7 Green's Functions for Ordinary Differential Equations

One of the most important applications of the δ-function is as a means to develop a systematic theory of Green's functions for ODEs. Consider a general linear second–order differential operator $\mathcal{L}$ on $[a, b]$ (which may be $\pm\infty$, respectively). We write

$$\mathcal{L}y(x) = \alpha(x)\frac{d^2}{dx^2}y + \beta(x)\frac{d}{dx}y + \gamma(x)y = f(x)\,, \tag{7.1}$$

where α, β, γ are continuous functions on $[a, b]$, and α is nonzero (except perhaps at a finite number of isolated points). We also require the forcing term $f(x)$ to be bounded in $[a, b]$. We now define the *Green's function* $G(x; \xi)$ of $\mathcal{L}$ to be the unique solution to the problem

$$\mathcal{L}G = \delta(x - \xi) \tag{7.2}$$

that satisfies homogeneous boundary conditions[29] $G(a; \xi) = G(b; \xi) = 0$.

The importance of the Green's function comes from the fact that, given our solution $G(x, \xi)$ to equation (7.2), we can immediately solve the more general problem $\mathcal{L}y(x) = f(x)$ of (7.1) for an arbitrary forcing term $f(x)$ by writing

$$y(x) = \int_a^b G(x; \xi)\, f(\xi)\, \mathrm{d}\xi\,. \tag{7.3}$$

To see that it does indeed solve (7.1), we compute

$$\begin{aligned}\mathcal{L}y(x) &= \mathcal{L}\left[\int_a^b G(x, \xi)\, f(\xi)\, \mathrm{d}\xi\right] = \int_a^b \left[\mathcal{L}G(x, \xi)\right]\, f(\xi)\, \mathrm{d}\xi \\ &= \int_a^b \delta(x - \xi)\, f(\xi)\, \mathrm{d}\xi = f(x)\,,\end{aligned} \tag{7.4}$$

since the Green's function is the only thing that depends on x. We also note that the solution (7.3) constructed this way obeys $y(a) = y(b) = 0$ as a direct consequence of these conditions on the Green's function.

The important point is that G depends on $\mathcal{L}$, but *not* on the forcing term $f(x)$. Once G is known, we will be able write down the solution to $\mathcal{L}y = f$ for an *arbitrary* force term. To put this differently, asking for a solution to the differential equation $\mathcal{L}y = f$ is asking to invert the differential operator $\mathcal{L}$, and we might formally write $y(x) = \mathcal{L}^{-1}f(x)$. Equation (7.3) shows what is meant by the inverse of the differential operator $\mathcal{L}$ is integration with the Green's function as the integral kernel.

7.1 Construction of the Green's function

We now give a constructive means for determining the Green's function. (We'll see later how this compares to the eigenfunction expansion for inverting Sturm–Liouville operators that we gave in 2.6.)

[29] Other homogeneous boundary conditions are also possible, but for clarity will will treat only the simplest case $G(a, \xi) = G(b, \xi) = 0$ here.

Our construction relies on the fact that whenever $x \neq \xi$, $\mathcal{L}G = 0$. Thus, both for $x < \xi$ and $x > \xi$ we can express G in terms of solutions of the homogeneous equation. Let us suppose that $\{y_1, y_2\}$ are a basis of linearly independent solutions to the second–order homogeneous problem $\mathcal{L}y = 0$ on $[a, b]$. We define this basis by requiring that $y_1(a) = 0$ whereas $y_2(b) = 0$. That is, each of $y_{1,2}$ obeys *one* of the homogeneous boundary conditions. On $[a, \xi)$ the Green's function obeys $\mathcal{L}G = 0$ and $G(a, \xi) = 0$. But any homogeneous solution to $\mathcal{L}y = 0$ obeying $y(a) = 0$ must be proportional to $y_1(x)$, with a proportionality constant that is independent of x. Thus we set

$$G(x, \xi) = A(\xi)\, y_1(x) \qquad \text{for } x \in [a, \xi)\,. \tag{7.5}$$

Similarly, on $(\xi, b]$ the Green's function must be proportional to $y_2(x)$ and so we set

$$G(x, \xi) = B(\xi)\, y_2(x) \qquad \text{for } x \in 9\xi, b]\,. \tag{7.6}$$

Note that the coefficient functions $A(\xi)$ and $B(\xi)$ may depend on the point ξ, but must be independent of x.

