

3 QFT in one dimension (= QM)

In one dimension there are two possible compact (connected) manifolds M : the circle S^1 and the interval I . We will parametrize the interval by $t \in [0, T]$ so that $t = 0$ and $t = T$ are the two point-like boundaries, while we will parametrize the circle by $t \in [0, T)$ with the identification $t \cong t + T$.

The most important example of a field on M is a map $x : M \rightarrow N$ to a Riemannian manifold (N, G) which we will take to have dimension n . That is, for each point t on our ‘space-time’ M , $x(t)$ is a point in N . It’s often convenient to describe N using coordinates. If an open patch $U \subset N$ has local co-ordinates x^a for $a = 1, \dots, n$, then we let $x^a(t)$ denote the coordinates of the image point $x(t)$. More precisely, $x^a(t)$ are the pullbacks to M of coordinates on U by the map x .

With these fields, the standard choice of action is

$$S[x] = \int_M \left[\frac{1}{2} G_{ab}(x) \dot{x}^a \dot{x}^b + V(x) \right] dt, \quad (3.1)$$

where $G_{ab}(x)$ is the pullback to M of the Riemannian metric on N , t is worldline time, and $\dot{x}^a = dx^a/dt$. We’ve also included in the action a choice of function $V : N \rightarrow \mathbb{R}$, or more precisely the pullback of this function to M , which is independent of worldline derivatives of x . Finally, when writing this action we chose the flat Euclidean metric $\delta_{tt} = 1$ on M ; we’ll examine other choices of metric on M in section 3.4.

Under a small variation δx of x the change in the action is

$$\begin{aligned} \delta S[x] &= \int_M \left[G_{ab}(x) \dot{x}^a \delta \dot{x}^b + \frac{1}{2} \frac{\partial G_{ab}(x)}{\partial x^c} \delta x^c \dot{x}^a \dot{x}^b + \frac{\partial V(x)}{\partial x^c} \delta x^c \right] dt \\ &= \int_M \left[-\frac{d}{dt} (G_{ac}(x) \dot{x}^a) + \frac{1}{2} \frac{\partial G_{ab}(x)}{\partial x^c} \dot{x}^a \dot{x}^b + \frac{\partial V(x)}{\partial x^c} \right] \delta x^c dt + G_{ab}(x) \dot{x}^a \delta x^b \Big|_{\partial M}. \end{aligned} \quad (3.2)$$

Requiring that the bulk term vanishes for arbitrary $\delta x^a(t)$ gives the Euler–Lagrange equations

$$\frac{d^2 x^a}{dt^2} + \Gamma_{bc}^a \dot{x}^b \dot{x}^c = G^{ab}(x) \frac{dV}{dx^b} \quad (3.3)$$

where $\Gamma_{bc}^a = \frac{1}{2} G^{ad} (\partial_b G_{cd} + \partial_c G_{bd} - \partial_d G_{bc})$ is the Levi–Civita connection on N , again pulled back to the worldline. If M has boundary, then the boundary term is the symplectic potential on the space of maps, where we note that $p_a = \delta \mathcal{L} / \delta \dot{x}^a = G_{ab}(x) \dot{x}^b$ is the momentum of the field.

3.1 Worldline quantum mechanics

The usual interpretation of all this is to image an arbitrary map $x(t)$ describes a possible *trajectory* a particle might in principle take as it travels through the space N . (See figure 4.) In this context, N is called the **target space** of the theory, while M (or its image $x(M) \subset N$) is known as the **worldline** of the particle. The field equation (3.3) says that when $V = 0$, classically the particle travels along a geodesic in (N, G) . V itself is then interpreted as a (non-gravitational) potential through which this particle moves. The absence of a

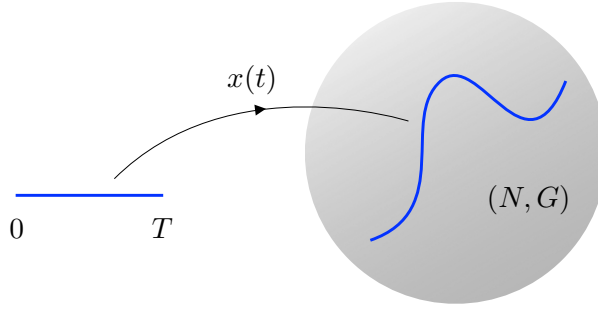


Figure 4: The theory (3.1) describes a map from an abstract worldline into the Riemannian target space (N, G) . The corresponding one-dimensional QFT can be interpreted as single particle Quantum Mechanics on N .

minus sign on the *rhs* of (3.3) is probably surprising, but follows from the action (3.1). This is actually the *correct* sign with a Euclidean worldsheet, because under the Wick rotation $t \rightarrow it$ back to a Minkowski signature worldline, the *lhs* of (3.3) acquires a minus sign. In other words, in Euclidean time $F = -ma!$

From this perspective, it's natural to think of the *target* space N as being the world in which we live, and computing the path integral for this action will lead us to single particle Quantum Mechanics, as we'll see below. However, we're really using this theory as a further warm-up towards QFT in higher dimensions, so I also want you to keep in mind the idea that the *worldline* M is actually 'our space-time' in a one-dimensional context, and the target space N can be some abstract Riemannian manifold unrelated to the space we see around us. For example, at physics of low-energy pions is described by a theory of this general kind, where M is our Universe and N is the coset manifold $(SU(2) \times SU(2))/SU(2)$.

3.1.1 The quantum transition amplitude

The usual way to do Quantum Mechanics is to pick a Hilbert space \mathcal{H} and a Hamiltonian H , which is a Hermitian operator $H : \mathcal{H} \rightarrow \mathcal{H}$. In the case relevant above, the Hilbert space would be $L^2(N)$, the space of square-integrable functions on N , and the Hamiltonian would usually be

$$H = -\frac{\hbar^2}{2}\Delta + V, \quad \text{where} \quad \Delta := \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^a} \left(\sqrt{G} G^{ab} \frac{\partial}{\partial x^b} \right) \quad (3.4)$$

is the Laplacian acting on functions in $L^2(N)$. The amplitude for the particle to travel from an initial point $y_0 \in N$ to a final point $y_1 \in N$ in Euclidean time T is given by

$$K_T(y_0, y_1) = \langle y_1 | e^{-HT/\hbar} | y_0 \rangle, \quad (3.5)$$

which is known as the **heat kernel**. (Here I've written the *rhs* in the Heisenberg picture, which I'll use below. In the Schrödinger picture where states depend on time we would instead write $K_T(y_0, y_1) = \langle y_1, T | y_0, 0 \rangle$.) The heat kernel is a function on $I \times N \times N$ which

may be defined to be the solution of the pde²⁸

$$\hbar \frac{\partial}{\partial t} K_t(x, y) + H K_t(x, y) = 0 \quad (3.6)$$

subject to the initial condition that $K_0(x, y) = \delta(x - y)$, the Dirac δ -function on the diagonal $N \subset N \times N$. (I remind you that we're in Euclidean worldline time here. Rotating to Minkowski signature by sending $t \mapsto it$, the heat equation becomes

$$i\hbar \frac{\partial}{\partial t} K_{it}(x, y) = H K_{it}(x, y) \quad (3.7)$$

which we recognize as Schrödinger's equation with Hamiltonian H .)

In the simplest example of $(N, G) \cong (\mathbb{R}^n, \delta)$ with vanishing potential $V \equiv 0$, the Hamiltonian is just

$$H = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^a \partial x_a} \quad (3.8)$$

and the heat kernel takes the familiar form

$$K_t(x, y) = \frac{1}{(2\pi\hbar t)^{n/2}} \exp\left(-\frac{\|x - y\|^2}{2\hbar t}\right) \quad (3.9)$$

where $\|x - y\|$ is the Euclidean distance between x and y . More generally, while the heat kernel on a Riemannian manifold (N, G) is typically very complicated, it can be shown that for small times it always has the asymptotic form

$$\lim_{\Delta t \rightarrow 0} K_{\Delta t}(x, y) \sim \frac{1}{(2\pi\hbar \Delta t)^{n/2}} a(x) \exp\left(-\frac{d(x, y)^2}{2\hbar \Delta t}\right), \quad (3.10)$$

where $d(x, y)$ is the distance between x and y measured along a geodesic of the metric G , and where

$$a(x) \sim \sqrt{G(x)} [1 + \text{Ric}_G(x) + \dots] \quad (3.11)$$

is an expression constructed from the Riemann curvature of G in a way that we won't need to be specific about.

