# Quantum Field Theory II

## University of Cambridge Part III Mathematical Tripos

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Abstract: These are the lecture notes for the Advanced Quantum Field Theory course given to students taking Part III Maths in Cambridge during Lent Term of 2018. The main aims are to discuss Path Integrals, the Renormalization Group, Wilsonian Effective Field Theory and non–Abelian Gauge Theories.

## Contents

**Acknowledgments**

Nothing in these lecture notes is original. In particular, my treatment is heavily influenced by several of the textbooks listed below, especially Vafa *et al.* and the excellent lecture notes of Neitzke in the early stages, then Schwartz and Weinberg's textbooks and Hollowood's lecture notes later in the course.

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## Preliminaries

This course is the second course on Quantum Field Theory offered in Part III of the Maths Tripos, so I'll feel free to assume you've already taken the first course in Michaelmas Term (or else an equivalent course elsewhere). You'll also find it helpful to know about groups and representation theory, say at the level of the *Symmetries, Fields and Particles* course last term. Last term's *General Relativity* and *Statistical Field Theory* courses may also be helpful, but I won't assume you attended these.

There may be some overlap between this course and certain other Part III courses this term. In particular, I'd expect the material here to complement the courses on *The Standard Model* and on *Applications of Differential Geometry to Physics* very well. In turn, I'd also hope this course is useful preparation for courses on *Supersymmetry* and *String Theory*.

### Books & Other Resources

There are many (too many!) textbooks and reference books available on Quantum Field Theory. Different ones emphasize different aspects of the theory, or applications to different branches of physics or mathematics – indeed, QFT is such a huge subject nowadays that it is probably impossible for a single textbook to give an encyclopedic treatment (and absolutely impossible for a course of 24 lectures to do so). Here are some of the ones I've found useful while preparing these notes; you might prefer different ones to me.

- **Nair, V.P.,** *Quantum Field Theory: A Modern Perspective*, Springer (2005).
  Although it isn't so well known, this is perhaps my favourite QFT book. It begins with a clear, concise discussion of all the standard perturbative material you'll find in any QFT course. However, unlike many books, it also makes clear that there's far more to QFT than just perturbation theory. Contains excellent discussions of the configuration space of field theories, ambiguities in quantization, approaches to strong coupling limits in QCD, and QFT at finite temperature.

This next list contains the stalwart QFT textbooks. You will certainly want to consult (at least) one of these repeatedly during the course. They'll also be very helpful for people taking the Standard Model course.

- **Peskin, M. and Schroeder, D.,** *An Introduction to Quantum Field Theory*, Addison–Wesley (1996).
  An excellent QFT textbook, containing extensive discussions of both gauge theories and renormalization. Many examples worked through in detail, with a particular emphasis on applications to particle physics.

- **Schwartz, M.,** *Quantum Field Theory and the Standard Model*, CUP (2014).
  The new kid on the block, honed during the author's lecture courses at Harvard. I really like this book – it strikes an excellent balance between formalism and applications (mostly to high energy physics), with fresh and clear explanations throughout.

- **Srednicki, M.,** *Quantum Field Theory*, CUP (2007).
  This is also an excellent, very clearly written and very pedagogical textbook, with clearly compartmentalised chapters breaking the material up into digestible chunks. However, our route through QFT in this course will follow a slightly different path.

- **Zee, A.,** *Quantum Field Theory in a Nutshell*, $2^{\text{nd}}$ edition, PUP (2010).
  QFT is notorious for containing many technical details, and its easy to get lost. This is a great book if you want to keep the big picture of what QFT is all about firmly in sight. It will put you joyfully back on track and remind you why you wanted to learn the subject in the first place. It's not the best place to work through detailed calculations, but that's not the point.

There are also a large number of books that are more specialized. Many of these are rather advanced, so I do not recommend you use them as a primary text. However, you may well wish to dip into them occasionally to get a deeper perspective on topics you particularly enjoy. This list is particularly biased towards my (often geometric) interests:

- **Banks, T.** *Modern Quantum Field Theory: A Concise Introduction*, CUP (2008).
  I particularly enjoyed its discussion of the renormalization group and effective field theories. As it says, this book is probably too concise to be a main text.

- **Cardy, J.,** *Scaling and Renormalization in Statistical Physics*, CUP (1996).
  A wonderful treatment of the Renormalization Group in the context in which it was first developed: calculating critical exponents for phase transitions in statistical systems. The presentation is extremely clear, and this book should help to balance the 'high energy' perspective of many of the other textbooks.

- **Coleman, S.,** *Aspects of Symmetry*, CUP (1988).
  Legendary lectures from one of the most insightful masters of QFT. Contains much material that is beyond the scope of this course, but so engagingly written that I couldn't resist including it here!

- **Costello, K.,** *Renormalization and Effective Field Theory*, AMS (2011).
  A pure mathematician's view of QFT. The main aim of this book is to give a rigorous definition of (perturbative) QFT via path integrals and Wilsonian effective field theory. Another major achievement is to implement this for gauge theories by combining BV quantization with the ERG. Repays the hard work you'll need to read it – for serious mathematicians only.

- **Deligne, P.,** *et al.*, *Quantum Fields and Strings: A Course for Mathematicians*, vols. 1 & 2, AMS (1999).
  Aimed at professional mathematicians wanting an introduction to QFT. They thus require considerable mathematical maturity to read, but most certainly repay the effort. Almost everything here is beyond the level of this course, but I can promise you're appreciation of QFT will be deepened immeasurably by reading the lectures

of Deligne & Freed on *Classical Field Theory* (vol. 1), Gross on the *Renormalization Group* (vol. 1), Gadwezki on CFTs (vol. 2), and especially Witten on *Dynamics of QFT* (vol. 2). (I recommend you read Witten on *anything*.)

- **Polyakov, A.,** *Gauge Fields and Strings*, Harwood Academic (1987).
  A very original and very deep perspective on QFT, building a form of synthesis of Polyakov's approach to strongly coupled QCD. Several of the most important developments in theoretical physics over the past couple of decades have been (directly or indirectly) inspired by ideas in this book.

- **Schweber, S.,** *QED and the Men Who Made It: Dyson, Feynman, Schwinger and Tomonaga*, Princeton (1994).
  Not a textbook, but a tale of the times in which QFT was born, and the people who made it happen. It doesn't aim to dazzle you with how very great these heroes were[1], but rather shows you how puzzled they were, how human their misunderstandings, and how tenaciously they had to fight to make progress. Inspirational stuff.

