6 Perturbative Renormalization

6.1 One–loop renormalization of $\lambda\phi^4$ theory

Consider the scalar theory

$$S_{\Lambda_0}[\phi] = \int \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right] \mathrm{d}^4x \tag{6.1}$$

with initial couplings m^2 and λ, defined for momentum modes $\leq \Lambda_0$. (The mass coupling is dimensionful here.) From the analysis of the previous chapter, we expect that near the Gaussian fixed point, the mass parameter is relevant, while the quartic coupling is marginally irrelevant. Let's see how these expectations are borne out in perturbative calculations.

Firstly, the mass term receives a correction from the Feynman diagram

If we're integrating out all momentum modes $|p| \leq \Lambda_0$ then this diagram is given by the Euclidean signature loop integral

$$\begin{aligned} -\frac{1}{2}\lambda \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2+m^2} &= -\lambda_0 \frac{\mathrm{Vol}(S^3)}{2(2\pi)^4}\int_0^{\Lambda_0} \frac{p^3\,\mathrm{d}p}{p^2+m^2} \\ &= -\frac{\lambda\, m^2}{32\pi^2}\int_0^{\Lambda_0^2/m^2} \frac{x\,\mathrm{d}x}{1+x} \\ &= \frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2\ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right]\,, \end{aligned} \tag{6.2}$$

where the factor of $1/2$ is the symmetry factor of the diagram, and we have used the fact that $\mathrm{Vol}(S^3) = 2\pi^2$. As expected, this result shows that the mass parameter is relevant: There's a quadratic divergence as we try to take the continuum limit $\Lambda_0 \to \infty$ (as well as a subleading logarithmic one).

If we wish to obtain finite results in the continuum limit, then we must tune the couplings (m^2, λ) in our scale-Λ_0 action so as to make (6.2) finite in the limit. Achieving such a tuning directly from (6.2) is complicated, but fortunately we don't need to do this. Recall from section 5.2.2 that we tune by modifying the action to include counterterms:

$$S_{\Lambda_0}[\phi] \to S_{\Lambda_0}[\phi] + \hbar S^{\mathrm{CT}}[\phi, \Lambda_0] \tag{6.3}$$

where in this case the counterterm action is

$$S^{\mathrm{CT}}[\phi, \Lambda_0] = \int \left[\frac{1}{2}\delta Z(\partial\phi)^2 + \frac{1}{2}\delta m^2\phi^2 + \frac{1}{4!}\delta\lambda\,\phi^4\right] \mathrm{d}^4x \tag{6.4}$$

with $(\delta Z, \delta m^2, \delta\lambda)$ representing our ability to adjust the couplings in the original action (including the coupling to the kinetic term — or wavefunction renormalization). These counterterm couplings will depend explicitly on Λ_0, as they represent the tuning that must be performed starting from the Λ_0 cut–off theory.

The fact that the counterterm action is proportional to $\hbar$ means that classical contributions from $S^{\rm CT}$ contribute to the same order in $\hbar$ as 1-loop diagrams from $S_{\Lambda_0}[\phi]$. Thus the mass term also receives a correction at order $\hbar^0$ from the tree diagram

where the cross represents the insertion of the counterterm δm^2 treated as a vertex. The full quantum contribution to the mass term is thus

$$\delta m^2 + \frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2 \ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right] \tag{6.5}$$

to 1-loop accuracy.

6.1.1 The on–shell renormalization scheme

The *raison d'etre* of counterterms is to ensure (6.5) has a finite continuum limit, so that they must cancel the part of (6.2) that diverges as $\Lambda_0 \to \infty$. This still leaves us a lot of freedom in choosing how much of the *finite* part of the loop integral can also be absorbed by the counterterms. There's no preferred way to do this, and any such choice is called a *renormalization scheme*. Ultimately, all physically measurable quantities (such as cross–sections, branching ratios, particle lifetimes *etc.*) should be independent of the choice of renormalization scheme.

One physically motivated choice is called the *on-shell scheme*. In this scheme, we fix the mass counterterm δm^2 by asking that, once we take the continuum limit, the pole in the exact propagator $\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}p\cdot x}\langle\phi(x)\phi(0)\rangle$ in momentum space occurs at some experimentally measured value. (Recall that the cross-section σ has a peak at the location of poles (in the complex momentum plane) in the S-matrix, so this location is an experimentally measurable quantity.) For example, it would be natural to try to set up our original action so that the coupling m^2 is indeed this experimentally measured value. If we denote

$$M^2(p^2) = \text{sum of all 1PI contributions to the mass}^2 \text{ term} \tag{6.6}$$

then the exact propagator can be written as a geometric series

$$= \frac{1}{p^2 + m^2 + M^2(p^2)} \, .$$

Taking into account both the loop integral and the counterterm, to 1-loop accuracy we'd find the propagator

$$= p^2 + m^2 + \delta m^2 + \frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2 \ln\left(1 + \frac{\Lambda_0^2}{m^2}\right)\right] \tag{6.7}$$

Consequently, in this scheme we should choose

$$\delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2\left[\ln\left(1 + \frac{\Lambda_0^2}{m^2}\right) - 1\right]\right] \tag{6.8}$$

so as to completely cancel the 1-loop contribution to the mass term.

Notice that we cannot sensibly take the limit $\Lambda_0 \to \infty$ in either the 1-loop correction or the counterterm separately, reflecting the fact that the path integral measure $D\phi$ over all modes in the continuum does not exist. However, we can take the continuum limit of the correlation function $\langle \phi(x)\phi(y) \rangle$ (or it's momentum space equivalent) *after* having computed the path integral. The combined contribution of the 1-loop diagram using the initial action and the tree–level diagram involving the counterterm (6.8) remains finite in the continuum. Of course, while our tuning (**??**) is good to 1-loop accuracy, if we computed the path integral to higher order in perturbation theory, including the contribution of higher–loop Feynman diagrams, we would have to make further tunings in δm^2 (proportional to higher powers of the coupling λ) so as to still retain a finite limiting result.