This construction gives us families of Green's function for $x \in [a, b] - \{\xi\}$, in terms of the functions A and B. We must now determine how these two solutions are to be joined together at $x = \xi$. Suppose first that $G(x, \xi)$ was discontinuous at $x = \xi$, with the discontinuity modelled by a step function. Then $\partial_x G \propto \delta(x - \xi)$ and consequently $\partial_x^2 G \propto \delta'(x - \xi)$. However, the form of equation (7.2) shows that $\mathcal{L}G$ involves no generalized functions beyond $\delta(x - \xi)$, and in particular contains no derivatives of δ-functions. Thus we conclude that $G(x, \xi)$ must be continuous throughout $[a, b]$ and in particular at $x = \xi$.

However, integrating equation (7.2) over an infinitesimal neighbourhood of $x = \xi$ we learn that

$$\int_{\xi-\epsilon}^{\xi+\epsilon} \left[\alpha(x)\frac{\partial^2 G}{\partial x^2}\,\mathrm{d}x + \beta(x)\frac{\partial G}{\partial x}\,\mathrm{d}x + \gamma(x) G\right] \mathrm{d}x = \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x - \xi)\,\mathrm{d}x = 1\,. \tag{7.7}$$

We have already seen that $G(x, \xi)$ is continuous, and all three coefficient functions α, β, γ are bounded by assumption, so the final term on the *lhs* contributes zero as we make the integration region infinitesimally thin. Also, since G is continuous, $\partial_x G$ must be bounded so the term $\beta \partial_x G$ also cannot contribute as the integration region shrinks to zero size. Finally, since α is continuous we have

$$\lim_{\epsilon \to 0^+} \int_{\xi-\epsilon}^{\xi+\epsilon} \alpha(x)\frac{\partial^2 G}{\partial x^2}\,\mathrm{d}x = \alpha(\xi)\left[\left.\frac{\partial G}{\partial x}\right|_{x=\xi^+} - \left.\frac{\partial G}{\partial x}\right|_{x=\xi^-}\right]\,. \tag{7.8}$$

To summarize, we must glue the Green's functions (7.5) & (7.6) according to the conditions

$$\begin{aligned} G(x, \xi)|_{x=\xi^-} &= G(x, \xi)|_{x=\xi^+} && \text{continuity} \\ \left.\frac{\partial G}{\partial x}\right|_{x=\xi^-} - \left.\frac{\partial G}{\partial x}\right|_{x=\xi^-} &= \frac{1}{\alpha(\xi)} && \text{jump in derivative.} \end{aligned} \tag{7.9}$$

In terms of (7.5) & (7.6) these conditions become

$$A(\xi)\, y_1(\xi) = B(\xi)\, y_2(\xi) \qquad \text{and} \qquad A(\xi)\, y_1'(\xi) - B(\xi)\, y_2'(\xi) = \frac{1}{\alpha(\xi)}\,. \tag{7.10}$$

These are two linear equations for A and B, determining them to be

$$A(\xi) = \frac{y_2(\xi)}{\alpha(\xi)W(\xi)} \qquad \text{and} \qquad B(\xi) = \frac{y_1(\xi)}{\alpha(\xi)W(\xi)}\,, \tag{7.11}$$

where

$$W(x) \equiv y_1 y_2' - y_2 y_1' \tag{7.12}$$

is known as the *Wronskian* of y_1 and y_2. Note that the Wronskian is evaluated at $x = \xi$ in equation (7.11).