- **Vafa, C., and Zaslow, E.,** (eds.), *Mirror Symmetry*, AMS (2003).
  A huge book comprising chapters written by different mathematicians and physicists with the aim of understanding Mirror Symmetry in the context of string theory. Chapters 8 – 11 give an introduction to QFT in low dimensions from a perspective close to the one we will start with in this course. The following chapters could well be useful if you're taking the String Theory Part III course.

- **Weinberg, S.,** *The Quantum Theory of Fields*, vols. 1 & 2, CUP (1996).
  Penetrating insight into everything it covers and packed with many detailed examples. The perspective is always deep, but it requires strong concentration to follow a story that sometimes plays out over several chapters. Weinberg's thesis is that QFT is the inevitable consequence of marrying Quantum Mechanics, Relativity and the Cluster Decomposition Principle (that distant experiments yield uncorrelated results). In this telling, particles play a primary role, with fields coming later; for me, that's backwards.

- **Zinn–Justin, J.** *Quantum Field Theory and Critical Phenomena*,
  4$^{\text{th}}$ edition, OUP (2002).
  Contains a very insightful discussion of the Renormalization Group and also a lot of information on Gauge Theories. Most of its examples are drawn from either Statistical or Condensed Matter Physics.

Textbooks are expensive. Fortunately, there are lots of excellent resources available freely online. I like these:

- **Dijkgraaf, R.,** *Les Houches Lectures on Fields, Strings and Duality*,
  http://arXiv.org/pdf/hep-th/9703136.pdf

[1] I should say 'are'; even now in 2017, Freeman Dyson still works at the IAS almost every day.

An modern perspective on what QFT is all about, and its relation to string theory. For the most part, the emphasis is on more mathematical topics (*e.g.* TFT, dualities) than we will cover in the lectures, but the first few sections are good for orientation.

- **Hollowood, T.,** *Six Lectures on QFT, RG and SUSY*,
  http://arxiv.org/pdf/0909.0859v1.pdf
  An excellent mini–series of lectures on QFT, given at a summer school aimed at end–of–first–year graduate students from around the UK. They put renormalization and Wilsonian Effective Theories centre stage. While the final two lectures on SUSY go beyond this course, I found the first three very helpful when preparing the current notes. We'll follow parts of these notes closely.

- **Neitzke, A.,** *Applications of Quantum Field Theory to Geometry*,
  https://www.ma.utexas.edu/users/neitzke/teaching/392C-applied-qft/
  Lectures aimed at introducing mathematicians to Quantum Field Theory techniques that are used in computing Seiberg–Witten invariants. I very much like the perspective of these lectures, and we'll Neitzke's notes closely for the first part of the course.

- **Osborn, H.,** *Advanced Quantum Field Theory*,
  http://www.damtp.cam.ac.uk/user/ho/Notes.pdf
  The lecture notes for a previous incarnation of this course, delivered by Prof. Hugh Osborn. They cover similar material to the current ones, but from a rather different perspective. If you don't like the way I'm doing things, or for extra practice, take a look here!

- **Polchinski, J.,** *Renormalization and Effective Lagrangians*,
  http://www.sciencedirect.com/science/article/pii/0550321384902876

- **Polchinksi, J.,** *Dualities of Fields and Strings*,
  http://arxiv.org/abs/1412.5704
  The first paper gives a very clear description of the 'exact renormalization group' and its application to scalar field theory. The second is a recent survey of the idea of 'duality' in QFT and beyond. We'll explore this if we get time.

- **Segal, G.,** *Quantum Field Theory* lectures,
  YouTube lectures
  Recorded lectures aiming at an axiomatization of QFT by one of the deepest thinkers around. I particularly recommend the lectures "*What is Quantum Field Theory?*" from Austin, TX, and "*Three Roles of Quantum Field Theory*" from Bonn (though the blackboards are atrocious!).

- **Tong, D.,** *Quantum Field Theory*,
  http://www.damtp.cam.ac.uk/user/tong/qft.html
  The lecture notes from the Michaelmas QFT course in Part III. If you feel you're

missing some background from last term, this is an excellent place to look. There are also some video lectures from when the course was given at Perimeter Institute.

- **Weinberg, S.,** *What Is Quantum Field Theory, and What Did We Think It Is?*, `http://arXiv.org/pdf/hep-th/9702027.pdf`
- **Weinberg, S.,** *Effective Field Theory, Past and Future*, `http://arXiv.org/pdf/0908.1964.pdf`
  These two papers provide a fascinating account of the origins of effective field theories in current algebras for soft pion physics, and how the Wilsonian picture of Renormalization gradually changed our whole perspective of what QFT is about.
- **Wilson, K., and Kogut, J.** *The Renormalization Group and the $\epsilon$-Expansion*, Phys. Rep. **12** 2 (1974),
  `http://www.sciencedirect.com/science/article/pii/0370157374900234`
  One of the first, and still one of the best, introductions to the renormalization group as it is understood today. Written by somone who changed the way we think about QFT. Contains lots of examples from both statistical physics and field theory.

That's a huge list, and only a real expert in QFT would have mastered everything on it. I provide it here so you can pick and choose to go into more depth on the topics you find most interesting, and in the hope that you can fill in any background you find you are missing.

# 1 Introduction

Quantum Field Theory is, to begin with, exactly what it says it is: the quantum version of a field theory. But this simple statement hardly does justice to what is the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology, geometry, combinatorics, astrophysics and cosmology. It's also extremely closely related to statistical field theory, probability and from there even to (quasi–)stochastic systems such as finance.