Feynman's intuition was that the amplitude for a particle to be found at y_0 at $t = 0$ and at y_1 at $t = T$ could be expressed in terms of the product of the amplitude for it to start at y_0 at $t = 0$, then be found at some other location x at an intermediate time $t \in (0, T)$, before finally being found at y_1 on schedule at $t = T$. Since we did not measure what the particle was doing at the intermediate time, we should sum (*i.e.* integrate) over all possible intermediate locations x in accordance with the linearity of quantum mechanics. Iterating this procedure, as in figure 5 we break the time interval $[0, T]$ into N chunks, each

²⁸Like the factor of $1/2$ in front of the Laplacian in (3.4), I've included a factor of \hbar in this equation for better agreement with the conventions of quantum mechanics, rather than Brownian motion. If you wish, you can imagine we're studying the usual heat equation in a medium with thermal conductivity $\hbar/2$.



Figure 5: Feynman’s approach to quantum mechanics starts by breaking the time evolution of a particle’s state into many chunks, then summing over all possible locations (and any other quantum numbers) of the particle at intermediate times.

of duration $\Delta t = T/N$. We then write

$$\begin{aligned}
\langle y_1 | e^{-HT/\hbar} | y_0 \rangle &= \langle y_1 | e^{-H\Delta t/\hbar} e^{-H\Delta t/\hbar} \dots e^{-H\Delta t/\hbar} | y_0 \rangle \\
&= \int d^n x_1 \dots d^n x_{N-1} \langle y_1 | e^{-H\Delta t/\hbar} | x_{N-1} \rangle \dots \langle x_2 | e^{-H\Delta t/\hbar} | x_1 \rangle \langle x_1 | e^{-H\Delta t/\hbar} | y_0 \rangle \\
&= \int \prod_{i=1}^{N-1} d^n x_i K_{\Delta t}(y_1, x_{N-1}) \dots K_{\Delta t}(x_2, x_1) K_{\Delta t}(x_1, y_0).
\end{aligned} \tag{3.12}$$

In the second line here we’ve inserted the identity operator $\int d^n x_i |x_i\rangle \langle x_i|$ on \mathcal{H} in between each evolution operator; in the present context this can be understood as the concatenation identity

$$K_{t_1+t_2}(x_3, x_1) = \int d^n x_2 K_{t_2}(x_3, x_2) K_{t_1}(x_2, x_1) \tag{3.13}$$

obeyed by convolutions of the heat kernel.

This more or less takes us to the path integral. The virtue of splitting up the time interval $[0, T]$ into many chunks is that we can now use the asymptotic form (3.10) to write

$$\langle y_1 | e^{-HT/\hbar} | y_0 \rangle \sim \frac{1}{(2\pi\hbar\Delta t)^{n/2}} \int \prod_{i=1}^{N-1} \frac{d^n x_i}{(2\pi\hbar\Delta t)^{n/2}} a(x_i) \exp \left[-\frac{1}{\hbar} \sum_{i=0}^N \frac{\Delta t}{2} \left(\frac{d(x_{i+1}, x_i)}{\Delta t} \right)^2 \right]. \tag{3.14}$$

We now consider take the limit that $N \rightarrow \infty$ with T fixed (so $\Delta t \rightarrow 0$). We might then hope that we can define our path integral measure to be

$$\mathcal{D}x \stackrel{?}{:=} \lim_{N \rightarrow \infty} \left(\frac{1}{2\pi\hbar\Delta t} \right)^{\frac{nN}{2}} \prod_{i=1}^{N-1} d^n x_i a(x_i) \tag{3.15}$$

as an integral over the values of the fields $x^a(t)$ at each time $t \in [0, T]$. Similarly, if the limiting trajectory is at least once differentiable then as $\Delta t \rightarrow 0$, $(d(x_{i+1}, x_i)/\Delta t)^2$

converges to $g_{ab} \dot{x}^a \dot{x}^b$ while the sum can be replaced by an integral, so we would have

$$\lim_{N \rightarrow \infty} \left[\sum_{i=0}^N \frac{\Delta t}{2} \left(\frac{d(x_{i+1}, x_i)}{\Delta t} \right)^2 \right] \stackrel{?}{=} \int_0^T \frac{1}{2} G_{ab}(x) \dot{x}^a \dot{x}^b dt. \quad (3.16)$$

This recovers the action (3.1), with $V = 0$. (A more general heat kernel can be used to incorporate a non-zero potential.) We'll investigate these limits further below; accepting them for now, combining (3.15) & (3.16) allows us to represent the heat kernel as an integral

$$\langle y_1 | e^{-HT/\hbar} | y_0 \rangle = K_T(y_0, y_1) = \int_{C_T[y_0, y_1]} \mathcal{D}x e^{-S[x]/\hbar} \quad (3.17)$$

taken over a space $C_T[y_0, y_1]$ of maps $x : [0, T] \rightarrow N$ that are constrained to obey the boundary conditions $x(0) = y_0$ and $x(T) = y_1$. A given map is called a **path** and the integral over all such paths is the **path integral**. We'll investigate exactly what sort of maps we should allow (smooth? differentiable? continuous?) in more detail below. Note that from our $d = 1$ QFT perspective, the path integral gives the amplitude for a field configuration $x = y_0$ on an initial codimension-1 slice (*i.e.* the point $t = 0$) to evolve through $M = [0, T]$ and emerge as the field configuration $x = y_1$ on the final codimension-1 slice (*i.e.* the point T). Thus, it's a sort of scattering amplitude $y_0 \rightarrow y_1$ in our one dimensional universe. (The name 'path integral' is also used in higher dimensional QFT.)

3.1.2 The partition function

In the operator approach to quantum mechanics, the partition function is defined to be the trace of the time evolution operator over the Hilbert space:

$$\mathcal{Z}(T) = \text{Tr}_{\mathcal{H}}(e^{-TH}). \quad (3.18)$$

In the case of a single particle moving on \mathbb{R}^n , we can take the position eigenstates $|y\rangle$ to be a (somewhat formal) 'basis' of $\mathcal{H} = L^2(\mathbb{R}^n, d^n y)$, in which case the partition function becomes

$$\mathcal{Z}(T) = \int d^n y \langle y | e^{-HT} | y \rangle = \int_N d^n y \int_{C_T[y, y]} \mathcal{D}x e^{-S} \quad (3.19)$$

where the last equality uses our path integral expression (3.17) for the heat kernel. Because we're taking the trace, the path integral here should be taken over maps $x : [0, T] \rightarrow N$ such that the endpoints are both mapped to the *same* point $y \in N$. We then integrate y everywhere over N ²⁹, erasing the memory of the particular point y . As long as we're being vague about the degree of differentiability of our map, this is (plausibly) the same thing as integrating over maps $x : S^1 \rightarrow N$ where the worldline has become a circle of circumference T . This shows that

$$\mathcal{Z}_{S^1}[T] = \text{Tr}_{\mathcal{H}}(e^{-TH}) = \int_{C_{S^1}} \mathcal{D}x e^{-S/\hbar}, \quad (3.20)$$

²⁹In flat space, the heat kernel (3.9) obeys $K_T(y, y) = K_T(0, 0)$ so is independent of y . Thus if $N \cong \mathbb{R}^n$ with a flat metric, this final y integral does not converge. This is an 'infra-red' effect that arises because (\mathbb{R}^n, δ) is non-compact. The partition function does converge if N is compact, which we can achieve by imposing that we live in a large box, or on a torus *etc.*, whilst still keeping a flat metric.

which was our earlier definition of the partition function on the compact universe $M = S^1$. In higher dimensions this formula will be the basis of the relation between QFT and Statistical Field Theory, and is really the origin of the name ‘partition function’ for \mathcal{Z} in physics.

