1B Methods – Example Sheet 3

Please email me with any comments, particularly if you spot an error. Problems marked with an asterisk (∗) are optional; only attempt them if you have time.

1. Suppose $\psi(x)$ is once continuously differentiable. By considering the inner product with a test function, justify the formula

$$\psi(x)\,\delta'(x) = \psi(0)\,\delta'(x) - \psi'(0)\,\delta(x)\,.$$

Find a similar formula for $\psi(x)\,\delta^{(n)}(x)$ in the case that ψ is n times continuously differentiable.

2. Suppose $x \in [-\pi, \pi]$. Do the Fourier series of $\delta(x)$ and $|p|\,\delta(px)$ agree? Why / why not?

3. The reading $\theta(t)$ of an ammeter satisfies

$$\ddot{\theta}(t) + 2p\,\dot{\theta}(t) + (p^2 + q^2)\theta(t) = f(t)\,,$$

where p, q are constants with $p > 0$. The ammeter is set so that $\theta(0) = \dot{\theta}(0) = 0$. Assuming $q \neq 0$, show by constructing the Green's function that

$$\theta(t) = \frac{1}{q}\int_0^t \mathrm{e}^{-p(t-\tau)} \sin[q(t-\tau)]\, f(\tau)\, d\tau\,.$$

Derive the same result by taking the Fourier transform of the original equation, showing that the transfer function for this system is

$$\tilde{R}(\omega) = \frac{1}{2qi}\left[\frac{1}{i\omega + p - iq} - \frac{1}{i\omega + p + iq}\right]\,.$$

4. Obtain the Green's function $G(x, \xi)$ satisfying

$$\frac{d^2G}{dx^2} - \lambda^2 G = \delta(x-\xi) \qquad \text{and} \qquad G(0,\xi) = G(1,\xi) = 0\,,$$

where λ is real, and $x \in [0, 1]$ with $\xi \in (0, 1)$. Show that the solution to the equation

$$\frac{d^2y}{dx^2} - \lambda^2 y = f(x)\,,$$

subject to the same boundary conditions, is

$$y = -\frac{1}{\lambda \sinh\lambda}\left[\sinh\lambda x \int_x^1 f(\xi)\sinh\lambda(1-\xi)\,d\xi + \sinh\lambda(1-x)\int_0^x f(\xi)\sinh\lambda\xi\,d\xi\right]\,.$$

5. A linear differential operator is defined by

$$L_x y = -\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + y\,.$$

By writing $y = z/x$ or otherwise, find those solutions of $L_x y = 0$ which are either (a) bounded as $x \to 0$, or (b) bounded as $x \to \infty$. Find the Green's function $G(x,a)$ satisfying

$$L_x G(x,a) = \delta(x-a)\,,$$

and both conditions (a) and (b). Use $G(x,a)$ to solve

$$L_x y(x) = \begin{cases} 1 & \text{for } x \in [0,R] \\ 0 & \text{for } x > R \end{cases}$$

subject to conditions (a) and (b). Show that the solution has the form

$$y(x) = \begin{cases} 1 + \dfrac{A}{x}\sinh x & \text{for } x \in [0,R] \\ \dfrac{B}{x}\,e^{-x} & \text{for } x > R \end{cases}$$

for suitable constants A, B.

6. By using differentiation and shift properties, calculate the Fourier transform of the Gaussian distribution $f(x) = \exp[-n^2(x-\mu)^2]$ for constants n and μ.

Now let $\mu = 0$, and consider $\delta_n(x) = (n/\sqrt{\pi})f(x)$. Sketch $\delta_n(x)$ and $\widetilde{\delta_n}(k)$ for small and large n. Evaluate

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx\,.$$

What is happening as $n \to \infty$?

7. Let

$$f(x) = \begin{cases} \mathrm{e}^{-x} & \text{for } x \in (0,\infty) \\ 0 & \text{for } x \in (-\infty,0) \end{cases}\,.$$

Show that the Fourier transform

$$\tilde{f}(k) = \frac{1-ik}{1+k^2}\,.$$

What value does the inverse Fourier transform of $\tilde{f}(k)$ take at $x = 0$? Explain this as fully as you can. (Inversion for general x is really straightforward with Complex Methods.)

8. By considering the Fourier transform of the function

$$f(x) = \begin{cases} \cos x & \text{for } |x| < \pi/2 \\ 0 & \text{for } |x| \geq \pi/2\,, \end{cases}$$

and the Fourier transform of its derivative, show that

$$\int_0^\infty \frac{\frac{\pi^2}{4}\cos^2 t}{\left(\frac{\pi^2}{4}-t^2\right)^2}dt = \frac{\pi}{4} \qquad \text{and that} \qquad \int_0^\infty \frac{t^2\cos^2 t}{\left(\frac{\pi^2}{4}-t^2\right)^2}dt = \frac{\pi}{4}\,.$$

9. Show that, for $\alpha \in \mathbb{R}$, the inverse Fourier transform of the function

$$\tilde{f}_\alpha(k) = \begin{cases} e^{k\alpha} - e^{-k\alpha} & \text{for } |k| \leq 1 \\ 0 & \text{for } |k| > 1 \end{cases}$$

is

$$f_\alpha(x) = \frac{2i}{\pi(\alpha^2 + x^2)}\left(\alpha\cosh\alpha\sin x - x\cos x\sinh\alpha\right).$$

Now let $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 1\}$. Suppose that $\phi : \Omega \to \mathbb{R}$ solves Laplace's equation $\nabla^2\phi = 0$ inside Ω and obeys the boundary conditions

$$\phi(x, 0) = f_1(x) \qquad \text{and} \qquad \phi(x, 1) = 0\,,$$

where $f_1(x)$ is the function given above at $\alpha = 1$. By taking the Fourier transform of Laplace's equation (wrt x), find ϕ.

 d.b.skinner@damtp.cam.ac.uk