3 Laplace's Equation

In the previous chapter, we learnt that there are a set of orthogonal functions associated to *any* second order self-adjoint operator $\mathcal{L}$, with the sines and cosines (or complex exponentials) of Fourier series arising just as the simplest case $\mathcal{L} = -d^2/dx^2$. While this is true, the important – or at least commonly occurring – such functions arise not from Sturm–Liouville operators with randomly chosen (real) coefficient functions $p(x)$ and $q(x)$, but from those SL operators that have some special geometric significance. This is what we'll investigate over the next few chapters.

We'll start by considering Laplace's equation,

$$\nabla^2\psi \equiv \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}\psi = 0 \tag{3.1}$$

where d is the number of spatial dimensions. The Laplace equation is one of the most fundamental differential equations in all of mathematics, pure as well as applied. A function $\psi : M \to \mathbb{R}$ obeying $\nabla^2\psi = 0$ is called *harmonic*, and harmonic analysis is a huge area of study (particularly when M is a smooth manifold or a compact group). In theoretical physics, Laplace's equation is ubiquitous throughout both electromagnetism and gravity. In particular, whenever we have a *conservative* force $\mathbf{F} = -\nabla\psi$ obeying Gauss' law that the flux of the force through any closed surface is proportional to the net charge contained within that surface, then in empty space ψ satisfies Laplace's equation. In this context, ψ is known as a *potential* for the force. If that wasn't enough, Laplace's equation also arises as a limiting case of both the heat and wave equations, $\partial\psi/\partial t = K\nabla^2\psi$ and $\partial^2\psi/\partial t^2 = c^2\nabla^2\psi$, when ψ is independent of time.

We'll meet some of these applications later, but for now we'll concentrate just on solving Laplace's equation everywhere within a bounded domain $\Omega \subset \mathbb{R}^d$ subject to some condition on the behaviour of our solution at the boundary $\partial\Omega$ of our domain. In the case of *Dirichlet* boundary conditions, we require that our solution takes some pre-determined shape on the boundary. So in this case we're given a function $f : \partial\Omega \to \mathbb{R}$ and we require that

$$\psi(\mathbf{x}) = f(\mathbf{x}) \qquad \text{at each point } \mathbf{x} \in \partial\Omega\,. \tag{3.2}$$

If we can find such a solution, then it must be unique, because if ψ_1 and ψ_2 both obey Laplace's equation in Ω and $\psi_1|_{\partial\Omega} = \psi_2|_{\partial\Omega} = f$, then setting $\delta\psi \equiv \psi_1 - \psi_2$ and integrating by parts we have

$$0 = \int_\Omega \delta\psi\, \nabla^2\delta\psi\, \mathrm{d}V = -\int_\Omega (\nabla\delta\psi)\cdot(\nabla\delta\psi)\, \mathrm{d}V + \int_{\partial\Omega} \delta\psi\, \mathbf{n}\cdot(\nabla\delta\psi)\, \mathrm{d}S\,. \tag{3.3}$$

The boundary term vanishes because $\delta\psi|_{\partial\Omega} = 0$ by assumption. The *rhs* is thus the integral of a non–negative quantity $(\nabla\delta\psi)\cdot(\nabla\delta\psi)$, and so the only way the integral can be zero is for $\nabla\delta\psi = 0$ throughout Ω. Hence $\delta\psi = $ *const.* Finally, since $\delta\psi$ vanishes on the boundary, this constant must be zero so that $\psi_1 = \psi_2$ everywhere and our solution is unique.

The other case to consider is that of *Neumann* boundary conditions. Here, we instead require that the normal derivative $\partial\psi/\partial n = \mathbf{n}\cdot\nabla\psi$ takes some specific form on the boundary. That is, we are given a function $g : \partial\Omega \to \mathbb{R}$ and we ask that

$$\mathbf{n}\cdot\nabla\psi(\mathbf{x}) = g(\mathbf{x}) \qquad \text{at each point } \mathbf{x}\in\partial\Omega\,, \tag{3.4}$$

where $\mathbf{n}$ is the *outward* normal. Notice that the same uniqueness argument as above still shows that $\delta\psi = const.$, but that now the boundary conditions no longer force this constant to vanish. Thus imposing Neumann boundary conditions determines our solution only up to the addition of a constant.

3.1 Laplace's equation on a disc

In two dimensions, a powerful method for solving Laplace's equation is based on the fact that we can think of $\mathbb{R}^2$ as the complex plane $\mathbb{C}$. For $(x,y)\in\mathbb{R}^2$ we introduce $z = x+\mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$, whereupon Laplace's equation becomes

$$\frac{\partial^2\psi}{\partial z\,\partial\bar{z}} = 0\,. \tag{3.5}$$

The general solution of this is $\psi(x,y) = \phi(z) + \chi(\bar{z})$ where $\phi(z)$ is holomorphic (*i.e.* $\partial\phi/\partial\bar{z} = 0$) and $\chi(\bar{z})$ is antiholomorphic ($\partial\chi/\partial z = 0$).

Suppose that we wish to solve Laplace's equation inside the unit disc $|z|\leq 1$, subject to the condition that ψ is regular throughout this disc and obeys $\psi = f(\theta)$ on the boundary of the disc, where $f(\theta)$ is some choice of function $f : S^1 \to \mathbb{C}$ that we assume is sufficiently well–behaved that its Fourier expansion exists. Noting that when $|z| = 1$ we can write $\mathrm{e}^{\mathrm{i}n\theta}$ both as z^n and $\bar{z}^{-n}$, this Fourier expansion is

$$f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}_n\,\mathrm{e}^{\mathrm{i}n\theta} = \hat{f}_0 \;+\; \sum_{n=1}^{\infty}\hat{f}_n\,z^n\Bigg|_{|z|=1} \;+\; \sum_{n=1}^{\infty}\hat{f}_{-n}\,\bar{z}^n\Bigg|_{|z|=1}\,. \tag{3.6}$$

The advantage of writing the Fourier series in this second form is that it may now be extended throughout the unit disc; since no negative powers of z or $\bar{z}$ arise, the extended function remains finite throughout the disc. Furthermore, the extension of the *rhs* is manifestly of the form of the sum of a holomorphic and an antiholomorphic function (the constant $\hat{f}_0$, being both holomorphic and antiholomorphic, may be included with either). Therefore we find our desired solution to be

$$\begin{aligned}\psi(x,y) &= \hat{f}_0 + \sum_{n=1}^{\infty}\left(\hat{f}_n\,z^n + \hat{f}_{-n}\,\bar{z}^n\right)\\ &= \hat{f}_0 + \sum_{n=1}^{\infty}\left(\hat{f}_n\,\mathrm{e}^{\mathrm{i}n\theta} + \hat{f}_{-n}\,\mathrm{e}^{-\mathrm{i}n\theta}\right)r^n\,.\end{aligned} \tag{3.7}$$

Note that since $r\leq 1$ everywhere on the unit disc, this expansion will certainly converge throughout the domain whenever the Fourier expansion of the boundary function $f(\theta)$ converges.

If instead we'd asked for a solution just in the annulus $a \leq |z| \leq b$, then we would require information about the behaviour of ψ at each boundary in order to decide how to split up the sum into holomorphic and antiholomorphic pieces. You'll learn much more about the isomorphism $\mathbb{R}^2 \cong \mathbb{C}$ in both the Complex Methods and Complex Analysis courses next term. The amazing power of complex analyticity is one of the true jewels of mathematics.