3.1.3 Operators and correlation functions

As in zero dimensions, we can also use the path integral to compute correlation functions of operators. A **local operator** is one which depends on the field only at one point of the worldline. The simplest type of local operator comes from a function on the target space. If $\mathcal{O} : N \rightarrow \mathbb{R}$ is a real-valued function on N , let $\hat{\mathcal{O}}$ denote the corresponding operator on \mathcal{H} . That is, \mathcal{O} depends only on the local coordinates x^a and $\hat{\mathcal{O}} = \mathcal{O}(\hat{x}^a)$ is the same function of the position operator \hat{x}^a acting on \mathcal{H} . Then for any fixed time $t \in (0, T)$ we have

$$\langle y_1 | e^{-HT/\hbar} \hat{\mathcal{O}}(t) | y_0 \rangle = \langle y_1 | e^{-H(T-t)/\hbar} \hat{\mathcal{O}} e^{-Ht/\hbar} | y_0 \rangle \quad (3.21)$$

in the Heisenberg picture. Inserting a complete set of position eigenstates, this is

$$\begin{aligned} \int d^n x \langle y_1 | e^{-H(T-t)/\hbar} \mathcal{O}(\hat{x}) | x \rangle \langle x | e^{-Ht/\hbar} | y_0 \rangle &= \int d^n x \mathcal{O}(x) \langle y_1 | e^{-H(T-t)/\hbar} | x \rangle \langle x | e^{-Ht/\hbar} | y_0 \rangle \\ &= \int d^n x \mathcal{O}(x) K_{T-t}(y_1, x) K_t(x, y_0), \end{aligned} \quad (3.22)$$

where we note that in the final two expressions $\mathcal{O}(x)$ is just a number; the eigenvalue of the operator $\hat{\mathcal{O}}$ acting on the state $|x\rangle$.

Using (3.17), everything on the rhs of this equation can now be written in terms of path integrals. We have

$$\begin{aligned} \langle y_1 | e^{-H(T-t)/\hbar} \hat{\mathcal{O}} e^{-Ht/\hbar} | y_0 \rangle &= \int d^n x_t \left[\int_{C_{T-t}[y_1, x_t]} e^{-S[x]/\hbar} \times \mathcal{O}(x_t) \times \int_{C_t[x_t, y_0]} e^{-S[x]/\hbar} \right] \\ &= \int_{C_T[y_1, y_0]} \mathcal{D}x e^{-S[x]/\hbar} \mathcal{O}(x(t)), \end{aligned} \quad (3.23)$$

where to reach the second line we again note that integrating over all continuous maps $x : [0, t] \rightarrow N$ with endpoint $x(t) = x_t$, then over all continuous maps $x : [t, T] \rightarrow N$ with initial point $x(t)$ again fixed to x_t and finally integrating over all points $x_t \in N$, is the same thing as integrating over all continuous maps $x : [0, T] \rightarrow N$ with endpoints y_0 and y_1 .

More generally, we can insert several such operators. If $0 < t_1 < t_2 < \dots < t_n < T$ then exactly the same arguments give

$$\begin{aligned} \langle y_1 | \hat{\mathcal{O}}_n(t_n) \cdots \hat{\mathcal{O}}_2(t_2) \hat{\mathcal{O}}_1(t_1) | y_0 \rangle &= \langle y_1 | e^{-H(T-t_n)/\hbar} \mathcal{O}_n(\hat{x}) \cdots \mathcal{O}_2(\hat{x}) e^{-H(t_2-t_1)/\hbar} \mathcal{O}_1(\hat{x}) e^{-Ht_1/\hbar} | y_0 \rangle \\ &= \int_{C_T[y_0, y_1]} \mathcal{D}x e^{-S[x]/\hbar} \prod_{i=1}^n \mathcal{O}_i(x(t_i)) \end{aligned} \quad (3.24)$$

for the n -point correlation function. The hats on the $\hat{\mathcal{O}}_i$ remind us that the *lhs* involves *operators* acting on the Hilbert space \mathcal{H} (which we may or may not choose to describe using the position representation). By contrast, the objects \mathcal{O}_i inside the path integral are just ordinary functions, evaluated at the points $x(t_i) \in N$ ³⁰.

Notice that in order to run our argument, it was very important that the insertion times t_i obeyed $t_i < t_{i+1}$: we would not have been able to interpret the *lhs* in the Heisenberg picture had this not been the case³¹. On the other hand, the insertions $\mathcal{O}_i(x(t_i))$ in the path integral are just functions and have no notion of ordering. Thus the expression on the right doesn't have any way to know which insertion times was earliest. For this to be consistent, for a general set of times $\{t_i\} \in (0, T)$ we must actually have

$$\int_{C_T[y_0, y_1]} \mathcal{D}x \left(e^{-S[x]/\hbar} \prod_{i=1}^n \mathcal{O}_i(x(t_i)) \right) = \langle y_1 | \mathcal{T} \{ \prod_i \hat{\mathcal{O}}_i(t_i) \} | y_0 \rangle \quad (3.25)$$

where the symbol \mathcal{T} on the *rhs* is defined by

$$\begin{aligned} \mathcal{T} \hat{\mathcal{O}}_1(t_1) &:= \mathcal{O}_1(t_1), \\ \mathcal{T} \{ \hat{\mathcal{O}}_1(t_1) \hat{\mathcal{O}}_2(t_2) \} &:= \Theta(t_2 - t_1) \hat{\mathcal{O}}_2(t_2) \hat{\mathcal{O}}_1(t_1) + \Theta(t_1 - t_2) \hat{\mathcal{O}}_1(t_1) \hat{\mathcal{O}}_2(t_2) \end{aligned} \quad (3.26)$$

and so on, where $\Theta(t)$ is the Heaviside step function and the operators are in the Heisenberg picture. By construction, these step functions mean that the *rhs* is now completely symmetric with respect to a permutation of the orderings. However, for any given choice of times t_i , only one term on the *rhs* can be non-zero. In other words, insertions in the path integral correspond to the **time-ordered product** of the corresponding operators in the Heisenberg picture.

The derivative terms in the action play an important role in evaluating these correlation functions. For suppose we'd chosen our action to be just a potential term $\int V(x(t)) dt$, independent of derivatives $\dot{x}(t)$. Then, regularizing the path integral by dividing M into many small intervals as before, we'd find that neighbouring points on the worldline completely decouple: unlike in (3.14) where the geodesic distance $d(x_{i+1}, x_i)^2$ in the heat kernel provides cross-terms linking neighbouring points together, we would obtain simply a product of independent integrals at each time step. Inserting functions $\mathcal{O}_i(x(t_i))$ that are likewise independent of derivatives of x into such a path integral would not change this conclusion. Thus, without the derivative terms in the action, we'd find

$$\langle \mathcal{O}_1(t_1) \mathcal{O}_2(t_2) \rangle = \langle \mathcal{O}_1(t_1) \rangle \langle \mathcal{O}_2(t_2) \rangle \quad (3.27)$$

for all such insertions. In other words, there would be no possible non-trivial correlations between objects at different points of our (one-dimensional) Universe. This would be a very boring world: without derivatives, the number of people sitting in the lecture theatre

³⁰A more precise statement would be that they are functions on the space of fields $C_T[y_0, y_1]$ obtained by pullback from a function on N by the evaluation map at time t_i .

³¹It's a good exercise to check you understand what goes wrong if we try to compute $\langle y_1 | e^{+tH/\hbar} | y_1 \rangle$ with $t > 0$.

would have nothing at all to do with whether or not a lecture was actually going on, and what you're thinking about right now would have nothing to do with what's written on this page.

This conclusion is a familiar result in perturbation theory. The kinetic terms in the action allow us to construct a **propagator**, and using this in Feynman diagrams enables us to join together interaction vertices at different points in space–time. As the name suggests, we interpret this propagator as a particle *traveling* between these two space–time interactions and the ability for particles to move is what allows for non–trivial correlation functions. Here we've obtained the same result directly from the path integral.

