c_s^2$. Non-relativistic matter fluctuations have a very small sound speed, so the relativistic correction in the Euler equation (4.2.120) is much smaller than the redshifting term. The limit $P \ll \rho$ then reproduces the Euler equation (4.1.25) of the linearised Newtonian treatment.

Eqs. (4.2.115) and (4.2.120) apply for the total matter and velocity, and *also separately* for any non-interacting components so that the individual stress-energy tensors are separately conserved. Once an equation of state of the matter (and other constitutive relations) are specified, we just need the gravitational potentials Ψ and Φ to close the system of equations. Equations for Ψ and Φ follow from the perturbed Einstein equations.

Perturbed Einstein Equations

Let us now compute the linearised Einstein equation in Newtonian gauge. We require the perturbation to the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, so we first need to calculate the perturbed Ricci tensor $R_{\mu\nu}$ and scalar R.

Ricci tensor.—We recall that the Ricci tensor can be expressed in terms of the connection as

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \ . \tag{4.2.121}$$

Substituting the perturbed connection coefficients (4.2.100)–(4.2.105), we find

$$R_{00} = -3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi' + \Psi') + 3\Phi'' \ , \tag{4.2.122}$$

$$R_{0i} = 2\partial_i \Phi' + 2\mathcal{H}\partial_i \Psi \ , \tag{4.2.123}$$

$$\begin{aligned} R_{ij} = &\left[\mathcal{H}' + 2\mathcal{H}^2 - \Phi'' + \nabla^2\Phi - 2(\mathcal{H}' + 2\mathcal{H}^2)(\Phi + \Psi) - \mathcal{H}\Psi' - 5\mathcal{H}\Phi'\right]\delta_{ij} \\ &+ \partial_i\partial_j(\Phi - \Psi) \ . \end{aligned} \tag{4.2.124}$$

I will derive R_{00} here and leave the others as an exercise.

Example.—The 00 component of the Ricci tensor is

$$R_{00} = \partial_\rho \Gamma^\rho_{00} - \partial_0 \Gamma^\rho_{0\rho} + \Gamma^\alpha_{00}\Gamma^\rho_{\alpha\rho} - \Gamma^\alpha_{0\rho}\Gamma^\rho_{0\alpha} \ . \tag{4.2.125}$$

When we sum over ρ, the terms with $\rho = 0$ cancel so we need only consider summing over $\rho = 1, 2, 3$, i.e.

$$\begin{aligned} R_{00} &= \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{0i} + \Gamma^\alpha_{00}\Gamma^i_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^i_{0\alpha} \\ &= \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{0i} + \Gamma^0_{00}\Gamma^i_{0i} + \underbrace{\Gamma^j_{00}\Gamma^i_{ji}}_{\mathcal{O}(2)} - \underbrace{\Gamma^0_{0i}\Gamma^i_{00}}_{\mathcal{O}(2)} - \Gamma^j_{0i}\Gamma^i_{0j} \\ &= \nabla^2\Psi - 3\partial_0(\mathcal{H} - \Phi') + 3(\mathcal{H} + \Psi')(\mathcal{H} - \Phi') - (\mathcal{H} - \Phi')^2 \delta^j_i \delta^i_j \\ &= -3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi' + \Psi') + 3\Phi'' \ . \end{aligned} \tag{4.2.126}$$

Exercise.—Derive eqs. (4.2.123) and (4.2.124).

Ricci scalar.—It is now relatively straightforward to compute the Ricci scalar

$$R = g^{00} R_{00} + 2 \underbrace{g^{0i} R_{0i}}_{0} + g^{ij} R_{ij} \ . \tag{4.2.127}$$

