4 Exact Properties of the Path Integral

In this chapter we begin our study of Quantum Field Theory proper: we take the dimension d of our space to be generic (certainly $d > 1$, and typically $d = 4$). Now, as you're no doubt aware, these theories are typically rather complicated. Except in very special circumstances, we can usually only hope to find a perturbative approximation to the path integral (or else we must seek to evaluate it numerically). As in the $d = 0$ example earlier, such perturbative expansions provide us with an asymptotic series that approximates the true path integral arbitrarily accurately as $\hbar \to 0$.

Given this situation, it'll be valuable to know what can be said *without* making any perturbative approximations. We'll start by taking a look at the role of symmetry in QFT and understanding how we can use it to constrain correlation functions — symmetries constrain the possible forms correlation functions may take *exactly*.

4.1 Symmetries of the quantum theory

One of the most important results in classical mechanics and classical field theory is *Noether's theorem*, stating that local symmetries of the action corresponds to a conserved charge. Let's recall how to derive this.

Consider the transformation

$$\delta_\epsilon \phi(x) = \epsilon f(\phi, \partial_\mu \phi) \tag{4.1}$$

where ϵ is an infinitesimal parameter, and $f(\phi, \partial_\mu \phi)$ is some function of the fields and their derivatives. The transformation is local if the function f depends on the values of the field and its derivatives only at the point $x \in M$, in which case you can think of it as being generated by the vector

$$V_f := \int_M \mathrm{d}^d x \sqrt{g}\; f(\phi, \partial\phi) \frac{\delta}{\delta \phi(x)} \tag{4.2}$$

acting on the infinite dimensional space of fields. The transformation (4.1) is a *symmetry* if the action is invariant, $\delta S[\phi] = 0$, whenever the parameter ϵ is constant. Because it is invariant when ϵ is constant, if ϵ is now allowed to depend on position the change in the action must be proportional to the derivative of ϵ. In other words[13]

$$\delta_\epsilon S[\phi] = -\int_M *j \wedge d\epsilon = -\int_M g^{\mu\nu}\, j_\mu(x)\, \partial_\nu \epsilon(x) \sqrt{g}\, \mathrm{d}^d x \tag{4.3}$$

for some function $j_\mu(x)$ known as the *current*.

However, when the equations of motion hold, the action is invariant under *any* small change in the fields. In particular, on the support of the equations of motion,

$$\delta_\epsilon S[\phi] = 0 \tag{4.4}$$

[13] The minus sign is a convention, included for later convenience. The first expression here is written in the language of differential forms, where we treat $j \in \Omega^1_M$ is a 1-form and where $* : \Omega^p_M \to \Omega^{d-p}_M$ is the Hodge star on the d-dimensional Riemannian manifold (M, g). The second expression is just the same thing written in a local coordinate patch. Below, I'll often work in the sleeker language of forms, but I'll be sure to give the main results both ways. If you're uncomfortable with differential geometry, I recommend you repeat the derivations for yourself in the case $(M, g) = (\mathbb{R}^d, \delta)$.

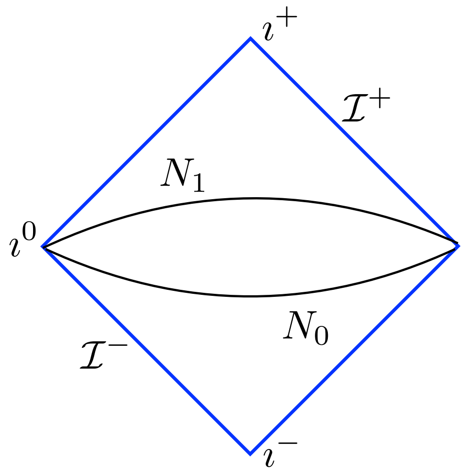

Figure 5: *Classically, the charge* (4.6) *associated to a local symmetry is independent of the hypersurface over which it is integrated.*

under the change (4.1) even if the parameter ϵ depends on position. Integrating by parts and choosing our function $\epsilon(x)$ to have compact support, we find that

$$d * j = 0 \qquad \text{or equivalently} \qquad \partial_\nu(\sqrt{g}\, g^{\mu\nu} j_\nu) = 0 \tag{4.5}$$

whenever the equations of motion are satisfied.

