1B Methods – Example Sheet 2

Please email me with any comments, particularly if you spot an error. Problems marked with an asterisk (∗) are optional; only attempt them if you have time.

1. If y_m and y_n are real eigenfunctions of the Sturm-Liouville equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (\lambda - q(x))y = 0 \qquad \text{for} \qquad a < x < b$$

satisfying the normalisation condition $\int_a^b y_m^2\,dx = \int_a^b y_n^2\,dx = 1$, show that (subject to boundary conditions which you should state)

$$\int_a^b (p\,y_m'\;y_n' + q\,y_m y_n)\,dx = \lambda_m \delta_{mn}$$

(no summation). With P_n a Legendre polynomial, use this result to evaluate

$$\int_{-1}^{1} (1-x^2)P_m'(x)P_n'(x)\,dx\,.$$

2. Show that

$$\frac{\sin{(n+1)\theta}}{\sin\theta} = \sum_{\ell=0}^{n} P_\ell(\cos\theta)P_{n-\ell}(\cos\theta)\,.$$

3. Show that whenever $\mathbf{r} \neq \mathbf{r}'$, the function $1/|\mathbf{r}-\mathbf{r}'|$ satisfies Laplace's equation. Find the potential inside a spherical region bounded by two (conducting) hemispheres upon which the potential takes the values $\pm V$ respectively. [*Hint:* Recall the orthogonality relation $\int_{-1}^{1} P_\ell(x)\,P_m(x)\,dx = \frac{2}{2\ell+1}\delta_{\ell m}$.]

4. The potential ϕ satisfies Laplace's equation inside the unit disc $D = \{(r,\theta) \in \mathbb{R}^2\,|\,r \leq 1\}$, with boundary condition

$$\phi(r=1,\theta) = \begin{cases} \pi/2 & 0 \leq \theta < \pi\,, \\ -\pi/2 & \pi \leq \theta < 2\pi\,. \end{cases}$$

Using the method of separation of variables show that

$$\phi(r,\theta) = 2\sum_{n=1}^{\infty} r^{2n-1}\frac{\sin(2n-1)\theta}{2n-1}\,.$$

Sum the series using the substitution $z = re^{i\theta}$. Interpret your solution geometrically in terms of the angle between the lines to the two points on the boundary where the data jumps.

5*. Suppose $\psi : \Omega \to \mathbb{R}$ obeys Laplace's equation throughout a domain $\Omega \subset \mathbb{R}^n$, and that ψ is continuous over $\partial\Omega$. Also let $B_r(x)$ be a ball of radius r, centred on x, that is entirely contained inside Ω.

(a) By writing the integral

$$\frac{1}{\mathrm{vol}(B_r)}\int_{B_r(x)} \nabla^2\psi \; d^n x$$

as a boundary term, show that $\psi(x)$ equals its average value over the sphere $\partial B_r(x)$. (This is the *mean value theorem* for Laplace's equation.)

(b) Let $M = \max\{\psi(y)\,|\,y \in \partial\Omega\}$. By considering the integral

$$\frac{1}{\mathrm{vol}(B_r)}\int_{B_r(x)} [\psi(x) - M]\, d^n x$$

show that $\psi(x) \leq M$ at all points inside Ω, with equality iff ψ is constant everywhere. [*Hint*: look for a contradiction.] (This is the *maximum principle* for Laplace's equation.)

(c) Show also that $\psi(x) \geq m = \min\{\psi(y)\,|\,y \in \partial\Omega\}$ whenever x is inside Ω.

