1B Methods – Example Sheet 1

Please email me with any comments, particularly if you spot an error. Problems marked with an asterisk (∗) are optional; only attempt them if you have time.

1. (a) Sketch the 2π-periodic function defined by $f(\theta) = (\theta^2 - \pi^2)^2$ when $\theta \in [-\pi, \pi)$. Find the Fourier series of this function. For what values of θ does this Fourier series converge to $f(\theta)$?

   (b) Obtain the Fourier series of the 2π-periodic function defined by $f(\theta) = \mathrm{e}^{\theta}$ for $\theta \in [-\pi, \pi)$. Deduce that
$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}\left(\pi \coth \pi - 1\right) .$$

   (c) Find the Fourier series of $f(\theta) = \theta \mathrm{e}^{\mathrm{i}\theta}$. Use your result to write down real Fourier series for $\theta \cos\theta$ and $\theta \sin\theta$.

2. Can you tell whether a function is real by looking at its complex Fourier coefficients? How about if it's even / odd ?

3. A certain function $\vartheta(x,t)$ obeys the conditions
$$\begin{aligned} \vartheta(x+1,t) &= \vartheta(x,t) \\ \vartheta(x+\mathrm{i}t,t) &= \mathrm{e}^{\pi t - 2\pi \mathrm{i} x}\, \vartheta(x,t) \\ \int_0^1 \vartheta(x,t)\, dx &= 1 \,. \end{aligned}$$

   (a) Using the first condition, represent $\vartheta(x,t)$ as a Fourier series with some unknown, t-dependent coefficients.

   (b) Use the remaining conditions to fix these coefficients. For what range of t does your series converge?

   (c) Show that
$$\frac{\partial \vartheta(x,t)}{\partial t} = \frac{1}{4\pi} \frac{\partial^2 \vartheta(x,t)}{\partial x^2} \,.$$

   (d*) What is the initial value $\lim_{t\to 0^+} \vartheta(x,t)$?

4. The *sawtooth function* is defined to be the function
$$f(\theta) = \theta$$
for $\theta \in [-\pi, \pi)$.

   (a) Compute the Fourier series of the sawtooth function and comment on its value at $\theta = \pi$.

(b) By applying Parseval's identity to the sawtooth function, show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}\,.$$

(c) The *Riemann ζ-function* is defined by the infinite sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

whenever $\mathrm{Re}(s) > 1$. Show that if m is a positive integer, $\zeta(2m) = q\pi^{2m}$ where q is rational. [*Hint: Induction from part (b).*]

5.(*) The *square wave function* is given by

$$f(\theta) = \begin{cases} 1 & \text{for } \theta \in (0,\pi) \\ 0 & \text{for } \theta \in (-\pi,0) \end{cases}\,.$$

(a) Sketch $f(\theta)$ and show that its Fourier series is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{2n-1}\,.$$

(b) Defining the partial sum of this series to be $S_N(\theta) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{N} \frac{\sin(2n-1)\theta}{2n-1}$, show that

$$S_N(\theta) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\theta} \frac{\sin 2Nt}{\sin t}\,\mathrm{d}t\,.$$

[*Hint: Consider $S_N'(\theta)$ for the two expressions.*]

(c) Deduce that $S_N(\theta)$ has extrema at $\theta = m\pi/2N$, where $m = 1, 2, \ldots, 2N-1, 2N+1, \ldots$ (*i.e.*, m is any natural number except for even multiples of N), and that for large N, the height of the first maximum is approximately

$$S_N(\pi/2N) \approx \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin u}{u}\,\mathrm{d}u \qquad (\approx\ 1.089)\,.$$

Comment on the accuracy of Fourier series at discontinuities. [*This question takes you through some important steps used in the proof of Fourier's theorem — refer, for example, to chapter 14 of Jeffreys & Jeffreys.*]

6. In the boundary value problem

$$y'' + \lambda y = 0 \qquad \text{with} \qquad y(0) = 0, \qquad y(1) + y'(1) = 0,$$

show that the eigenvalue λ can take infinitely many values $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$. Indicate roughly the behaviour of λ_n as $n \to \infty$.

7. Express the following equations in Sturm–Liouville form:

   (a) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$,

   (b) $xy'' + (b-x)y' - ay = 0$,

   where n, a and b are constants. Find the eigenvalues and eigenfunctions for

$$y'' + 4y' + 4y = -\lambda y,$$

   where $y(0) = y(1) = 0$. What is the orthogonality relation for these eigenfunctions?

8. Show that the eigenvalues of the Sturm-Liouville problem

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) = -\lambda x u \qquad x \in (0,1)$$

   with $u(x)$ bounded as $x \to 0$ and $u(1) = 0$ are $\lambda = a_n^2$ for $n \in \mathbb{N}$, where a_n is the location of the n^{th} zero of the Bessel function $J_0(x)$; *i.e.* $J_0(a_n) = 0$. [*Recall that* $J_0(x)$ *is the unique solution of* $(xu'(x))' + xu(x) = 0$ *that is regular at* $x = 0$].

   (a) Using integration by parts on the differential equations obeyed by $J_0(\alpha x)$ and $J_0(\beta x)$, show that

$$\int_0^1 J_0(\alpha x) J_0(\beta x)\, x\, dx = \frac{\beta J_0(\alpha) J_0'(\beta) - \alpha J_0(\beta) J_0'(\alpha)}{\alpha^2 - \beta^2}$$

   and

$$\int_0^1 J_0(a_m x) J_0(a_n x)\, x\, dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2}[J_0'(a_n)]^2 & m = n. \end{cases}$$

   (b) Assume that the inhomogeneous equation

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \tilde{\lambda} x u = x f(x)\,,$$

   where $\tilde{\lambda}$ is not an eigenvalue, has a unique solution obeying $u(1) = 0$ and $u(x)$ bounded as $x \to 0$. In the case that $f(x)$ obeys the same boundary conditions as $u(x)$, obtain the expansion of $u(x)$ in terms of the $J_0(a_n x)$, assuming that these form a complete set.