So far, we've just been considering insertions that are functions of position only. A wider class of local path integral insertions depend not just on x but also on its worldline derivatives \dot{x}, \ddot{x} etc.. In the canonical framework, with Lagrangian L we have

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = G_{ab} \dot{x}^b \quad (3.28)$$

where the last equality is for our action (3.1). Thus we might imagine replacing a general *operator* $\mathcal{O}(\hat{x}^a, \hat{p}_b)$ in the canonical quantization framework by the *function* $\mathcal{O}(x^a, G_{bc}(x)\dot{x}^c)$ of x and its derivative in the path integral. From the path integral perspective, however, something smells fishy here. Probably the first thing you learned about QM was that $[\hat{x}^a, \hat{p}_b] \neq 0$. If we replace \hat{x}^a and \hat{p}_b by x^a and $G_{bc}\dot{x}^c$ in the path integral, how can these *functions* fail to commute, even when $G_{ab} = \delta_{ab}$? To understand what's going on, we'll need to look into the definition of our path integral in more detail.

By the way, we should note that there's an important other side to this story, revealing ambiguities in the canonical approach to quantum mechanics. Suppose we're given some function $\mathcal{O}(x^a, p_b)$ on a classical phase space, corresponding to some observable quantity. If we wish to 'quantize' this classical system, it may not be obvious to decide what operator to use to represent our observable as the replacement

$$\mathcal{O}(x^a, p_b) \rightarrow \mathcal{O}(\hat{x}^a, \hat{p}_b)$$

is plagued by ordering ambiguities. For example, if we represent p_a by³² $-\hbar\partial/\partial x^a$, then should we replace

$$x^a p_a \rightarrow -x^a \hbar \frac{\partial}{\partial x^a}$$

or should we take

$$x^a p_a \rightarrow -\hbar \frac{\partial}{\partial x^a} x^a = -\hbar n - x^a \hbar \frac{\partial}{\partial x^a}$$

or perhaps something else? According to Dirac, if two classical observables f and g have Poisson bracket $\{f, g\} = h$ for some other function h , then we should quantize by finding a Hilbert space on which the corresponding operators \hat{f} and \hat{g} act irreducibly and *ii*) obey $[\hat{f}, \hat{g}] = i\hbar \hat{h}$. Unfortunately, even in flat space quantum mechanics with $(N, G, V) = (\mathbb{R}^n, \delta, 0)$, the **Groenewald–Van Hove theorem** states that we cannot generally achieve this, even for functions that are polynomial in position and momenta, of degree higher than 2.

³²The absence of a factor of i on the *rhs* here is again a consequence of having a Euclidean worldline.

(We can make progress if some extra structure is present, such as if the operators represent the action of some finite dimensional Lie group, or if the phase space is provided with a complex structure, making it a Kähler manifold.) The idea of ‘quantizing’ a classical system thus remains ambiguous in general.

3.2 The continuum limit

In this section, we’ll take a closer look at the origin of non-commutativity of x^a and p_b from the path integral perspective. Doing so will lead to a deeper understanding of the subtleties involved in taking the naive continuum limit of the path integral measure and action.

3.2.1 Discretization and non-commutativity

Non-commutativity is present in quantum mechanics right from the beginning, so it will suffice to consider the simplest case of a free particle travelling in one dimension. We thus pick $(N, G) = (\mathbb{R}, \delta)$ and $V = 0$. Then if $0 < t_- < t < t_+ < T$ we have

$$\int_{C_T[y_0, y_1]} \mathcal{D}x \, e^{-S[x]/\hbar} x(t) \dot{x}(t_-) = \langle y_1 | e^{-H(T-t)/\hbar} \hat{x} e^{-H(t-t_-)/\hbar} \hat{p} e^{-Ht_-/\hbar} | y_0 \rangle, \quad (3.29a)$$

when the insertion of \hat{p} is earlier than that of \hat{x} , and

$$\int_{C_T[y_0, y_1]} \mathcal{D}x \, e^{-S[x]/\hbar} x(t) \dot{x}(t_+) = \langle y_1 | e^{-H(T-t_+)/\hbar} \hat{p} e^{-H(t_+-t)/\hbar} \hat{x} e^{-Ht/\hbar} | y_0 \rangle \quad (3.29b)$$

when \hat{p} is inserted at a later time than \hat{x} . Taking the limits $t_+ \rightarrow t$ from above and $t_- \rightarrow t$ from below, the difference between the *rhs* of (3.29a) & (3.29b) is

$$\langle y_1 | e^{-H(T-t)/\hbar} [\hat{x}, \hat{p}] e^{-Ht}/\hbar | y_0 \rangle = \hbar \langle y_1 | e^{-HT}/\hbar | y_0 \rangle \quad (3.30)$$

which does not vanish. By contrast, the difference of the *lhs* seems to be automatically zero. What have we missed?

In handling the *lhs* of (3.29a)-(3.29b) we need to be careful. Our arguments allowed us to be confident of the relation between the canonical and path integral approaches only when working with some discretization of $M = [0, T]$. Taking the continuum limit to obtain a path integral measure $\mathcal{D}x$ and action $S[x]$ was a formal operator and we did not check that these limits actually make sense.

To stay on safe ground, let’s regularize the path integrals in (3.29a)-(3.29b) by chopping $[0, T]$ into many chunks, each of width Δt . With this discretization, we cannot pretend we are bringing the x and \dot{x} insertions any closer to each other than Δt without also taking account of the discretization of the whole path integral. Thus we replace

$$\lim_{t_- \uparrow t} [x(t) \dot{x}(t_-)] - \lim_{t_+ \downarrow t} [x(t) \dot{x}(t_+)]$$

by the discretized version

$$x_t \frac{x_t - x_{t-\Delta t}}{\Delta t} - x_t \frac{x_{t+\Delta t} - x_t}{\Delta t} \quad (3.31)$$

where we stop the limiting procedure as soon as x coincides with any part of the discretized derivative. As always with path integrals, the order of the factors of x_t and $x_{t\pm\Delta t}$ here doesn't matter; they're just ordinary integration variables.

Now consider the integral over x_t . Apart from the insertion of (3.31), the only dependence of the discretized path integral on this variable is in the heat kernels $K_{\Delta t}(x_{t+\Delta t}, x_t)$ and $K_{\Delta t}(x_t, x_{t-\Delta t})$ that describe the evolution from neighbouring chunks of our discretized worldline. Using the explicit form (3.9) of these kernels in flat space we have

$$\begin{aligned} & \int dx_t K_{\Delta t}(x_{t+\Delta t}, x_t) \left(x_t \frac{x_t - x_{t-\Delta t}}{\Delta t} - x_t \frac{x_{t+\Delta t} - x_t}{\Delta t} \right) K_{\Delta t}(x_t, x_{t-\Delta t}) \\ &= -\hbar \int dx_t x_t \frac{\partial}{\partial x_t} \left(K_{\Delta t}(x_{t+\Delta t}, x_t) K_{\Delta t}(x_t, x_{t-\Delta t}) \right) \\ &= \hbar \int dx_t K_{\Delta t}(x_{t+\Delta t}, x_t) K_{\Delta t}(x_t, x_{t-\Delta t}) = \hbar K_{2\Delta t}(x_{t+\Delta t}, x_{t-\Delta t}) \end{aligned} \quad (3.32)$$

where the first step recognizes the two insertions as being \hbar times the x_t derivatives of $K_{\Delta t}(x_{t+\Delta t}, x_t)$ and $K_{\Delta t}(x_t, x_{t-\Delta t})$, respectively. The second step is a simple integration by parts and the final equality uses the concatenation property (3.13). The integration over x_t thus removes all the insertions from the path integral, and the remaining integrals can be done using concatenation as before. We are thus left with $\hbar K_T(y_1, y_0) = \hbar \langle y_1 | e^{-HT} | y_0 \rangle$ in agreement with the operator approach.

There's an important point to notice about this calculation. If we'd assumed that, in the continuum limit, our path integral included only maps $x : [0, T] \rightarrow N$ whose first derivative was everywhere continuous, then the limiting value of (3.31) would necessarily vanish when $\Delta t \rightarrow 0$, contradicting the operator calculation. Non-commutativity arises in the path integral approach to quantum mechanics precisely because we're forced to include *non-differentiable* paths, *i.e.* our map $x \in C^0(M, N)$ but $x \notin C^1(M, N)$. In fact, since we want to recover the non-commutativity no matter at which time t we insert \hat{x} and \hat{p} , we need paths that are *nowhere* differentiable.