6.1.2 Dimensional regularization

While the idea of integrating out momenta only up to a cut–off Λ_0 is very intuitive, in more complicated examples it becomes very cumbersome to perform the loop integrals over $|p|$ with a finite upper limit. In studying the local potential approximation in different dimensions (such as for the Wilson–Fisher fixed point), we saw that different couplings change their behaviour, becoming either relevant, marginal or irrelevant, as the dimension of the space is changed. Since the irrelevant couplings always have vanishing continuum limits, whereas (untuned) relevant ones diverge in the continuum, this suggests that we can regularize our loop integrals by analytically continuing the dimension d of our space. I stress that this is purely a convenient device for regularizing loop integrals — there is no suggestion that Nature 'really' lives in non–integer dimensions. Furthermore, dimensional regularization is only a *perturbative* regularization scheme: whilst we shall see that it does allow us to regulate individual loop integrals over the full range $|p| \in [0, \infty)$, unlike imposing a cut–off or a lattice regularization, dimensional regularization does *not* provide any definition of a finite–dimensional path integral measure. Despite these conceptual shortcomings, its practical convenience makes it an essential tool in perturbative calculations, especially in gauge theories as we shall see later.

In d–dimensions, the quartic coupling λ has non–zero mass dimension $4 - d$, so we replace $\lambda \to \mu^{4-d}\lambda$ where the new λ is dimesionless, and μ is an arbitrary mass scale.

Thus, in dimensional regularization, we obtain the 1-loop correction to the mass coupling

$$\frac{1}{2}\lambda\mu^{4-d}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^2+m^2}=\lambda\mu^{4-d}\frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d}\int_0^\infty\frac{p^{d-1}\,\mathrm{d}p}{p^2+m^2}\,, \tag{6.9}$$

where $\mathrm{Vol}(S^{d-1})$ is the surface volume of a unit sphere in d dimensions. To compute this, note that

$$\begin{aligned}\pi^{\frac{d}{2}}&=\int_{\mathbb{R}^d}\mathrm{e}^{-(x,x)}\,\mathrm{d}^d x=\mathrm{Vol}(S^{d-1})\int_0^\infty \mathrm{e}^{-r^2}\,r^{d-1}\,\mathrm{d}r\\&=\frac{\mathrm{Vol}(S^{d-1})}{2}\int_0^\infty \mathrm{e}^{-u}\,u^{\frac{d}{2}-1}\,\mathrm{d}u\end{aligned} \tag{6.10}$$

and thus

$$\mathrm{Vol}(S^{d-1})=\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\,. \tag{6.11}$$

The remaining integral is

$$\begin{aligned}\mu^{4-d}\int_0^\infty\frac{p^{d-1}\,\mathrm{d}p}{p^2+m^2}&=\frac{1}{2}\mu^{4-d}\int_0^\infty\frac{(p^2)^{d/2-1}\,\mathrm{d}(p^2)}{p^2+m^2}\\&=\frac{m^2}{2}\left(\frac{\mu}{m}\right)^{4-d}\int_0^1(1-u)^{\frac{d}{2}-1}\,u^{-\frac{d}{2}}\,\mathrm{d}u\\&=\frac{m^2}{2}\left(\frac{\mu}{m}\right)^{4-d}\frac{\Gamma(\frac{d}{2})\,\Gamma(1-\frac{d}{2})}{\Gamma(1)}\,,\end{aligned} \tag{6.12}$$

where $u:=m^2/(p^2+m^2)$ and we have used the defintion of the Euler beta–function

$$B(s,t)=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}=\int_0^1 u^{s-1}(1-u)^{t-1}\,\mathrm{d}u\,. \tag{6.13}$$

Combining the pieces the 1-loop contribution to the mass–shift is

$$\frac{m^2\lambda}{2}\frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d}\left(\frac{\mu}{m}\right)^{4-d}\frac{\Gamma(\frac{d}{2})\,\Gamma(1-\frac{d}{2})}{\Gamma(\frac{d}{2})}=\frac{m^2\lambda}{2(4\pi)^{d/2}}\left(\frac{\mu}{m}\right)^{4-d}\Gamma\left(1-\frac{d}{2}\right) \tag{6.14}$$

Expanding around $d=4$ by setting $d=4-\epsilon$, this is

$$-\frac{m^2\lambda}{32\pi^2}\left[\frac{2}{\epsilon}-\gamma+\log\left(\frac{4\pi\mu^2}{m^2}\right)\right]+\mathcal{O}(\epsilon) \tag{6.15}$$

as $\epsilon\to 0$, where we've used the basic properties

$$\Gamma(z)=\frac{\Gamma(z+1)}{z}\qquad\text{and}\qquad\Gamma(z)\sim\frac{1}{z}-\gamma+\mathcal{O}(\epsilon)$$

of the Gamma function, with $\gamma\approx 0.577$ being the Euler–Mascheroni constant. The divergence we saw as $\Lambda_0\to\infty$ in the cut–off regularization has become a *pole* in $d=4$ in dimensional regularization.

The simplest renormalization scheme is *minimal subtraction* (MS) — one simply chooses the counterterm

$$\delta m^2=-\frac{m^2\lambda}{16\pi^2\epsilon}\qquad\qquad \mathrm{MS} \tag{6.16}$$

so as to remove the purely divergent parts of the loop diagrams. A more common scheme is *modified minimal subtraction* ($\overline{\mathrm{MS}}$) in which one also removes the Euler–Mascheroni constant and the $\log 4\pi$ term, so that

$$\delta m^2 = -\frac{m^2\lambda}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right) \qquad \overline{\mathrm{MS}}\,. \tag{6.17}$$

We shall see later that these minimal subtraction schemes are rather different in character from the on–shell renormalization scheme. Nonetheless, it has become the most frequently used renormalization scheme in the literature.