It follows that

$$\begin{aligned}
a^2 R &= (1-2\Psi)R_{00} - (1+2\Phi)\delta^{ij}R_{ij} \\
&= (1-2\Psi)\left[-3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi'+\Psi') + 3\Phi''\right] \\
&\quad - 3(1+2\Phi)\left[\mathcal{H}' + 2\mathcal{H}^2 - \Phi'' + \nabla^2\Phi - 2(\mathcal{H}'+2\mathcal{H}^2)(\Phi+\Psi) - \mathcal{H}\Psi' - 5\mathcal{H}\Phi'\right] \\
&\quad - (1+2\Phi)\nabla^2(\Phi-\Psi)\ .
\end{aligned} \tag{4.2.128}$$

Dropping non-linear terms, we find

$$a^2 R = -6(\mathcal{H}'+\mathcal{H}^2) + 2\nabla^2\Psi - 4\nabla^2\Phi + 12(\mathcal{H}'+\mathcal{H}^2)\Psi + 6\Phi'' + 6\mathcal{H}(\Psi'+3\Phi')\ . \tag{4.2.129}$$

Einstein tensor.—Computing the Einstein tensor is now just a matter of collecting our previous results. The 00 component is

$$\begin{aligned}
G_{00} &= R_{00} - \frac{1}{2}g_{00}R \\
&= -3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi'+\Psi') + 3\Phi'' + 3(1+2\Psi)(\mathcal{H}'+\mathcal{H}^2) \\
&\quad - \frac{1}{2}\left[2\nabla^2\Psi - 4\nabla^2\Phi + 12(\mathcal{H}'+\mathcal{H}^2)\Psi + 6\Phi'' + 6\mathcal{H}(\Psi'+3\Phi')\right]\ .
\end{aligned} \tag{4.2.130}$$

Most of the terms cancel leaving the simple result

$$G_{00} = 3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi'\ . \tag{4.2.131}$$

The $0i$ component of the Einstein tensor is simply R_{0i} since $g_{0i} = 0$ in Newtonian gauge:

$$G_{0i} = 2\partial_i(\Phi' + \mathcal{H}\Psi)\ . \tag{4.2.132}$$

The remaining components are

$$\begin{aligned}
G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R \\
&= \left[\mathcal{H}' + 2\mathcal{H}^2 - \Phi'' + \nabla^2\Phi - 2(\mathcal{H}'+2\mathcal{H}^2)(\Phi+\Psi) - \mathcal{H}\Psi' - 5\mathcal{H}\Phi'\right]\delta_{ij} + \partial_i\partial_j(\Phi-\Psi) \\
&\quad - 3(1-2\Phi)(\mathcal{H}'+\mathcal{H}^2)\delta_{ij} \\
&\quad + \frac{1}{2}\left[2\nabla^2\Psi - 4\nabla^2\Phi + 12(\mathcal{H}'+\mathcal{H}^2)\Psi + 6\Phi'' + 6\mathcal{H}(\Psi'+3\Phi')\right]\delta_{ij}\ .
\end{aligned} \tag{4.2.133}$$

This neatens up (only a little!) to give

$$\begin{aligned}
G_{ij} &= -(2\mathcal{H}'+\mathcal{H}^2)\delta_{ij} + \left[\nabla^2(\Psi-\Phi) + 2\Phi'' + 2(2\mathcal{H}'+\mathcal{H}^2)(\Phi+\Psi) + 2\mathcal{H}\Psi' + 4\mathcal{H}\Phi'\right]\delta_{ij} \\
&\quad + \partial_i\partial_j(\Phi-\Psi)\ .
\end{aligned} \tag{4.2.134}$$

Einstein Equations

Substituting the perturbed Einstein tensor, metric and stress-energy tensor into the Einstein equation gives the equations of motion for the metric perturbations and the zeroth-order Friedmann equations:

- Let us start with the trace-free part of the ij equation, $G_{ij} = 8\pi G T_{ij}$. Since we have dropped anisotropic stress there is no source on the right-hand side. From eq. (4.2.134), we get

$$\boxed{\partial_{\langle i}\partial_{j\rangle}(\Phi-\Psi) = 0}\ . \tag{4.2.135}$$

Had we kept anisotropic stress, the right-hand side would be $-8\pi G a^2 \Pi_{ij}$. In the absence of anisotropic stress[4] (and assuming appropriate decay at infinity), we get[5]

$$\Phi = \Psi \ . \tag{4.2.136}$$

There is then only one gauge-invariant degree of freedom in the metric. In the following, we will write all equations in terms of Φ.