This non-differentiability is the familiar stochastic ('jittering') behaviour of a particle undergoing Brownian motion. It's closely related to a very famous property of random walks: that after a time interval t , one has moved through a net distance proportional to \sqrt{t} rather than $\propto t$ itself. More specifically, averaging with respect to the one-dimensional heat kernel

$$K_t(x, y) = \frac{1}{\sqrt{2\pi\hbar t}} e^{-(x-y)^2/2\hbar t},$$

in time t , the mean squared displacement is

$$\langle (x - y)^2 \rangle = \int_{-\infty}^{\infty} K_t(x, y) (x - y)^2 dx = \int_{-\infty}^{\infty} K_t(u, 0) u^2 du = \hbar t \quad (3.33)$$

so that the rms displacement from the starting point after time t is $\propto \sqrt{t}$. Similarly, our regularized path integrals yield a finite result because the average value of

$$x_{t+\Delta t} \frac{x_{t+\Delta t} - x_t}{\Delta t} - x_t \frac{x_{t+\Delta t} - x_t}{\Delta t} = \Delta t \left(\frac{x_{t+\Delta t} - x_t}{\Delta t} \right)^2,$$

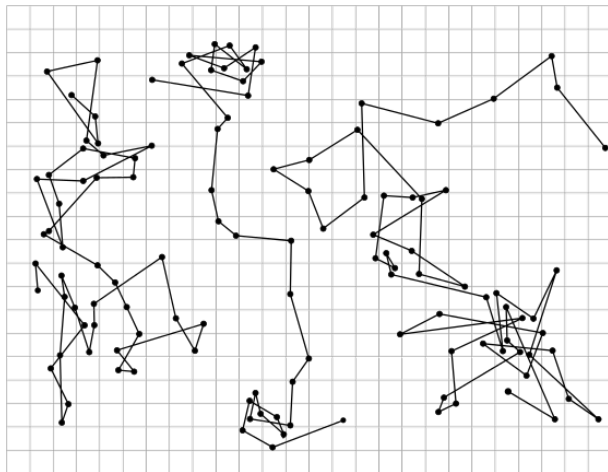


Figure 6: Stimulated by work of Einstein and Smoluchowski, Jean-Baptiste Perron made many careful plots of the locations of hundreds of tiny particles as they underwent Brownian motion. Understanding their behaviour played a key role in confirming the existence of atoms. A particle undergoing Brownian motion moves an average (rms) distance of \sqrt{t} in time t , a fact that is responsible for non-trivial commutation relations in the (Euclidean) path integral approach to Quantum Mechanics.

which for a differentiable path would vanish as $\Delta t \rightarrow 0$, here remains finite.

The importance of nowhere-differentiable paths has a further very important consequence. Since we cannot assign any sensible meaning to

$$\lim_{\Delta t \rightarrow 0} \frac{x_{t+\Delta t} - x_t}{\Delta t},$$

we cannot sensibly claim that

$$\lim_{N \rightarrow \infty} \exp \left[-\frac{\Delta t}{\hbar} \sum_{i=0}^N \frac{1}{2} \left(\frac{x_{t_{i+1}} - x_{t_i}}{\Delta t} \right)^2 \right] \stackrel{??}{=} \exp \left[-\frac{1}{\hbar} \int_0^T \frac{1}{2} \dot{x}^2 dt \right]$$

and thus we do not really have any continuum action. Naively, we might have thought that the presence of $e^{-S[x]/\hbar}$ damps out the contribution of wild field configurations. However, this cannot be the case: nowhere-differentiable paths are essential if we wish our path integral to know about even basic quantum properties.

3.2.2 The path integral measure

Having realized that we need to include nowhere-differentiable fields, and that the continuum action does not exist — even for a free particle — we now return to consider the limit of the measure. You probably won't be surprised to hear that this doesn't exist either.

First recall that for vector space V of finite dimension D , $d\mu$ is a **Lebesgue measure** on V if

- i) it assigns a strictly positive volume $\text{vol}(U) = \int_U d\mu > 0$ to every non-empty open set $U \subset V$,



Figure 7: In a D -dimensional vector space, an open hypercube of finite linear dimension L contains 2^D open hypercubes of linear dimension $L/2 - \epsilon$ for any $L/2 > \epsilon > 0$. We choose the side length to be slightly less than half the original length to ensure these smaller hypercubes are open and non-overlapping.

- ii) $\text{vol}(U') = \text{vol}(U)$ whenever U' may be obtained from U by translation, and
- iii) for every $p \in V$ there exists at least one open neighbourhood U_p , containing p , for which $\text{vol}(U_p) < \infty$.

The standard example of a Lebesgue measure is of course $d\mu = dx$ on $V = \mathbb{R}^D$.

Now let's return to consider the path integral measure. To keep things simple, we again work just with the case that the target space $N = \mathbb{R}^n$ with a flat metric. In the continuum, the space of fields is naturally an infinite dimensional *vector* space, where addition is given by pointwise addition of the fields at each t on the worldline. In the previous section we identified this infinite dimensional space as the space $C^0(M, \mathbb{R}^n)$ of continuous maps $x : M \rightarrow \mathbb{R}^n$. We certainly want our measure to be strictly positive, since (in Euclidean signature) it has the interpretation of a probability measure. Also, we used translational invariance of the measure throughout our discussion in earlier chapters, for example in completing the square and shifting $\phi \rightarrow \tilde{\phi} = \phi + M^{-1}(J, \cdot)$ to write the partition function in the presence of sources as $\mathcal{Z}(J) = e^{M^{-1}(J, J)/2\hbar} \mathcal{Z}(0)$. So we'd like our measure $\mathcal{D}x$ to be translationally invariant, too.

But it's easy to prove that *there is no non-trivial Lebesgue measure on an infinite dimensional vector space*. Let $C_x(L)$ denote the open (hyper)cube centered on x and of side length L . This cube contains 2^D smaller cubes $C_{x_n}(L/2 - \epsilon)$ of side length $L/2 - \epsilon$, all of which are disjoint (see figure 7). Then

$$\text{vol}(C_x(L)) \geq \sum_{n=1}^{2^D} \text{vol}(C_{x_n}(L/2 - \epsilon)) = 2^D \text{vol}(C_x(L/2 - \epsilon)) \quad (3.34)$$

where the first inequality uses the fact that the measure is positive-definite and the smaller hypercubes are open and non-overlapping, and the final equality uses translational invariance. We see that as $D \rightarrow \infty$, the only way the *rhs* can remain finite is if

$\text{vol}(C_x(L/2 - \epsilon)) \rightarrow 0$ for any finite L . So the measure must assign zero volume to any infinite dimensional hypercube of finite linear size. Finally, provided our vector space V is of *countably* infinite dimension (which the discretizes path integral makes plain), we can cover *any* open $U \subset V$ using at most countably many such cubes, so $\text{vol}(U) = 0$ for any U and the measure must be identically zero. In particular, the limit

$$\mathcal{D}x \stackrel{??}{=} \lim_{N \rightarrow \infty} \prod_{i=1}^N \left[\frac{d^n x_i}{(2\pi\hbar \Delta t)^{n/2}} \right]$$

does not exist, and there is no measure $\mathcal{D}x$ in the continuum limit of the path integral.

In fact, in one dimension, while neither $\mathcal{D}x$ nor $e^{-S[x]/\hbar}$ themselves have any continuum meaning, the limit

$$d\mu_W := \lim_{N \rightarrow \infty} \left[\prod_{i=1}^N \frac{d^n x_{t_i}}{(2\pi\Delta t)^{n/2}} \exp \left[-\frac{\Delta t}{2} \left(\frac{x_{t_{i+1}} - x_{t_i}}{\Delta t} \right)^2 \right] \right] \quad (3.35)$$

of the standard measures $d^n x_{t_i}$ on \mathbb{R}^n at each time-step *together with* the factor e^{-S_i} does exist as a measure on $C^0(M, \mathbb{R}^n)$. The limit $d\mu_W$ is known as the **Wiener measure** and, as you might imagine from our discussion above, it plays a central role in the mathematical theory of Brownian motion. The presence of the factor $e^{-S_i/\hbar}$ means that this measure is Gaussian, rather than translationally invariant in the fields, avoiding the above no-go theorem. However tempting it may be to interpret this as ‘obviously’ the product of a Gaussian factor and a usual Lebesgue measure, we know from above that this cannot be true in the continuum limit (though it is true *before* taking the limit).