3.2 Separation of variables

The use of complex variables is very pretty, but beyond two dimensions it isn't generically available[16]. Suppose Ω is the three dimensional region

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \; : \; 0 \leq x \leq a, \;\; 0 \leq y \leq b, \;\; z \geq 0\} \tag{3.8}$$

and that we wish to find a function $\psi : \Omega \to \mathbb{R}$ that obeys Laplace's equation $\nabla^2\psi = 0$ throughout the interior of Ω, and that satisfies the Dirichlet boundary conditions

$$\begin{aligned} &\psi(0, y, z) = 0 && \psi(a, y, z) = 0 \\ &\psi(x, 0, z) = 0 && \psi(x, b, z) = 0 \\ &\psi(x, y, 0) = f(x, y) && \psi(x, y, z) \to 0 \text{ as } z \to \infty \end{aligned} \tag{3.9}$$

for some given function $f : [0, a] \times [0, b] \to \mathbb{R}$.

The fundamental idea that allows us to make progress is to assume that ψ takes the form

$$\psi(x, y) = X(x)Y(y)Z(z) \,. \tag{3.10}$$

This is known as *separation of variables*. Inserting this ansatz into Laplace's equation we find $0 = \nabla^2\psi = Y(y)Z(z)X''(x) + X(x)Z(z)Y''(y) + X(x)Y(y)Z''(z)$, where the primes denote differentiation with respect to the arguments. At any point $(x, y, z) \in \Omega$ where $\psi \neq 0$, we can divide Laplace's equation by ψ to find

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0 \,. \tag{3.11}$$

The key point is that each ratio here depends on only one of the variables, and a different one in each case. So in order for the equation to hold as we move around in Ω, it must be that they are each constant. For example, we could choose to move out along the x-direction while staying at constant y and z. The terms Y''/Y and Z''/Z cannot change along this path, because they don't depend on x. But then since $X''/X = -Y''/Y - Z''/Z$ it must be that X''/X is also independent of x and hence constant. Arguing similarly for the other terms, we have

$$X'' = -\lambda X \,, \qquad Y'' = -\mu Y \,, \qquad Z'' = (\lambda + \mu)Z \tag{3.12}$$

[16]In fact, there is a version in four dimensions, which is one way of viewing what Penrose's twistor theory is about. And no, it doesn't have much to do with quaternions.

for some constants λ and μ[17]. Note that the final constant here is not independent because they must all sum to zero by Laplace's equation.

These equations are simple to solve. If $\lambda < 0$ there are no solutions obeying the boundary conditions. When $\lambda \geq 0$ then $X(x) = a\sin(\sqrt{\lambda}x) + b\cos(\sqrt{\lambda}x)$ solves $X'' = \lambda X$ and similarly $Y(y) = c\sin(\sqrt{\mu}y) + d\cos(\sqrt{\mu}y)$ while $Z = r\mathrm{e}^{-z\sqrt{\lambda+\mu}} + s\,\mathrm{e}^{+z\sqrt{\lambda+\mu}}$. Thus, after relabelling the constants,

$$\psi(x,y,z) = A\left(\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)\right)\left(\sin(\sqrt{\mu}y) + C\cos(\sqrt{\mu}y)\;\right)\left(\mathrm{e}^{-z\sqrt{\lambda+\mu}} + D\,\mathrm{e}^{+z\sqrt{\lambda+\mu}}\right) \tag{3.13}$$

is a solution of Laplace's equation for arbitrary λ and μ. We will assume ψ does not vanish everywhere, so that $A \neq 0$.

We must now try to impose the boundary conditions. We'll begin with the homogeneous ones (*i.e.* those that don't involve $f(x,y)$.) The condition $\psi(0,y,z) = 0$ tells us that $B = 0$. The condition $\psi(a,y,z) = 0$ now tells us that $\sqrt{\lambda} = n\pi/a$ for[18] $n \in \mathbb{Z}^*$, or in other words that λ must take one of the values

$$\lambda_n \equiv \frac{n^2\pi^2}{a^2}\,, \qquad n = 1,2,3,\ldots\,. \tag{3.14}$$

Likewise, the boundary condition $\psi(x,0,z) = 0$ tells us that $C = 0$ while the condition $\psi(x,b,z) = 0$ restricts μ to be

$$\mu_m \equiv \frac{m^2\pi^2}{b^2}\,, \qquad m = 1,2,3,\ldots\,. \tag{3.15}$$

The condition that ψ falls to zero as $z \to +\infty$ immediately tells us that $D = 0$.

At this point, we have a family of solutions

$$\psi_{n,m}(x,y,z) \equiv \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\exp\left(-\pi z\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\right) \tag{3.16}$$

labelled by the pair of integers (n,m), all of which obey the boundary conditions we've so far considered. Since Laplace's equation is linear and the boundary conditions we've so far imposed are homogeneous, any linear combination of these solutions is also a solution. Thus our general solution is

$$\psi(x,y) = \sum_{n,m=1}^{\infty} A_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\exp\left(-\pi z\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\right) \tag{3.17}$$

for some constants $A_{n,m}$.

The final step is to try to choose the $A_{n,m}$ so as to obey the final boundary condition. Setting $z = 0$ we require $\psi(x,y,0) = \sum_{m,n=1}^{\infty} A_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$ to equal the given function $f(x,y)$, or in other words that

$$f(x,y) = \sum_{m,n=1}^{\infty} A_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\,, \tag{3.18}$$

[17] The signs here are purely for later convenience with the given boundary conditions: check for yourself that nothing changes at the end of the day if you don't include them.

[18] Here $\mathbb{Z}^*$ denotes the non-zero integers $\{\cdots,-2,-1,+1,+2,\ldots\}$.

This looks just like a Fourier (sine) series expansion – in both x and y – for the boundary function $f(x, y)$! Using orthogonality relations

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \mathrm{d}x = \frac{a}{2}\delta_{m,n} \tag{3.19}$$

we see that the constants $A_{n,m}$ are fixed to be

$$A_{n,m} = \frac{4}{ab} \int_{[0,a]\times[0,b]} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)\ f(x, y)\ \mathrm{d}x\,\mathrm{d}y \tag{3.20}$$

in equation (3.17). We note (in passing!) that the function $f(x, y)$ should be smooth enough so that this Fourier series both converges and is (at least) twice differentiable, so that it does indeed define a solution of the Laplace equation.

As a simple example, suppose $f(x, y) = 1$. Then we have

$$A_{n,m} = \frac{4}{ab} \int_{[0,a]\times[0,b]} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \mathrm{d}x\,\mathrm{d}y = \begin{cases} \dfrac{16}{\pi^2}\dfrac{1}{mn} & \text{if } n \text{ and } m \text{ are both odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

Therefore, the solution satisfying these boundary conditions is

$$\psi(x, y, z) = \frac{16}{\pi^2} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\sin\left[\frac{(2k-1)\pi x}{a}\right]}{(2k-1)} \frac{\sin\left[\frac{(2\ell-1)\pi y}{b}\right]}{(2\ell-1)} \exp\left[-s_{k,\ell}\,\pi z\right] \tag{3.22}$$

where $s_{k,\ell}^2 = \frac{(2k-1)^2}{a^2} + \frac{(2\ell-1)^2}{b^2}$. We see that at fixed z, the function $\psi(x, y, z)$ is larger near the middle of the region $[0, a] \times [0, b]$, and that for large z the lowest harmonics – where k and ℓ are small – dominate.

Note that in this example, we obtained a Fourier sine series because of the homogeneous Dirichlet boundary conditions on x and y. If instead we'd imposed Neumann boundary conditions $\partial\psi/\partial x = 0$ at $y = 0, b$ and $\partial\psi/\partial y = 0$ at $x = 0, a$, then we would instead find Fourier *co*sine series. Finally if, instead of allowing our domain Ω to extend to infinity in the z direction, we had imposed a boundary conditon $\psi(x, y, c) = g(x, y)$ at some finite location $z = c$, then both the positive and negative exponential terms in the function $Z(z)$ would contribute.