6.1.3 Renormalization of the quartic coupling

The 1-loop correction to the quartic vertex is given by the three Feynman diagrams

which lead to the momentum space integrals

$$\frac{\lambda^2\mu^{4-d}}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^2+m^2}\,\frac{1}{(p+k_1+k_2)^2+m^2} \;+\; \text{other channels} \tag{6.18}$$

in dimensional regularization. Because these Feynman diagrams involve two separate vertices, they will lead to non–local contributions; indeed, expanding (6.18) in powers of the momenta k_i generates new derivative interactions such as $\sim \phi^2(\partial\phi)^2$. You should check that all such derivative terms are irrelevant as expected — they remain finite in the limit $d \to 4$ in dimensional regularization, and would in fact vanish in the continuum limit $\Lambda_0 \to \infty$ had we worked with a hard cut–off.

The k-independent part of each of the three loop diagrams involves the integral

$$\begin{aligned}
\mu^{4-d}\int_0^\infty \frac{p^{d-1}\,\mathrm{d}p}{(p^2+m^2)^2} &= \frac{1}{2}\int_0^\infty \frac{(p^2)^{(d-2)/2}}{(p^2+m^2)^2}\,\mathrm{d}(p^2) \\
&= \frac{1}{2}\left(\frac{\mu}{m}\right)^{4-d}\int_0^1 u^{1-\frac{d}{2}}(1-u)^{\frac{d}{2}-1}\,\mathrm{d}u \\
&= \frac{1}{2}\left(\frac{\mu}{m}\right)^{4-d}\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(2)}
\end{aligned} \tag{6.19}$$

Combining all three diagrams, we have a total 1-loop contribution

$$3\lambda^2\frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d}\,\frac{1}{2}\left(\frac{\mu}{m}\right)^{4-d}\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(2)} = \frac{3\lambda^2}{2(4\pi)^{d/2}}\left(\frac{\mu}{m}\right)^{4-d}\Gamma\left(2-\frac{d}{2}\right) \tag{6.20}$$

using our result (6.11) for $\mathrm{Vol}(S^{d-1})$. Setting $d = 4-\epsilon$, this becomes

$$\frac{3\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{m^2}\right) + \mathcal{O}(\epsilon)\,. \tag{6.21}$$

Consequently, in the $\overline{\text{MS}}$ scheme we choose our counterterm $\delta\lambda$ to be

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right) \tag{6.22}$$

removing both the pole as $\epsilon \to 0$ and the $\gamma - \log 4\pi$ terms.

The remaining 1-loop correction to the quartic vertex leads to a β-function

$$\beta = \mu\frac{\partial\lambda}{\partial\mu} = \frac{3\lambda^2}{16\pi^2} \tag{6.23}$$

agreeing (to this order) with what we found in (**??**) for the local potential approximation. The fact that the β-function is positive shows that the quartic coupling is (marginally) irrelevant in $d = 4$. Thus, no matter how small we choose the interaction to be at some scale μ, if it is non–zero, then there is a scale $\mu' > \mu$ in the UV at which the coupling diverges. Of course, our perturbative treatment is not powerful enough to really say what occurs as this happens, but more sophisticated treatments indeed show that $\lambda\phi^4$ does not exist as a continuum QFT.

Notice that there are no 1-loop diagrams that can contribute to the wavefunction renormalization factor Z_Λ here — to obtain a correction to the kinetic term $(\partial\phi)^2$ we would need a momentum space diagram involving precisely two external ϕ fields. With a purely quartic interaction, the only such 1-loop diagram is the one relevant for the mass shift. However, the momentum running around the loop in this diagram does not involve any momentum being brought in by the external fields, and therefore cannot contribute the factor of k^2 necessary to be interpreted as a correction to the kinetic term. Put differently, this loop diagram is a purely *local* contribution, so does not affect any derivative terms. Thus $Z_\Lambda = 1$ to 1-loop accuracy. There are non-trivial wavefunction renormalization factors beginning at 2-loops.

6.2 One–loop renormalization of QED

The action for QED describes a massive charged Dirac spinor coupled to the electromagnetic field is

$$S_{\text{QED}}[A,\psi] = \int \mathrm{d}^d x\,\left[\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \bar\psi \not{D}\psi + m\bar\psi\psi\right] \tag{6.24}$$

where the covariant derivative in the fermion kinetic term is $\not{D}\psi = \gamma^\mu(\partial_\mu + \mathrm{i}A_\mu)\psi$, and the Dirac matrices γ_μ obey $\{\gamma_\mu, \gamma_\nu\} = +2\delta_{\mu\nu}$ in Euclidean signature. (We'll understand more about covariant derivatives when we look at Yang–Mills theory.) In order for the covariant derivative $D_\mu = \partial_\mu + \mathrm{i}A_\mu$ to make sense, the gauge field must have mass dimension 1 even in d dimensions. Thus, the electric charge e has dimensions $(4-d)/2$, so is relevant when $d < 4$, irrelevant in $d > 4$ and marginal in $d = 4$, at least to leading order. Introducing an arbitrary mass scale μ as before we introduce a dimensionless coupling $g(\mu)$ as

$$e^2 = \mu^{4-d}\, g^2(\mu)\,. \tag{6.25}$$

To do perturbation theory, we'd like the kinetic terms to be canonically normalized, so we introduce a rescaled photon field $A_\mu^{\rm new} = eA_\mu^{\rm old}$. In terms of this rescaled field the action becomes

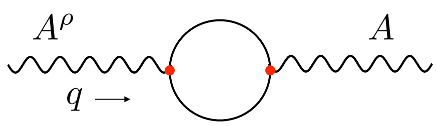
$$S_{\rm QED}[A^{\rm new}, \psi] = \int \mathrm{d}^d x \, \left[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(\not{\partial} + m)\psi + \mathrm{i}\mu^{2-\frac{d}{2}} g \, \bar{\psi} \not{A} \psi \right] \tag{6.26}$$

with $g(\mu)$ appearing only in the electron–photon vertex, as befits a coupling. Notice that the new photon field has mass dimension $(d-2)/2$, just like a scalar field.

6.2.1 Vacuum polarization: loop calculation

We'll now take a look at the simplest, and probably also the most important 1-loop graph in QED; the photon self–energy graph, also known as vacuum polarization.