- Next, we consider the 00 equation, $G_{00} = 8\pi G T_{00}$. Using eq. (4.2.131), we get

$$\begin{aligned} 3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi' &= 8\pi G\, g_{0\mu} T^\mu{}_0 \\ &= 8\pi G \left(g_{00} T^0{}_0 + g_{0i} T^i{}_0 \right) \\ &= 8\pi G a^2 (1+2\Phi)(\bar\rho + \delta\rho) \\ &= 8\pi G a^2 \bar\rho (1 + 2\Phi + \delta) \ . \end{aligned} \tag{4.2.137}$$

The zeroth-order part gives

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \bar\rho \ , \tag{4.2.138}$$

which is just the Friedmann equation. The first-order part of eq. (4.2.137) gives

$$\nabla^2\Phi = 4\pi G a^2 \bar\rho \delta + 8\pi G a^2 \bar\rho \Phi + 3\mathcal{H}\Phi' \ . \tag{4.2.139}$$

which, on using eq. (4.2.138), reduces to

$$\boxed{\nabla^2\Phi = 4\pi G a^2 \bar\rho \delta + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi)} \ . \tag{4.2.140}$$

- Moving on to $0i$ equation, $G_{0i} = 8\pi G T_{0i}$, with

$$T_{0i} = g_{0\mu} T^\mu{}_i = g_{00} T^0{}_i = \bar g_{00} T^0{}_i = -a^2 q_i \ . \tag{4.2.141}$$

It follows that

$$\partial_i(\Phi' + \mathcal{H}\Phi) = -4\pi G a^2 q_i \ . \tag{4.2.142}$$

If we write $q_i = (\bar\rho + \bar P)\partial_i v$ and assume the perturbations decay at infinity, we can integrate eq. (4.2.142) to get

$$\boxed{\Phi' + \mathcal{H}\Phi = -4\pi G a^2 (\bar\rho + \bar P) v} \ . \tag{4.2.143}$$

- Substituting eq. (4.2.143) into the 00 Einstein equation (4.2.140) gives

$$\boxed{\nabla^2\Phi = 4\pi G a^2 \bar\rho \Delta} \ , \qquad \text{where} \quad \bar\rho\Delta \equiv \bar\rho\delta - 3\mathcal{H}(\bar\rho + \bar P) v \ . \tag{4.2.144}$$

[4] In reality, neutrinos develop anisotropic stress after neutrino decoupling (i.e. they do not behave like a perfect fluid). Therefore, Φ and Ψ actually differ from each other by about 10% in the time between neutrino decoupling and matter-radiation equality. After the universe becomes matter-dominated, the neutrinos become unimportant, and Φ and Ψ rapidly approach each other. The same thing happens to photons after photon decoupling, but the universe is then already matter-dominated, so they do not cause a significant $\Phi - \Psi$ difference.

[5] In Fourier space, eq. (4.2.135) becomes

$$\left(k_i k_j - \tfrac{1}{3}\delta_{ij} k^2\right)(\Phi - \Psi) = 0 \ .$$

For finite k, we therefore must have $\Phi = \Psi$. For $k = 0$, $\Phi - \Psi = const.$ would be a solution. However, the constant must be zero, since the mean of the perturbations vanishes.

This is of the form of a *Poisson equation*, but with source density given by the gauge-invariant variable Δ of eq. (4.2.87) since $B = 0$ in the Newtonian gauge. Let us introduce *comoving hypersurfaces* as those that are orthogonal to the worldlines of a set of observers comoving with the total matter (i.e. they see $q^i = 0$) and are the constant-time hypersurfaces in the *comoving gauge* for which $q^i = 0$ and $B_i = 0$. It follows that Δ is the fractional overdensity in the comoving gauge and we see from eq. (4.2.144) that this is the source term for the gravitational potential Φ.

