5 Structure Formation

In the previous chapter, we derived the evolution equations for all matter and metric perturbations. In principle, we could now solve these equations. The complex interactions between the different species (see fig. 5.1) means that we get a large number of coupled differential equations. This set of equations is easy to solve numerically and this is what is usually done. However, our goal in this chapter is to obtain some analytical insights into the basic qualitative features of the solutions.

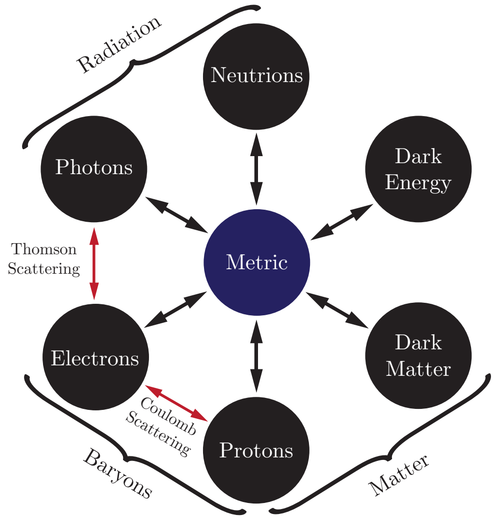

Figure 5.1: *Interactions between the different forms of matter in the universe.*

5.1 Initial Conditions

Any mode of interest for observations today was outside the Hubble radius if we go back sufficiently far into the past. Inflation sets the initial condition for these superhorizon modes. The prediction from inflation (see Ch. 6) is presented most conveniently in terms of a spectrum of fluctuations for the curvature perturbation $\mathcal{R}$. Eq. (4.4.175) relates this to the gravitational potential Φ in Newtonian gauge

$$\mathcal{R} = -\Phi - \frac{2}{3(1+w)}\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right) , \tag{5.1.1}$$

where w is the equation of state of the background. For adiabatic perturbations, we have $c_s^2 \approx w$ and a combination of Einstein equations imply a closed form evolution equation for the gravitational potential

$$\boxed{\Phi'' + 3(1+w)\mathcal{H}\Phi' + wk^2\Phi = 0} \, . \tag{5.1.2}$$

Notice that in deriving (5.1.2) we have assumed a constant equation of state. It therefore only applies if a single component dominates the universe. For the more general case, you should consult (4.4.171).

Exercise.—Derive eq. (5.1.2) from the Einstein equations.

5.1.1 Superhorizon Limit

On superhorizon scales, $k \ll \mathcal{H}$, we can drop the last term in (5.1.2). The growing-mode solution then is

$$\Phi = const. \qquad \text{(superhorizon)}\ . \tag{5.1.3}$$

Notice that this superhorizon solution is independent of the equation of state w (as long as $w = const.$). In particular, the gravitational potential is frozen outside the horizon during both the radiation and matter eras.

The Poisson equation (4.4.169) relates the gravitational potential to the *total* Newtonian-gauge density contrast

$$\delta = -\frac{2}{3}\frac{k^2}{\mathcal{H}^2}\,\Phi - \frac{2}{\mathcal{H}}\,\Phi' - 2\Phi\ , \tag{5.1.4}$$

where we have used $\frac{3}{2}\mathcal{H}^2 = 4\pi G a^2 \bar{\rho}$. On superhorizon scales, only the decaying mode contributes to Φ'. The first and second term in (5.1.4) then are of the same order and both are much smaller than the third term. We therefore get

$$\delta \approx -2\Phi = const.\ , \tag{5.1.5}$$

so δ is also frozen on superhorizon scales. For adiabatic initial conditions, we can relate the primordial potential Φ to the fluctuations in both the matter and the radiation:

$$\delta_m = \frac{3}{4}\delta_r \approx -\frac{3}{2}\Phi_{\rm RD}\ , \tag{5.1.6}$$

where we have used that $\delta_r \approx \delta$ for adiabatic perturbations during the radiation era. On superhorizon scales, the density perturbations are therefore simply proportional to the curvature perturbation set up by inflation.