Electromagnetic forces between charged particles are mediated by photon exchange. Quantum corrections modify the form of this propagator, for example by the 1-loop graph



where a virtual e^+e^- pair is formed and then reabsorbed. If we let $\Pi^{\rho\sigma}_{\rm loop}(q)$ denote the 1PI contributions to the photon self–energy, then at one loop order the only contribution to $\Pi^{\rho\sigma}_{\rm loop}$ is from the diagram above. The Feynman rules following from (6.26) give

$$\Pi^{\rho\sigma}_{1\,\rm loop}(q) = \mu^{4-d} g^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, \frac{\mathrm{tr}\left((-\mathrm{i}\not{p} + m)\gamma^\rho(-\mathrm{i}(\not{p} - \not{q}) + m)\gamma^\sigma\right)}{(p^2 + m^2)((p-q)^2 + m^2)} \,, \tag{6.27}$$

and we must now evaluate this integral.

To begin, note that

$$\frac{1}{AB} = \frac{1}{B-A}\left[\frac{1}{A} - \frac{1}{B}\right] = \int_0^1 \frac{\mathrm{d}x}{[(1-x)A + xB]^2} \tag{6.28}$$

so that we can combine the two propagators in (6.27) as

$$\begin{aligned} \int_0^1 \frac{\mathrm{d}x}{\left[(p^2+m^2)(1-x) + ((p-q)^2 + m^2)x\right]^2} &= \int_0^1 \frac{\mathrm{d}x}{[p^2 + m^2 - 2xp\cdot q + q^2 x]^2} \\ &= \int_0^1 \frac{\mathrm{d}x}{[(p-qx)^2 + m^2 + q^2 x(1-x)]^2} \,. \end{aligned} \tag{6.29}$$

If we now change variables $p \to p' = p + qx$ then (6.27) becomes (dropping the prime)

$$\Pi^{\rho\sigma}_{1\,\rm loop}(q) = \mu^{4-d} g^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{\mathrm{tr}\left((-\mathrm{i}(\not{p} + \not{q}x) + m)\gamma^\rho(-\mathrm{i}(\not{p} - \not{q}(1-x)) + m)\gamma^\sigma\right)}{[p^2 + \Delta]^2} \,, \tag{6.30}$$

where $\Delta := m^2 + q^2 x(1-x)$.

The next step is to perform the traces over the Dirac matrices. We'll do this treating the Dirac spinors as having 4 components[29] as appropriate for our final goal of $d = 4$. Thus

$$\begin{aligned} \mathrm{tr}(\gamma^\rho\gamma^\sigma) &= 4\,\delta^{\rho\sigma} \\ \mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) &= 4(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}) \end{aligned} \tag{6.31}$$

in Euclidean signature, so that

$$\begin{aligned} &\mathrm{tr}\left((-\mathrm{i}(\not{p} + \not{q}x) + m)\gamma^\rho(-\mathrm{i}(\not{p} - \not{q}(1-x)) + m)\gamma^\sigma\right) \\ &= \ 4\left[-(p+qx)^\rho(p - q(1-x))^\sigma + (p+qx)\cdot(p-q(1-x))\delta^{\rho\sigma}\right. \\ &\qquad \left. -(p+qx)^\sigma(p-q(1-x))^\rho + m^2\delta^{\rho\sigma}\right]\,. \end{aligned} \tag{6.32}$$

Thus the loop integral becomes

$$\begin{aligned} \Pi^{\rho\sigma}_{1\,\text{loop}}(q) = 4\mu^{4-d}g^2 &\int \frac{\mathrm{d}^d p}{(2\pi)^d}\int_0^1 \mathrm{d}x\ \frac{1}{[p^2+\Delta]^2} \\ &\times\left[-(p+qx)^\rho(p-q(1-x))^\sigma + (p+qx)\cdot(p-q(1-x))\delta^{\rho\sigma}\right. \\ &\qquad\left. -(p+qx)^\sigma(p-q(1-x))^\rho + m^2\delta^{\rho\sigma}\right]\,, \end{aligned} \tag{6.33}$$

which would be quadratically divergent in $d = 4$.

We're now ready to perform the loop integral. Observing that whenever $d \in \mathbb{N}$, any term involving an odd number of powers of momentum would vanish, we drop these terms. For the same reason, we replace

$$p^\mu p^\nu \to \frac{1}{d}\delta^{\mu\nu}p^2 \qquad \text{and} \qquad p^\mu p^\nu p^\rho p^\sigma \to \frac{(p^2)^2}{d(d+2)}\left[\delta^{\mu\nu}\,\delta^{\rho\sigma} + \delta^{\mu\rho}\,\delta^{\nu\sigma} + \delta^{\mu\sigma}\,\delta^{\nu\rho}\right]$$

where the tensor structure is fixed by Lorentz invariance and permutation symmetry, and the numerical factors are determined by contracting both sides with metrics. Finally, since the integrand now depends only on p^2, the angular integrals may be performed trivially to obtain

$$\frac{\mathrm{d}^d p}{(2\pi)^d} = \mathrm{Vol}(S^{d-1})\,\frac{p^{d-1}\,\mathrm{d}p}{(2\pi)^d} = \frac{1}{(4\pi)^{d/2}\,\Gamma(d/2)}\,(p^2)^{\frac{d}{2}-1}\,\mathrm{d}(p^2) \tag{6.34}$$

as in section 6.1.2. Thus (6.33) becomes

$$\begin{aligned} &\Pi^{\rho\sigma}_{1\,\text{loop}}(q) = 4\mu^{4-d}\,\frac{g^2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})} \\ &\times\int_0^1 \mathrm{d}x\int_0^\infty \mathrm{d}(p^2)\,(p^2)^{\frac{d}{2}-1}\left[\frac{p^2(1-\frac{2}{d})\,\delta^{\rho\sigma} + (2q^\rho q^\sigma - q^2)x(1-x) + m^2\delta^{\rho\sigma}}{[p^2+\Delta]^2}\right]\,. \end{aligned} \tag{6.35}$$

To go further, we use the integrals

$$\begin{aligned} \int_0^\infty \mathrm{d}(p^2)\frac{(p^2)^{\frac{d}{2}-1}}{(p^2+\Delta)^2} &= \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\frac{\Gamma(2-\frac{d}{2})\,\Gamma(\frac{d}{2})}{\Gamma(2)} \\ \int_0^\infty \mathrm{d}(p)^2\,\frac{(p^2)^{\frac{d}{2}}}{(p^2+\Delta)^2} &= \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}}\frac{\Gamma(1+\frac{d}{2})\,\Gamma(1-\frac{d}{2})}{\Gamma(2)} \end{aligned} \tag{6.36}$$

[29] In certain supersymmetric theories, it is often convenient to work instead with d–dimensional spinors, which is known as *dimensional reduction*, rather than dimensional regularization.

that can be evaluated using the substition $u = \Delta/(p^2+\Delta)$ and the definition of the Euler B–function, just as we did in section 6.1.2.