To conclude, we have found that the solution $G(x, \xi)$ of $\mathcal{L}G = \delta(x-\xi)$ obeying $G(a, \xi) = G(b, \xi) = 0$ is given by

$$\begin{aligned} G(x;\xi) &= \begin{cases} \dfrac{y_1(x)y_2(\xi)}{\alpha(\xi)W(\xi)} & a \le x < \xi \\ \dfrac{y_2(x)y_1(\xi)}{\alpha(\xi)W(\xi)} & \xi < x \le b \end{cases} \\ &= \frac{1}{\alpha(\xi)W(\xi)}\left[\Theta(\xi - x)\, y_1(x)y_2(\xi) + \Theta(x-\xi)\, y_2(x)y_1(\xi)\right] \end{aligned} \tag{7.13}$$

where Θ is again the step function. Hence the solution to $\mathcal{L}y = f$ is

$$\begin{aligned} y(x) &= \int_a^b G(x;\xi)\, f(\xi)\, \mathrm{d}\xi \\ &= y_2(x)\int_a^x \frac{y_1(\xi)}{\alpha(\xi)W(\xi)} f(\xi)\, \mathrm{d}\xi + y_1(x)\int_x^b \frac{y_2(\xi)}{\alpha(\xi)W(\xi)} f(\xi)\, \mathrm{d}\xi\,. \end{aligned} \tag{7.14}$$

The integral over ξ here is separated at x into two parts, (*i*) $\int_a^x$ and (*ii*) $\int_x^b$. In the range of (*i*) we have $\xi < x$ so the *second* line of equation (7.13) for $G(x;\xi)$ applies, even though this expression incorporates the boundary condition at $x = b$. For (*ii*) we have $x > \xi$ so we use the $G(x;\xi)$ expression from the *first* line of equation (7.13) that incorporates the boundary condition at $x = a$.

As an example of the use of Green's functions, suppose we wish to solve the forced problem

$$\mathcal{L}y = -y'' - y = f(x) \tag{7.15}$$

on the interval $[0, 1]$, subject to the boundary conditions $y(0) = y(1) = 0$. We follow our procedure above. The general homogeneous solution is $c_1 \sin x + c_2 \cos x$ so we can take $y_1(x) = \sin x$ and $y_2(x) = \sin(1-x)$ as our homogeneous solutions satisfying the boundary conditions at $x = 0$ and $x = 1$, respectively. Then

$$G(x;\xi) = \begin{cases} A(\xi)\sin x & 0 \le x < \xi \\ B(\xi)\sin(1-x) & \xi < x \le 1\,. \end{cases} \tag{7.16}$$

Applying the continuity condition we get

$$A\sin\xi = B\sin(1-\xi) \tag{7.17}$$

while the jump condition gives

$$B(-\cos(1-\xi)) - A\cos\xi = -1\,. \tag{7.18}$$

where we note that $\alpha = -1$. Solving these two equations for A and B gives the Green's function

$$G(x;\xi) = \frac{1}{\sin 1}\left[\Theta(\xi - x)\,\sin(1-\xi)\sin x + \Theta(x-\xi)\,\sin(1-x)\sin\xi\right] \tag{7.19}$$

Using this Green's function we are immediately able to write down the complete solution to $-y'' - y = f(x)$ with $y(0) = y(1) = 0$ as

$$y(x) = \frac{\sin(1-x)}{\sin 1}\int_0^x f(\xi)\,\sin\xi\,\mathrm{d}\xi + \frac{\sin x}{\sin 1}\int_x^1 f(\xi)\,\sin(1-\xi)\,\mathrm{d}\xi. \tag{7.20}$$

where again only the *second* term for G in (7.19) contributes in the first integration region where $\xi > x$, while only the *first* term for G contributes to the integral over the region $\xi < x$.

7.2 Physical interpretation of the Green's function

We can think of the expression

$$y(x) = \int_a^b G(x;\xi)\,f(\xi)\,\mathrm{d}\xi \tag{7.21}$$

as a 'summation' (or integral) of individual point source effects, each of strength $f(\xi)$, with $G(x;\xi)$ describing the effect at x of a unit point source placed at ξ.