5.1.2 Radiation-to-Matter Transition

We have seen that the gravitational potential is frozen on superhorizon scales as long as the equation of state of the background is constant. However, unlike the curvature perturbation $\mathcal{R}$, the gravitational doesn't stay constant when the equation of state changes. To follow the evolution of Φ through the radiation-to-matter transition, we exploit the conservation of $\mathcal{R}$.

In the superhorizon limit, the comoving curvature perturbation (4.4.175) becomes

$$\boxed{\mathcal{R} = -\frac{5+3w}{3+3w}\,\Phi} \qquad \text{(superhorizon)}\ . \tag{5.1.7}$$

This provides an important link between the source term for the evolution of fluctuations (Φ) and the primordial initial conditions set up by inflation ($\mathcal{R}$). Evaluating (5.1.7) for $w = \frac{1}{3}$ and

$w = 0$ relates the amplitudes of Φ during the radiation era and the matter eta

$$\mathcal{R} = -\frac{3}{2}\Phi_{\rm RD} = -\frac{5}{3}\Phi_{\rm MD} \quad \Rightarrow \quad \Phi_{\rm MD} = \frac{9}{10}\Phi_{\rm RD} \ , \tag{5.1.8}$$

where we have used that $\mathcal{R} = const.$ throughout. We see that the gravitational potential decreases by a factor of 9/10 in the transition from radiation-dominated to matter-dominated.

5.2 Evolution of Fluctuations

We wish to understand what happens to the superhorizon initial conditions, when modes enter the horizon. We will first study the evolution of the gravitational potential (§5.2.1), and then the perturbations in radiation (§5.2.2), matter (§5.2.3) and baryons (§5.2.4).

5.2.1 Gravitational Potential

To determine the evolution of Φ during both the radiation era and the matter era, we simply have to specialise (5.1.2) to $w = \frac{1}{3}$ and $w = 0$, respectively.

Radiation Era

In the radiation era, $w = \frac{1}{3}$, we get

$$\Phi'' + \frac{4}{\tau}\Phi' + \frac{k^2}{3}\Phi = 0 \ . \tag{5.2.9}$$

This equation has the following exact solution

$$\Phi_{\boldsymbol{k}}(\tau) = A_{\boldsymbol{k}}\,\frac{j_1(x)}{x} + B_{\boldsymbol{k}}\,\frac{n_1(x)}{x} \ , \qquad x \equiv \frac{1}{\sqrt{3}}k\tau \ , \tag{5.2.10}$$

where the subscript $\boldsymbol{k}$ indicates that the solution can have different amplitudes for each value of $\boldsymbol{k}$. The size of the initial fluctuations as a function of wavenumber will be a prediction of inflation. The functions $j_1(x)$ and $n_1(x)$ in (5.2.10) are the spherical Bessel and Neumann functions

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} = \frac{x}{3} + \mathcal{O}(x^3) \ , \tag{5.2.11}$$

$$n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} = -\frac{1}{x^2} + \mathcal{O}(x^0) \ . \tag{5.2.12}$$

Since $n_1(x)$ blows up for small x (early times), we reject that solution on the basis of initial conditions, i.e. we set $B_{\boldsymbol{k}} \equiv 0$. We match the constant $A_{\boldsymbol{k}}$ to the primordial value of the potential, $\Phi_{\boldsymbol{k}}(0) = -\frac{2}{3}\mathcal{R}_{\boldsymbol{k}}(0)$. Using (5.2.11), we find

$$\Phi_{\boldsymbol{k}}(\tau) = -2\mathcal{R}_{\boldsymbol{k}}(0)\left(\frac{\sin x - x\cos x}{x^3}\right) \qquad \text{(all scales)} \ . \tag{5.2.13}$$

Notice that (5.2.13) is valid on all scales. Outside the (sound) horizon, $x = \frac{1}{\sqrt{3}}k\tau \ll 1$, the solution approaches $\Phi = const.$, while on subhorizon scales, $x \gg 1$, we get

$$\Phi_{\boldsymbol{k}}(\tau) \approx -6\mathcal{R}_{\boldsymbol{k}}(0)\,\frac{\cos\left(\frac{1}{\sqrt{3}}k\tau\right)}{(k\tau)^2} \qquad \text{(subhorizon)} \ . \tag{5.2.14}$$

During the radiation era, subhorizon modes of Φ therefore oscillate with frequency $\frac{1}{\sqrt{3}}k$ and an amplitude that decays as $\tau^{-2} \propto a^{-2}$ (see fig. 5.2). Remember this.