We define the *charge* Q corresponding to this transformation by

$$Q[N] := \int_N *j = \int_N j_\mu n^\mu \sqrt{g_N}\, \mathrm{d}^{d-1}x \tag{4.6}$$

where N is any codimension–one hypersurface in the space–time M, n^μ is a unit normal vector to N (so $g(n,n) = 1$ and $g(n,v) = 0$ for any $v \in TN$), g_N denotes the restriction of the metric on M to N, and thus $\sqrt{g_N}\,\mathrm{d}^{d-1}x$ is the $(d-1)$–dimensional volume element on N. If $N_{0,1}$ are two such hypersurfaces bounding a region $M' \subset M$ of our space, then

$$Q[N_1] - Q[N_0] = \int_{N_1} *j - \int_{N_0} *j = \int_{\partial M'} *j = \int_{M'} d * j = 0 \tag{4.7}$$

where the third equality is Stokes' theorem and the final equality follows by the conservation equation (4.5). Thus $Q[N]$ depends on the choice of N only through its homology class. The most common application of this result is to take the surfaces $N_{1,0}$ to be constant time slices of Minkowski space–time[14], as in figure 5. The statement that $Q[N_1] = Q[N_0]$ then becomes the statement that the charge Q is conserved under time evolution.

As a simple example, consider the action

$$S[\phi] = \frac{1}{2}\int_M d\bar{\phi} \wedge *d\phi + *V(|\phi|^2) \tag{4.8}$$

[14]Since the constant time slices $N_{0,1}$ are non–compact in this case, for our derivation to hold we should ensure that the current j decays sufficiently rapidly as we head towards spatial infinity so that the integrals defining $Q[N_{0,1}]$ converge.

for a complex scalar field. This action is invariant under the $U(1)$ transformation $\phi \mapsto \mathrm{e}^{\mathrm{i}\alpha}\phi$ that rotates the phase of ϕ by a constant amount α. Taking α to be infinitesimally small, we have

$$\delta\phi = \mathrm{i}\alpha\phi\,, \qquad \delta\bar{\phi} = -\mathrm{i}\alpha\bar{\phi} \tag{4.9}$$

The corresponding current is $j_\mu = \mathrm{i}\left(\partial_\mu\bar{\phi}\,\phi - \bar{\phi}\,\partial_\mu\phi\right)$ and so the charge associated with a hypersurface N is

$$Q[N] = \mathrm{i}\int_N *(d\bar{\phi}\,\phi - \bar{\phi}\,d\phi)\,. \tag{4.10}$$

In particular, if we are in flat space–time $(M,g) = (\mathbb{R}^{1,3},\eta)$ and the fields decay rapidly as we approach spatial infinity, then the charge on a constant time hypersurface is

$$Q = \mathrm{i}\int (\partial_0\bar{\phi}\,\phi - \bar{\phi}\,\partial_0\phi)\,\mathrm{d}^3x \tag{4.11}$$

From the original action we have that the momentum conjugate to the field is

$$\pi = \frac{\delta\mathcal{L}}{\delta\partial_0\phi} = \partial_0\bar{\phi} \qquad \text{and} \qquad \bar{\pi} = \frac{\delta\mathcal{L}}{\delta\partial_0\bar{\phi}} = \partial_0\phi \tag{4.12}$$

and hence the charge can be written as

$$Q = \mathrm{i}\int (\phi\pi - \bar{\phi}\bar{\pi})\,\mathrm{d}^3x \tag{4.13}$$

which indeed generates the transformations (4.9) via Poisson brackets.

4.1.1 Ward identities

The derivation of Noether's theorem used the classical equations of motion, so we must re–examine this in quantum theory. Suppose that some local field transformation $\phi \to \phi'(\phi)$ leaves the product of the action and path integral measure invariant, *i.e.*,

$$\mathcal{D}\phi\,\mathrm{e}^{-S[\phi]} = \mathcal{D}\phi'\,\mathrm{e}^{-S[\phi']}\,. \tag{4.14}$$

In most cases, the symmetry transformation will actually leave *both* the action and measure invariant separately, but the weaker condition (4.14) is sufficient (and necessary).