To illustrate this with a physical example, consider again the wave equation for a horizontal elastic string with ends fixed at $x = 0, L$. If $y(x,t)$ represents the small vertical displacement transverse to the string, we found that $T\partial_x^2 y = \mu\,\partial_t^2 y$. Also including the effect of gravity acting in the vertical direction leads to

$$T\frac{\partial^2 y}{\partial x^2} - \mu g = \mu\frac{\partial^2 y}{\partial t^2} \tag{7.22}$$

for $x \in [0, L]$ with $y(0) = y(L) = 0$. Here, T is the constant tension in the string and μ is the mass density per unit length, which may vary with x.

When the string is at rest, its profile obeys the steady state equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu(x) g}{T}\,, \tag{7.23}$$

whose solution describes the shape of a (non–uniform) string hanging under gravity. We'll be interested in these steady state solutions. We consider three cases. Firstly, suppose μ is a (non–zero) constant then equation (7.23) is easily integrated and we find the parabolic shape

$$y(x) = \frac{\mu g}{2T}\,x(x-L)\,. \tag{7.24}$$

that obeys $y(0) = y(L) = 0$.

In the second case, suppose instead that the string itself is very light, but that it has a metal bead attached at a point $x = \xi$. We treat the bead as a point mass m, and assume it is not too heavy. To find its location, let θ_1 and θ_2 be the angles the string makes on either side of the bead. Resolving forces vertically, the equilibrium condition is

$$mg = T(\sin\theta_1 + \sin\theta_2) \approx T(\tan\theta_1 + \tan\theta_2) \tag{7.25}$$

where the small angle approximation $\sin\theta \approx \tan\theta$ will hold provided the mass m is sufficiently small. (Note also that $y < 0$ since the bead pulls the string down.) Thus the point mass is located at $(x, y) = (\xi, y(\xi))$ where

$$y(\xi) = \frac{mg}{T}\frac{\xi(\xi - L)}{L}\,. \tag{7.26}$$

Since the string is effectively massless on either side of the bead, gravity does not act there, so the only force felt by the string at $x \neq \xi$ is the (tangential) tension. Thus the string must be straight either side of the point mass and so

$$y(x) = \frac{mg}{T} \times \begin{cases} \dfrac{x(\xi - L)}{L} & \text{for } 0 \leq x < \xi, \\ \dfrac{\xi(x - L)}{L} & \text{for } \xi < x \leq L \end{cases} \tag{7.27}$$

gives the steady–state shape of this string.

We obtained this answer from physical principles; let's now rederive it using the Green's function. For the case of a point mass at $x = \xi$, we take the mass density to be $\mu(x) = m\,\delta(x - \xi)$ so that the steady–state equation becomes

$$\frac{\partial^2 y}{\partial x^2} = \frac{mg}{T}\,\delta(x - \xi)\,. \tag{7.28}$$

The differential operator $\partial^2/\partial x^2$ is a (very simple) self–adjoint operator and the *rhs* is a forcing term. We look for a Green's function $G(x, \xi)$ that obeys

$$\frac{\partial^2 G}{\partial x^2} = \delta(x - \xi) \tag{7.29}$$

subject to the boundary conditions $G(0, \xi) = G(L, \xi) = 0$. Following our usual procedure, we have the general solutions

$$\begin{aligned} G(x, \xi) &= A(\xi)x + B(\xi) && \text{when } \ 0 \leq x < \xi, \\ G(x, \xi) &= C(\xi)(1 - x) + D(\xi) && \text{when } \ \xi < x \leq L \end{aligned} \tag{7.30}$$

on either side of the point mass. The boundary conditions at 0 and L enforce $B(\xi) = D(\xi) = 0$, and continuity (the string does not break!) at $x = \xi$ fixed $C(\xi) = A(\xi)\xi/(\xi - L)$. Finally, the jump condition on the derivative (with $\alpha = 1$) gives $A(\xi) = (\xi - L)/L$. Thus our Green's function is

$$G(x, \xi) = \begin{cases} \dfrac{x(\xi - L)}{L} & \text{for } \ 0 \leq x < \xi, \\ \dfrac{\xi(x - L)}{L} & \text{for } \ \xi < x \leq L. \end{cases} \tag{7.31}$$

Rescaling this Green's function by mg/T gives exact agreement with the string profile in (7.27).