Matter Era

In the matter era, $w = 0$, the evolution of the potential is

$$\Phi'' + \frac{6}{\tau}\Phi' = 0 \ , \tag{5.2.15}$$

whose solution is

$$\Phi \propto \begin{cases} const. \\ \tau^{-5} \propto a^{-5/2} \end{cases} . \tag{5.2.16}$$

We conclude that the gravitational potential is frozen on *all scales* during matter domination.

Summary

Fig. 5.2 shows the evolution of the gravitational potential for difference wavelengths. As predicted, the potential is constant when the modes are outside the horizon. Two of the modes enter the horizon during the radiation era. While they are inside the horizon during the radiation era their amplitudes decrease as a^{-2}. The resulting amplitudes in the matter era are therefore strongly suppressed. During the matter era the potential is constant on all scales. The longest wavelength mode in the figure enters the horizon during the matter era, so its amplitude is only suppressed by the factor of $\frac{9}{10}$ coming from the radiation-to-matter transition.

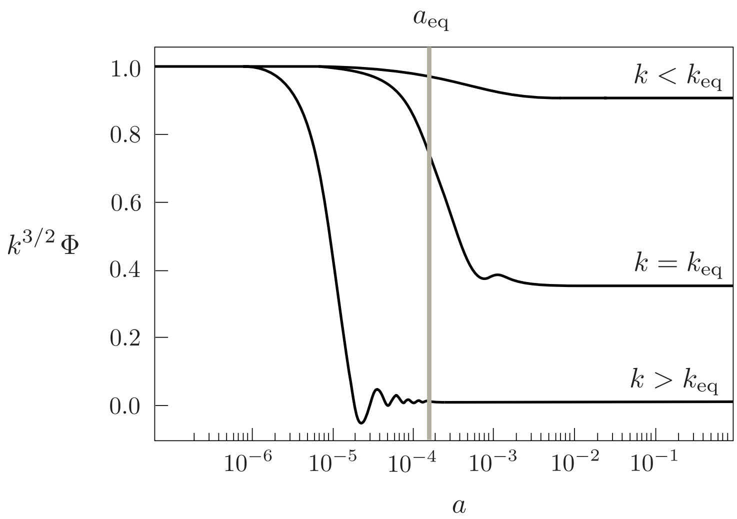

Figure 5.2: *Numerical solutions for the linear evolution of the gravitational potential.*

5.2.2 Radiation

In this section, we wish to determine the evolution of perturbations in the radiation density.

Radiation Era

In the radiation era, perturbations in the radiation density dominate (for adiabatic initial conditions). Given the solution (5.2.13) for Φ during the radiation era, we therefore immediately obtain a solution for the density contrast of radiation (δ_r or Δ_r) via the Poisson equation

$$\delta_r = -\frac{2}{3}(k\tau)^2\Phi - 2\tau\Phi' - 2\Phi \ , \tag{5.2.17}$$

$$\Delta_r = -\frac{2}{3}(k\tau)^2\Phi \ . \tag{5.2.18}$$

We see that while δ_r is constant outside the horizon, Δ_r grows as $\tau^2 \propto a^2$. Inside the horizon,[1]

$$\delta_r \approx \Delta_r = -\frac{2}{3}(k\tau)^2\Phi = 4\mathcal{R}(0)\cos\left(\frac{1}{\sqrt{3}}k\tau\right) \ , \tag{5.2.19}$$

which is the solution to

$$\boxed{\delta_r'' - \frac{1}{3}\nabla^2\delta_r = 0} \ . \tag{5.2.20}$$

We see that subhorizon fluctuations in the radiation density oscillate with constant amplitude around $\delta_r = 0$.