To summarize, the method of separation of variables starts by writing ψ as a product of functions that depend only on one variable each. We use this ansatz to reduce Laplace's PDE to a system of ODEs that depend on a number of constants (here λ and μ). Since Laplace's equation was a second order linear equation, these ODEs will always be of Sturm–Liouville type; the constants will appear as eigenvalues of the SL equation and the equations will be solved by the eigenfunctions of the SL operator. After solving these SL equations, we use the homogeneous boundary conditions to impose restrictions on the possible values of the eigenvalues. The solution for a fixed permissible choice of the eigenvalues is known as a *normal mode* of the system. By linearity, the general solution is a linear combination of these normal modes. The final step is to use the inhomogeneous boundary conditions to determine which linear combination we should take; this will require using the orthogonality property of the eigenfunctions of the SL operator.

3.3 The Laplacian in spherical polar coordinates

In the previous example, our Sturm–Liouville equations were of the form $X'' = -\lambda X$, and we were led to the *Fourier* expansion of $f(x, y)$ on the boundary $z = 0$. This occurred because the domain of that example was a (non-compact) rectangular cuboid, and it was natural to treat this using Cartesian coordinates. We're now going to consider Laplace's equation in spherical polar coordinates. Separation of variables now leads to a more interesting SL equation with a non–constant coefficient function $p(x)$.

Recall that a point $p \in \mathbb{R}^3$ with position vector $\mathbf{r}$ may be described in terms of polar coordinates (r, θ, ϕ). The radial coordinate $r = |\mathbf{r}|$ is the distance of point p from the origin. In particular $r \geq 0$. Whenever $r > 0$ we define θ to be the angle that $\mathbf{r}$ makes with the (chosen) positive z axis, and take $0 \leq \theta \leq \pi$. Finally, whenever $r > 0$ and $0 < \theta < \pi$ we define ϕ to be the angle that the projection of $\mathbf{r}$ into the plane $z = 0$ makes with the positive x axis, measured counterclockwise. Thus $0 \leq \phi < 2\pi$. It follows from the above that

$$x = r\,\sin\theta\,\cos\phi\,, \qquad y = r\,\sin\theta\,\sin\phi\,, \qquad z = r\,\cos\theta \tag{3.23}$$

and hence that the volume element is

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{3.24}$$

in spherical polar coordinates.

The Laplacian operator becomes

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{3.25}$$

as you check by the chain rule. In this course, for simplicity, we'll restrict attention to axisymmetric situations where all our functions are assumed independent of the angle ϕ. In these circumstances we can omit the final term in the Laplacian (3.25), but you can read about the more general situation in *e.g.* Arfken & Weber or Boas' books.

We now seek solutions of Laplace's equation $\nabla^2\psi(r, \theta) = 0$ in the interior of the spherical domain $\Omega = \{(r, \theta, \phi) \in \mathbb{R}^3 \;:\; r \leq a\}$ for some constant a, where we'll demand that our solution remains finite everywhere in Ω. Once again we separate variables by writing $\psi(r, \theta) = R(r)\Theta(\theta)$ and by our now standard argument find that Laplace's equation is equivalent to the system of ordinary differential equations

$$\begin{aligned} \frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + \lambda\sin\theta\,\Theta &= 0 \\ \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda R &= 0 \end{aligned} \tag{3.26}$$

where $\lambda \in \mathbb{R}$ is our separation constant. Each of these equations is of Sturm–Liouville type. Let's examine them in more detail.

3.3.1 Legendre polynomials

We'll start with the angular equation. We can simplify its form with the substitution $x = \cos\theta$, although note that this x is nothing to do with our original Cartesian coordinate in $\mathbb{R}^3$. Since $0 \le \theta \le \pi$ we have $-1 \le x \le 1$ and

$$\frac{d}{d\theta} = -\sin\theta \frac{d}{dx}\,.$$

Therefore the angular part of Laplace's equation becomes

$$-\sin\theta \frac{d}{dx}\left[\sin\theta\left(-\sin\theta\frac{d\Theta}{dx}\right)\right] + \lambda\sin\theta\,\Theta = 0\,, \tag{3.27}$$

or in other words

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] = -\lambda\Theta\,. \tag{3.28}$$

This is known as *Legendre's equation*. It is a standard Sturm–Liouville eigenfunction problem on $\Omega = [-1,1]$, with coefficient functions $p(x) = (1-x^2)$ and $q(x) = 0$, and where the weight function $w(x) = 1$. When we checked in chapter 2 whether a Sturm–Liouville operator $\mathcal{L}$ on the domain $[-1,1]$ was self–adjoint, we met the condition

$$(f, \mathcal{L}g) = (\mathcal{L}f, g) + \left[p(x)(f^{*\prime}g - f^*g')\right]_{-1}^{1}\,.$$

In the case of Legendre's equation we see that $p(x) = 0$ at $x = \pm 1$, so the boundary condition required for self–adjointness is simply that the functions and their derivatives remain *regular* on $\partial\Omega$.

Let's now look for a regular solution of (3.28) throughout the interior of Ω that obeys the boundary condition that it remains regular on $\partial\Omega$. We seek a power series solution to (3.28) of the form $\Theta(x) = \sum_{n=0}^{\infty} a_n\, x^n$, where only non–negative powers of x appear since we want the solution to be regular at the origin. Substituting this into (3.28) gives

$$0 = (1-x^2)\sum_{n=2}^{\infty} a_n\, n(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} a_n\, nx^n + \lambda\sum_{n=0}^{\infty} x^n \tag{3.29}$$

which must hold it must hold for each power of x separately as it must hold throughout the open set $x \in (-1,1)$. This implies that the coefficients must obey the recursion relation

$$0 = a_{n+2}(n+2)(n+1) - a_n\, n(n-1) - 2a_n\, n + \lambda a_n\,,$$

or equivalently

$$a_{n+2} = \left[\frac{n(n+1)-\lambda}{(n+1)(n+2)}\right] a_n\,. \tag{3.30}$$

The recursion relation relates a_{n+2} to a_n, so we can pick a_0 and a_1 freely and write

$$\Theta(x) = a_0\Theta_0(x) + a_1\Theta_1(x)\,, \tag{3.31}$$

where

$$\begin{aligned}\Theta_0(x) &= 1 + \frac{(-\lambda)}{2!}x^2 + \frac{(-\lambda)(6-\lambda)}{4!}x^4 + \frac{(-\lambda)(6-\lambda)(20-\lambda)}{6!}x^6 + \cdots \\ \Theta_1(x) &= x + \frac{(2-\lambda)}{3!}x^3 + \frac{(2-\lambda)(12-\lambda)}{5!}x^5 + \cdots .\end{aligned} \tag{3.32}$$

Note that $\Theta_0(-x) = \Theta_0(x)$ while $\Theta_1(-x) = -\Theta_1(x)$. Thus, for any value of λ, we've found two independent solutions of Legendre's second order equation.

We now consider the boundary conditions. The recurrence relation (3.30) shows that as n becomes large

$$\frac{a_{n+2}}{a_n} = 1 - \frac{2}{n} + \frac{4-\lambda}{n^2} . \tag{3.33}$$

Thus, by the ratio test the series will always converge for $|x| < 1$. However, *for generic values of* λ the series diverges at $x = \pm 1$, violating our boundary condition[19]. The only way to avoid this divergence is if the power series for $\Theta(x)$ somehow *terminates*. Looking back at the recurrence relation, this will occur iff λ takes the form

$$\lambda = \ell(\ell+1) \qquad \text{for } \ell \in \mathbb{Z}_{\geq 0} . \tag{3.34}$$

Only for these eigenvalues will (both) the series we found in (3.32) terminate, leaving us with a $\Theta(x)$ that is polynomial of degree ℓ. The situation is exactly analogous to our familiar sines and cosines. For any $\lambda \in \mathbb{R}$, we can solve $y'' = -\lambda y$ on $[-1, 1]$ as $y = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x$, but if we impose the boundary condition $y(-1) = y(1)$ then the eigenvalue must be restricted to $\lambda = (2\pi n)^2$ for some $n \in \mathbb{Z}$.