For our final case, we now imagine that we have several point masses m_i at positions $x_i \in [0, L]$. We can simply sum the solutions to obtain

$$y(x) = \sum_i G(x, x_i)\, \frac{m_i\, g}{T}\,. \tag{7.32}$$

To take the continuum limit we can imagine there are a large number of masses m_i placed at equal intervals $x_i = iL/N$ along the string, with $i \in \{1, 2, \ldots, N-1\}$. Setting $m_i = \mu(\xi_i)\Delta\xi$ where $\xi_i = i\Delta\xi = iL/N$, then by Riemann's definition of integrals, as $N \to \infty$ equation (7.32) becomes

$$y(x) = \int_0^L G(x, \xi)\, \frac{g\, \mu(\xi)}{T}\, \mathrm{d}\xi\,. \tag{7.33}$$

If μ is constant this function reproduces the parabolic result of case 1, as you should check by direct integration. (*Exercise!* – take care with the limits of integration.)

7.3 Green's functions for inhomogeneous boundary conditions

Our construction of the solution to the forced problem relied on the Green's function obeying *homogeneous* boundary conditions. This is because the integral in equation (7.3) represents a "continuous superposition" of solutions for individual values of ξ. In order to treat problems with inhomogeneous boundary conditions using Green's functions, we must proceed as follows.

First, find *any* particular solution $y_p(x)$ to the *homogeneous* equation $\mathcal{L}y = 0$ that satisfies the *in*homgeneous boundary conditions. This step is usually easy because we're not looking for the most general solution, just any simple solution. Since the differential operator $\mathcal{L}$ is linear, the general solution of $\mathcal{L}y = f$ obeying inhomogeneous boundary conditions is simply

$$y(x) = y_p(x) + \int_a^b G(x, \xi)\, f(\xi)\, \mathrm{d}\xi\,, \tag{7.34}$$

where the term involving the Green's function ensures that $\mathcal{L}y$ indeed equals the forcing term $f(x)$, but does not disturb the boundary values.

As an example, suppose again we wish to solve $-y'' - y = f(x)$, but now with inhomogeneous boundary conditions $y(0) = 0$ and $y(1) = 1$. We already have the Green's function solution to the homogeneous problem in (7.20), so we simply need to find a solution to the homogeneous equation $-y'' - y = 0$ that obeys the boundary conditions. The general solution of this homogeneous equation $c_1 \cos x + c_2 \sin x$ and the inhomogeneous boundary conditions require $c_1 = 0$ and $c_2 = 1/\sin 1$. Therefore $y_p(x) = \sin x/\sin 1$ is the desired particular solution and the general solution is

$$y(x) = \frac{\sin x}{\sin 1} + \frac{\sin(1-x)}{\sin 1} \int_0^x f(\xi)\, \sin\xi\, \mathrm{d}\xi + \frac{\sin x}{\sin 1} \int_x^1 f(\xi)\, \sin(1-\xi)\, \mathrm{d}\xi \tag{7.35}$$

using the result (7.20).

7.4 Equivalence of eigenfunction expansion of $G(x;\xi)$

For self–adjoint differential operators, we have now discovered two different expressions for the Green's function with homogeneous boundary conditions. On the one hand, we have

$$G(x,\xi) = \frac{1}{\alpha(\xi)W(\xi)}\left[\Theta(\xi-x)\, y_1(x)y_2(\xi) + \Theta(x-\xi)\, y_2(x)y_1(\xi)\right] \tag{7.36}$$

as in equation (7.13). On the other hand, in section 2.6 we showed that the Green's function for a self–adjoint operator could be written as

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\, Y_n(x)\, Y_n^*(\xi) \tag{7.37}$$

in terms of the eigenfunctions $\{Y_n(x)\}$ and eigenvalues $\{\lambda_n\}$ of the Sturm–Liouville operator.