Thus far, we’ve considered only the path integral for a free particle travelling in \mathbb{R}^n . Kac was able to show that the Wiener measure could also be used to provide a rigorous definition of Feynman’s path integral for *interacting* quantum mechanical models. That is, suppose our quantum particle feels a potential $V : \mathbb{R}^n \rightarrow \mathbb{R}$ which contributes to its Hamiltonian. Then, provided V is sufficiently nice³³, as a path integral we have

$$(e^{-T\hat{H}/\hbar} \psi)(x_0) = \int_{C_{x_0}([0,T]; \mathbb{R}^n)} \exp \left[-\frac{1}{\hbar} \int_0^T V(x(s)) ds \right] \psi(x(T)) d\mu_W, \quad (3.36)$$

where $C_{x_0}([0,t]; \mathbb{R}^n)$ is the space of continuous maps $x : [0,t] \rightarrow \mathbb{R}^n$ with $x(0) = x_0$, and where $d\mu_W$ is the Wiener measure on $\mathcal{C}([0,t]; \mathbb{R}^n)$. I won’t prove this result here, but if you’re curious you can consult *e.g.* B. Simon, *Functional Integration and Quantum Physics*, 2nd ed, AMS (2005), or B. Hall, *Quantum Theory for Mathematicians*, Springer (2013), which also gives a fuller discussion of many of the issues we’ve considered in this section. Note that, when evaluating an asymptotic series for the path integral using Feynman graphs, all we ever really needed was the Gaussian measure describing the free theory: all interaction vertices or operator insertions were treated perturbatively and evaluated using integration against the path integral measure of the free theory.

³³Technically, V must be the sum of a function in $L^2(\mathbb{R}^n, d^n x)$ and a bounded function.

3.3 Effective quantum mechanics and locality

We've seen that naïve interpretations of path integrals over infinite dimensional spaces can be very misleading. Rather than try to deal directly with the infinite dimensional space of continuous maps $C^0(M, N)$ (and the even larger, wilder spaces that arise in QFT in $d > 1$) it may seem safer to always work with a regularized path integral, delaying taking the continuum limit until the end of the calculation. However, there are any number of finite dimensional approximations to an infinite dimensional space, and it's far from clear exactly *which* of these we should choose to define our regularized integral.

Up until now, we've reduced the path integral to a finite dimensional integral by discretising our worldline M , but there many other ways to regularize. For example, even if our field $x(t)$ is nowhere differentiable, we can represent it as a Fourier series

$$x^a(t) = \sum_{k \in \mathbb{Z}} \tilde{x}_k^a e^{2\pi i k t / T}.$$

We might choose to regularize by truncating this series to a finite sum with $|k| \leq N$. The (free) action for the truncated field is

$$S_N(\tilde{x}_k) = \frac{2\pi}{T} \sum_{|k| \leq N} k^2 \delta_{ab} \tilde{x}_k^a \tilde{x}_{-k}^b \quad (3.37a)$$

and we can take the path integral over these finitely many Fourier coefficients with measure

$$\mathcal{D}x_N = \prod_{k=-N}^N \frac{d^n \tilde{x}_k}{(2\pi)^{n/2}} \quad (3.37b)$$

If we try to include all infinitely many Fourier modes, then the sum (3.37a) will diverge and the measure (3.37b) ceases to exist. However, with a finite cutoff N , we will obtain perfectly sensible answers.

The problem, of course, is that these answers will depend on the details of how we chose to regularize. This is not just the question of how they depend on the precise value of N , or the precise scale of the discretization. Rather, how can we be sure whether the results we obtain by discretizing our universe are compatible with those we'd obtain by instead imposing a cut-off on the Fourier modes of the fields? Or with any other way of regularizing that we might dream up? The answer to this will be the subject of (Wilsonian) renormalization in the next chapter, but we can get some flavour of it even here in $d = 1$.

We imagine we have two different fields x and y on the same worldline M , that I'll take to be a circle. We'll start with the action

$$S[x, y] = \int_{S^1} \left[\frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 + V(x, y) \right] dt \quad (3.38)$$

where the potential

$$V(x, y) = \frac{1}{2}(m^2 x^2 + M^2 y^2) + \frac{\lambda}{4} x^2 y^2. \quad (3.39)$$

In terms of the one-dimensional QFT, x and y look like interacting fields with masses m and M , while from the point of view of the target space \mathbb{R}^2 you should think of them as two harmonic oscillators with frequencies m and M , coupled together in a particular way. Of course, this coupling has been chosen to mimic what we did in section 2.4.2 in zero dimensions. If we're interested in perturbatively computing correlation functions of (local) operators, then we could proceed by directly using (3.38) to construct Feynman diagrams. We have the momentum space Feynman rules (with $\hbar = 1$)

$$\begin{array}{ccc} \text{---} x \text{---} & \text{---} y \text{---} & \text{---} \lambda \text{---} \\ 1/(k^2 + m^2) & 1/(k^2 + M^2) & -\lambda \end{array}$$

where k is the one-dimensional worldline momentum, which on a circle of circumference T is quantized in units of $2\pi/T$.

However, if we're interested purely in correlators of operators that depend only on the field x , such as $\langle x(t_2)x(t_1) \rangle$, then we saw in section 2.4.2 that it's expedient to first integrate out the y field, obtaining an effective action for the x that takes the quantum behaviour of y into account. Let's repeat that calculation here. As in zero dimensions, we expect our effective action will contain infinitely many new self-interactions of x . As far as the path integral over $y(t)$ is concerned, x is just a fixed background field so we have formally

$$\int \mathcal{D}y \exp \left[-\frac{1}{2} \int_{S^1} y \left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda}{2} x^2 \right) y \right] = \left[\det \left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda}{2} x^2 \right) \right]^{-1/2}, \quad (3.40)$$

where (for fixed $x(t)$) the determinant of the differential operator can be understood as

Accordingly, the effective action for x is

$$S_{\text{eff}}[x] = \int_{S^1} \left[\frac{1}{2} \dot{x}^2 + \frac{m^2}{2} x^2 \right] dt + \frac{1}{2} \ln \det \left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda}{2} x^2 \right). \quad (3.41)$$

Note the factor of $1/2$ in front of the logarithm, which comes because we got a square root when performing the Gaussian integral over each mode of the real field $y(t)$. Note also that because the effective action is defined by $e^{-S_{\text{eff}}[x]/\hbar} = \int \mathcal{D}y e^{-S[x,y]/\hbar}$ the fact that the square root of the determinant appeared in the *denominator* after performing the Gaussian integral over y means that the logarithm contributes *positively* to the effective action. Had we integrated out a fermionic field, following the rules of Berezin integration would lead to a determinant in the *numerator*, which thus contributes *negatively* to S_{eff} .

Now let's try to understand the effect of this term. First, using

$$\ln \det(AB) = \ln(\det A \det B) = \ln \det A + \ln \det B = \text{tr} \ln A + \text{tr} \ln B$$

we write

$$\ln \det \left(-\frac{d^2}{dt^2} + M^2 + \frac{\lambda}{2} x^2 \right) = \text{tr} \ln \left(-\frac{d^2}{dt^2} + M^2 \right) + \text{tr} \ln \left(1 - \frac{\lambda}{2} \left(\frac{d^2}{dt^2} - M^2 \right)^{-1} x^2 \right). \quad (3.42)$$

The first term on the *rhs* is independent of the field x ; it will drop out if we normalize our calculations by the partition function of the free ($\lambda = 0$) theory. As in $d = 0$, this term is related to the cosmological constant problem and we will consider it further later, but for now our main interest is in the second, x -dependent term.