In practice, we have much more familiarity with classical actions that with path integral measures, so one tends to look for symmetries of the action first and then hope these can be extended to symmetries of the measure. For example, if some action $S[\phi]$ depends on a field only through its derivatives $\partial^r\phi$ with $r \geq 1$, then this action is invariant under shifts of the field

$$\phi(x) \to \phi(x) + \phi_0 \tag{4.15}$$

where ϕ_0 is constant over M. Naïvely, the path integral measure $\mathcal{D}\phi$ instructs us to integrate over *all* continuous maps $\phi \in C^0(M,\mathbb{R})$ and we'd expect such an instruction not to care about a constant translation. However, to actually *define* what we mean by $\mathrm{D}\phi$ we must first pick a regularization and if we wish (4.14) to be valid under (4.15) then this regularization procedure must also be compatible with constant field translations. For example, if $M = T^d$ we might expand $\phi(x)$ in terms of a Fourier series, and declare that we are only going to integrate over finitely many Fourier coefficients. The constant translation (4.15) only affects the lowest Fourier mode, and our regularized measure integrates over all values of this coefficient, so indeed (4.14) will hold.

As a further example, a theory living on $(M, g) = (T^d, \delta)$ will be invariant under SO(d) rotations

$$x \to Lx\,, \qquad \phi(x) \to \phi'(x) = \phi(Lx) \tag{4.16}$$

if the action is built from SO(d) invariant combinations of the fields and their derivatives. We can regularize the path integral by integrating over all Fourier modes with the SO(d)-invariant quantity $p^\mu p_\mu$ is less than some chosen cut-off[15]. However, if we chose instead to regularize our theory by replacing space by a simple cubic lattice $\Lambda \subset T^d$ and integrating over the values of the fields at each lattice site, then SO(d) invariance would be broken to the discrete group of lattice symmetries.

When a symmetry of the classical action is broken by the regularized path integral measure, there are two possible outcomes. In the one hand, it may be that *some* regularization procedure that would have preserved the symmetry does exist — it's just that we didn't use it, whether through choice or ignorance. This was the case above with, where SO(d) transformations were broken by the lattice, but not by the cut–off regularization. In this case, it turns out that if one computes any physical object such as a correlation function or scattering amplitude and then removes the regularization at the end of the calculation[16], then the symmetry will be restored. Exactly this situation occurs for lattice treatments of gauge theories such as QCD. The only difficulty is in the intermediate steps of the calculations, where the symmetries are absent.

On the other hand, it may turn out that *no regularization procedure which preserves the symmetry exists.* In this case, the symmetry of the classical field theory is genuinely absent at the quantum level. Such symmetries are said to be *anomalous*; we'll consider them in more detail at the end of the course[17]. For now, we'll assume we've found a regulator that preserves our classical symmetry, so that (4.14) holds.

One of the main uses of symmetry is to deduce important restrictions on correlation functions. Consider a class of operators whose only variation under the transformation $\phi \to \phi'$ comes from their dependence on ϕ itself (such as scalar operators under rotations). Such operators transform as $\mathcal{O}(\phi) \to \mathcal{O}(\phi')$. At least on a compact manifold M we have

$$\begin{aligned}\int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]}\, \mathcal{O}_1(\phi(x_1)) \cdots \mathcal{O}_n(\phi(x_n)) &= \int \mathcal{D}\phi'\, \mathrm{e}^{-S[\phi']}\, \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n)) \\ &= \int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]}\, \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n))\end{aligned} \tag{4.17}$$

The first equality here is a triviality: we've simply relabeled ϕ by ϕ' as a dummy variable in the path integral. The second equality is non–trivial and uses the assumed symmetry (4.14)

[15] In Minkowski signature, the corresponding statements for SO$(1, d-1)$ go through in essentially the same way for massive particles, but are more subtle when massless states are present.