The polynomials we've found are known as *Legendre polynomials of order* ℓ, and denoted by $P_\ell(x)$. One can show using the series (3.32) that the first four Legendre polynomials are given by

$$P_0(x) = 1 , \qquad P_1(x) = x , \qquad P_2(x) = \frac{1}{2}(3x^2 - 1) , \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x) .$$

where we've fixed the overall factor by requiring $P_\ell(1) = 1$. You can find plots of the first few Legendre polynomials in figure 3. It turns out that they can be usefully represented as

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell , \tag{3.35}$$

which is known as *Rodrigues' formula*. The numerical prefactor ensures that $P_\ell(1) = 1$; to see this use Leibnitz' rule to compute

$$\begin{aligned}P_\ell(x) &= \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \left[(x-1)^\ell (x+1)^\ell \right] \\ &= \frac{1}{2^\ell \ell!} \left[\ell!(x+1)^\ell + \text{terms proportional to } (x-1) \right]\end{aligned} \tag{3.36}$$

[19] Since $\lim_{n\to\infty} \frac{a_{n+2}}{a_n} = 1$, the ratio test is inconclusive and we must examine the higher order terms. Gauss' test for convergence states that if the ratio of successive coefficients behaves at large n like $1 + \frac{h}{n} + \frac{C_n}{n^2}$ with C_n bounded and $h < 1$ then the series will diverge. So the infinite series $\Theta_0(x)$ and $\Theta_1(x)$ do indeed diverge at $x = \pm 1$. A proof of Gauss' test can be found here.

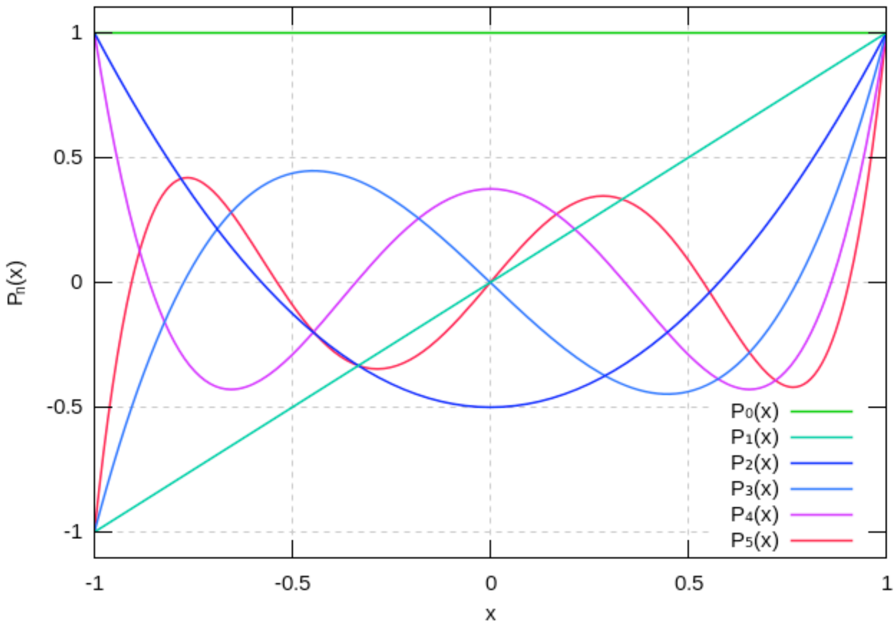

Figure 3. *Plots of the first five Legendre polynomials $P_\ell(x)$ for $x \in [-1, 1]$. Note that $P_\ell(x)$ is even (odd) if ℓ is even (odd), that $P_\ell(1) = 1$, and that $P_\ell(x)$ has ℓ real roots in between $x = -1$ and 1.*

and evaluate at $x = 1$. Notice also that since $(x^2 - 1)^\ell$ is a polynomial in x of degree 2ℓ containing only even powers of x, $P_\ell(x)$ is a polynomial of degree ℓ that is an even (odd) function of x when ℓ is even (odd), in agreement with the expectation from our recurrence relation.

The Legendre polynomials $P_\ell(x)$ are the basic orthogonal polynomials on $[-1, 1]$ with weight function $w(x) = 1$. They have many beautiful properties. To prove some of them, it is helpful to first note that if $\ell \geq r \geq 0$ then $(d^r/dx^r)(x^2 - 1)^\ell = (x^2 - 1)^{\ell - r}\, Q_{\ell,r}(x)$ where $Q_{\ell,r}(x)$ is some polynomial of degree r. This follows by induction: it is true when $r = 0$, and assuming it is true for some r then differentiating $r + 1$ times

$$\begin{aligned}
\frac{d^{r+1}}{dx^{r+1}}(x^2 - 1)^\ell &= \frac{d}{dx}\left[(x^2 - 1)^{\ell - r}\, Q_{\ell,r}\right] \\
&= (x^2 - 1)^{\ell - r - 1}\left[2x(\ell - r)Q_{\ell,r}(x) + (x^2 - 1)Q'_{\ell,r}(x)\right] .
\end{aligned}$$

If $Q_{\ell,r}(x)$ is a polynomial of degree r then $Q'_{\ell,r}(x)$ is a polynomial of degree $r - 1$ and the content of the square brackets is a polynomial of degree $r + 1$. Thus the claim holds at $r + 1$. A consequence of this lemma is that

$$\frac{d^r}{dx^r}(x^2 - 1)^\ell \bigg|_{x=\pm 1} = 0 \tag{3.37}$$

whenever $r < \ell$ (but $r \geq 0$). This fact is useful in showing that functions $P_\ell(x)$ defined in (3.35) are orthogonal for different values of ℓ. Explicitly, in considering $\int_{-1}^{1} P_m(x)\, P_\ell(x)\, \mathrm{d}x$ for $m \neq \ell$ then without loss of generality we can assume that $m < \ell$. Then repeatedly integrating by parts $m+1$ times we have

$$\begin{aligned}
2^{\ell+m}\ell!\,m!\int_{-1}^{1}P_m(x)\,P_\ell(x)\,\mathrm{d}x &= \int_{-1}^{1}P_m(x)\,\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\,\mathrm{d}x \\
&= \left[P_m(x)\,\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right]_{-1}^{1} - \int_{-1}^{1}\frac{d}{dx}P_m(x)\,\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\,\mathrm{d}x \\
&= -\int_{-1}^{1}\frac{d^{m+1}}{dx^{m+1}}P_m(x)\,\frac{d^{\ell-m-1}}{dx^{\ell-m-1}}(x^2-1)^\ell\,\mathrm{d}x \\
&= 0
\end{aligned} \tag{3.38}$$

where in performing the integrations by parts we note that all the boundary terms vanish by our lemma (3.37), and in going to the final line we use the fact that $P_m(x)$ is a polynomial of degree m. Thus Legendre polynomials with different eigenvalues are indeed orthogonal on $[-1,1]$ in agreement with the general results of Sturm–Liouville theory.