6. Consider the unit disc, with initial temperature distribution $\psi_0(r,\theta)$. Require the boundary of the disc to be held at (wlog) zero temperature $\psi(1,\theta,t) = 0$ for all $t > 0$. By assuming that the temperature satisfies the diffusion equation in the disc (with unit diffusion coefficient) show that the solution is

$$\psi = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty} c_{nk}\, J_n(j_{nk} r)\exp[in\theta - j_{nk}^2 t],$$

where j_{nk} is the k^{th} smallest (positive) zero of the n^{th} order Bessel function of the first kind, (*i.e.* $J_n(j_{nk}) = 0$). Present an appropriate expression for c_{nk}, showing explicitly that

$$\int_0^1 J_n(j_{nk}r)J_n(j_{nl}r)r dr = \delta_{kl}\frac{[J_n'(j_{nk})]^2}{2} = \delta_{kl}\frac{J_{n+1}^2(j_{nk})}{2}\,.$$

Suppose now that the initial temperature $\psi_0(r,\theta) = \Psi_0$ is constant for all $r < 1$. Show that the only non-zero coefficients are

$$c_{0k} = \frac{2\Psi_0}{j_{0k}J_1(j_{0k})}\,.$$

What is the spatial structure of the temperature distribution as $t \to \infty$?

[*Hint:* The recursion relations $[z^{-\nu}J_\nu(z)]' = -z^{-\nu}J_{\nu+1}(z)$ and $[z^{\nu+1}J_{\nu+1}(z)]' = z^{\nu+1}J_\nu(z)$ may be useful.]

7. A uniform string of line density μ and tension T undergoes small transverse vibrations of amplitude $y(x,t)$. The string is fixed at $x = 0$ and $x = L$, and satisfies the initial conditions

$$y(x,0) = 0$$
$$\frac{\partial y}{\partial t}(x,0) = \frac{4V}{L^2}\,x(L-x)$$

for $0 < x < L$. Using the fact that $y(x,t)$ is a solution of the wave equation, find the amplitudes of the normal modes and deduce the kinetic and potential energies of the string at time t.

8. A uniform stretched string of length L, density per unit length μ and tension $T = \mu c^2$ is fixed at both ends. The motion of the string is resisted by the surrounding medium, the resistive force on unit length being $-2k\mu\,(\partial y/\partial t)$, where y is the transverse displacement and the constant $k = \pi c/L$.

   (a) Show that the equation of motion of the string is

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} + 2k\frac{\partial y}{\partial t}\,,$$

   and find $y(x,t)$, given that

$$y(x,0) = A\sin(\pi x/L) \qquad \text{and} \qquad \frac{\partial y}{\partial t}(x,0) = 0\,.$$

   (b) If an extra transverse force $F_0\sin(\pi x/L)\cos(\pi c t/L)$ per unit length acts on the string, find the resulting forced oscillation. [*Hint:* You may find it useful to guess a particular solution to combine with the general homogeneous solution that you derived in (i).]

9. A string of uniform density is stretched along the x-axis under tension T and undergoes small transverse oscillations in the (x,y) plane so that its displacement $y(x,t)$ satisfies

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}\,, \qquad (\star)$$

   where c is a constant.

   (a) Show that if a mass M is fixed to the string at $x = 0$ then its equation of motion can be written

$$\frac{M}{T}\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=0} = \left.\frac{\partial y}{\partial x}\right|_{x=0_+} - \left.\frac{\partial y}{\partial x}\right|_{x=0_-}\,.$$

   (b) Suppose that a wave $\exp[i\omega(t - x/c)]$ is incident from $x = -\infty$. Obtain the amplitudes and phases of the reflected and transmitted waves, and comment on their values when $\lambda = M\omega c/T$ is large or small.

10. The displacement $y(x,t)$ of a uniform string stretched between the points $x = 0$ and $x = L$ satisfies the wave equation $(\star)$ given above, but with the boundary conditions

$$y(0,t) = y(L,t) = 0\,.$$

   For $t < 0$ the string oscillates in its fundamental mode and $y(x,0) = 0$. A musician strikes the midpoint of the string impulsively at time $t = 0$ so that the change in $\partial y/\partial t$ at $t = 0$ is $\lambda\,\delta(x - L/2)$. Find $y(x,t)$ for $t > 0$.