1B Methods – Example Sheet 3

Please email me with any comments, particularly if you spot an error. Problems marked with an asterisk (*) are optional; only attempt them if you have time.

1. Suppose $\psi(x)$ is once continuously differentiable. By considering the inner product with a test function, justify the formula

$$\psi(x) \delta'(x) = \psi(0) \delta'(x) - \psi'(0) \delta(x).$$

Find a similar formula for $\psi(x) \delta^{(n)}(x)$ in the case that $\psi$ is $n$ times continuously differentiable.

2. Suppose $x \in [-\pi, \pi]$. Do the Fourier series of $\delta(x)$ and $|p| \delta(px)$ agree? Why / why not?

3. The reading $\theta(t)$ of an ammeter satisfies

$$\ddot{\theta}(t) + 2p \dot{\theta}(t) + (p^2 + q^2)\theta(t) = f(t),$$

where $p, q$ are constants with $p > 0$. The ammeter is set so that $\theta(0) = \dot{\theta}(0) = 0$. Assuming $q \neq 0$, show by constructing the Green’s function that

$$\theta(t) = \frac{1}{q} \int_0^t e^{-p(t-\tau)} \sin[q(t-\tau)] f(\tau) d\tau.$$

Derive the same result by taking the Fourier transform of the original equation, showing that the transfer function for this system is

$$\tilde{R}(\omega) = \frac{1}{2qi} \left[ \frac{1}{i\omega + p - iq} - \frac{1}{i\omega + p + iq} \right].$$

4. Obtain the Green’s function $G(x, \xi)$ satisfying

$$\frac{d^2 G}{dx^2} - \lambda^2 G = \delta(x - \xi) \quad \text{and} \quad G(0, \xi) = G(1, \xi) = 0,$$

where $\lambda$ is real, and $x \in [0,1]$ with $\xi \in (0,1)$. Show that the solution to the equation

$$\frac{d^2 y}{dx^2} - \lambda^2 y = f(x),$$

subject to the same boundary conditions, is

$$y = -\frac{1}{\lambda \sinh \lambda} \left[ \sinh \lambda x \int_x^1 f(\xi) \sinh \lambda(1 - \xi) d\xi + \sinh \lambda(1 - x) \int_0^x f(\xi) \sinh \lambda \xi d\xi \right].$$
5. A linear differential operator is defined by
\[ L_x y = -\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) + y. \]

By writing \( y = z/x \) or otherwise, find those solutions of \( L_x y = 0 \) which are either (a) bounded as \( x \to 0 \), or (b) bounded as \( x \to \infty \). Find the Green’s function \( G(x, a) \) satisfying
\[ L_x G(x, a) = \delta(x - a), \]
and both conditions (a) and (b). Use \( G(x, a) \) to solve
\[ L_x y(x) = \begin{cases} 1 & \text{for } x \in [0, R] \\ 0 & \text{for } x > R \end{cases} \]
subject to conditions (a) and (b). Show that the solution has the form
\[ y(x) = \begin{cases} 1 + \frac{A}{x} \sinh x & \text{for } x \in [0, R] \\ \frac{B}{x} e^{-x} & \text{for } x > R \end{cases} \]
for suitable constants \( A, B \).

6. By using differentiation and shift properties, calculate the Fourier transform of the Gaussian distribution \( f(x) = \exp[-n^2(x - \mu)^2] \) for constants \( n \) and \( \mu \).

Now let \( \mu = 0 \), and consider \( \delta_n(x) = (n/\sqrt{\pi})f(x) \). Sketch \( \delta_n(x) \) and \( \tilde{\delta}_n(k) \) for small and large \( n \). Evaluate
\[ \int_{-\infty}^{\infty} \delta_n(x) \, dx. \]

What is happening as \( n \to \infty \)?

7. Let
\[ f(x) = \begin{cases} e^{-x} & \text{for } x \in (0, \infty) \\ 0 & \text{for } x \in (-\infty, 0) \end{cases}. \]

Show that the Fourier transform
\[ \tilde{f}(k) = \frac{1 - ik}{1 + k^2}. \]

What value does the inverse Fourier transform of \( \tilde{f}(k) \) take at \( x = 0 \)? Explain this as fully as you can. (Inversion for general \( x \) is really straightforward with Complex Methods.)

8. By considering the Fourier transform of the function
\[ f(x) = \begin{cases} \cos x & \text{for } |x| < \pi/2 \\ 0 & \text{for } |x| \geq \pi/2, \end{cases} \]
and the Fourier transform of its derivative, show that
\[
\int_0^\infty \frac{\pi^2 \cos^2 t}{(\pi^2 - t^2)^2} t \, dt = \frac{\pi}{4}
\text{ and that } \int_0^\infty \frac{t^2 \cos^2 t}{(\pi^2 - t^2)^2} \, dt = \frac{\pi}{4}.
\]

9. Show that, for \( \alpha \in \mathbb{R} \), the inverse Fourier transform of the function
\[
\tilde{f}_\alpha(k) = \begin{cases} 
 e^{\alpha k} - e^{-\alpha k} & \text{for } |k| \leq 1 \\
 0 & \text{for } |k| > 1
\end{cases}
\]
is
\[
f_\alpha(x) = \frac{2i}{\pi(\alpha^2 + x^2)} (\alpha \cosh \alpha \sin x - x \cos x \sinh \alpha).
\]
Now let \( \Omega = \{ (x, y) \in \mathbb{R}^2 | 0 \leq y \leq 1 \} \). Suppose that \( \phi : \Omega \to \mathbb{R} \) solves Laplace’s equation \( \nabla^2 \phi = 0 \) inside \( \Omega \) and obeys the boundary conditions
\[
\phi(x, 0) = f_1(x) \quad \text{and} \quad \phi(x, 1) = 0,
\]
where \( f_1(x) \) is the function given above at \( \alpha = 1 \). By taking the Fourier transform of Laplace’s equation (wrt \( x \)), find \( \phi \).