To make sense of this second term, let $G(t, t')$ be the worldline propagator (or Green's function), defined by

$$\left(\frac{d^2}{dt^2} - M^2\right) G(t, t') = \delta(t - t'), \quad (3.43)$$

so that $G(t, t')$ is the inverse of the free kinetic operator $d^2/dt^2 - M^2$ on the worldline. Then

$$\left(\left(\frac{d^2}{dt^2} - M^2\right)^{-1} x^2\right)(t) = \int_{S^1} G(t, t') x^2(t') dt' \quad (3.44)$$

Explicitly, the Green's function is

$$G(t, t') = \frac{1}{2M} \sum_{r \in \mathbb{Z}} e^{-M|t-t'+rT|} \quad (3.45)$$

where the sum over $r \in \mathbb{Z}$ allows the propagator to travel r times around the circle on its way from t' to t . With this understanding of the inverse of the differential operator $(d^2/dt^2 - M^2)$ we can expand the second term in (3.42) as an asymptotic series valid as $\lambda \rightarrow 0$. From the standard Taylor series of $\ln(1 + \epsilon)$ we have

$$\begin{aligned} & \text{tr} \ln \left(1 - \frac{\lambda}{2} \left(\frac{d^2}{dt^2} - M^2 \right)^{-1} x^2 \right) \\ & \sim - \sum_{n=1}^{\infty} \frac{\lambda^n}{2^n n} \int_{(S^1)^{\times n}} dt_1 \cdots dt_n G(t_n, t_1) x^2(t_1) G(t_1, t_2) x^2(t_2) \cdots G(t_{n-1}, t_n) x^2(t_n) \\ & = - \frac{\lambda}{2} \int_{S^1} dt G(t, t) x^2(t) - \frac{\lambda^2}{8} \int_{S^1 \times S^1} dt dt' G(t', t) x^2(t) G(t, t') x^2(t') + \cdots \end{aligned} \quad (3.46)$$

As expected, integrating out y has generated both a new contribution to the quadratic term in x^2 and also an infinite series of new interactions, just as it did in $d = 0$. However, here there's a new feature: except for the leading $O(\lambda)$ term, *these interactions are now non-local*. They involve the value of the field x integrated over several (or infinitely many) copies of the worldline.

It's instructive to see why this non-locality arises. The first two terms in the series (3.46) represent the Feynman diagrams

$$-\frac{\lambda}{2} x^2(t) \text{---} \text{---} G(t, t) \quad -\frac{\lambda}{2} x^2(t) \text{---} \text{---} G(t, t') \text{---} \text{---} G(t', t) \text{---} \text{---} \frac{\lambda}{2} x^2(t')$$

that arise in the perturbative evaluation of the y path integral. (In these diagrams, the green dot represents the vertex $-\lambda x^2/2$ with x treated as a fixed ‘source’ just as we did in zero dimensions. Just as before, the second diagram comes with a symmetry factor of $1/2$, as the two propagators are interchangeable.) Unlike the trivial case of zero dimensions, here the y field is dynamical; the worldline propagator $G(t, t')$ allows it to move around, and the insertions of x^2 are at independent points in our one-dimensional universe.

Non-locality is generally bad news in physics: the equations of motion we’d obtain from $S_{\text{eff}}[x]$ would be integro-differential equations stating that in order to work out the behaviour of the field x here, we first have to add up what it’s doing everywhere else in the (one-dimensional) Universe. But we don’t want the results of our experiment in CERN to depend on what Ming the Merciless may or may not be having for breakfast over on the far side of the Galaxy.

So how bad is it here? From the explicit form (3.45) of the Green’s function we see that $G(t, t')$ decays exponentially quickly when $t \neq t'$, with a scale set by the inverse mass M^{-1} of y . This suggests that the effects of non-locality will be small *provided we restrict attention to fields that vary slowly on scales $\sim M^{-1}$* . More specifically, expanding $x(t')$ around $t' = t$, the second term in (3.46), involving a total of four powers of the field x , becomes

$$\begin{aligned} & \int dt dt' G(t, t')^2 x^2(t) x^2(t') \\ &= \int dt dt' G(t, t')^2 x^2(t) [x^2(t) + 2x(t)\dot{x}(t)(t' - t) + (\dot{x}^2(t) + x(t)\ddot{x}(t))(t' - t)^2 + \dots] \\ &= \int dt \left[\frac{\alpha}{M^3} x^4(t) + \frac{\beta}{M^5} \left(x^2 \dot{x}^2 + \frac{1}{2} x^3 \ddot{x} \right) + \frac{\gamma}{M^7} (\text{four-derivative terms}) + \dots \right]. \end{aligned} \tag{3.47}$$

In going to the last line we’ve performed the t' integral, noting that terms that are odd in $(t - t')$ will not contribute. The remaining terms are obtained by noting that the Green’s function $G(t, t')$ involves an explicit factor of $1/M$, and depends on t' only through the dimensionless combination $u = M(t' - t)$. Thus, if we replace the factor $(t' - t)^p$ in the p^{th} order term in the Taylor expansion by $(u/M)^p$ and change variables $dt' = du/M$ to integrate over the dimensionless quantity u , the remaining integrals will just yield some dimensionless numbers $\alpha, \beta, \gamma, \dots$. (The precise values of these constants don’t matter for the present discussion.)

The important point is that every new derivative of x in these vertices is *suppressed* by a further power of the mass M of the y field. Thus, so long as $\dot{x}, \ddot{x}, \ddot{\ddot{x}}, \dots$ are all small in units of M^{-1} , we should have a controllable expansion. Similarly, terms further down the expansion in (3.46) that involve higher powers of x also come with further powers of $G(t_i, t_j)$ and further integrals, and so are again suppressed by higher powers of M . If we truncate at any finite order both the expansion (3.46) itself and the derivative expansion of each term in (3.46), we’ll regain an apparently local effective action. This truncation is justified provided we restrict to processes where the momentum of the x field is $\ll M$.

However, once we start to probe energies $\sim M$ something will go badly wrong with our truncated theory. Assuming the original action (3.38) defined a unitary theory (with

a Minkowski signature worldline $M = [0, T]$), simply performing the exact path integral over y must preserve unitarity. This is because we haven't yet made any approximations, just taken the first step to performing the full $\mathcal{D}x \mathcal{D}y$ path integral. All the possible states of the y field are still secretly there, encoded in the infinite series of non-local interactions for x . However, the approximation to keep just the first few terms in S_{eff} *can't* be unitary, because we're rejecting by hand various pieces of Feynman diagrams: we're throwing away some of the things y might have been doing.

The weak interactions are responsible for many important things, from the formation of light elements such as deuterium in the early Universe, to powering stars such as our Sun, to the radioactive β -decay of ^{14}C used in radiocarbon dating. Since the 1960s physicists have known that these weak interactions are mediated by a field called the W-boson and in 1983, the UA1 experiment at CERN discovered this field and measured its mass to be $M_W \simeq 80 \text{ GeV}$. Typically, β -decay takes place at much lower energies, so to describe them it makes sense to integrate out the dynamics of the W boson leaving us with an effective action for the proton, neutron, electron and neutrino that participate in the interaction. This effective action contains an infinite series of terms, suppressed by higher and higher powers of the large mass M_W . Truncating this infinite effective action to its first few terms leads to Fermi's theory of β -decay which gives excellent results at low energies. However, if one extrapolates the results obtained using this truncated action to high energies, one finds a violation of unitarity. The non-unitarity in Fermi's theory is what lead physicists to suspect the existence of the W-boson in the first place.

3.4 Quantum gravity in one dimension

The heat kernels, partition functions and correlation functions we've computed depend on the choices we made in setting up our theory, including in particular the worldline metric g . So far, we've fixed this to be $g = \delta$, but it's interesting to see what happens if we also allow ourselves to couple to a general worldline metric. Even more

We start by rewriting our original action (3.1) describing maps $x : (M, g) \rightarrow (N, G)$ in a way that makes it invariant under diffeomorphisms of the worldline M . We have

$$S[g, x] = \int_M \sqrt{g} \left[\frac{1}{2} G_{ab}(x) g^{tt}(t) \partial_t x^a \partial_t x^b + \frac{1}{2} V(x) \right] dt \quad (3.48)$$

where we've emphasized that this action now depends on the worldline metric g . Note that $g_{tt}(t)$ is a 1×1 positive symmetric matrix, so is specified by just a single positive function $e^2 : M \rightarrow \mathbb{R}_{>0}$. We have $\sqrt{g} = |e|$ and $g^{tt} = e^{-2}$. Also, there is no notion of Riemann curvature since $[\nabla_t, \nabla_t] \equiv 0$ as we only have one direction. Thus there is no analogue of the Einstein-Hilbert term $\int_M \sqrt{g} \text{Ric}(g)$ which provides the kinetic terms for the metric, so gravity in $d = 1$ is non-dynamical. Varying the action (3.48) with respect to g we obtain the Einstein equation

$$T_{tt} := \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L})}{\delta g^{tt}} = \frac{1}{|e|} \left[G_{ab}(x) \dot{x}^a \dot{x}^b - e^2(t) V(x) \right] = 0 \quad (3.49)$$

which just says that the worldline stress tensor must vanish. In particular, this Einstein equation fixes the metric to be

$$g_{tt}(t) = e^2(t) = \frac{1}{V(x)} G_{ab}(x) \dot{x}^a \dot{x}^b \quad (3.50)$$

which is positive definite provided $V(x) > 0$ and (N, G) is Riemannian. Since the metric is non-dynamical, we can use this equation to eliminate it from the action (3.48) obtaining

$$S[x] = \sqrt{V_0} \int_M \sqrt{G_{ab}(x) \dot{x}^a \dot{x}^b} dt \quad (3.51)$$

in the special case that $V(x) = V_0$ is constant. We recognize this as just the proper length of the image curve $x(M) \subset N$, which is the most geometrically natural action in $d = 1$.