To fix the normalization we similarly note that

$$\begin{aligned}
2^{2\ell}(\ell!)^2\int_{-1}^{1}P_\ell(x)\,P_\ell(x)\,\mathrm{d}x &= \int_{-1}^{1}P_\ell(x)\,\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\,\mathrm{d}x \\
&= -\int_{-1}^{1}(x^2-1)^\ell\,\frac{d^\ell}{dx^\ell}P_\ell(x)\,\mathrm{d}x = -\int_{-1}^{1}(x^2-1)^\ell\,\frac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^\ell\,\mathrm{d}x
\end{aligned} \tag{3.39}$$

The only term in $(x^2-1)^\ell$ that survives being differentiated 2ℓ times is the highest power $x^{2\ell}$ and since $(d^r/dx^r)x^r = r!$ we have

$$2^{2\ell}(\ell!)^2\int_{-1}^{1}P_\ell(x)\,P_\ell(x)\,\mathrm{d}x = (2\ell)!\int_{-1}^{1}(x^2-1)^\ell\,\mathrm{d}x \equiv (2\ell)!\,I_\ell\,. \tag{3.40}$$

Yet again integrating by parts we find that for $\ell > 0$

$$\begin{aligned}
I_\ell \equiv \int_{-1}^{1}(x^2-1)^\ell\,\mathrm{d}x &= \Big[x(x^2-1)^\ell\Big]_{-1}^{1} - 2\ell\int_{-1}^{1}x^2(x^2-1)^{\ell-1}\,\mathrm{d}x \\
&= -2\ell\,I_\ell + 2\ell\,I_{\ell-1}\,,
\end{aligned} \tag{3.41}$$

or in other words $I_\ell = \frac{2\ell}{2\ell+1}I_{\ell-1}$. Since $I_0 = 2$ we find inductively that

$$I_\ell = (-1)^\ell\frac{2^{\ell+1}\ell!}{\prod_{r=0}^{\ell}(2r+1)}\,.$$

Using this in equation (3.39) gives finally

$$\int_{-1}^{1}P_m(x)\,P_\ell(x)\,\mathrm{d}x = \frac{2}{2\ell+1}\,\delta_{m,\ell} \tag{3.42}$$

as the orthogonality relations among the Legendre polynomials.

Any degree ℓ polynomial has ℓ complex roots, but in fact all ℓ roots of $P_\ell(x)$ are real, and remarkably they all lie in $x \in (-1,1)$. To see this, assume for a contradiction that it's false. Then $P_\ell(x)$ has only $k < \ell$ real roots in between -1 and 1. Suppose these are at points $x_1, x_2, \ldots, x_k$ and construct the polynomial $Q_k(x) \equiv \prod_{r=1}^{k}(x - x_r)$. Then for $x \in [-1, 1]$ the product $P_\ell(x)\, Q_k(x)$ is either always positive or always negative, because by our assumptions $P_\ell(x)$ and $Q_k(x)$ change sign simultaneously. Thus, on the one hand

$$\int_{-1}^{1} P_\ell(x)\, Q_k(x)\, \mathrm{d}x \;\neq\; 0$$

since the integrand always has definite sign. On the other hand, we can always expand as $Q_k(x) = \sum_{r=0}^{k} \hat{Q}_r P_r(x)$ in a basis of Legendre polynomials. Since $k < \ell$ by assumption each term in this sum is orthogonal to $P_\ell(x)$, so the above integral must vanish, giving a contradiction.

The Legendre polynomials have many other curious properties, some of which you will explore in the problem sets and many of which are explained in Arfken & Weber or in Boas. I recommend that you browse through a few of these, but the most important thing to remember is simply that the Legendre polynomials are the solutions of Legendre's equation (3.28) with eigenvalue $\lambda = \ell(\ell+1)$ for $\ell \in \mathbb{Z}_{\geq 0}$. By standard SL theory they form a complete set of orthogonal functions on $x \in [-1, 1]$ or equivalently on $\theta \in [0, \pi]$ where $x = \cos\theta$.

3.3.2 The Cosmic Microwave Background

In 1964, Arno Penzias and Robert Wilson were trying to clean their radio telescope of various pigeon droppings. They were hoping this was the cause of an annoying background noise that was stymying their attempts to measure weak radio waves bouncing off various satellite balloons NASA had launched into the upper atmosphere. But the noise did not go away. They asked around their friends and colleagues to find out if anyone had a clue what could be causing this mysterious microwave background, that seemed to be come evenly from all directions and constant in time.

One of them recalled a recent paper of Dicke, Peebles and Wilkinson which predicted that, had the Universe started in a hot, dense state then some radiation from that time should be around now and would be redshifted down to microwave frequencies. This Cosmic Microwave Background would be, to excellent approximation, homogeneous and isotropic and provided the perfect explanation for the blackbody spectrum of temperature $\sim$ 3K measured by Penzias and Wilson.

The discovery and measurement of the CMB is one of the key pieces of evidence we have for the Big Bang. But it's not completely isotropic. The CMB anisotropies were first measured by the COBE satellite in the early 1990s and have been intensively studied by many telescopes ever since. The best one to date is the Planck satellite – orbiting right now – which has a strong Cambridge involvement. These satellites produce detailed maps of the CMB, a (low resolution) example being shown in figure 4.

To understand what these pictures are trying to tell us, we need to process it a little. Cosmologists are particularly interested in the two–point function

$$C(\theta) \equiv \left\langle \frac{\delta T}{T}(\hat{\mathbf{r}}_1) \frac{\delta T}{T}(\hat{\mathbf{r}}_2) \right\rangle$$

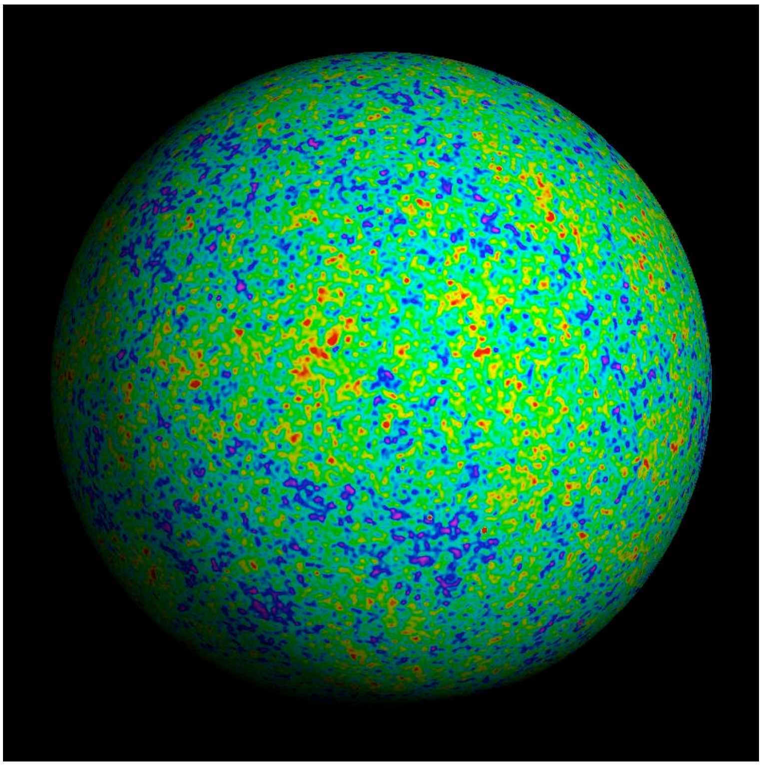

Figure 4. *A map of the CMB, produced by the WMAP satellite. The picture depicts fluctuations around the average temperature $T = 2.725K$ in different directions in the sky, with red and yellow being hotspots while blue and purple are cold. The fluctuations are very small, with $\delta T/T \sim 10^{-5}$.*

defined as the temperature difference of the CMB when looking out in different directions $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$ with $\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2 = \cos\theta$, averaged over all points on the sky. The resulting function depends on $\theta \in [0, \pi]$ and so you can expand it in Legendre polynomials as

$$C(\theta) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell + 1)\, C_\ell\, P_\ell(\cos\theta)\,. \tag{3.43}$$

If you do this you'll find the graph shown in figure 3.3.2. The peaks and troughs of this graph contain a vast wealth of information about the history of the very early universe. If you want to know more, take the Part II Cosmology course.