## 1.1 Choosing a QFT

To build a QFT, we start by picking the space on which it lives. Usually, this will be some smooth, Riemannian (or pseudo–Riemannian) manifold $(M, g)$ of dimension $\dim(M) = d$. For example, for most applications to particle physics, we'd choose $(M, g) = (\mathbb{R}^4, \eta)$ where $\eta$ is the Minkowski metric. However, this is far from being the only interesting choice. For many applications in condensed matter, one sets either $(M, g) = (\mathbb{R}^3, \delta)$ with $\delta$ the flat Euclidean metric, or perhaps $M = U \subset \mathbb{R}^3$ to study field theory living in a sample of material that occupies some region $U$. As a further example, the worldsheet description of string theory involves a QFT living on a Riemann surface $(\Sigma, [g])$ where only the conformal class

$$[g] = \{g \in \mathrm{Met}(\Sigma) \text{ with } g \sim \mathrm{e}^{2\sigma} g \text{ for } \sigma : \Sigma \to \mathbb{R}\}$$

of the metric needs to be specified, while applications of QFT to topological problems such as knot invariants make use of a certain gauge theory (known as Chern–Simons theory) living on an arbitrary orientable three–manifold $M$ with no metric at all. Whatever choice we make, in QFT the metric $g$ is regarded as fixed – studying what happens when the metric itself has quantum fluctuations requires quantum gravity.

Having decided which universe we live in, our next choice is to pick which objects we wish to study. That is, we must choose the **fields**. The simplest choice is a scalar field, which is just a function on $M$. It'll often be useful to think of this a map

$$\phi : M \to \mathbb{R}, \mathbb{C}, \dots,$$

according to whether the scalar is real– or complex–valued. More generally, $\phi$ could describe a map

$$\phi : M \to N$$

from our space to some other (Riemannian) manifold $(N, G)$, known as the **target space**. For example, we'll see that we can think of ordinary non–relativistic Quantum Mechanics in terms of a $d = 1$ QFT living on an interval $I = [0, 1]$ known as the **worldline**, where the fields describe a map $\phi : I \to \mathbb{R}^3$. In particle physics, the pion field $\pi(x)$ describes a map $M \to G/H$ where $M$ is our Universe and $G$ and $H$ are Lie groups. (In the specific case of pions in the Standard Model, it turns out that $G/H = (SU(2) \times SU(2))/SU(2)$.) In string

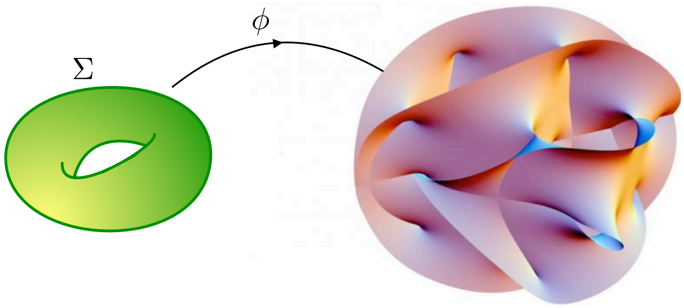


**Figure 1**: *String Theory involves a QFT describing maps from a Riemann surface $\Sigma$ to a Calabi–Yau manifold.*

theory, some of the worldsheet fields are scalars describing a map $\phi : \Sigma \to N$ embedding the worldsheet in a certain special type of Riemannian manifold $N$ called a Calabi–Yau manifold.

There are many further options. In a gauge theory, as we'll see in chapter 8, the basic field is a connection $\nabla$ on a principal $G$-bundle $P \to M$. We could also choose to include charged matter, described mathematically in terms of sections of vector bundles $E \to M$ associated to $P \to M$ by a choice of representation. For example, scalar QED involves a photon $A_\mu$ and a scalar $\phi$, defined up to the gauge transformations

$$A_\mu \sim A_\mu + \partial_\mu \lambda \qquad\qquad \phi \sim \mathrm{e}^{\mathrm{i}\lambda}\phi$$

This is just the local description of a connection on a principal $U(1)$ bundle, together with a section of a rank one complex vector bundle $E \to M$ whose fibres are equipped with a Hermitian metric. As you learned if you took the General Relativity course, Riemannian manifolds naturally come along with various bundles, such as the tangent and cotangent bundles $TM$ and $T^*M$. Under mild topological conditions, we might also be able to define spin bundles over $M$. In physics, we'd think of sections of these bundles as being fields that transform non-trivially under Lorentz transformations; *i.e.* they carry non–zero 'spin' and are described (at least locally) by functions such as $V^\mu$, $\psi^{\dot\alpha}$, $B_{[\mu\nu]}$ and $\chi^{\dot\alpha}_\mu$ with various types of vector and/or spinor indices. I don't want to get into any details in the introduction — we'll explore these objects and what the mathematical words mean in detail as we go along. My only point here is there's lots of choice in what type fields we might like to include in our QFT, and that all the most common choices (certainly all the ones I expect you to have heard of so far, and all the ones we'll meet in this course) are very natural geometrical objects.

Whatever fields we pick, I'll let $\mathcal{C}$ denote the **space of field configurations** on $M$. That is, every point $\phi \in \mathcal{C}$ corresponds to a configuration of the field – a picture of what (every component of) the field looks like across the whole universe $M$. Since we allow our fields to have arbitrarily small bumps and ripples, $\mathcal{C}$ is typically an infinite dimensional function space. Trying to understand the geometry and topology of this infinite dimensional space of fields, and then trying to do something useful with it is fundamentally what makes QFT difficult, but it's also what makes it interesting and powerful.

The next ingredient we need is to specify the **action** for our theory. This is a function

$$S : \mathcal{C} \to \mathbb{R} \tag{1.1}$$

on the space of fields. In other words, given a field configuration, the action produces a real number. We often write $S[\phi]$ for this number, as opposed to $S(\phi)$, and say the action is a **functional**. The word is just to remind us that the domain $\mathcal{C}$ of $S$ is itself and infinite–dimensional function space. The critical set[2]

$$Crit_{\mathcal{C}}(S) = \{\phi \in \mathcal{C} \mid \delta S[\phi] = 0\} \tag{1.2}$$

correspond to fields that solve the classical field equations, the Euler–Lagrange equations. In the simplest circumstances, these critical points are isolated.

When setting up our QFT, we often assume that $S[\phi]$ is **local**, meaning that it can be written as

$$S[\phi] = \int_M \mathrm{d}^d x \sqrt{g}\ \mathcal{L}(\phi(x), \partial\phi(x), \ldots) \tag{1.3}$$

where the Lagrangian[3] $\mathcal{L}$ depends on the value of $\phi$ and finitely many derivatives at just a *single* point in $M$. As a consequence, the classical field equations become nonlinear pdes of an order determined by the number of derivatives of $\phi$ appearing in $\mathcal{L}$.