Using these integrals to evaluate (6.35), altogether one finds that the 1-loop contribution to vacuum polarization is given by

$$\begin{aligned}\Pi^{\rho\sigma}_{1\,\mathrm{loop}}(q) &= -\frac{4g^2\mu^{4-d}}{(4\pi)^{d/2}}\,\Gamma\left(2-\frac{d}{2}\right)\\ &\times\int_0^1 \mathrm{d}x\left[\frac{\delta^{\rho\sigma}(-m^2+x(1-x)q^2)+\delta^{\rho\sigma}(m^2+x(1-x)q^2)-2x(1-x)q^\rho q^\sigma}{\Delta^{2-\frac{d}{2}}}\right]\\ &:= (q^2\,\delta^{\rho\sigma}-q^\rho q^\sigma)\,\pi_{1\,\mathrm{loop}}(q^2)\,,\end{aligned} \tag{6.37}$$

where in the last line we have defined $\pi_{1\,\mathrm{loop}}(q^2)$ to be the dimensionless quantity

$$\pi_{1\,\mathrm{loop}}(q^2) = -\frac{8g^2(\mu)\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\,x(1-x)\left(\frac{\mu^2}{\Delta}\right)^{2-d/2}\,. \tag{6.38}$$

and we recall that $\Delta = m^2+q^2x(1-x)$ (and that $d<4$).

6.2.2 Counterterms in QED

The first thing to notice about our result (6.37) is that it, if all couplings remain constant, it will diverge in the physically interesting dimension because $\Gamma(2-\frac{d}{2})$ has a pole when $d=4$. To obtain a finite continuum result we must tune the initial couplings in the action, which as always we do by introducing counterterms. For QED the counterterms are

$$S^{\mathrm{CT}}[A,\psi,\epsilon] = \int \mathrm{d}^dx\,\left[\delta Z_3\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\delta Z_2\,\bar\psi \not{D}\psi+\delta m\,\bar\psi\psi\right]\,. \tag{6.39}$$

Adding these to the QED action (6.26) allows us to tune the initial values of the photon and electron wavefunction renormalizations, and the electron mass. The labels $(\delta Z_3,\delta Z_2)$ for the photon and electron wavefunction renormalization factors are conventional.

The fact that the entire kinetic term for the electron, including the gauge covariant derivative operator $\not{D} = \not{\partial}+\mathrm{i}e\not{A}$, receives only one counterterm assumes that the regularized path integral preserves gauge invariance: Provided our regularized path integral is indeed gauge invariant, then $\not{\partial}\psi$ and $\mathrm{i}\not{A}\psi$ cannot appear independently. Gauge invariance is maintained in lattice regularization, but would fail if one simply imposed a cut–off Λ_0, because the requirement that fields only contain Fourier modes with $|p|\le\Lambda_0$ is not preserved under the gauge transformation $\psi\to \mathrm{e}^{\mathrm{i}\chi(x)}\psi$, even if it is true of ψ and χ separately.[30] The desire to maintain manifest gauge invariance was one of the main motivation to use dimensional regularization in the first place, and our result (6.37) vindicates this decision: we see that the 1-loop correction $\Pi^{\rho\sigma}_{1\,\mathrm{loop}}(q^2)$ is proportional to $(q^2\delta^{\rho\sigma}-q^\rho q^\sigma)$, so

$$q_\rho\Pi^{\rho\sigma}_{1\,\mathrm{loop}}(q) = 0\,. \tag{6.40}$$

[30] In fact, the conceptually simple idea of integrating over modes only up to a cut–off *can* be done in gauge theory, but requires the introduction of a fair amount of technology beyond the scope of this course; see *e.g.* K. Costello's book cited in the introduction.

This signifies that the Ward identity for gauge transformations holds in the quantum theory (at least to one loop, but in fact it holds in general).

To fix the counterterm δZ_3, note that the loop diagram (6.37) diverges in the physical dimension since $\Gamma(2-d/2)$ has a pole as $d \to 4^-$. Indeed, setting $d = 4-\epsilon$ we have

$$\pi_{1\,\text{loop}}(q^2) \xrightarrow{d\to 4} -\frac{g^2(\mu)}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{\Delta}\right) + \mathcal{O}(\epsilon) \tag{6.41}$$

where again γ is the Euler–Mascheroni constant. The contribution

from $-\frac{1}{4}\delta Z_3\, F^{\rho\sigma}F_{\rho\sigma}$ must remove this pole, and in the $\overline{\text{MS}}$ scheme we'd set

$$\delta Z_3 = -\frac{g^2(\mu)}{12\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) \tag{6.42}$$

so as also to remove the contribution $\propto (-\gamma + \ln 4\pi)$. (To check that this counterterm does indeed cancel the pole, note that $\int_0^1 \mathrm{d}x\, x(1-x) = \frac{1}{6}$.) Thus the total contribution to the effective photon self–energy at one loop is

$$\Pi^{\rho\sigma}(q) = \left(q^2\delta^{\rho\sigma} - q^\rho q^\sigma\right)\pi(q^2) \tag{6.43}$$

where

$$\pi(q^2) = +\frac{g^2(\mu)}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\,\ln\left[\frac{m^2 + x(1-x)q^2}{\mu^2}\right] \tag{6.44}$$

in the $\overline{\text{MS}}$ scheme.