[16] We'll understand later how, and in which circumstances, this can be done.

[17] Time permitting!

under the transformation $\phi \to \phi'$. We see that the correlation function obeys

$$\langle \mathcal{O}_1(\phi(x_1)) \cdots \mathcal{O}_n(\phi(x_n)) \rangle = \langle \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n)) \rangle \tag{4.18}$$

so that it is *invariant* under the transformation.

For example, consider again the phase transformation

$$\phi \to \phi' = \mathrm{e}^{\mathrm{i}\alpha}\phi\,, \qquad \bar{\phi} \to \bar{\phi}' = \mathrm{e}^{-\mathrm{i}\alpha}\bar{\phi} \tag{4.19}$$

that we examined above. The path integral measure will be invariant under this symmetry provided we integrate over as many modes of $\bar{\phi}$ as we do of ϕ. The Ward identity then implies that correlation functions built from local operators of the form $\mathcal{O}_i = \phi^{r_i}\bar{\phi}^{s_i}$ must obey

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \mathrm{e}^{\mathrm{i}\alpha \sum_{i=1}^{n}(r_i - s_i)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle\,.$$

Allowing α to take different (constant) values shows that this correlator vanishes unless $\sum_i r_i = \sum_i s_i$. The symmetry thus imposes a **selection rule** on the operators we can insert if we wish to obtain a non–zero correlator.

For a second example, suppose $(M, g) = (\mathbb{R}^d, \delta)$ and consider a space–time translation $x \mapsto x' := x - a$ where a is a constant vector. Under this translation, we have

$$\phi(x) \mapsto \phi'(x) := \phi(x - a)\,. \tag{4.20}$$

If the action and path integral measure are translation invariant and the operators $\mathcal{O}_i$ depend on x only via their dependence on $\phi(x)$, then the Ward identity gives

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle \mathcal{O}_1(x - a) \cdots \mathcal{O}_n(x_n - a) \rangle$$

for any such vector a. Thus, having carried out the path integral, we'll be left with a function $f(x_1, x_2, \ldots, x_n)$ that depends only on the *relative* positions $(x_i - x_j)$. Similarly, if the action & measure are invariant under rotations (or Lorentz transformations) $x \to Lx$ then a correlation function of scalar operators will obey

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle \mathcal{O}_1(Lx_1) \cdots \mathcal{O}_n(Lx_n) \rangle\,.$$

Combining this with the previous result shows that the correlator can depend only on the rotational (or Lorentz) invariant distances $(x_i - x_j)^2$ between the insertion points.

4.1.2 Currents and charges

As in the classical theory, any continuous symmetry comes with an associated current. Suppose that $\phi \to \phi' = \phi + \delta_\epsilon\phi$ is a symmetry of the path integral when ϵ is an infinitesimal constant parameter. Then, as in Noether's theorem, the variation of the action *and path integral measure* must be proportional to $\partial_\mu\epsilon$ when ϵ depends on position, so that

$$\mathcal{Z} = \int \mathcal{D}\phi'\, \mathrm{e}^{-S[\phi']} = \int \mathcal{D}\phi\; \mathrm{e}^{-S[\phi]} \left[1 - \int_M *j \wedge d\epsilon\right] \tag{4.21}$$

to lowest order. Notice that the current here may include possible changes in the path integral measure as well as in the action. The zeroth order term agrees with the partition function on the left, so the first order term must vanish and we have

$$0 = -\int_M *\langle j(x)\rangle \wedge d\epsilon = \int_M \epsilon(x)\, d * \langle j(x)\rangle\,, \tag{4.22}$$

if either $\partial M = 0$ or the fields decay sufficiently rapidly that there is no boundary contribution. For this to hold for arbitrary ϵ we must have $\partial^\mu \langle j_\mu(x)\rangle = 0$ so that the *expectation value* of the current obeys a conservation law, just as in classical physics.