3.3.3 Laplace's equation on the sphere

After this long interlude, let's return to our problem of finding the general regular, axisymmetric solution to Laplace's equation $\nabla^2\psi = 0$ on the spherical domain $\Omega = \{r \le a\} \subset \mathbb{R}^3$. Using separation of variables we found $\psi(r, \theta) = R(r)\, \Theta(\theta)$ where R and Θ obey the odes (3.26). We've just seen that for a regular solution, we require the separation constant $\lambda = \ell(\ell + 1)$ for $\ell \in \mathbb{Z}_{\ge}$ and that in this case $\Theta = P_\ell(\cos\theta)$. Notice that regularity at $x = \pm 1$, necessary for self–adjointness of the Sturm–Liouville operator in Legendre's equation, amounts to regularity of Θ at $\theta = 0$ and π, or in other words along the z-axis of our

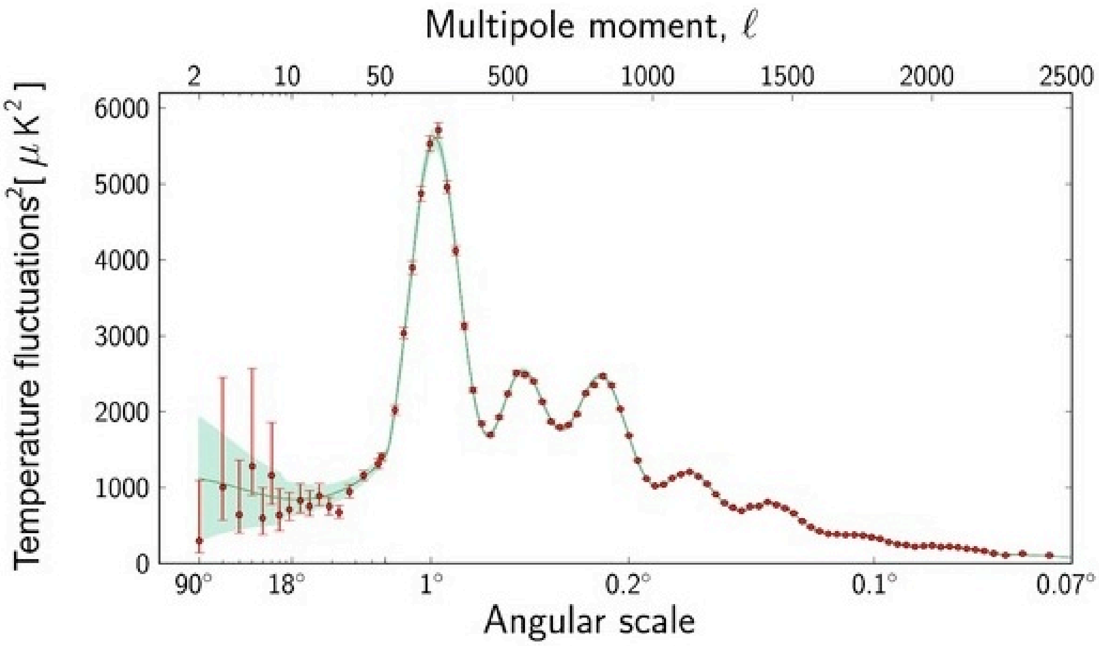

Figure 5. *The CMB power spectrum $|C_\ell|^2$ plotted against ℓ (called the multipole moment).*

original problem in $\mathbb{R}^3$. If we want a solution regular everywhere in the interior of $r \leq a$ then it certainly needs to be regular along the z-axis!

The remaining equation to consider is the radial equation, which with $\lambda = \ell(\ell+1)$ becomes

$$(r^2 R_\ell')' = \ell(\ell+1)R\,. \tag{3.44}$$

Trying a solution of the form $R(r) \propto r^\alpha$ for some power α we learn that

$$\alpha(\alpha+1) = \ell(\ell+1) \tag{3.45}$$

whose roots are $\alpha = \ell$ and $\alpha = -(\ell+1)$. Thus our general solution takes the form

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell\, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \tag{3.46}$$

where for each ℓ the A_ℓ and B_ℓ are (generically complex) constants. Note that we're now treating the Legendre polynomials as functions of θ; they are polynomials in $\cos\theta$. Since we require the solution to be regular everywhere inside $r = a$, we must set $B_\ell = 0$ for all ℓ. (If we were interested in solving Laplace's equation everywhere *outside* the sphere $r = a$, then for regularity as $r \to \infty$ we'd need $A_\ell = 0$ for $\ell > 0$. And, of course, if we wish for a regular solution valid inside the spherical shell $a \leq r \leq b$ then both the A_ℓ and B_ℓ can generically be present.)

As always, to pin down the remaining constants A_ℓ we must impose some boundary condition at $r = a$. For example, if we demand that $\psi(a,\theta) = f(\theta)$ for some axisymmetric function f on the sphere, then by the general results of Sturm–Liouville theory f has an expansion in terms of the Legendre polynomials as

$$f(\theta) = \sum_{\ell=0}^{\infty} F_\ell \, P_\ell(\cos\theta) \qquad \text{where} \qquad F_\ell = \frac{2\ell+1}{2} \int_{-\pi}^{\pi} P_\ell(\theta)\, f(\theta)\, \sin\theta \, \mathrm{d}\theta \, .$$

Note the presence of the factor $(2\ell + 1)/2$ in front of the integral; this comes from the normalization condition (3.39) of the Legendre polynomials. Laplace's equation is thus solved by

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} F_\ell \left(\frac{r}{a}\right)^{\ell} P_\ell(\cos\theta) \, . \tag{3.47}$$

with the choice $A_\ell = F_\ell \, a^{-\ell}$ ensuring that our boundary condition $\psi(a,\theta) = f(\theta)$ is met.

3.3.4 Multipole expansions

Consider the function

$$\frac{1}{|\mathbf{r} - \mathbf{k}|} = \frac{1}{\sqrt{1 + r^2 - 2r\cos\theta}} \, . \tag{3.48}$$

where $\mathbf{k}$ is a unit vector in the z-direction. You can check (exercise!) that this function satisfied Laplace's equation for all $\mathbf{r} \neq \mathbf{k}$ and that in particular it is regular at the origin $\mathbf{r} = \mathbf{0}$. So from what we've said above, it must be possible to expand it in terms of Legendre polynomials. That is, we must have

$$\frac{1}{|\mathbf{r} - \mathbf{k}|} = \sum_{\ell=0}^{\infty} a_\ell \, r^\ell \, P_\ell(\cos\theta) \tag{3.49}$$

for some coefficients a_ℓ. It's simple to determine these coefficients: Set $\theta = 0$ in equation (3.48) so that $\mathbf{r}$ also points along the z-axis, and Taylor expand to find

$$\frac{1}{\sqrt{1 + r^2 - 2r}} = \frac{1}{1-r} = \sum_{\ell=0}^{\infty} r^\ell \qquad \text{whenever } r < 1 \, . \tag{3.50}$$

If we recall that $P_\ell(1) = 1$ for all ℓ, then this is compatible with the expansion (3.49) iff $a_\ell = 1$, showing that

$$\frac{1}{|\mathbf{r} - \mathbf{k}|} = \sum_{\ell=0}^{\infty} r^\ell P_\ell(\cos\theta) \tag{3.51}$$

at general locations $\mathbf{r}$. More generally, for any vector $\mathbf{r}'$ we have

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r'} \sum_{\ell=0}^{\infty} \left(\frac{r}{r'}\right)^{\ell} P_\ell(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \tag{3.52}$$

whenever $r' > r$. This is known as the *multipole expansion*, with the $\ell = 0$ term known as the *monopole* term and the $\ell = 1$ term known as the *dipole*. In electrostatics, q times the *monopole* term $1/r'$ is the potential experienced at $\mathbf{r}'$ due to a point charge q at the origin. The *dipole* term $r/(r')^2 \cos\theta = \mathbf{r} \cdot \mathbf{r}'/(r')^3$ is likewise proportional to the potential experienced at $\mathbf{r}'$ due to two charges $\pm a$ placed at a separation $\mathbf{r}$ from eachother.