Strikingly, the loop correction to the photon propagator has created the logarithm[31]

$$\ln\left[m^2 + x(1-x)\,q^2\right]$$

in momentum space. This is quite unlike anything you've seen at tree–level, where Feynman diagrams are always rational functions of momenta, but it is very similar to the logarithms we obtained from integrating out fields in lower dimensional examples. In the present case, the logarithm has a branch cut in the region $m^2 + x(1-x)q^2 < 0$, or in other words when

$$x(1-x)\,q^2_{\text{Lorentz}} > m^2\,. \tag{6.45}$$

back in Lorentzian signature. Since $x(1-x) \leq 1/4$ for $x \in [0,1]$, the smallest value of q^2 at which this branch cut is reached is

$$q^4_{\text{Lorentz}} = 4m^2 \tag{6.46}$$

[31]Actually, it has produced a certain integral of a logarithm involving $x(1-x)$. This integral can be explicitly computed in terms of dilogarithms, but we won't need to know the result.

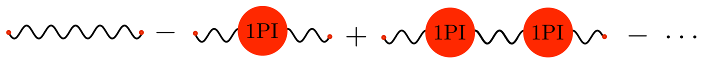
which is precisely the threshold energy for the creation of a *real* (as opposed to virtual) electron–positron pair.

At tree–level the photon propagator is

$$\Delta^0_{\mu\nu}(q) = \frac{1}{q^2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \tag{6.47}$$

in Lorenz (or Landau) gauge. In the quantum theory, the exact momentum space photon propagator $\Delta_{\mu\nu}(q)$ is then obtained by summing the geometric series

$\Delta_{\mu\nu} =$



$$= \Delta^0_{\mu\nu} - \Delta^0_{\mu\rho}\Pi^\rho{}_\sigma\Delta^{0\sigma}{}_\nu + \Delta^0_{\mu\rho}\Pi^\rho{}_\sigma\Delta^{0\sigma}{}_\kappa\Pi^\kappa{}_\lambda\Delta^{0\lambda}{}_\nu - \cdots$$

where $\Pi^{\mu\nu}$ denotes the sum of *all* one particle irreducible graphs with just two external photon lines, and the minus signs arise because we are in Euclidean signature with path integrals weighted by e^{-S}.

We've just found that

$$\Pi^\rho{}_\sigma = q^2\left(\delta^\rho{}_\sigma - \frac{q^\rho q_\sigma}{q^2}\right)\pi(q^2) \tag{6.48}$$

where, to 1-loop accuracy, $\pi(q^2)$ is given by (6.44) in the $\overline{\mathrm{MS}}$ scheme. In fact, one can show that (6.48) holds to all orders in the electromagnetic coupling as a consequence of the Ward identity: only the explicit form of $\pi(q^2)$ is modified. The factor in brackets projects onto the polarization states transverse to q^σ and obeys $P^\rho{}_\sigma P^\sigma{}_\kappa = P^\rho{}_\kappa$ as for any projection operator. Therefore the exact photon propagator may be written

$$\begin{aligned}\Delta_{\mu\nu}(q) &= \Delta^0_{\mu\nu} - \Delta^0_{\mu\rho}\Pi^\rho{}_\sigma\Delta^{0\sigma}{}_\nu + \Delta^0_{\mu\rho}\Pi^\rho{}_\sigma\Delta^{0\sigma}{}_\kappa\Pi^\kappa{}_\lambda\Delta^{0\lambda}{}_\nu - \cdots \\ &= \Delta^0_{\mu\nu}\left(1 - \pi(q^2) + \pi^2(q^2) - \pi^3(q^2)\cdots\right) \\ &= \frac{\Delta^0_{\mu\nu}}{1+\pi(q^2)}\end{aligned} \tag{6.49}$$

by summing this geometric series.

Let's think about what this exact propagator tells us about the effective action for the photon that we'd obtain after performing the path integral over the electron field. The classical Maxwell action is

$$\begin{aligned}\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}\,\mathrm{d}^dx &= \frac{1}{4}\int -\mathrm{i}\left(q^\mu\tilde{A}^\nu(-q) - q^\nu\tilde{A}^\mu(-q)\right)\mathrm{i}\left(q_\mu\tilde{A}_\nu(q) - q_\nu\tilde{A}_\mu(q)\right)\mathrm{d}^dq \\ &= \frac{1}{2}\int q^2\left(\delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\tilde{A}_\mu(-q)\tilde{A}_\nu(q)\,\mathrm{d}^dq\end{aligned} \tag{6.50}$$

when written in momentum space. Note that $\Delta^0_{\mu\nu}$ is indeed the inverse of this momentum space kinetic term for polarizations transverse to q_μ. Thus, the exact photon propagator would follow from a momentum space quadratic term

$$S^{(2)}_{\mathrm{eff}}[\tilde{A}] = \frac{1}{4}\int [1+\pi(q^2)]\,q^2\left(\delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\tilde{A}_\mu(-q)\tilde{A}_\nu(q)\,\mathrm{d}^dq \tag{6.51}$$

where the electron loop effects are incorporated in the factor $[1-\pi(q^2)]$. The part $\pi(0)$ of $\pi(q^2)$ that is independent of q^2 just provides an overall factor multiplying the classical action, and so corresponds straightforwardly to the position space term

$$S^{(2)}_{\rm eff}[A] = \frac{1+\pi(0)}{4}\int F^{\mu\nu}(z)F_{\mu\nu}(z)\,{\rm d}^4z = \frac{1}{4}\left[1+\frac{g^2}{2\pi^2}\int_0^1 {\rm d}x\,x(1-x)\ln\frac{m^2}{\mu^2}\right]\int F^{\mu\nu}F_{\mu\nu}\,{\rm d}^4z \tag{6.52}$$

where the second expression uses our result (6.44) for the 1-loop contribution to $\pi(0)$ in the $\overline{\rm MS}$ scheme in $d=4$. As expected, this is a contribution to photon wavefunction renormalization. Expanding $\pi(q^2)$ as a power series in q^2/m^2 shows that the remaining, q^2-dependent terms correspond to an infinite series of higher derivative interactions of the schematic form $\partial^n F^{\mu\nu}\partial^n F_{\mu\nu}$. All these higher derivative couplings are irrelevant in $d=4$ and may be expected to be small at energies much lower than the electron mass. (In particular, the original loop integrals for these terms were finite in $d=4$.)