- Finally, we consider the trace-part of the ij equation, i.e. $G^i{}_i = 8\pi G T^i{}_i$. We compute the left-hand side from eq. (4.2.134) (with $\Phi = \Psi$),

$$\begin{aligned} G^i{}_i &= g^{i\mu} G_{\mu i} \\ &= g^{ik} G_{ki} \\ &= -a^{-2}(1+2\Phi)\delta^{ik}\left[-(2\mathcal{H}' + \mathcal{H}^2)\delta_{ki} + \left(2\Phi'' + 6\mathcal{H}\Phi' + 4(2\mathcal{H}' + \mathcal{H}^2)\Phi\right)\delta_{ki}\right] \\ &= -3a^{-2}\left[-(2\mathcal{H}' + \mathcal{H}^2) + 2\left(\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi\right)\right] . \end{aligned} \tag{4.2.145}$$

We combine this with $T^i{}_i = -3(\bar{P} + \delta P)$. At zeroth order, we find

$$2\mathcal{H}' + \mathcal{H}^2 = -8\pi G a^2 \bar{P} , \tag{4.2.146}$$

which is just the second Friedmann equation. At first order, we get

$$\boxed{\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P} . \tag{4.2.147}$$

Of course, the Einstein equations and the energy and momentum conservation equations form a redundant (but consistent!) set of equations because of the Bianchi identity. We can use whichever subsets are most convenient for the particular problem at hand.

4.3 Conserved Curvature Perturbation

There is an important quantity that is *conserved* on super-Hubble scales for adiabatic fluctuations irrespective of the equation of state of the matter: the *comoving curvature perturbation*. As we will see below, the comoving curvature perturbation provides the essential link between the fluctuations that we observe in the late-time universe (Chapter 5) and the primordial seed fluctuations created by inflation (Chapter 6).

4.3.1 Comoving Curvature Perturbation

In some arbitrary gauge, let us work out the *intrinsic curvature* of surfaces of constant time. The *induced metric*, γ_{ij}, on these surfaces is just the spatial part of eq. (4.2.41), i.e.

$$\gamma_{ij} \equiv a^2\left[(1+2C)\delta_{ij} + 2E_{ij}\right] . \tag{4.3.148}$$

where $E_{ij} \equiv \partial_{\langle i}\partial_{j\rangle}E$ for scalar perturbations. In a tedious, but straightforward computation, we derive the three-dimensional Ricci scalar associated with γ_{ij},

$$a^2\, R_{(3)} = -4\nabla^2\left(C - \frac{1}{3}\nabla^2 E\right) . \tag{4.3.149}$$

In the following insert I show all the steps.

Derivation.—The connection corresponding to γ_{ij} is

$$ {}^{(3)}\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right) \ , \tag{4.3.150} $$

where γ^{ij} is the inverse of the induced metric,

$$ \gamma^{ij} = a^{-2}\left[(1-2C)\delta^{ij} - 2E^{ij}\right] = a^{-2}\delta^{ij} + \mathcal{O}(1) \ . \tag{4.3.151} $$

In order to compute the connection to first order, we actually only need the inverse metric to zeroth order, since the spatial derivatives of the γ_{ij} are all first order in the perturbations. We have

$$ \begin{aligned} {}^{(3)}\Gamma^i_{jk} &= \delta^{il}\partial_j\left(C\delta_{kl} + E_{kl}\right) + \delta^{il}\partial_k\left(C\delta_{jl} + E_{jl}\right) - \delta^{il}\partial_l\left(C\delta_{jk} + E_{jk}\right) \\ &= 2\delta^i_{(j}\partial_{k)}C - \delta^{il}\delta_{jk}\partial_l C + 2\partial_{(j}E_{k)}{}^i - \delta^{il}\partial_l E_{jk} \ . \end{aligned} \tag{4.3.152} $$