You've doubtless been writing down local actions for so long – motivated by either classical mechanics or classical field theories such as electromagnetism – that you now do it without thinking. However, it's worth pointing out that, purely from the point of view of functions on $\mathcal{C}$, locality on $M$ is actually a very strong restriction. Even a monomial function on $\mathcal{C}$ generically looks like

$$\int_{M^{\otimes n}} \mathrm{d}^d x_1\, \mathrm{d}^d x_2 \cdots \mathrm{d}^d x_n\ \Lambda(x_1, x_2, \ldots, x_n)\, \phi(x_1)\phi(x_2)\cdots\phi(x_n)$$

involving the integral of the field at many different points, with some choice of function $\Lambda : M^{\otimes n} \to \mathbb{R}$. (You can think of this as an infinite dimensional analogue of a monomial

$$\sum_{ijk\cdots l} \Lambda_{ijk\ldots l} z^i z^j z^k \cdots z^l$$

[2]Here, $\delta$ is properly viewed as the exterior derivative on $\mathcal{C}$ and obeys $\delta^2 = 0$. Thus $\delta = \int_M \delta\phi(x)\, \delta/\delta\phi(x)$ where $\delta\phi(x)$ is a one-form on $\mathcal{C}$ and the derivative $\delta/\delta\phi(x)$ on $\mathcal{C}$ acts *e.g.* as

$$\frac{\delta}{\delta\phi(x)}\,\phi(y) = \delta^d(x-y)\,, \qquad \frac{\delta}{\delta\phi(x)}\int_M \mathrm{d}^d y\, \phi(y)^2 = 2\phi(x)\,,$$
$$\frac{\delta}{\delta\phi(x)}\int_M \mathrm{d}^d y\, \partial^\mu\phi\, \partial_\mu\phi = 2\int_M \mathrm{d}^d y\ \partial^\mu \delta^d(x-y)\ \partial_\mu\phi = -2\Box\phi(x)$$

(the last example holding in the case that there is no boundary term).

[3]What I'm calling the Lagrangian here is really the Lagrangian *density*, with the Lagrangian itself being the integral $\mathcal{L}$ over a Cauchy surface in $M$. The abuse of terminology is standard.

in finitely many variables $z^i$, with the function $\Lambda(x_1, x_2, \ldots, x_n)$ playing the role of the 'coefficients'.) Locality means that we restrict to 'functions' $\Lambda$ of the form

$$\Lambda(x_1, x_2, \ldots, x_n) = \lambda(x)\, \partial^{(p_1)}\delta^d(x_1 - x_2)\ \partial^{(p_2)}\delta^d(x_2 - x_3)\ \cdots\ \partial^{(p_{n-1})}\delta^d(x_{n-1} - x_n) \tag{1.4}$$

that are supported on the main diagonal $M \subset M^{\otimes n}$, with finitely many derivatives allowed to act on the $\delta$–functions. Integrating by parts if necessary, these derivatives can be made to act on the fields, leaving us with an expression of the general form

$$\int_M \mathrm{d}^d x\ \lambda(x)\, \partial^{(q_1)}\phi(x)\ \partial^{(q_2)}\phi(x)\ \cdots\ \partial^{(q_n)}\phi(x)$$

of a monomial of degree $n$ in the fields, acted on again by some derivatives, with all fields and derivatives evaluated at the same $x \in M$. The only remnant of the original $\Lambda : M^{\otimes n} \to \mathbb{R}$ is the function $\lambda : M \to \mathbb{R}$. If this monomial is higher than quadratic in the fields, it leads to a non–linear term in the classical field equations, meaning we no longer have superposition of solutions (*i.e.* the space of solutions to the Euler–Lagrange equations will no longer be expected to be a vector space). Physically, we interpret this as an **interaction**, either between several different fields or between a field and itself. In many cases, we restrict further and choose the functions $\lambda$ to be constant, $\lambda(x) = \lambda$ with the constant $\lambda$ known as a **coupling constant**.

If we *were* to allow multi-local terms in our action, the resulting classical field equations would be integro–differential equations, so the behaviour of our field at one point $x \in M$ would depend what the field configuration looks like across all of $M$. This 'action at a distance' is usually thought to be unphysical, at least in classical physics. However, we'll see later that QFT forces us to consider certain non-local terms even if we try to rule them out when setting up the theory.

### 1.2 What do we want to compute?

In this course, the main tools we'll use to study QFT are **path integrals**. Heuristically, these are integrals such as

$$\int_{\mathcal{C}} [\mathcal{D}\phi]\ \exp\left(-\frac{1}{\hbar} S[\phi]\right) \tag{1.5}$$

that are taken over the infinite dimensional space of fields $\mathcal{C}$, with some sort of measure $[\mathcal{D}\phi]\, \mathrm{e}^{-S[\phi]/\hbar}$ that weights the contribution of each field configuration $\phi \in \mathcal{C}$ by $\mathrm{e}^{-S/\hbar}$. The vague idea of this measure is for the exponential to suppress field configurations that are 'wild', so that for example we might optimistically hope that configurations in which $\phi$ jumps around rapidly between very different values (perhaps even being discontinuous, or worse) play a 'negligible' role.

However, it's very far from clear that this hope will be realised. Even if you have only an anecdotal knowledge of functional analysis, you likely know that there are vastly more discontinuous functions than continuous ones, vastly more continuous than continuously differentiable, vastly more functions that are $C^k$ than there are functions that are $C^{k+1}$, and vastly more smooth functions than analytic ones[4]. Going in the other direction, there are vastly more distributions than even discontinuous functions. Despite the best efforts of the suppression factor $\mathrm{e}^{-S/\hbar}$, the contribution of the enormous variety of 'wild' field configurations can easily overwhelm the much smaller set of 'nice' (*e.g.* smooth) fields.

This idea is familiar in statistical mechanics, where the contribution of any given configuration (*e.g.* the spin state of electrons located at each site of a lattice) is weighted by $\mathrm{e}^{-\beta E(\phi)}$ with $\beta$ the inverse temperature of the system. However, there are typically many more 'disordered' states (*e.g.* with random alignments of neighbouring spins) than ordered ones (*e.g.* all spins aligned) and, depending on the details of the model (*e.g.* types of interactions allowed between nearby / distant spins, and the dimensionality and connectivity of the lattice on which they sit) there can be a complicated structure of phase transitions as parameters such as the temperature $\beta^{-1}$ is varied. In the poetic words of Bryce de Witt, the balance between the tendency of the exponential factor to suppress rapidly varying configurations and the fact that there are simply many more of them is "the eternal struggle between energy and entropy".