Incidentally, we derived (7.37) in section 2.6 without any mention of δ-functions, but it may also be quickly derived using the eigenfunction expansion

$$\delta(x-\xi) = w(x)\sum_{n\in\mathbb{Z}}^{\infty} Y_n(x)\, Y_n^*(\xi) \tag{7.38}$$

as in equation (6.36). Viewing ξ as a parameter we can write an eigenfunction expansion of the Green's function as

$$G(x,\xi) = \sum_{n\in\mathbb{Z}} \hat{G}_n(\xi)\, Y_n(x)\,. \tag{7.39}$$

Applying the self–adjoint operator $\mathcal{L}$ we have

$$\mathcal{L}G = \sum_{n\in Z} \hat{G}_n(\xi)\, \mathcal{L}Y_n(x) = w(x)\sum_{n\in\mathbb{Z}} \hat{G}_n(\xi)\, \lambda_n\, Y_n(x) \tag{7.40}$$

and for this to agree with $\delta(x-\xi)$, so that the expansion (7.39) obeys the defining equation $\mathcal{L}G = \delta(x-\xi)$ for the Green's function, we need

$$\hat{G}_n(\xi) = \frac{1}{\lambda_n}\, Y_n^*(\xi) \tag{7.41}$$

as can be checked by multiplying both sides of $w(x)\sum_{n\in\mathbb{Z}} \hat{G}_n(\xi)\, \lambda_n\, Y_n(x) = \delta(x-\xi)$ by $Y_m^*(x)$, integrating from a to b and using the orthogonality of the Sturm–Liouville eigenfunctions with weight function w. Thus we have recovered the eigenfunction expansion (7.37) of the Green's function. Note that the expression (7.37) requires that all eigenvalues λ_n be *non–zero*. This means that the homogeneous equation $\mathcal{L}y = 0$ — which is the eigenfunction equation when $\lambda = 0$ — should have no non–trivial solutions satisfying the boundary conditions. The existence of such solutions would certainly be problematic for the concept of a Green's function: If such solutions exist, then the inhomogeneous equation $\mathcal{L}y = f$ does not have a unique solution, because if y is any solution then so too is $y + y_0$. The operator $\mathcal{L}$ is thus not invertible, and the Green's function cannot exist. This is just the infinite dimensional analogue of the familiar situation of a system of linear equations $\mathbf{M}\,\mathbf{u} = \mathbf{f}$ with non–invertible coefficient matrix $\mathbf{M}$. Indeed a matrix is non–invertible iff it has nontrivial eigenvectors with eigenvalue zero.

Since the Green's function is the *unique* solution to $\mathcal{L}G(x,\xi) = \delta(x-\xi)$ that obeys $G(a,\xi) = G(b,\xi) = 0$, it must be that the two expressions (7.36) and (7.37) are the same. To see that this is true, we first notice that for a self–adjoint operator (in Sturm–Liouville form) the first two coefficient functions in

$$\mathcal{L} = \alpha(x)\frac{\partial^2}{\partial x^2} + \beta(x)\frac{\partial}{\partial x} + \gamma(x)$$

are related by $\beta = d\alpha/dx$. In this case, the denominator $\alpha(\xi)W(\xi)$ in equation (**??**) for the Green's function is necessarily a (non–zero) *constant*. To show this, note that

$$\begin{aligned} \frac{d}{dx}(\alpha W) &= \alpha' W + \alpha W = \beta(y_1 y_2' - y_2 y_1') + \alpha(y_1 y_2'' - y_2 y_1'') \\ &= y_1 \mathcal{L} y_2 - y_2 \mathcal{L} y_1 = 0\,. \end{aligned} \tag{7.42}$$

Being constant, $\alpha(x)W(x)$ is independent of where we evaluate it and in particular is independent of ξ. We thus set $\alpha(x)W(x) = c$ and rewrite equation (7.36) as

$$G(x,\xi) = \frac{1}{c}\left[\Theta(\xi - x)\, y_1(x) y_2(\xi) + \Theta(x-\xi)\, y_2(x) y_1(\xi)\right]\,. \tag{7.43}$$

Like the eigenfunction expansion, this expression is now symmetric under exchange of x and ξ, so that $G(x;\xi) = G(\xi;x)$.