Now let's see how the current insertions affect more general correlation functions. Consider a class of local operators that transform under $\phi \mapsto \phi' := \phi + \epsilon\delta\phi$ as

$$\mathcal{O}(\phi) \mapsto \mathcal{O}(\phi + \epsilon\delta\phi) = \mathcal{O}(\phi) + \epsilon\delta\mathcal{O} \tag{4.23}$$

to lowest order in ϵ, where we've defined $\delta\mathcal{O} := \delta\phi\, \partial\mathcal{O}/\partial\phi$. Then, accounting for both the change in the action and measure as well as in the operators,

$$\begin{aligned}
&\int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]} \prod_{i=1}^n \mathcal{O}_i(\phi(x_i)) = \int \mathcal{D}\phi'\, \mathrm{e}^{-S[\phi']} \prod_{i=1}^n \mathcal{O}_i(\phi'(x_i)) \\
&= \int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]} \left[1 - \int_M *j \wedge d\epsilon\right] \left[\prod_{i=1}^n \mathcal{O}_i(x_i) + \sum_{i=1}^n \epsilon(x_i)\delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j)\right] .
\end{aligned} \tag{4.24}$$

Again, the first equality is a triviality and the second follows upon expanding both $\mathcal{D}\phi'\, \mathrm{e}^{-S[\phi']}$ and the operators to first order in the variable parameter $\epsilon(x)$. The ϵ-independent term on the *rhs* exactly matches the *lhs*, so the remaining terms must cancel. To first order in ϵ this gives

$$\int_M \epsilon(x) \wedge d * \left\langle j(x) \prod_{i=1}^n \mathcal{O}_i(x_i)\right\rangle = -\sum_{i=1}^n \left\langle \epsilon(x_i)\delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j)\right\rangle , \tag{4.25}$$

after an integration by parts with $\epsilon(x)$ of compact support. Note that the derivative on the *lhs* hits the whole correlation function.

We'd like to strip off the parameter $\epsilon(x)$ and obtain a relation purely among the correlation functions themselves. In order to do this, we must handle the fact that the operator variations on the *rhs* are located only at the points $\{x_1, \ldots, x_n\} \in M$. We thus write[18]

$$\epsilon(x_i)\delta\mathcal{O}_i(x_i) = \int_M * \delta^d(x - x_i)\, \epsilon(x)\, \delta\mathcal{O}_i(x_i) = \int_M \delta^d(x - x_i)\, \epsilon(x)\, \delta\mathcal{O}_i(x_i)\, \sqrt{g}\, \mathrm{d}^d x$$

as an integral, so that all terms in (4.25) are proportional to $\epsilon(x)$. Since this may be chosen arbtrarily, we obtain finally

$$d * \left\langle j(x) \prod_{i=1}^n \mathcal{O}_i(x_i)\right\rangle = - \;* \sum_{i=1}^n \delta^d(x - x_i) \left\langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j)\right\rangle , \tag{4.26}$$

[18] Here, for an open set $U \subset M$ in curved space, my δ-function is defined to obey $\int_U \delta^d(x - x_i)\, \sqrt{g}\, \mathrm{d}^d x = 1$ if $x_i \in U$ and 0 else, with an integration measure including the factor of $\sqrt{g}$.

or equivalently

$$\partial_\mu \left(g^{\mu\nu}\sqrt{g}\,\left\langle j^\mu(x)\prod_{i=1}^n \mathcal{O}_i(x_i)\right\rangle\right) = -\sum_{i=1}^n \delta^d(x-x_i)\left\langle \delta\mathcal{O}_i(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\right\rangle \qquad (4.27)$$

in terms of local co-ordinates on (M,g). This is known as the *Ward identity* for the symmetry represented by $\phi \to \phi + \delta\phi$. It states that the divergence of a correlation function involving a current j^μ vanishes everywhere except at the locations of other operator insertions, and is the modification of the conservation law $d*\langle j(x)\rangle = 0$ for the expectation value of the current itself. Again, note that the divergence is taken *after* computing the path integral.