In a field theory, the fact that we're dealing with infinite–dimensional spaces makes path integrals such as (1.5) even more delicate to study. Since we're integrating over an *infinite* dimensional space, it's far from obvious that we can get any sort of finite answer out of a path integral at all. Indeed, much of the hard work we need to do in this course is about understanding how to achieve this (even at low orders in perturbation theory). However, living on the edge is exciting, and when a path integral exists it has a rich character that is often goes far beyond what one can see in the classical action.

Incidentally, I've chosen the argument of the exponential in (1.5) to be $-S[\phi]/\hbar$, as appropriate when $(M, g)$ is Riemannian (such as Euclidean space). For a pseudo–Riemannian manifold (such as Minkowski space) we'd instead use $\mathrm{i}S[\phi]/\hbar$. It should be clear already that any difficulties we have in making sense of the doubtfully convergent integral (1.5) are only going to be worse for a doubtfully–conditionally convergent integral. For this reason, we'll mostly stick to Riemannian signature spaces in this course[5].

### 1.2.1 The partition function

Let's now take a look at the path integral in a little more detail, though still completely heuristically. If $M$ is closed and compact (such as a sphere $S^d$ or torus $T^d = S^1 \times S^1 \times \cdots \times S^1$), the most important object to compute in any QFT is the **partition function**

$$\mathcal{Z}_{(M,g)}(\lambda, \cdots) = \int_{\mathcal{C}} [\mathcal{D}\phi] \exp\left(-\frac{S[\phi]}{\hbar}\right) . \tag{1.6}$$

[4] Recall that a function on $M$ is in $C^k(M)$ if, at every $p \in M$, the function and its first $k$ derivatives exist and are each continuous. A continuous function is said to be in $C^0(M)$. A function is in $C^\infty(M)$ (smooth) if it is in $C^k(M)$ for all $k$. It is $C^\omega$ (analytic) if its Taylor expansion around any point converges to the function itself.

[5] You'll find that most QFT textbooks do too: even though they may claim to start out writing things in Minkowksi space, before any real calculation is done they will 'Wick rotate' to Euclidean space. Working in Euclidean space is also essentially the same thing as studying Statistical Field Theory, except here we'll take $d = \dim(M)$ to be the total space–time dimension.

As we've indicated here, the partition function depends on all the choices we made in setting up our theory, such as the space $(M,g)$ on which the theory lives and the values of the couplings $\lambda$, as well as of course on $\hbar$. Note however that $\mathcal{Z}$ does *not* depend on the fields! These are just dummy variables that we've integrated out in computing the partition function.

### 1.2.2 Correlation functions

After the partition function, the most important objects we wish to compute in any QFT are **correlation functions**. These are path integrals with further insertions, of the general form

$$\int_{\mathcal{C}} [\mathcal{D}\phi]\ \exp\left(-\frac{S[\phi]}{\hbar}\right)\ \prod_{i=1}^{n} \mathcal{O}_i[\phi] \tag{1.7}$$

where the insertions $\mathcal{O}_i$ are again functions on $\mathcal{C}$. We sometimes normalise the correlation functions by the partition function, writing

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i[\phi] \right\rangle = \frac{1}{\mathcal{Z}} \int_{\mathcal{C}} [\mathcal{D}\phi]\ \exp\left(-\frac{S[\phi]}{\hbar}\right)\ \prod_{i=1}^{n} \mathcal{O}_i[\phi]\,. \tag{1.8}$$

The idea of this normalisation (as we'll see in detail later) is both to ensure that $\langle 1\rangle = 1$ and to separate out the effect of inserting the operator $\mathcal{O}$ into the path integral from effects that are there in the basic partition function already. Mathematically, normalised correlation functions compute various **moments** of the probability distribution $[\mathcal{D}\phi]\,\mathrm{e}^{-S/\hbar}/\mathcal{Z}$. From the point of view of physics, we choose the functions we insert to correspond to some quantity of physical interest that we wish to measure; perhaps the energy of the quantum field in some region, or the total angular momentum carried by some electrons, or perhaps temperature fluctuations in the CMB at different angles on the night sky.

In the QFT context, we often call these extra functions **operator insertions** in the path integral, for reasons that will become apparent. The most common examples of operator insertions are **local operators** that depend on the value of the field (and perhaps finitely many derivatives) at a single point in $M$. Examples include

$$\mathcal{O}_i(x_i) = \phi^4(x_i)\,, \qquad \mathcal{O}(x_j) = \phi^3\,\partial^\mu\phi\,\partial_\mu\phi(x)\,, \qquad \mathcal{O}(x_k) = \mathrm{e}^{\phi(x_k)}$$

and very many more. It's also perfectly possible to have operator insertions such as

$$\int_M \mathrm{d}^d x\,(\partial^\mu\phi\,\partial_\mu\phi)^2$$

that depend on the value of the field over all of $M$. For example, if you take the Part III String Theory course, you'll often compute correlation functions of a mixture of operator insertions, some of which are inserted at points $x \in M = \Sigma$, and others that are integrated over all of the worldsheet $\Sigma$. As you'll learn, these correlation functions correspond to dynamical processes in the target space of the string. We can also have operators that depend on the value of the field along a curve $\Gamma \subset M$, or some other subspace $K \subset M$. Indeed, the operator

$$W_\Gamma[A] = \mathrm{tr}\,P\exp\left(-\oint A\right)\,,$$

that depends on the value of gauge field $A = A_\mu \, \mathrm{d}x^\mu$ along a curve $\Gamma$ is one of the most fundamental operators present in any gauge theory, known to physicists as a Wilson loop and to mathematicians as the trace of the holonomy of the connection. Note again that although the operator insertions depend on the values of the fields, the correlation functions themselves do not. Rather, our correlators are functions

$$F_{(M,g)}(x_1, \ldots, x_n; \lambda, \cdots) = \left\langle \prod_{i=1}^{n} \mathcal{O}_i(x_i) \right\rangle$$

that depend on all the same data as the partition function $\mathcal{Z}$, together with any extra choices (such as the choice of points $x_i \in M$ or subspaces $\Gamma_i$, or $K_i$ and general form of the operator) that were made in choosing which correlator to compute.[6]