Matter Era

In the matter era, radiation perturbations are subdominant. Their evolution then has to be determined from the conservation equations. On subhorizon scales, we have

$$\left.\begin{array}{ll} \text{(C)} & \delta_r' = -\frac{4}{3}\boldsymbol{\nabla}\cdot\boldsymbol{v}_r \\[2ex] \text{(E)} & \boldsymbol{v}_r' = -\frac{1}{4}\boldsymbol{\nabla}\delta_r - \boldsymbol{\nabla}\Phi \end{array}\right\} \quad \boxed{\delta_r'' - \frac{1}{3}\nabla^2\delta_r = \frac{4}{3}\nabla^2\Phi} = const. \tag{5.2.21}$$

This is the equation of motion of a harmonic oscillator with constant driving force. During the matter era, the subhorizon fluctuations in the radiation density therefore oscillate with constant amplitude around a shifted equilibrium point, $\delta_r = -4\Phi_{\rm MD}(k)$. Here, $\Phi_{\rm MD}(k)$ is the k-dependent amplitude of the gravitational potential in the matter era; cf. fig. 5.2.

Summary

The *acoustic oscillations* in the perturbed radiation density are what gives rise to the peaks in the spectrum of CMB anisotropies (see fig. 6.3 in §6.5.4) This will be analysed in much more detail in the *Advanced Cosmology* course next term.

5.2.3 Dark Matter

In this section, we are interested in the evolution of matter fluctuations from early times (during the radiation era) until late times (when dark energy starts to dominate).

Early Times

At early times, the universe was dominated by a mixture of radiation (r) and pressureless matter (m). For now, we ignore baryons (but see §5.2.4). The conformal Hubble parameter is

$$\mathcal{H}^2 = \frac{\mathcal{H}_0^2\Omega_m^2}{\Omega_r}\left(\frac{1}{y} + \frac{1}{y^2}\right) \ , \qquad y \equiv \frac{a}{a_{\rm eq}} \ . \tag{5.2.22}$$

[1] We see that well inside the horizon, the density perturbations in the comoving and Newtonian gauge coincide. This is indicative of the general result that there are no gauge ambiguities inside the horizon.

We wish to determine how matter fluctuations evolve on subhorizon scales from the radiation era until the matter era. We consider the evolution equations for the matter density contrast and velocity:

$$\left.\begin{aligned} &(\mathrm{C}) \quad \delta_m' = -\boldsymbol{\nabla}\cdot\boldsymbol{v}_m \\ &(\mathrm{E}) \quad \boldsymbol{v}_m' = -\mathcal{H}\boldsymbol{v}_m - \boldsymbol{\nabla}\Phi \end{aligned}\right\} \quad \delta_m'' + \mathcal{H}\delta_m' = \nabla^2\Phi\ . \tag{5.2.23}$$

In general, the potential Φ is sourced by the total density fluctuation. However, we have seen that perturbations in the radiation density oscillate rapidly on small scales. The *time-averaged* gravitational potential is therefore only sourced by the matter fluctuations, and the fluctuations in the radiation can be neglected (see Weinberg, astro-ph/0207375 for further discussion). The evolution of the matter perturbations then satisfies

$$\delta_m'' + \mathcal{H}\delta_m' - 4\pi G a^2 \bar{\rho}_m \delta_m \approx 0\ , \tag{5.2.24}$$

where $\mathcal{H}$ given by (5.2.22). On Problem Set 3, you will show that this equation can be written as the *Mészáros equation*

$$\boxed{\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} - \frac{3}{2y(1+y)}\delta_m = 0}\ . \tag{5.2.25}$$

You will also be asked to show that the solutions to this equation take the form

$$\delta_m \propto \begin{cases} 2+3y \\ (2+3y)\ln\left(\dfrac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 6\sqrt{1+y} \end{cases} .$$

In the limit $y \ll 1$ (RD), the growing mode solution is $\delta_m \propto \ln y \propto \ln a$, confirming the logarithmic growth of matter fluctuations in the radiation era. In the limit $y \gg 1$ (MD), we reproduce the expected solution in the matter era: $\delta_m \propto y \propto a$. Table 5.1 summarises the analytical limits for the evolution of the potential Φ and the matter density contrasts δ_m and Δ_m.