Going further in general requires a rather tedious procedure of expanding the step functions and solutions $y_1(x)$ and $y_2(x)$ (which we recall obey $y_1(a) = y_2(b) = 0$) in terms of the eigenfunctions. Instead, we'll content ourselves with an example and for lack of imagination we again take $\mathcal{L}y = -y'' - y$ on $[a,b] = [0,1]$, with boundary conditions $y(0) = y(1) = 0$. The normalized eigenfunctions and corresponding eigenvalues are easily calculated to be

$$Y_n(x) = \sqrt{2}\sin n\pi x \qquad\qquad \text{with} \quad \lambda_n = n^2\pi^2 - 1 \tag{7.44}$$

and the Green's function is given in terms of these eigenfunctions by

$$G(x,\xi) = 2\sum_{n=1}^{\infty}\frac{\sin n\pi x\, \sin n\pi\xi}{n^2\pi^2 - 1}\,. \tag{7.45}$$

On the other hand, in a previous example we constructed the expression

$$G(x,\xi) = \frac{1}{\sin 1}\left[\Theta(x-\xi)\, \sin(1-x)\, \sin\xi + \Theta(\xi - x)\, \sin x\, \sin(1-\xi)\right] \tag{7.46}$$

using homogeneous solutions. Standard trigonometric addition formulæ for $\sin(1-x)$ allow us to write this as

$$G(x,\xi) = \Theta(x-\xi)\, \cos x\, \sin\xi\, + \Theta(\xi - x)\, \sin x\, \cos\xi - \cot 1\, \sin x\, \sin\xi\,. \tag{7.47}$$

Viewing x as the independent variable and ξ as a parameter, we expand this function as a Fourier sine series

$$\Theta(x-\xi)\,\cos x\,\sin\xi\,+\Theta(\xi-x)\,\sin x\,\cos\xi-\cot 1\,\sin x\,\sin\xi=\sum_{n=1}^{\infty}\hat{g}_n(\xi)\,\sin n\pi x\,. \quad (7.48)$$

As usual, the Fourier coefficients are given by

$$\hat{g}_n(\xi)=2\int_0^1 \sin n\pi x\,[\Theta(x-\xi)\,\cos x\,\sin\xi\,+\Theta(\xi-x)\,\sin x\,\cos\xi-\cot 1\,\sin x\,\sin\xi]\,\mathrm{d}x \quad (7.49)$$

and a direct (though rather tedious calculation — try it as an exercise if you really must) gives

$$\hat{g}_n(\xi)=\frac{2\sin n\pi\xi}{n^2\pi^2-1}\,. \quad (7.50)$$

Comparing this to the eigenfunction expansion (7.45) we see that the two expressions for the Green's function agree, as expected.

7.5 Application of Green's functions to initial value problems

Green's functions can also be used to solve initial value problems. Let's take the independent variable to be time t, and suppose we wish to find the function $y:[t_0,\infty)\to\mathbb{R}$ that obeys the differential equation

$$\mathcal{L}y=f(t) \quad (7.51)$$

subject to the initial conditions $y(t_0)=0$ and $y'(t_0)=0$. The method for construction of the Green's function in this initial value problem is similar to the previous method in the case of a boundary value problem. As before, we want to find G such that $\mathcal{L}G=\delta(t-\tau)$, so that for each value of τ, the Green's function $G(t,\tau)$ will solve the homogeneous equation $\mathcal{L}G=0$ whenever $t\neq\tau$. We proceed as follows

- Construct G for $t_0\le t<\tau$ as a general solution of the homogeneous equation, so $G=Ay_1(t)+By_2(t)$. Here $\{y_1(t),y_2(t)\}$ is *any* basis of linearly independent homogeneous solutions.

- In contrast to the boundary value problem, we now apply *both* initial conditions to this solution. That is, we enforce

$$\begin{aligned}Ay_1(t_0)+By_2(t_0)&=0\,,\\ Ay_1'(t_0)+By_2'(t_0)&=0\,.\end{aligned} \quad (7.52)$$

  This pair of linear equations for A and B can be written as

$$\begin{pmatrix}y_1(t_0) & y_2(t_0)\\ y_1'(t_0) & y_2'(t_0)\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}=0 \quad (7.53)$$

  and since y_1 and y_2 are linearly independent, the determinant of the matrix (the Wronskian) is non-zero. Thus the only way to impose both initial conditions is to set $A=B=0$. This implies that $G(t,\tau)=0$ identically whenever $t\in[a,\tau)$!