Because local operators also appear in the action $S[\phi]$, correlation functions are closely related to the partition function. Indeed, if the action includes a term

$$\mathcal{O} = \lambda \int_M \phi^4(x) \, \mathrm{d}^d x$$

with coupling constant $\lambda$, then differentiating formally[7] we get

$$\begin{aligned} -\frac{\hbar}{\mathcal{Z}} \frac{\partial}{\partial \lambda} \mathcal{Z} &= -\frac{\hbar}{\mathcal{Z}} \frac{\partial}{\partial \lambda} \left( \int_{\mathcal{C}} [\mathcal{D}\phi] \, \mathrm{e}^{-S[\phi]/\hbar} \right) = \frac{1}{\mathcal{Z}} \int_{\mathcal{C}} \left( [\mathcal{D}\phi] \, \mathrm{e}^{-S[\phi]/\hbar} \int_M \phi^4 \, \mathrm{d}^d x \right) \\ &= \langle \mathcal{O} \rangle \ , \end{aligned} \tag{1.9}$$

so that the normalised correlator is ($-\hbar$ times) the derivative of $\ln \mathcal{Z}$ with respect to the coupling. Thus, knowing $\mathcal{Z}$ as a function of the all couplings in the action is equivalent to knowing all the correlators of operators appearing in the action.

The operators that appear in the action are integrated over all of $M$. It's often convenient to extend the idea above so as to obtain correlators of local operators $\mathcal{O}(x)$ that depend on the value of $\phi$ (and perhaps finitely many derivatives) just at one point $x \in M$. To do this, we include **source terms** such as

$$S[\phi] \to S[\phi] + \int_M \mathrm{d}^d x \, J_i(x) \mathcal{O}_i(x)$$

in the action. The **source** $J_i(x)$ is, like the field $\phi$, a function on $M$. Really, this is just another case of the choices we made in picking our (local) action, allowing the coupling

[6] Skipping very far ahead of our story, if our real interest is in objects such as these functions, which are independent of any fields, we might hope to avoid our troubles in actually defining the path integral by instead trying to give some other, perhaps axiomatic, way to compute and manipulate such functions. This turns out to be successful in various special cases such as topological QFT and minimal models (a special class of conformal field theory in two dimensions), but despite much effort no one has yet come up with a set of axioms that are rich and flexible enough to allow the great variety of phenomena we see in QFTs, whilst still being useful enough that we can actually calculate with them. The path integral is the best description we have.

[7] That is, without worrying about whether the differentiation and integration commute. Since we haven't properly defined our path integral and don't even know yet whether it actually exists, there's no point in worrying about this seriously at this stage anyway.

‘constant’ $\lambda \to \lambda(x)$ to still vary over $M$, but the name ‘source’ and use of the letter $J(x)$ is conventional.

We do not integrate over $J$ in performing the path integral, so the partition function itself becomes a functional

$$\mathcal{Z} \to \mathcal{Z}_{(M,g)}[J_i]$$

depending on the choice of functions $J_i$ in addition to the other data. Varying this partition function wrt the value of the source at some point $x_i \in M$, we obtain formally

$$\begin{aligned} -\hbar \frac{\delta}{\delta J_i(x_i)} \mathcal{Z}[J_i] &= \int_{\mathcal{C}} \left( [\mathcal{D}\phi]\, \mathrm{e}^{-(S[\phi]+\int J_j \mathcal{O}_j)/\hbar} \frac{\delta}{\delta J_i(x_i)} \int_M \mathrm{d}^d y\, J_i(y) \mathcal{O}_i(y) \right) \\ &= \int_{\mathcal{C}} \left( [\mathcal{D}\phi]\, \mathrm{e}^{-(S[\phi]+\int J_j \mathcal{O}_j)/\hbar}\, \mathcal{O}_i(x_i) \right) , \end{aligned} \tag{1.10}$$

and thus

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \, \cdots \, \mathcal{O}_n(x_n) \rangle = \frac{(-\hbar)^n}{\mathcal{Z}} \left. \frac{\delta^n \mathcal{Z}[J]}{\delta J_1(x_1)\, \delta J_2(x_2) \, \cdots \, \delta J_n(x_n)} \right|_{J=0} . \tag{1.11}$$

Probably the most common use of this formula is when the sources couple to single powers of the field, such as

$$S[\phi] \to S[\phi] + \int_M \mathrm{d}^d x\, J(x) \phi(x) \tag{1.12}$$

for just one field. Computing $\mathcal{Z}[J]$ here is equivalent to knowing all the correlation functions

$$\langle \phi(x_1)\, \phi(x_2) \, \cdots \, \phi(x_n) \rangle = \frac{(-\hbar)^n}{\mathcal{Z}} \left. \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1)\, \delta J(x_2) \, \cdots \, \delta J(x_n)} \right|_{J=0} . \tag{1.13}$$

of the field itself. However, there’s no reason we can’t choose the source to couple to a **composite operator** – *i.e.* a non–linear function of the fields. For example, since we chose a Riemannian metric $g$ to define our QFT, we could vary the partition function with respect to the value of this metric. You should recall from last term’s QFT lectures (or any course on classical field theory) that the stress tensor $T_{\mu\nu}(x)$ is defined by

$$\delta_g S[\phi] = \frac{1}{2} \int_M T_{\mu\nu}(x)\, \delta g^{\mu\nu}(x) \sqrt{g}\, \mathrm{d}^d x \tag{1.14}$$

in terms of the variation of the (matter part of the) action wrt the metric. The stress tensor is indeed a non-linear function of the fields and their derivatives. Varying inside the path integral we obtain

$$-\hbar\, \delta_g (\ln \mathcal{Z}_{(M,g)}) = \frac{1}{2} \int_M \langle T_{\mu\nu}(x) \rangle\, \delta g^{\mu\nu}(x) \sqrt{g}\, \mathrm{d}^d x \tag{1.15}$$

and hence

$$-\frac{2\hbar}{\sqrt{g}(x)} \frac{\delta \ln \mathcal{Z}_{(M,g)}}{\delta g^{\mu\nu}(x)} = \langle T_{\mu\nu}(x) \rangle_{(M,g)} \tag{1.16}$$

where on the rhs we emphasize that the correlation function is computed using the original metric $g$.