Let's integrate the Ward identity over some region $M' \subseteq M$ with boundary $\partial M' = N_1 - N_0$, just as we studied classically. We'll first choose M' to contain all the points $\{x_1,\ldots,x_n\}$ so that the integral receives contributions from all of the terms on the *rhs* of (4.26). Then

$$\langle Q[N_1]\prod_i \mathcal{O}_i(x_i)\rangle - \langle Q[N_0]\prod_i \mathcal{O}_i(x_i)\rangle = -\sum_{i=1}^n \langle \delta\mathcal{O}_i(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\rangle \qquad (4.28)$$

where the charge $Q[N] = \int_N *j$ just as in the classical case. In particular, if $M' = M$ and M is closed (*i.e.*, compact without boundary) then we obtain

$$0 = \sum_{i=1}^n \langle \delta\mathcal{O}_i(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\rangle \qquad (4.29)$$

telling us that if we perform the symmetry transform throughout space–time then the correlation function is simply invariant, $\delta\langle\prod_i \mathcal{O}_i\rangle = 0$. This is just the infinitesimal version of the result we had before in (4.18).

On the other hand, if only one some of the x_i lie inside M', then only some of the δ-functions will contribute. In particular, if $I \subseteq \{1,2,\ldots,n\}$ then we obtain

$$\langle Q[N_1]\prod_{i=1}^n \mathcal{O}_i(x_i)\rangle - \langle Q[N_0]\prod_{i=1}^n \mathcal{O}_i(x_i)\rangle = \sum_{i\in I} \langle \delta\mathcal{O}_i(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\rangle\,. \qquad (4.30)$$

whenever $x_i \in M'$ for $i \in I$. Only those operators enclosed in M' contribute to the changes on the *rhs*.

Note that the condition that M be closed cannot be relaxed lightly. On a manifold with boundary, to define the path integral we must specify some boundary conditions for the fields. The transformation $\phi \mapsto \phi'$ may now affect the boundary conditions, which lead to further contributions to the *rhs* of the Ward identity. For a relatively trivial example, the condition that the net charges of the operators we insert must be zero becomes modified to the condition that the *difference* between the charges of the incoming and outgoing states (boundary conditions on the fields) must be balanced by the charges of the operator insertions.

A much more subtle example arises when the space–time is non–compact and has infinite volume. In this case, the required boundary conditions as $|x| \to \infty$ are that our fields take some constant value ϕ_0 which lies at the minimum of the effective potential. Because of the suppression factor $e^{-S[\phi]}$, such field configurations will dominate the path integral on an infinite volume space–time. However, it may be that the (global) minimum of the potential is not unique; if $V(\phi)$ is minimized for any $\phi \in \mathcal{M}$ and our symmetry transformations move ϕ around in $\mathcal{M}$ the symmetry will be **spontaneously broken**. You'll learn much more about this story if you're taking the Part III Standard Model course.

4.1.3 The Ward–Takahashi identity in QED

Ward identities can be derived for any symmetry transformation, but the name is often associated to the transformations

$$\psi \mapsto \psi' := e^{i\alpha}\psi\,, \qquad \bar{\psi} \mapsto \bar{\psi}' := e^{-i\alpha}\bar{\psi}\,, \qquad A_\mu \mapsto A'_\mu := A_\mu \tag{4.31}$$

which for constant α are symmetries of the QED action

$$S_{\text{QED}}[A,\psi] = \int d^4x \left[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi\right]. \tag{4.32}$$

This was the case originally considered by Ward and Takahashi. The regularized path integral measure is also invariant under these transformations, *i.e.*,

$$\mathcal{D}\psi\,\mathcal{D}\bar{\psi} \mapsto \mathcal{D}\psi'\,\mathcal{D}\bar{\psi}' = \mathcal{D}\psi\,\mathcal{D}\bar{\psi} \tag{4.33}$$

provided our regularized measure involves integrating over an equal number of ψ and $\bar{\psi}$ modes. Thus these transformations are indeed symmetries of the path integral.

As above, we now promote α to a position–dependent parameter $\alpha(x)$, — this is *not* a gauge transformation because the gauge field A_μ itself remains unaffected. The path integral measure depends only on the fields ψ and $\bar{\psi}$, not their derivatives, so *if* our regularized measure also preserves the local symmetry[19], the only contribution to the current will come from the action. One finds $j^\mu = \bar{\psi}\gamma^\mu\psi$, which of course is just the charged current to which the photon couples in QED.