– For $t > \tau$, again construct the Green's function as a general solution of the homogeneous equation, so $G = Cy_1(t) + Dy_2(t)$.

– Finally, we apply the continuity and jump conditions at $t = \tau$. Since $G = 0$ for $t < \tau$ we obtain

$$\begin{aligned} Cy_1(\tau) + Dy_2(\tau) &= 0 \\ Cy_1'(\tau) + Dy_2'(\tau) &= \frac{1}{\alpha(\tau)} \end{aligned} \tag{7.54}$$

where, as usual, $\alpha(t)$ is the coefficient of the second derivative in the differential operator $\mathcal{L}$. These simultaneous equations determine $C(\tau)$ and $D(\tau)$, thus completing the construction of the Green's function $G(t;\tau)$.

We can again use our Green's function to solve the forced problem (7.51) as

$$y(t) = \int_{t_0}^{\infty} G(t,\tau)\, f(\tau)\, \mathrm{d}\tau = \int_{t_0}^{t} G(t,\tau)\, f(\tau)\, \mathrm{d}\tau\,. \tag{7.55}$$

Here, in the second equality we have used the fact that $G(t,\tau)$ vanishes for $\tau > t$. This equation shows that the solution obeys a *causality condition*: the value of y at time t depends only on the behaviour of the forcing function for *earlier* times $\tau \in [t_0, t]$.

As an example, consider the problem

$$\frac{d^2y}{dt^2} + y = f(t),\ y(0) = y'(0) = 0 \tag{7.56}$$

with initial conditions $y(0) = y'(0) = 0$. Following our procedure above we get

$$G(t,\tau) = \Theta(t-\tau)\left[C(\tau)\,\cos(t-\tau) + D(\tau)\,\sin(t-\tau)\right]\,, \tag{7.57}$$

where we've chosen the basis of linearly independent solutions to be $\{\cos(t-\tau), \sin(t-\tau)\}$ purely because they make it easy to impose the initial conditions. Continuity demands that $G(\tau,\tau) = 0$, so $C(\tau) = 0$. The jump condition (with $\alpha(\tau) = 1$ then enforces $D(\tau) = 1$. Therefore, the Green's function is

$$G(t,\tau) = \Theta(t-\tau)\,\sin(t-\tau) \tag{7.58}$$

and the general solution to $\mathcal{L}y = f(t)$ obeying $y(0) = y'(0) = 0$ is

$$y(t) = \int_0^t \sin(t-\tau)\, f(\tau)\, \mathrm{d}\tau\,. \tag{7.59}$$

Again, we see that this solution knows about what the forcing function was doing only at earlier times.

7.6 Higher order differential operators

We briefly mention that there is a natural generalization of Green's functions to higher order differential operators (and indeed to PDEs, as we shall see in the last part of the course). If $\mathcal{L}$ is a n^{th}-order ODE on $[a, b]$, with $n > 2$ then the general solution to the forced differential equation $\mathcal{L}y = f(x)$ obeying the homogeneous boundary conditions $y(a) = y(b) = 0$ is again given by

$$y(x) = \int_a^b G(x;\xi)\, f(\xi)\, \mathrm{d}\xi\,, \tag{7.60}$$

where G still obeys $\mathcal{L}G = \delta(x - \xi)$ subject to homogeneous boundary conditions $G(a, \xi) = G(b, \xi) = 0$, but where now both G and its first $n - 2$ derivatives are continuous at $x = \xi$, while

$$\left.\frac{\partial^{(n-1)}G}{\partial x^{(n-1)}}\right|_{x=x^+} - \left.\frac{\partial^{(n-1)}G}{\partial x^{(n-1)}}\right|_{x=x^-} = \frac{1}{\alpha(\xi)} \tag{7.61}$$

where we again assume that $\alpha(x)$ is the coefficient function of the highest derivative in $\mathcal{L}$. An example can be found on problem sheet 3.