Relations such as these show the close connection between correlation functions and the partition function. We see that correlators probe the response of the partition function to a change in the background structures we chose in setting up the theory.

### 1.2.3 Boundaries and Hilbert space

If $M$ has boundaries, say $\partial M = \cup_i B_i$, then to specify the path integral we must choose some boundary conditions for the fields on each component of $\partial M$. We'll see below that on each boundary component $B_i$, the possible configurations of the field naturally form a Hilbert space $\mathcal{H}_i$. Thus, on a manifold with boundary the path integral really defines a map

$$\otimes_i \mathcal{H}_i \to \mathbb{C}\,. \tag{1.17}$$

To compute this object, the idea is that once we decide what our fields look like on each $B_i$ (in other words once we pick a state in each $\mathcal{H}_i$) we obtain a complex number by performing the path integral

$$\int_{\phi|_{B_i}=\varphi_i} \mathcal{D}\phi\ \mathrm{e}^{-S[\phi]/\hbar} \tag{1.18}$$

over those fields on $M$ that agree with our chosen profiles $\varphi_i$ on each boundary component. As a very important special case, suppose $M = N \times I$, where $N$ is some $d-1$ dimensional manifold and $I$ is just an interval of length $T$ with respect to the metric $g$ on $M$. In this case the path integral gives us a map[8]

$$U(T) : \mathcal{H} \to \mathcal{H} \tag{1.19}$$

from the Hilbert space associated to the incoming boundary of $M$ to that associated to the outgoing boundary, where

$$\langle\varphi_1|U(T)|\varphi_0\rangle = \int_{\phi|_{N\times\{0\}}=\varphi_0}^{\phi|_{N\times\{T\}}=\varphi_1} \mathcal{D}\phi\ \mathrm{e}^{-S[\phi]/\hbar}\,. \tag{1.20}$$

evaluates the map acting on $|\varphi_0\rangle \in \mathcal{H}$ and ending on $|\varphi_1\rangle \in \mathcal{H}$.

The fact that Hilbert spaces are associated to boundaries of $M$ is completely natural – we'll see that it's exactly what happens in the path integral approach to Quantum Mechanics as well as QFT[9]. An important hint of this can already be seen in classical mechanics. In the case that $\partial M = \cup_i B_i$, varying the classical action leads to[10]

$$\delta S[\phi] = (\text{bulk eom})\,\delta\phi + \sum_i \int_{B_i} n_i^\mu \frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)}\,\delta\phi\,\sqrt{g}\,\mathrm{d}^{d-1}x \tag{1.21}$$

where (unlike usual) I haven't assumed the variation obeys $\delta\phi|_{\partial M} = 0$. We define the call the **field momentum** $\pi$ conjugate to $\phi$ along $B_i$ as the variation

$$\pi = \sqrt{g}\,n_i^\mu\,\delta\mathcal{L}/\delta(\partial^\mu\phi) \tag{1.22}$$

[8]In fact, in Minkowskian signature, this map is **unitary**. Unitarity is difficult to see from the path integral perspective and is why you spent time studying canonical quantization last term.

[9]It's also the starting point for Atiyah's axiomatic approach to topological QFTs, and to Segal's approach to $d = 2$ CFTs.

[10]In the boundary term, $n_i^\mu$ is the outward–pointing normal to the boundary component $B_i$; 'normal' means wrt $g$.

of the Lagrangian. The standard example you're probably most used to is when $\partial M$ is a constant time slice of flat Minkowski space, and $\sqrt{g}\, n^\mu \delta\mathcal{L}/\delta(\partial^\mu\phi) = \delta\mathcal{L}/\delta\dot{\phi}$, but the statement holds more generally.

Suppose $M = N \times I$ so that $\partial M$ has just two components. Like any exterior derivative, the exterior derivative $\delta$ on the space of fields obeys $\delta^2 = 0$. Thus, if the classical equations of motion are satisfied,

$$0 = \delta^2 S[\phi]\big|_{\text{eom}} = \int_{N\times\{T\}} \delta\pi \wedge \delta\phi \; \mathrm{d}^{d-1}x \; - \int_{N\times\{0\}} \delta\pi \wedge \delta\phi \; \mathrm{d}^{d-1}x \tag{1.23}$$

showing that the quantity

$$\Omega = \int_N \delta\pi \wedge \delta\phi \; d^{d-1}x \tag{1.24}$$

is conserved under evolution via the classical equations. (The minus sign arises because the natural orientation of $n^\mu$ points into $M$ at one end, and out at the other.) $\Omega$ is a 2-form on the space of boundary field configurations $\mathcal{C}[N]$ and is obviously closed. It is a **symplectic form on the space of fields**, and the fact that it is conserved is the usual fact that in classical field theory (as in classical mechanics) time evolution is symplectic. *It is this symplectic structure on the space of fields that we wish to quantise in QFT*, exactly as you quantised the symplectic structure $(\mathbb{R}^{2n}, \omega)$ with $\omega = \mathrm{d}p_i \wedge \mathrm{d}x^i$ in quantum mechanics.

Note that (for eoms that are 2$^{\text{nd}}$ order), specifying boundary values of $\phi$ and $\pi$ determines a unique solution to the classical field equations, so the space of boundary values $(\phi|_N, \pi)$ is isomorphic to the space of classical solutions (at least in a small neighbourhood of $N$). This is why, when studying canonical quantisation on $(\mathbb{R}^4, \eta)$ last term, you expanded fields in terms of modes

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E}} \left[ \mathrm{e}^{\mathrm{i}p_\mu x^\mu}\, a(\mathbf{p}) + \mathrm{e}^{-\mathrm{i}p_\mu x^\mu}\, a^\dagger(\mathbf{p}) \right] \tag{1.25}$$

that satisfied the equations of motion. In fact, since you were working perturbatively, fields satisfied the *free* equations of motion, meaning in the relativistic context that their energy was fixed in terms of their mass and momentum by $E = \sqrt{\mathbf{p}^2 + m^2}$. It's also why, when you introduced commutation relations

$$\begin{aligned} [\phi(\mathbf{x}), \phi(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \pi(\mathbf{y})] \\ [\phi(\mathbf{x}), \pi(\mathbf{y})] &= \mathrm{i}\, \delta^3(\mathbf{x} - \mathbf{y}) \end{aligned} \tag{1.26}$$

for the fields and their momenta, these were defined only when the fields were evaluated at *equal time*; the symplectic structure $\Omega$ whose form they reflect is only defined on a co-dimension 1 slice of $M$ (such as a boundary).