The intrinsic curvature is the associated Ricci scalar, given by

$$ R_{(3)} = \gamma^{ik}\partial_l {}^{(3)}\Gamma^l_{ik} - \gamma^{ik}\partial_k {}^{(3)}\Gamma^l_{il} + \gamma^{ik}\,{}^{(3)}\Gamma^l_{ik}\,{}^{(3)}\Gamma^m_{lm} - \gamma^{ik}\,{}^{(3)}\Gamma^m_{il}\,{}^{(3)}\Gamma^l_{km} \ . \tag{4.3.153} $$

To first order, this reduces to

$$ a^2\,R_{(3)} = \delta^{ik}\partial_l {}^{(3)}\Gamma^l_{ik} - \delta^{ik}\partial_k {}^{(3)}\Gamma^l_{il} \ . \tag{4.3.154} $$

This involves two contractions of the connection. The first is

$$ \begin{aligned} \delta^{ik\,(3)}\Gamma^l_{ik} &= \delta^{ik}\Big(2\delta^l_{(i}\partial_{k)}C - \delta^{jl}\delta_{ik}\partial_j C\Big) + \delta^{ik}\Big(2\partial_{(i}E_{k)}{}^l - \delta^{jl}\partial_j E_{ik}\Big) \\ &= 2\delta^{kl}\partial_k C - 3\delta^{jl}\partial_j C + 2\partial_i E^{il} - \delta^{jl}\partial_j \underbrace{\left(\delta^{ik}E_{ik}\right)}_{0} \\ &= -\delta^{kl}\partial_k C + 2\partial_k E^{kl} \ . \end{aligned} \tag{4.3.155} $$

The second is

$$ \begin{aligned} {}^{(3)}\Gamma^l_{il} &= \delta^l_l\partial_i C + \delta^l_i\partial_l C - \partial_i C + \partial_l E_i{}^l + \partial_i E_l{}^l - \partial_l E_i{}^l \\ &= 3\,\partial_i C \ . \end{aligned} \tag{4.3.156} $$

Eq. (4.3.154) therefore becomes

$$ \begin{aligned} a^2\,R_{(3)} &= \partial_l\left(-\delta^{kl}\partial_k C + 2\partial_k E^{kl}\right) - 3\delta^{ik}\partial_k\partial_i C \\ &= -\nabla^2 C + 2\partial_i\partial_j E^{ij} - 3\nabla^2 C \\ &= -4\nabla^2 C + 2\partial_i\partial_j E^{ij} \ . \end{aligned} \tag{4.3.157} $$

Note that this vanishes for vector and tensor perturbations (as do all perturbed scalars) since then $C = 0$ and $\partial_i\partial_j E^{ij} = 0$. For scalar perturbations, $E_{ij} = \partial_{\langle i}\partial_{j\rangle}E$ so

$$ \begin{aligned} \partial_i\partial_j E^{ij} &= \delta^{il}\delta^{jm}\partial_i\partial_j\left(\partial_l\partial_m E - \frac{1}{3}\delta_{lm}\nabla^2 E\right) \\ &= \nabla^2\nabla^2 E - \frac{1}{3}\nabla^2\nabla^2 E \\ &= \frac{2}{3}\nabla^4 E \ . \end{aligned} \tag{4.3.158} $$

Finally, we get eq. (4.3.149).

We define the *curvature perturbation* as $C - \frac{1}{3}\nabla^2 E$. The *comoving curvature perturbation* $\mathcal{R}$

is the curvature perturbation evaluated in the comoving gauge ($B_i = 0 = q^i$). It will prove convenient to have a gauge-invariant expression for $\mathcal{R}$, so that we can evaluate it from the perturbations in any gauge (for example, in Newtonian gauge). Since B and v vanish in the comoving gauge, we can always add linear combinations of these to $C - \frac{1}{3}\nabla^2 E$ to form a gauge-invariant combination that equals $\mathcal{R}$. Using eqs. (4.2.58)–(4.2.60) and (4.2.85), we see that the correct gauge-invariant expression for the comoving curvature perturbation is

$$\boxed{\mathcal{R} = C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B+v)}\ . \tag{4.3.159}$$

Exercise.—Show that $\mathcal{R}$ is gauge-invariant.