6.2.3 The β-function of QED

Knowing the effective action allows us to read off the β-function for the electric charge. To relate the photon kinetic term – involving wavefunction renormalization – to the β function for the electromagnetic coupling e, we first undo our rescaling $A^{\rm old}_\mu = eA^{\rm new}_\mu$ and work back in terms of the original gauge field $A^{\rm old}_\mu$. Then the quadratic term (6.52) in the effective action becomes

$$S^{(2)}_{\rm eff}[A^{\rm old}] = \frac{1}{4}\left[\frac{1}{g^2}+\frac{1}{2\pi^2}\int_0^1 {\rm d}x\,x(1-x)\ln\frac{m^2}{\mu^2}\right]\int F^{\mu\nu}F_{\mu\nu}\,{\rm d}^4z\,. \tag{6.53}$$

In this way, we can view the vacuum polarization as a quantum correction to the value of $1/g^2$. Therefore

$$\mu\frac{\partial}{\partial\mu}\left(\frac{1}{g^2}\right) = -\frac{2}{g^3}\beta(g) = -\frac{1}{\pi^2}\int_0^1 {\rm d}x\,x(1-x) \tag{6.54}$$

so that the β function for $g(\mu)$ is

$$\beta(g) := \mu\frac{\partial g}{\partial\mu} = \frac{g^3}{12\pi^2}\,. \tag{6.55}$$

Solving this $g(\mu)$ gives

$$\frac{1}{g^2(\mu)} = C - \frac{1}{6\pi^2}\ln\mu\,, \tag{6.56}$$

or equivalently

$$g^2(\mu) = \frac{g^2(\mu')}{1-g^2(\mu')/6\pi^2\,\ln(\mu/\mu')} \tag{6.57}$$

which fixes the coupling $g(\mu)$ at arbitrary scales in terms of its value at some arbitrary reference scale μ'. For example, we could choose μ' to be the scale of the (physcial) electron mass, at which $g^2(m_e)/4\pi \approx 1/137$ is found experimentally.

As for the quartic coupling of $\lambda\phi^4$ theory, the fact that the β-function in QED is positive shows that the electromagnetic coupling is marginally irrelevant in $d=4$, at least near the

Gaussian critical point $g = 0$. Consequently, it is believed that *pure QED does not exist as a continuum QFT* in four dimensions! This fact has no immediate phenomenological consequences, because taking $g^2(m_e)/4\pi \approx 1/137$ from experiment, the scale μ at which the coupling (6.57) diverges is fixed to be $\sim 10^{286}$ GeV, well beyond any point at which we claim to even vaguely trust QFT as a description of Nature. Nonetheless, the lesson from the β–function is that pure QED can only exist as a low–energy effective theory — in our world it unifies with the weak interactions at around ~ 100 GeV, where the physics of non–Abelian gauge theories comes into play.

Finally, I wish to point out a small peculiarity inherent in the $\overline{\text{MS}}$ renormalization scheme. In studying the local potential approximation in section 5.3.2 we discovered that the quantum contributions to the β-functions due to high energy states circulating in the loop became suppressed at scales much lower than the masses of these states. This made good sense: the heavy states decoupled from low–energy physics. However, in the $\overline{\text{MS}}$ scheme the β-function (6.54) shows no suppression at any scale and the coupling (6.57) still runs even at scales $\mu \ll m_e$. There is nothing wrong with this: as $\mu \to 0$ there is a balance in the effective action (6.51) between $e^2(\mu) \to 0$ and the growing effect of the loop contribution $\propto \ln(1/\mu^2) \to \infty$ so that the actual *observable physics* does remain constant. However, it's strange to have loop effects dominating the classical ones, so for some purposes it is better to proceed as follows. We consider *two* different theories. One includes the electron and is valid at scales $\mu \geq m_e$, while the second does not and is valid at scales $\mu \leq m_e$. Physical quantities in the two theories are matched at $\mu = m_e$. The effects of the electron (or other heavy particle) are then manually frozen out as we continue on to our other theory at lower scales. In particular, since pure Maxwell theory is free, for any $\mu \leq m_e$ the fine structure constant $\alpha(\mu) = g^2(\mu)/4\pi$ will remain frozen at its value $\approx 1/137$ at the electron mass.

We could avoid the need to decouple by hand had we used a renormalization scheme, such as on–shell renormalization, that fixes the value of the counterterm δZ_3 in terms of $\pi_{1\,\text{loop}}(q^2)$ at some definite scale $q^2 = \mu'^2$. That is, we set

$$\delta Z_3 = -\frac{g^2}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{m^2 - x(1-x)\mu'^2}\right) \tag{6.58}$$

In this scheme, the β-function instead becomes

$$\beta(g) = \mu\frac{\partial g}{\partial \mu} = \frac{g^3}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\, \ln\frac{x(1-x)\mu'^2}{m^2 - x(1-x)\mu'^2} \tag{6.59}$$

in four dimensions. This does indeed approach zero when $\mu' \ll m_e$. However, for most purposes (particularly in more complicated theories such as Yang–Mills theory, or the full Standard Model) the $\overline{\text{MS}}$ scheme is so convenient that it's worth paying the price of having to decouple the electron by hand.

6.2.4 Physical interpretation of vacuum polarization

When light propagates through a region containing an insulating medium with no relevant degrees of freedom, on general grounds we expect the low–energy effective field theory to be governed by an action that modifies the coefficients of the electric and magnetic fields in the usual Maxwell action by terms that respect the microscopic symmetries of the medium. In the present case, the medium is simply *the vacuum itself*! Since the vacuum is Lorentz invariant, these modifications must be proportional to the Lorentz invariant combination $E^2 - B^2 = F^{\mu\nu}F_{\mu\nu}$. In (6.52) we see explicitly that this is true. If we place a medium such as water in the presence of an electric field, it will become polarized due to the large dipole moment of the H_2O molecules. Likewise, at distances $\gtrsim 1/m$ the vacuum itself becomes a dielectric medium in which virtual electron–positron pairs form dipoles, polarizing the vacuum.