Now consider the correlation function $\langle\psi(x_1)\bar{\psi}(x_2)\rangle$. Since $\delta\psi \propto \psi$ the Ward identity becomes

$$\partial_\mu\langle j^\mu(x)\psi(x_1)\psi(x_2)\rangle = -\delta^d(x-x_1)\langle\psi(x_1)\bar{\psi}(x_2)\rangle + \delta^d(x-x_2)\langle\psi(x_1)\bar{\psi}(x_2)\rangle \tag{4.34}$$

so that the vector $f_\mu(x,x_1,x_2) = \langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle$ is divergence free everywhere except at the insertions of the electron field.

[19]Note that in any case, the change in the measure here will be field independent.

The identity (4.40) is traditionally viewed in momentum space. We Fourier transform the two-point function of the electron field:

$$\begin{aligned}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}k_1\cdot x_1}\,\mathrm{e}^{-\mathrm{i}k_2\cdot x_2}\,\langle\psi(x_1)\bar\psi(x_2)\rangle \\ = \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}k_1\cdot(x_1-x_2)}\,\mathrm{e}^{\mathrm{i}(k_1-k_2)\cdot x_2}\,\langle\psi(x_1-x_2)\bar\psi(0)\rangle \\ = \delta^4(k_1-k_2)\,S(k_1)\end{aligned} \tag{4.35}$$

where the first equality follows from translational invariance of the correlation function. Note that (unlike a Feynman diagram for scattering amplitudes) there's no requirement that the momenta here are on–shell; they're just the Fourier transforms of the insertion points. The previous equation defines the **exact electron propagator**,

$$S(k) := \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}k\cdot x}\langle\psi(x)\bar\psi(0)\rangle \tag{4.36}$$

in momentum space. In perturbation theory, it represents the sum of all possible connected[20] Feynman diagrams that can be drawn in connecting the $\psi\bar\psi$ insertions together. (Thus, like the 2–point function $\langle\psi\bar\psi\rangle$, $S(k)$ carries a pair of Dirac spinor indices, which we've suppressed in the notation.) Specifically,

where the first line contains all possible connected contributions to the two–point function, and the second line writes these in terms of **one particle irreducible** (1PI) graphs: those connected graphs which cannot be turned into a disconnected graph by cutting any single internal line. The sum of such 1PI contributions is usually called the **electron self–energy** and denoted $\Sigma(\not{k})$. (This is also a matrix in spin space.) The exact electron propagator is related to $\Sigma(\not{k})$ by

$$S(k) = \frac{1}{\mathrm{i}\not{k} + m - \Sigma(\not{k})} \tag{4.37}$$

by summing the geometric series above.

[20]Recall that our correlation functions are normalized by the partition function, which is the sum of all *vacuum* diagrams.

In a similar way, we introduce the **exact electromagnetic vertex** $\Gamma_\mu(k_1, k_2)$ by the Fourier transform

$$\begin{aligned}
&\int \mathrm{d}^4x\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}p\cdot x}\,\mathrm{e}^{\mathrm{i}k_1\cdot x_1}\,\mathrm{e}^{-\mathrm{i}k_2\cdot x_2}\,\langle j_\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle \\
&\quad = \int \mathrm{d}^4x\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}p\cdot(x-x_2)}\,\mathrm{e}^{\mathrm{i}k_1\cdot(x_1-x_2)}\,\mathrm{e}^{\mathrm{i}(p+k_1-k_2)\cdot x_2}\,\langle j_\mu(x-x_2)\psi(x_1-x_2)\bar{\psi}(0)\rangle \\
&\quad =: \delta^4(p+k_1-k_2)\,S(k_1)\Gamma_\mu(k_1,k_2)S(k_2)\,.
\end{aligned} \tag{4.38}$$