		RD		MD	
		Φ	δ_m (Δ_m)	Φ	δ_m (Δ_m)
$k \gg k_{\rm eq}$:	superhorizon	*const.*	*const.* (a^2)	–	–
	subhorizon	a^{-2}	$\ln a$	*const.*	a
$k \ll k_{\rm eq}$:	superhorizon	*const.*	*const.* (a^2)	*const.*	*const.* (a)
	subhorizon	–	–	*const.*	a

Table 5.1: *Analytical limits of the solutions for the potential Φ and the matter density contrasts δ_m and Δ_m.*

Intermediate Times

The solution in the matter era also follows directly from the solution (5.2.16) for the gravitational potential, which determines the comoving density contrast

$$\Delta_m = \frac{\nabla^2 \Phi}{4\pi G a^2 \bar{\rho}} \propto \begin{cases} a \\ a^{-3/2} \end{cases} , \tag{5.2.26}$$

just as in the Newtonian treatment [cf. eq. (4.1.30)], but now valid on all scales. Notice that the growing mode of Δ_m grows as a outside the horizon, while δ_m is constant. Inside the horizon, $\delta_m \approx \Delta_m$ and the density contrasts in both gauges evolve as a.

Late Times

At late times, the universe is a mixture of pressureless matter (m) and dark energy (Λ). Since dark energy doesn't have fluctuations, we still have

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho}_m \Delta_m \ . \tag{5.2.27}$$

Pressure fluctuations are negligible, so the Einstein equations give

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 0 \ . \tag{5.2.28}$$

To get an evolution equation for Δ_m, we use a neat trick. Since $a^2\bar{\rho}_m \propto a^{-1}$, we have $\Phi \propto \Delta_m/a$. Hence, eq. (5.2.28) implies

$$\partial_\tau^2(\Delta_m/a) + 3\mathcal{H}\partial_\tau(\Delta_m/a) + (2\mathcal{H}' + \mathcal{H}^2)(\Delta_m/a) = 0 \ , \tag{5.2.29}$$

which rearranges to

$$\Delta_m'' + \mathcal{H}\Delta_m' + (\mathcal{H}' - \mathcal{H}^2)\Delta_m = 0 \ . \tag{5.2.30}$$

> *Exercise.*—Show that (5.2.30) follows from (5.2.29). Use the Friedmann and conservation equations to show that
>
> $$\mathcal{H}' - \mathcal{H}^2 = -4\pi G a^2(\bar{\rho} + \bar{P}) = -4\pi G a^2 \bar{\rho}_m \ . \tag{5.2.31}$$

Using (5.2.31), eq. (5.2.30) becomes

$$\boxed{\Delta_m'' + \mathcal{H}\Delta_m' - 4\pi G a^2 \bar{\rho}_m \Delta_m = 0} \ . \tag{5.2.32}$$

This is the conformal-time version of the Newtonian equation (4.1.36), but now valid on all scales. So we recover the usual suppression of the growth of structure by Λ, but now on all scales (see also Problem Set 3).

Summary

Fig. 5.3 shows the evolution of the matter density contrast δ_m for the same modes as in fig. 5.2. Fluctuations are frozen until they enter the horizon. Subhorizon matter fluctuations in the radiation era only grow logarithmically, $\delta_m \propto \ln a$. This changes to power-law growth, $\delta_m \propto a$