The first effect is that vacuum polarization leads to a measureable change in the Coulomb potential. Recall that in the non-relativistic limit (in Lorentzian signature), the Fourier transform of the (Feynman gauge) photon propagator $\delta^{\rho\sigma}/q^2$ is the Coulomb potential $V(r) = e^2/4\pi r$, as I hope is familiar from Rutherford scattering. Let's compute the 1-loop quantum corrections to this result. We consider a scattering process in which two spin $\frac{1}{2}$ charged particles interact electromagnetically. The Feynman diagrams

make contributions of the form

$$S(1,2 \to 1',2') = \frac{-e_1e_2}{4\pi^2 q^2}\,\delta^4(p_1+p_2-p_{1'}-p_{2'})\,\left[1+\pi(q^2)\right]\,\bar{u}_{1'}\gamma^\mu u_1\,\bar{u}_{2'}\gamma_\mu u_2 \tag{6.60}$$

where $u_{1,2}$ are the on–shell Dirac wavefunctions of the incoming particles and $\bar{u}_{1',2'}$ are the on–shell Dirac wavefunctions for the outgoing particles. The 1–loop diagram modifies the classical answer by the factor $[1+\pi(q^2)]$. Non–relativistically, the energy transfer $q^0 \ll |\mathbf{q}|$ and

$$\bar{u}_{1'}\gamma^\mu u_1 \approx \begin{pmatrix} -\mathrm{i}\delta_{m_1,m_{1'}} \\ \mathbf{0} \end{pmatrix},$$

where the factor of $\delta_{mm'}$ enforces that the spins of the two particles should be aligned. Thus, in the non–relativistic limit we have

$$S(1,2 \to 1',2') \approx \frac{-e_1e_2}{4\pi^2 \mathbf{q}^2}\,\delta^4(p_1+p_2-p_{1'}-p_{2'})\,\left[1+\pi(\mathbf{q}^2)\right]\,\delta_{m_1m_{1'}}\,\delta_{m_2,m_{2'}}\,. \tag{6.61}$$

We can compare this result to the calculation of scattering in non–relativistic quantum mechanics for a potential $V(\mathbf{r})$ in the Born approximation, where

$$S_{\text{Born}}(1,2 \to 1',2') = \frac{-e_1e_2}{4\pi^2}\,\delta^4(p_1+p_2-p_{1'}-p_{2'})\,\delta_{m_1,m_{1'}}\,\delta_{m_2,m_{2'}}\int \mathrm{d}^3\mathbf{r}\,V(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}} \tag{6.62}$$

This shows that the 1-loop corrected amplitude looks just like the amplitude we would find from Born level scattering off a modified classical potential $V_1(\mathbf{r})$ whose Fourier transform is $e_1 e_2[1+\pi(\mathbf{q})^2]/\mathbf{q}^2$, or in other words

$$V_1(\mathbf{r}) = \frac{e_1 e_2}{(2\pi)^3}\int \mathrm{d}^3\mathbf{q}\,\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\,\frac{1+\pi(\mathbf{q}^2)}{\mathbf{q}^2}\,. \tag{6.63}$$

To lowest order in $\pi(\mathbf{q}^2)$, this is just the potential energy

$$V(|\mathbf{r}|) = \int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{y}\,\frac{\rho_1(\mathbf{x})\,\rho_2(\mathbf{y})}{|\mathbf{x}-\mathbf{y}+\mathbf{r}|} \tag{6.64}$$

produced at point $\mathbf{r}$ from the electrostatic interaction of two charge distributions $\rho_1(\mathbf{x})$ and $\rho_2(\mathbf{y})$ defined by

$$\rho_{1,2}(\mathbf{r}) = e_{1,2}\,\delta^3(\mathbf{r}) + \frac{e_{1,2}}{2(2\pi)^3}\int \mathrm{d}^3\mathbf{q}\,\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\,\pi(\mathbf{q}^2)\,. \tag{6.65}$$

In particular, we see that

$$\begin{aligned}\int \mathrm{d}^3\mathbf{r}\,\rho_{1,2}(\mathbf{r}) &= e_{1,2}\left[1+\frac{1}{2(2\pi)^3}\int \mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{q}\,\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\,\pi(\mathbf{q}^2)\right]\\ &= e_{1,2}\left[1+\frac{1}{2}\pi(0)\right]\\ &= e_{1,2}\end{aligned} \tag{6.66}$$

where the last line uses the on–shell renormalization scheme result (**??**). Thus the total charge seen by the long–range part of the Coulomb potential is the same as the charge governing the interaction in Feynman diagrams. After a contour integration using the on–shell scheme result for $\pi(\mathbf{q}^2)$, one finds

$$\frac{\rho_{1,2}(\mathbf{r})}{e_{1,2}} = (1+L)\delta^3(\mathbf{r}) - \frac{e^2}{8\pi^3 r^3}\int_0^1 \mathrm{d}x\,x(1-x)\left[1+\frac{mr}{\sqrt{x(1-x)}}\right]\exp\left(\frac{-mr}{\sqrt{x(1-x)}}\right) \tag{6.67}$$

where L is the integral

$$L = \frac{e^3}{8\pi^3}\int \mathrm{d}^3\mathbf{r}\frac{1}{r^3}\left[\int_0^1 \mathrm{d}x\,x(1-x)\left[1+\frac{mr}{\sqrt{x(1-x)}}\right]\exp\left(\frac{-mr}{\sqrt{x(1-x)}}\right)\right]\,. \tag{6.68}$$

This integral diverges at short distances $r \to 0$. The interpretation is that the bare point charge of strength $e_1(1+L)$ sitting at $\mathbf{r} = 0$ polarizes the vacuum, attracting virtual particles of opposite charge towards it and repelling their antiparticles as they circulate around the loop. Thus the bare charge is partially *shielded* and we see only a finite charge e_1.