Let's understand this definition. $\langle\psi(x_1)j_\mu(x)\bar{\psi}(x_2)\rangle$ will be given by the sum of all Feynman graphs connecting the electron field insertions at $x_{1,2}$ to the current at x. Recalling that $j_\mu = \bar{\psi}\gamma_\mu\psi$, we see that the leading contribution will simply come from a pair of propagators connecting $\psi(x_1)$ to $\bar{\psi}(x)$, and $\psi(x)$ to $\bar{\psi}(x_2)$ respectively. Further contributions will come from diagrams that correct each of these free propagators, turning them into the exact electron propagators on each side; *i.e.*

$$\langle\psi(x_1)j_\mu(x)\bar{\psi}(x_2)\rangle \supset \langle\psi(x_1)\bar{\psi}(x)\rangle\,\gamma_\mu\,\langle\psi(x)\bar{\psi}(x_2)\rangle\,. \tag{4.39}$$

These diagrams tell us nothing new about the *vertex*; they're already part of the exact electron propagator. We thus include factors of $S(k_1)$ and $S(k_2)$ in our definition.

The remaining contributions are the ones we care about. They involve graphs that connect the two exact electron propagators together in some way. For example, at leading order, we have the diagram

This class of diagrams is what contributes to $\Gamma_\mu(k_1, k_2)$, so

$$\Gamma_\mu(k_1,k_2) = \gamma_\mu + \text{quantum corrections}\,, \tag{4.40}$$

where we note that all the corrections to γ_μ come from loop diagrams.

Now let's return to our Ward identity (4.40). Taking the Fourier transform of the complete equation, we obtain

$$(k_1-k_2)_\mu\,S(k_1)\Gamma^\mu(k_1,k_2)S(k_2) = \mathrm{i}S(k_1) - \mathrm{i}S(k_2) \tag{4.41}$$

or equivalently

$$(k_1-k_2)_\mu\,\Gamma^\mu(k_1,k_2) = \mathrm{i}S(k_2)^{-1} - \mathrm{i}S(k_1)^{-1} \tag{4.42}$$

by acting with $S^{-1}(k_1)$ on the left and $S^{-1}(k_2)$ on the right. This identity was obtained by Takahashi; taking the limit $k_2 \to k_1$ gives

$$\Gamma_\mu(k,k) = -\mathrm{i}\frac{\partial}{\partial k^\mu} S^{-1}(k) \tag{4.43}$$

which was the form originally obtained by Ward.

In previous chapters, we've seen that integrating out (high–energy) fields generically shifts the values of couplings in the (low–energy) effective action. Anticipating our story slightly, in QED, we'd expect that we can generate new contributions to both the electron kinetic term $\int \mathrm{i}\bar{\psi}\partial\!\!\!/\psi\, \mathrm{d}^4x$ and the electron–photon vertex $\int \bar{\psi} A\!\!\!/\psi\, \mathrm{d}^4x$ (as well as the electron mass term). The significance of the Ward identity is that, provided the regulated path integral measure is compatible with the symmetry (4.37), *the quantum corrections to these two terms must be related.* In particular, using (4.43) in (4.48) we have

$$\begin{aligned}(k_1-k_2)_\mu\Gamma^\mu(k_1,k_2) &= \mathrm{i}\,(\mathrm{i}k\!\!\!/_2 + m - \Sigma(k\!\!\!/_2) - \mathrm{i}k\!\!\!/_1 - m + \Sigma(k\!\!\!/_1)) \\ &= (k_1-k_2)_\mu\gamma^\mu + \mathrm{i}(\Sigma(k\!\!\!/_1) - \Sigma(k\!\!\!/_2))\end{aligned} \tag{4.44}$$

where we note that the 'inverse electron propagator' $S^{-1}(k)$ is nothing but the electron kinetic term in the action, written in momentum space. The fact that quantum corrections treat the whole covariant derivative $\int \mathrm{i}\bar{\psi}D\!\!\!\!/\,\psi\, \mathrm{d}^4x$ together is important in ensuring that the quantum theory respects gauge transformations, as you'll explore further in the problem sets. In the early days of QED, before renormalization was systematically understood, the Ward identity provided a good check that the regularized theory was compatible with gauge invariance.