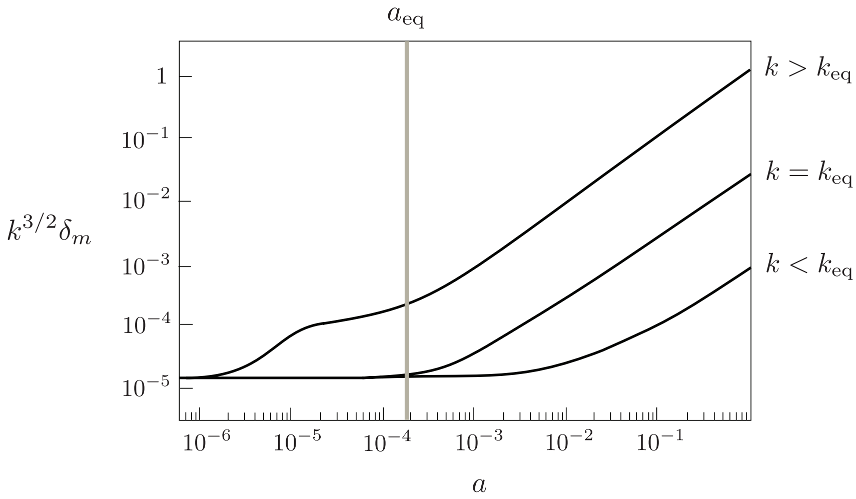

Figure 5.3: *Evolution of the matter density contrast for the same modes as in fig. 5.2.*

when the universe becomes matter dominated. When the universe becomes dominated by dark energy, perturbations stop growing.

The effects we discussed above lead to a post-processing of the primordial perturbations. This evolution is often encoded in the so-called *transfer function*. For example, the value of the matter perturbation at redshift z is related to the primordial perturbation $\mathcal{R}_{\boldsymbol{k}}$ by

$$\Delta_{m,\boldsymbol{k}}(z) = T(k,z)\,\mathcal{R}_{\boldsymbol{k}} \ . \tag{5.2.33}$$

The transfer function $T(k,z)$ depends only on the magnitude k and not on the direction of $\boldsymbol{k}$, because the perturbations are evolving on a homogeneous and isotropic background. The square of the Fourier mode (5.2.33) defines that matter *power spectrum*

$$P_\Delta(k,z) \equiv |\Delta_{m,\boldsymbol{k}}(z)|^2 = T^2(k,z)\,|\mathcal{R}_{\boldsymbol{k}}|^2 \ . \tag{5.2.34}$$

Fig. 5.4 shows predicted matter power spectrum for scale-invariant initial conditions, $k^3|\mathcal{R}_{\boldsymbol{k}}|^2 =$ *const*. (see Chapter 6).

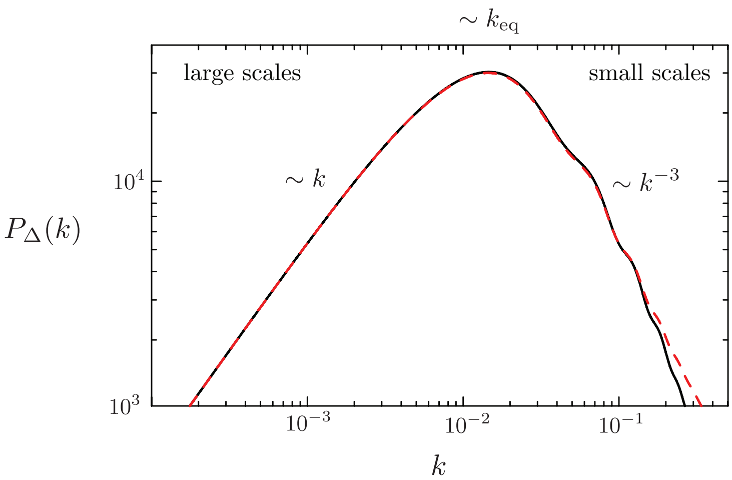

Figure 5.4: *The matter power spectrum $P_\Delta(k)$ at $z = 0$ in linear theory (solid) and with non-linear corrections (dashed). On large scales, $P_\Delta(k)$ grows as k. The power spectrum turns over around $k_{\rm eq} \sim 0.01\,{\rm Mpc}^{-1}$ corresponding to the horizon size at matter-radiation equality. Beyond the peak, the power falls as k^{-3}. Visible are small amplitude baryon acoustic oscillations in the spectrum.*

Exercise.—Explain the asymptotic scalings of the matter power spectrum

$$P_\Delta(k) = \begin{cases} k & k < k_{\rm eq} \\ k^{-3} & k > k_{\rm eq} \end{cases} . \tag{5.2.35}$$

5.2.4 Baryons*

Let us say a few (non-examinable!) words about the evolution of baryons.