3.4 Laplace's equation in cylindrical polar coordinates

The final case we'll consider in this course is problems with cylindrical symmetry. Here, separation of variables leads to a SL equation that has a non–constant weight function as well as a non–constant SL coefficient function $p(x)$.

Recall that in cylindrical polar coordinates $(x, y, z) = (r\cos\theta, r\sin\theta, z)$ the Laplacian operator

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\,. \tag{3.53}$$

Suppose we're interested in solving Laplace's equation $\nabla^2\psi = 0$ on the cylinder $\Omega = \{(r,\theta,z) \in \mathbb{R}^3 \ :\ r \leq a\,,\ z \geq 0\}$, subject to the Dirichlet boundary conditions that $\psi(\mathbf{r})$ is single valued and finite throughout Ω and decays as $z \to \infty$. We also assume that ψ vanishes on the curved edge $r = a$ of the cylinder, but that $\psi(r,\theta,0) = f(r,\theta)$ for some given function f on the base of the cylinder.

Again we try separating variables by writing $\psi = R(r)\Theta(\theta)Z(z)$, learning that

$$\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right) + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = 0 \tag{3.54}$$

wherever $\psi \neq 0$. We now argue that since Z''/Z depends only on z but cannot vary as z is changed with (r,θ) held fixed, in fact $Z''/Z = \mu$ for some constant μ. This being so, we can now multiply (3.54) through by r^2 and notice that by the same argument, $\Theta''/\Theta = -\lambda$ must also be constant. Therefore, in cylindrical coordinates Laplace's equation reduces to the system of ODEs

$$\begin{aligned} \Theta'' &= -\lambda\Theta \qquad\qquad Z'' = \mu Z \\ 0 &= r^2R'' + rR' + (\mu r^2 - \lambda)R \end{aligned} \tag{3.55}$$

If we want $\psi(\mathbf{r})$ to be single valued, then we must have $\Theta(\theta + 2\pi) = \Theta(\theta)$, so λ must be one of the values $\lambda_n \equiv n^2$ for $n \in \mathbb{Z}$, whereupon we have the usual solution

$$\Theta(\theta) = \Theta_n(\theta) \equiv a_n \sin n\theta + b_n \cos n\theta\,. \tag{3.56}$$

(Note that if $\lambda = 0$ then the equation $\Theta'' = 0$ is solved by $\Theta = a_0\theta + b_0$. Periodicity requires $a_0 = 0$ and the remaining constant term is just what we'd find by putting $n = 0$ in (3.56).) The equation for $Z(z)$ is equally straightforward. If $\mu < 0$ we'd again find solutions in terms of sines and cosines, but let's suppose that $\mu > 0$ and that we require $\psi(\mathbf{r}) \to 0$ as $z \to \infty$. Then the only possibilities are $Z(z) = Z_\mu(z) \equiv c_\mu \exp\left(-z\sqrt{\mu}\right)$ for some $\mu \in \mathbb{R}^+$, where c_μ is a constant.

3.4.1 Bessel functions

We now turn to the radial equation. Multiplying this through by r and noting that $rR'' + R' = (rR')'$ we obtain the standard Sturm–Liouville form

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r}R = -\mu r R \tag{3.57}$$

where the SL coefficients are $p(r) = r$ and $q(r) = -\lambda_n/r = -n^2/r$, while the weight function

$$w(r) = r \tag{3.58}$$

multiplies the eigenvalue $-\mu$. We can actually eliminate μ from this equation by introducing the rescaled radial coordinate[20] $x = r\sqrt{\mu}$ whereupon we find

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0 \tag{3.59}$$

This equation is known as *Bessel's equation of order n*. As with any second order ODE, this equation has two linearly independent solutions. They're denoted $J_n(x)$ and $Y_n(x)$ and are respectively known as *Bessel functions of the first (second) kind, or order n*, or often just 'Bessel functions' for short. The first few Bessel functions are plotted in figures 6–7. You should think of them as analogues of sines and cosines for the radial equation (3.59) instead of for the Cartesian equation $X'' = -\lambda X$.

Bessel functions of the first kind $J_n(x)$ are *regular* at the origin $x = 0$, and in fact all but $J_0(x)$ actually vanish at the origin. By contrast, the Bessel functions of the second kind $Y_n(x)$ are *singular* at the origin. This property means that if we are interested in solutions to Laplace's equation that are well–behaved within some radius r_0, then the Y_n functions cannot arise. On the other hand, if we're interested not in a solid cylinder, but in a cylindrical shell $r_0 \leq r \leq r_1$ then both types of Bessel functions generically do occur.

In the eighteenth century, winters were long and there was no Facebook, so people spent their time working out all sorts of properties of these Bessel functions. Here are some of the more prominent ones, none of which I'm going to prove and none of which I expect you to remember:

– Using the Frobenius method of power series, one can show

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k}$$

whenever n is a non-negative integer. In fact Bessel's equation is of Sturm–Liouville type even if $n \in \mathbb{R}$ rather than $n \in \mathbb{Z}$. This formula still holds provided we replace the factorials by Gamma functions as $k!(k+n)! \rightsquigarrow \Gamma(k+1)\,\Gamma(k+n+1)$.

[20] This x of course has nothing to do with a Cartesian coordinate on the original $\mathbb{R}^3$.

– At small x, one finds

$$\begin{aligned} J_n(x) &= \frac{1}{n!}\left(\frac{x}{2}\right)^n + O(x^{n+2}) && \text{when } n \in \mathbb{Z}_{\geq 0}. \\ Y_0(x) &= O(\ln x) \\ Y_n(x) &= O(x^{-n}) && \text{when } n \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

– At large x, the asymptotic behaviour of the Bessel functions is

$$\begin{aligned} J_n(x) &= \left(\frac{2}{\pi x}\right)^{1/2} \cos\left[x - \frac{n\pi}{2} - \frac{\pi}{4}\right] + O(x^{-3/2}) \\ Y_n(x) &= \left(\frac{2}{\pi x}\right)^{1/2} \sin\left[x - \frac{n\pi}{2} - \frac{\pi}{4}\right] + O(x^{-3/2}) \end{aligned}$$

In particular, this shows that both $J_n(x)$ and $Y_n(x)$ have an infinite number of zeros and turning points. If we recall that $x = \sqrt{\mu}r$, where r was the radial coordinate in Laplace's equation, then we see that the location of these zeros in the radial direction of $\mathbb{R}^3$ depends on the eigenvalue μ.

You can find derivations of these and many more properties of Bessel functions in the books by Arfken & Weber or by Boas that I recommended earlier. As with the Legendre polynomials, I recommend you take a look through some of these, but again the most important fact about the n^{th} order Bessel functions are simply that they are eigenfunctions of a Sturm–Liouville operator that arises from the Laplacian in cylindrical polar coordinates when the angular equation $\Theta'' = -n^2\Theta$ has eigenvalue n^2.