4.3.2 A Conservation Law

We now want to prove that the comoving curvature perturbation $\mathcal{R}$ is indeed conserved on large scales and for adiabatic perturbations. We shall do so by working in the *Newtonian gauge*, in which case

$$\mathcal{R} = -\Phi + \mathcal{H}v\ , \tag{4.3.160}$$

since $B = E = 0$ and $C \equiv -\Phi$. We can use the $0i$ Einstein equation (4.2.143) to eliminate the peculiar velocity in favour of the gravitational potential and its time derivative:

$$\mathcal{R} = -\Phi - \frac{\mathcal{H}(\Phi' + \mathcal{H}\Phi)}{4\pi G a^2(\bar\rho + \bar P)}\ . \tag{4.3.161}$$

Taking a time derivative of (4.3.161) and using the evolution equations of the previous section, we find

$$\boxed{-4\pi G a^2(\bar\rho + \bar P)\,\mathcal{R}' \;=\; 4\pi G a^2 \mathcal{H}\,\delta P_{\rm nad} + \mathcal{H}\frac{\bar P'}{\bar\rho'}\nabla^2\Phi}\ , \tag{4.3.162}$$

where we have defined the *non-adiabatic pressure perturbation*

$$\delta P_{\rm nad} \equiv \delta P - \frac{\bar P'}{\bar\rho'}\,\delta\rho\ . \tag{4.3.163}$$

*Derivation.**—We differentiate eq. (4.3.161) to find

$$\begin{aligned} -4\pi G a^2(\bar\rho + \bar P)\,\mathcal{R}' \;&=\; 4\pi G a^2(\bar\rho + \bar P)\Phi' + \mathcal{H}'(\Phi' + \mathcal{H}\Phi) + \mathcal{H}(\Phi'' + \mathcal{H}'\Phi + \mathcal{H}\Phi') \\ &\quad + \mathcal{H}^2(\Phi' + \mathcal{H}\Phi) + 3\mathcal{H}^2\frac{\bar P'}{\bar\rho'}(\Phi' + \mathcal{H}\Phi)\ , \end{aligned} \tag{4.3.164}$$

where we used $\bar\rho' = -3\mathcal{H}(\bar\rho + \bar P)$. This needs to be cleaned up a bit. In the first term on the right, we use the Friedmann equation to write $4\pi G a^2(\bar\rho + \bar P)$ as $\mathcal{H}^2 - \mathcal{H}'$. In the last term, we use the Poisson equation (4.2.140) to write $3\mathcal{H}(\Phi' + \mathcal{H}\Phi)$ as $(\nabla^2\Phi - 4\pi G a^2\bar\rho\delta)$. We then find

$$\begin{aligned} -4\pi G a^2(\bar\rho + \bar P)\,\mathcal{R}' \;&=\; (\mathcal{H}^2 - \mathcal{H}')\Phi' + \mathcal{H}'(\Phi' + \mathcal{H}\Phi) + \mathcal{H}(\Phi'' + \mathcal{H}'\Phi + \mathcal{H}\Phi') \\ &\quad + \mathcal{H}^2(\Phi' + \mathcal{H}\Phi) + \mathcal{H}\frac{\bar P'}{\bar\rho'}\left(\nabla^2\Phi - 4\pi G a^2\bar\rho\delta\right)\ . \end{aligned} \tag{4.3.165}$$

Adding and subtracting $4\pi G a^2 \mathcal{H}\delta P$ on the right-hand side and simplifying gives

$$\begin{aligned} -4\pi G a^2(\bar\rho+\bar P)\,\mathcal{R}' &= \mathcal{H}\left[\Phi''+3\mathcal{H}\Phi'+(2\mathcal{H}'+\mathcal{H}^2)\Phi-4\pi G a^2\delta P\right] \\ &\quad + 4\pi G a^2\mathcal{H}\,\delta P_{\rm nad}+\mathcal{H}\frac{\bar P'}{\bar\rho'}\nabla^2\Phi\ , \end{aligned} \tag{4.3.166}$$

where $\delta P_{\rm nad}$ was defined in (4.3.163). The first term on the right-hand side vanishes by eq. (4.2.147), so we obtain eq. (4.3.162).