Before Decoupling

At early times, $z > z_{dec} \approx 1100$, photons and baryons are coupled strongly to each other via Compton scattering. We can therefore treat the photons and baryons a single fluid, with $\boldsymbol{v}_\gamma = \boldsymbol{v}_b$ and $\delta_\gamma = \frac{4}{3}\delta_b$. The pressure of the photons supports oscillations on small scales (see fig. 5.5). Since the dark matter density contrast δ_c grows like a after matter-radiation equality, it follows that just after decoupling, $\delta_c \gg \delta_b$. Subsequently, the baryons fall into the potential wells sourced mainly by the dark matter and $\delta_b \to \delta_c$ as we shall now show.

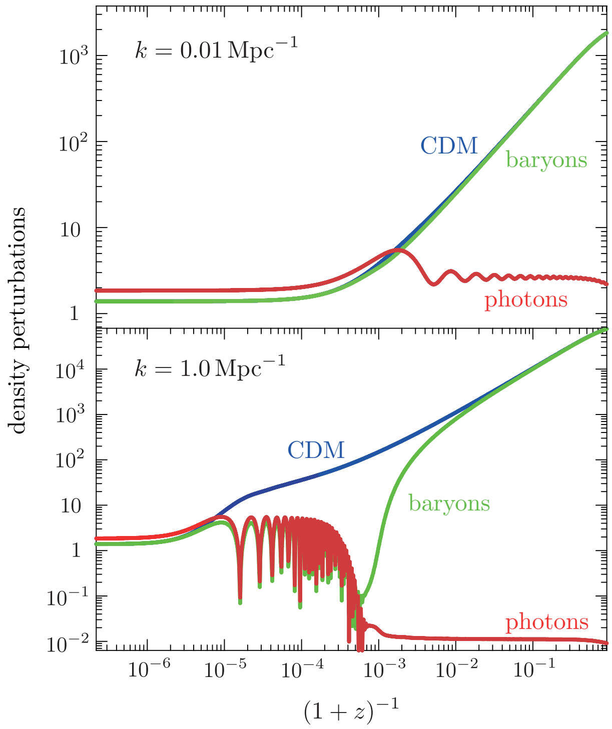

Figure 5.5: *Evolution of photons, baryons and dark matter.*

After Decoupling

After decoupling, the baryons lose the pressure support of the photons and gravitational instability kicks in. Ignoring baryon pressure, the coupled dynamics of the baryon fluid and the dark matter fluid after decoupling is approximately given by

$$\delta_b'' + \mathcal{H}\delta_b' = 4\pi G a^2(\bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c) \ , \tag{5.2.36}$$

$$\delta_c'' + \mathcal{H}\delta_c' = 4\pi G a^2(\bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c) \ . \tag{5.2.37}$$

The two equations are coupled via the gravitational potential which is sourced by the total density contrast $\bar{\rho}_m\delta_m = \bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c$. We can decouple these equations by defining $D \equiv \delta_b - \delta_c$. Subtracting eqs. (5.2.36) and (5.2.37), we find

$$D'' + \frac{2}{\tau}D' = 0 \quad \Rightarrow \quad D \propto \begin{cases} const. \\ \tau^{-1} \end{cases} , \tag{5.2.38}$$

while the evolution of δ_m is governed

$$\delta_m'' + \frac{2}{\tau}\delta_m' - \frac{6}{\tau^2}\delta_m = 0 \quad \Rightarrow \quad \delta_m \propto \begin{cases} \tau^2 \\ \tau^{-3} \end{cases} . \tag{5.2.39}$$

Since

$$\frac{\delta_b}{\delta_c} = \frac{\bar{\rho}_m\delta_m + \bar{\rho}_c D}{\bar{\rho}_m\delta_m - \bar{\rho}_b D} \to \frac{\delta_m}{\delta_m} = 1 \ , \tag{5.2.40}$$

we see that δ_b approaches δ_c during matter domination (see fig. 5.2).

The non-zero initial value of δ_b at decoupling, and, more importantly δ_b', leaves a small imprint in the late-time δ_m that oscillates with scale. These *baryon acoustic oscillations* have recently been detected in the clustering of galaxies.