3.4.2 Boundary value problems in cylindrical coordinates

Armed with the Bessel functions, we now return to our boundary value problem. We've found

$$\psi_{\mu,n}(r,\theta,\phi) = (a_n \sin n\theta + b_n \cos n\theta))\, \mathrm{e}^{-z\sqrt{\mu}}\, (J_n(r\sqrt{\mu}) + B_n Y_n(r\sqrt{\mu}))$$

for $n \in \mathbb{Z}_{\geq 0}$ and $\mu \in \mathbb{R}^+$ provides a solution of Laplace's equation that decays as $z \to +\infty$. Since we want the solution to be regular throughout the cylinder, in particular along the z-axis, we must set $B_n = 0$. The boundary condition that $\psi(\mathbf{r}) = 0$ at $r = a$ requires $J_n(a\sqrt{\mu}) = 0$. Thus, for any given n, this fixes μ to be one of the values

$$\sqrt{\mu} = \frac{k_{ni}}{a} \qquad \text{for } i = 1, 2, 3, \ldots \tag{3.60}$$

where i labels the roots $J_n(k_{ni}) = 0$ of the n^{th} Bessel function. This is just like we found for the sinusoidal case: the homogeneous boundary conditions fix the allowed eigenvalues. Since we've so far imposed only the homogeneous boundary conditions, we can consider an arbitrary linear combination of these normal modes. Relabelling constants, we have

$$\psi(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} (A_{ni} \sin n\theta + B_{ni} \cos n\theta)\, J_n(k_{ni} r/a)\, \mathrm{e}^{-k_{ni} z/a} \tag{3.61}$$

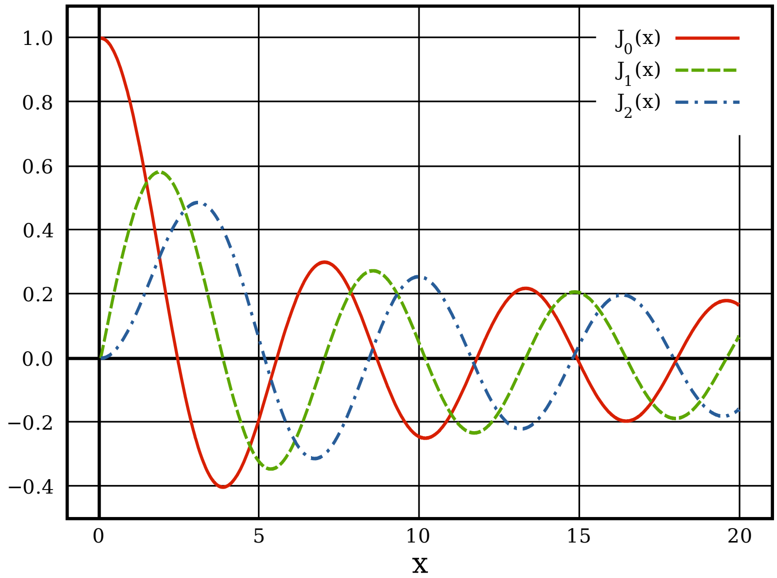

Figure 6. *Plots of some Bessel functions of the first kind* $J_n(x)$*, for order* $n = 0, 1, 2$*. The location of the first zero in* $x > 0$ *increases as the order of the Bessel function increases, and* $J_n(x)$ *is falling as it passes through this zero. Note that the zeros are not evenly spaced.* $J_0(0) = 1$ *while all other* $J_n(x)$ *vanish at the origin.*

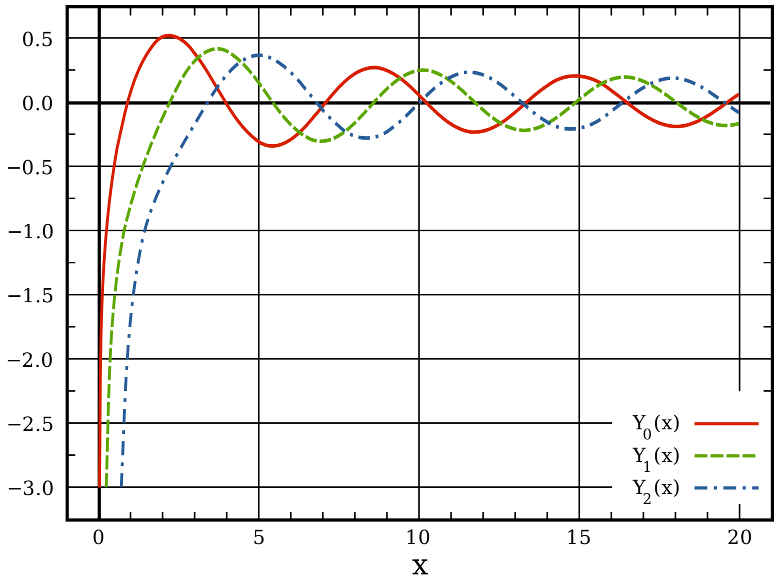

Figure 7. *Plots of some Bessel functions of the second kind* $Y_n(x)$*, for order* $n = 0, 1, 2$*. Again, the location of the first zero increases as the order of the Bessel function, but now* $Y_n(x)$ *rises as it passes through zero. All the* $Y_n(x)$ *are singular at the origin.*

as our general solution.

The final step is to fix the constants A_{ni} and B_{ni} by imposing the inhomogeneous boundary condition $\psi(r,\theta,0) = f(r,\theta)$ when $z = 0$. Again, this is done using the Sturm–Liouville orthogonality conditions

$$\int_0^a J_n(k_{mj}r/a)\, J_n(k_{ni}r/a)\, r\, \mathrm{d}r = \frac{a^2}{2}\, \delta_{i,j}\, \left[J'_n(k_{ni})\right]^2 = \frac{a^2}{2}\, \delta_{i,j}\, [J_{n+1}(k_{ni})]^2 \tag{3.62}$$

for the Bessel functions. There are two things to note about this orthogonality relation. First, in (3.58) we identified the weight function in Bessel's equation as $w(r) = r$; it appears here. The second point note is that the Bessel functions $J_n(k_{ni}r)$ satisfy orthogonality relations *for each fixed* n*, but between different values* i*,* j *of the index labelling the roots.* You'll derive this relation in the problems sheets, and it's just what we need here. We can use the orthogonality relations

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin m\theta\, \sin n\theta\, \mathrm{d}\theta = \delta_{m,n}\,,\ \ \frac{1}{\pi}\int_{-\pi}^{\pi} \cos m\theta\, \cos n\theta\, \mathrm{d}\theta = \delta_{m,n}\,,\ \ \frac{1}{\pi}\int_{-\pi}^{\pi} \sin m\theta\, \cos n\theta\, \mathrm{d}\theta = 0$$

among the trigonometric functions to fix a value of the n index, determining which order of Bessel function we're considering. For example, setting $z = 0$ and integrating (3.61) against $\cos m\theta$ gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos m\theta\, f(r,\theta)\, \mathrm{d}\theta = \sum_{i=1}^{\infty} B_{mi}\, J_m(k_{mi}r/a) \tag{3.63}$$

involving only the m^{th} order Bessel function of the first kind, but with k_{mi} still summed over all the roots. The orthogonality condition (3.62) for the Bessel functions then gives

$$B_{mj} = \frac{2}{\pi a^2}\frac{1}{[J'_m(k_{mj})]^2}\int_0^a \left[J_m(k_{mj}r/a)\int_{-\pi}^{\pi} \cos m\theta\, f(r,\theta)\, \mathrm{d}\theta\right] r\, \mathrm{d}r \tag{3.64}$$

which fixes the constants B_{mj} in terms of the function f in the boundary condition. The B_{mj} are determined similarly.

As an example, let's suppose $f(r,\theta) = C$, a constant. Since this is in particular independent of θ, we see immediately from (3.64) that $B_{mj} = 0$ whenever $m \neq 0$. (The A_{mj} are similarly all zero.) The only non-zero coefficients are thus B_{0j}, multiplying a function with trivial angular dependence. These are

$$B_{0j} = \frac{2C}{a^2}\frac{1}{[J_1(k_{0j})]^2}\int_0^a J_0(k_{0j}r/a)\, r\, \mathrm{d}r = \frac{2C}{k_{0j}}\frac{1}{J_1(k_{0j})} \tag{3.65}$$

where the second equality is something you'll prove in the problem sets.