Exercise.—Show that $\delta P_{\rm nad}$ is gauge-invariant.

The non-adiabatic pressure $\delta P_{\rm nad}$ vanishes for a barotropic equation of state, $P = P(\rho)$ (and, more generally, for adiabatic fluctuations in a mixture of barotropic fluids). In that case, the right-hand side of eq. (4.3.162) scales as $\mathcal{H}k^2\Phi \sim \mathcal{H}k^2\mathcal{R}$, so that

$$\frac{d\ln\mathcal{R}}{d\ln a} \sim \left(\frac{k}{\mathcal{H}}\right)^2\ . \tag{4.3.167}$$

Hence, we find that $\mathcal{R}$ doesn't evolve on super-Hubble scales, $k \ll \mathcal{H}$. This means that the value of $\mathcal{R}$ that we will compute at horizon crossing during inflation (Chapter 6) survives unaltered until later times.

4.4 Summary

We have derived the linearised evolution equations for scalar perturbations in Newtonian gauge, where the metric has the following form

$$ds^2 = a^2(\tau)\left[(1+2\Psi)d\tau^2-(1-2\Phi)\delta_{ij}dx^i dx^j\right]\ . \tag{4.4.168}$$

In these lectures, we won't encounter situations where anisotropic stress plays a significant role, so we will always be able to set $\Psi = \Phi$.

- The Einstein equations then are

$$\nabla^2\Phi - 3\mathcal{H}(\Phi'+\mathcal{H}\Phi) = 4\pi G a^2\,\delta\rho\ , \tag{4.4.169}$$

$$\Phi'+\mathcal{H}\Phi = -4\pi G a^2(\bar\rho+\bar P)v\ , \tag{4.4.170}$$

$$\Phi''+3\mathcal{H}\Phi'+(2\mathcal{H}'+\mathcal{H}^2)\Phi = 4\pi G a^2\,\delta P\ . \tag{4.4.171}$$

  The source terms on the right-hand side should be interpreted as the sum over all relevant matter components (e.g. photons, dark matter, baryons, etc.). The Poisson equation takes a particularly simple form if we introduce the comoving gauge density contrast

$$\nabla^2\Phi = 4\pi G a^2\bar\rho\,\Delta\ . \tag{4.4.172}$$

- From the conservation of the stress-tensor, we derived the relativistic generalisations of the continuity equation and the Euler equation

$$\delta'+3\mathcal{H}\left(\frac{\delta P}{\delta\rho}-\frac{\bar P}{\bar\rho}\right)\delta = -\left(1+\frac{\bar P}{\bar\rho}\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{v}-3\Phi'\right)\ , \tag{4.4.173}$$

$$\boldsymbol{v}'+3\mathcal{H}\left(\frac{1}{3}-\frac{\bar P'}{\bar\rho'}\right)\boldsymbol{v} = -\frac{\boldsymbol{\nabla}\delta P}{\bar\rho+\bar P}-\boldsymbol{\nabla}\Phi\ . \tag{4.4.174}$$

These equations apply for the total matter and velocity, and also separately for any non-interacting components so that the individual stress-energy tensors are separately conserved.

- A very important quantity is the comoving curvature perturbation

$$\mathcal{R} = -\Phi - \frac{\mathcal{H}(\Phi' + \mathcal{H}\Phi)}{4\pi G a^2(\bar{\rho} + \bar{P})} \,. \tag{4.4.175}$$

We have shown that $\mathcal{R}$ doesn't evolve on super-Hubble scales, $k \ll \mathcal{H}$, unless non-adiabatic pressure is significant. This fact is crucial for relating late-time observables, such as the distributions of galaxies (Chapter 5), to the initial conditions from inflation (Chapter 6).