States in which the fields take definite values on the boundary are the analogue of position eigenstates in Quantum Mechanics. Just as $|x\rangle$ represents a quantum mechanical state in which the particle is definitely located at $x$, so $|\varphi_i\rangle$ represents a state in quantum field theory in which the field on the boundary component $B_i$ definitely takes some profile $\varphi_i$. In quantum mechanics we can have more general states, written in Dirac notation as

$$|\psi\rangle = \int \mathrm{d}^n x \, |x\rangle\langle x|\psi\rangle \,, \tag{1.27}$$

where $\psi(x) = \langle x|\psi\rangle \in L^2(\mathbb{R}^n, d^n x)$ is a **wavefunction**. So too in QFT we can have more general states

$$|\Psi\rangle = \int_{\mathcal{C}[B]} [d\varphi]\, |\varphi\rangle\langle\varphi|\Psi\rangle \tag{1.28}$$

where the integral is taken over the space $\mathcal{C}[B]$ of all possible boundary field configurations, and $\Psi[\varphi] = \langle\varphi|\Psi\rangle$ is, heuristically, a wavefunction *on this space of fields.*

Again, you saw this already in last term's course when studying canonical quantisation of a field theory. There, rather than general functions $\Psi[\phi]$ you studied *polynomials* on $\mathcal{C}[N]$, with a monomial

$$\int_{N^{\otimes n}} \psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)\, \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n)$$

of degree $n$ in the fields being interpreted as an $n$-particle state. Via (1.25), in canonical quantisation the fields themselves were written in terms of creation and annihilation operators, whilst the 'co-efficients' of $\psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ this monomial were interpreted as the wavefunction of an $n$-particle quantum *mechanical* state. Indeed, if $V$ is the Hilbert space associated to a single particle, then you decomposed the Hilbert space $\mathcal{H}$ of QFT as[11]

$$\begin{aligned} \mathcal{H} &= \mathbb{C} \oplus V \oplus \mathrm{Sym}^2 V \oplus \mathrm{Sym}^3 V \oplus \cdots \\ &= \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n V\,. \end{aligned} \tag{1.29}$$

where $\mathbb{C}$ represents the vacuum state, $V$ the one–particle state, $\mathrm{Sym}^2 V$ the two–particle state, and so on. This is known as the **Fock basis** of $\mathcal{H}$.

Restricting to polynomials is somewhat like expanding a general state of the quantum harmonic oscillator in Hermite polynomials, which are square–integrable wrt to a Gaussian measure (*i.e.* they form a basis of the Hilbert space $L^2(\mathbb{R}, \mathrm{e}^{-x^2/2}\, \mathrm{d}x)$). The difficulties of defining what we mean by the infinite–dimensional path integral are reflected in the canonical quantisation approach to QFT in the difficulties of defining what is meant by the 'Hilbert space' $L^2(\mathcal{C}[B], \mathrm{d}\mu)$ for functions on the infinite–dimensional space of boundary field configurations $\mathcal{C}[B]$.

### 1.2.4 Scattering amplitudes

If $M$ is non–compact then there may be a region that is asymptotically far away in the metric $g$, such as the region $\|\mathbf{x}\| \to \infty$ in $\mathbb{R}^d$, or the asymptotic past and future in Minkowski space. On a non–compact manifold case, to define the path integral we have to specify asymptotic values of the fields, and the result of the path integral then depends on our choice of these asymptotic values. The simplest choice is just to ask that $\phi \to 0$ in this asymptotic region, and we use this to define the partition function $\mathcal{Z}$ on a non–compact space. Another standard example is the case of Minkowski space, where we choose initial

[11] Sym means symmetric power, *e.g.* $a\otimes b + b\otimes a \in \mathrm{Sy}^2 V \subset V\otimes V$. I'm assuming the field is bosonic so that the wavefunction of identical particles is symmetrized; multi-particle fermion states involve antisymmetric powers of $V$.

and field profiles

$$\begin{aligned} \phi &\to \phi_{\mathrm{i}} \qquad \text{as } t \to -\infty \\ \phi &\to \phi_{\mathrm{f}} \qquad \text{as } t \to +\infty \end{aligned}$$

in the asymptotic past and future, and obtain a path integral written as

$$\langle \phi_{\mathrm{f}} | \phi_{\mathrm{i}} \rangle = \frac{1}{\mathcal{Z}} \int_{\mathcal{C}(\phi_{\mathrm{i}},\phi_{\mathrm{f}})} [\mathcal{D}\phi]\, \mathrm{e}^{\mathrm{i}S[\phi]/\hbar} \tag{1.30}$$

taken over the space of fields that approach these asymptotic profiles, normalised by the partition function computed again using the 'trivial' profiles. This is known as the **scattering amplitude**, and represents the quantum amplitude for a state that initially looks like the field is $\phi_{\mathrm{i}}$ to evolve throughout space–time and emerge looking like $\phi_{\mathrm{f}}$.

Of course, even on a non–compact manifold we can still define correlation functions, where we assume $\phi \to 0$ in the asymptotic region. More generally still **form factors**, which are simply correlation functions taken in the presence of non–trivial asymptotic conditions on the fields (and thus are a sort of mixture of scattering amplitudes and correlation functions). Remarkably, as we'll learn later in the course, scattering amplitudes are themselves related to correlation functions of the form $\langle \phi(x_1)\,\phi(x_2)\,\cdots\,\phi(x_n)\rangle$ through the LSZ theorem. This is a very useful theorem, as it's typically easier to understand correlation functions than to work with boundary conditions on the path integral.

Naively, you might think it's easier to work with a QFT on $\mathbb{R}^d$ than on a compact space $M$, particularly if we just require $\phi \to 0$ as $\|\mathbf{x}\| \to \infty$. Whilst there's some truth in this (and we'll certainly mostly just consider $(M, g) = (\mathbb{R}^d, \delta)$ in this course), you should be aware that putting QFT on a non–compact space introduces new difficulties in defining the path integral, in part associated with infra–red divergences. The issue is somewhat like an infinite–dimensional analogue of the fact that convergence of Fourier integrals is even more subtle than convergence of Fourier series.