With the action (3.48) the momentum conjugate to the field x^a is

$$p_a = \frac{\delta \mathcal{L}}{\delta \dot{x}^a} = \frac{1}{|e|} G_{ab}(x) \dot{x}^b \quad (3.52)$$

and consequently the Einstein equation (3.49) says that

$$G^{ab}(x) p_a p_b + V(x) = 0 \quad (3.53)$$

under canonical quantization $p_a \rightarrow -\partial/\partial x^a$ this becomes

$$(-\eta^{ab} \partial_a \partial_b + m^2) \Psi(x) = 0 \quad (3.54)$$

which is just the Klein–Gordon equation for a particle of mass m .

Let's consider the case $(N, G) = (\mathbb{R}^{n-1,1}, \eta)$ and $V(x) = m^2$, a constant which plays the role of a cosmological constant in our $d = 1$ universe. By inserting complete sets of momentum eigenstates, we have

$$\begin{aligned} \langle y | e^{-HT/\hbar} | x \rangle &= \int d^n p d^n q \langle y | p \rangle \langle p | e^{-HT/\hbar} | q \rangle \langle q | x \rangle \\ &= \int \frac{d^n p}{(2\pi\hbar)^n} e^{ip \cdot (x-y)/\hbar} e^{-T(p^2+m^2)/2} \end{aligned} \quad (3.55)$$

and so the path integral over the matter fields becomes

$$\int_{C_I[x,y]} \mathcal{D}x e^{-S[x]/\hbar} = \int \frac{d^n p}{(2\pi\hbar)^n} e^{ip \cdot (x-y)/\hbar} e^{-T(p^2+m^2)/2} . \quad (3.56)$$

(An alternative way to obtain the same result is to write the flat space heat kernel (3.9) as its inverse Fourier transform.)

If we're doing quantum gravity, we should now integrate this expression over all possible metrics on M , chosen upto diffeomorphisms, which plays the role of gauge equivalence in General Relativity.³⁴ Under a general coordinate transformation $t \mapsto t'(t)$ the worldline metric

$$g_{tt}(t) \mapsto g'_{t't'}(t') = \frac{dt}{dt'} \frac{dt}{dt'} g_{tt}(t) = \left(\frac{dt}{dt'} \right)^2 g_{tt}(t) . \quad (3.57)$$

³⁴In general for a gauge theory, we always take the path integral over the space of gauge fields considered upto gauge transformations, as we'll see in more detail in chapter ??.

Thus we can always find a diffeomorphism that rescales the value of the metric to anything we like at each t . In particular, we are always free to choose $g = \delta$ locally in one dimension. However, we cannot quite erase all trace of the original metric, because the total volume

$$T = \int_I dt \sqrt{g_{tt}} = \int_I dt' \sqrt{g_{t't'}} \quad (3.58)$$

is unchanged by diffeomorphisms. (In fact, since there is no notion of curvature, this total volume is the only invariant of a $d = 1$ Riemannian manifold.) Consequently, the space $\text{Met}(I)/\text{Diff}(I)$ of metrics upto diffeomorphism is simply the space of possible total lengths of our worldline, or in other words all possible values $T \in [0, \infty)$. Rather grandly, this is known as the **moduli space** of Riemannian metrics on the interval I and, in this context, the proper length T is sometimes known as a **Schwinger parameter**. Integrating the result (3.56) of the matter path integral over this moduli space thus gives

$$\int_0^\infty dT \int \frac{d^n p}{(2\pi\hbar)^n} e^{ip \cdot (x-y)/\hbar} e^{-T(p^2+m^2)/2} = 2 \int \frac{d^n p}{(2\pi\hbar)^n} \frac{e^{ip \cdot (x-y)/\hbar}}{p^2 + m^2} \quad (3.59)$$

which we recognise as (twice) the Euclidean space propagator $D(x, y)$ for a scalar field $\Phi(x)$ of mass m on the *target* space \mathbb{R}^n . In other words, the position space propagator can be written

$$D(x, y) = \int_{\text{Met}(I)/\text{Diff}(I)} \mathcal{D}g \int_{C_I[x, y]} \mathcal{D}x e^{-S[x, g]/\hbar} \quad (3.60)$$

as a path integral in worldline quantum gravity. Choosing more elaborate matter content (*e.g.* fermions) for our worldline QFT similarly leads to propagators for particles of different spin in the target space (N, G) . We'll meet this way of thinking about field theory propagators again in chapter 7, where we put it to work calculating propagators in the presence of background fields.

Feynman realized that one could describe several such particles interacting with one another if one replaced the *worldline* I by a *worldgraph* Γ . For example, to obtain a perturbative evaluation of the r -point correlation function

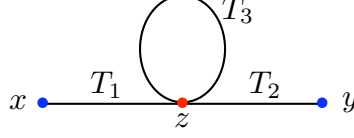
$$\langle \Phi(x_1) \Phi(x_2) \dots \Phi(x_r) \rangle$$

of a massive scalar field $\Phi(x)$ in $\lambda\Phi^4$ theory on \mathbb{R}^n , one could start by considering a 1-dimensional QFT living on a 4-valent graph Γ with r end-points. This QFT is described by the action (??), where x is constrained to map each end-point of the graph to a different one of the Φ insertion points $x_i \in \mathbb{R}^n$. We assign independent Schwinger parameters T_e to each edge e of the graph and take the path integral over all maps $x : \Gamma \rightarrow \mathbb{R}^n$ as well as integrating over all the Schwinger parameters.

Part of what is meant by an ‘integral over all maps $x : \Gamma \rightarrow \mathbb{R}^n$ ’ includes an integral over the location in \mathbb{R}^n to which each vertex of Γ is mapped. When we perform this integral, the factors of $e^{ip \cdot (x-y)}$ in the path integral (3.56) for each edge lead to a target space momentum conserving δ -function at each vertex. As in (3.59), integrating over the Schwinger parameters generates a propagator $1/(p^2 + m^2)$ for each edge of the graph. Thus,

after including a factor of $(-\lambda)^{|v(\Gamma)|}$ and dividing by the symmetry factor of the graph, our 1-dimensional QFT has generated the same expression as we would have obtained from Feynman rules for $\lambda\Phi^4$ on \mathbb{R}^n .

For example, the 4-valent graph with two end-points shown here:



corresponds to the path integral expression

$$\begin{aligned} & \frac{-\lambda}{4} \int_0^\infty dT_1 \int_{C_{T_1}[x,z]} \mathcal{D}x e^{-S} \times \int_0^\infty dT_2 \int_{C_{T_2}[y,z]} \mathcal{D}x e^{-S} \times \int_0^\infty dT_3 \int_{C_{T_3}[z,z]} \mathcal{D}x e^{-S} \\ &= \frac{-\lambda}{4} \int d^n z \frac{d^n p}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{d^n \ell}{(2\pi)^n} \frac{e^{ip \cdot (x-z)}}{p^2 + m^2} \frac{e^{iq \cdot (y-z)}}{q^2 + m^2} \frac{e^{i\ell \cdot (z-z)}}{\ell^2 + m^2} \\ &= \frac{-\lambda}{4} \int \frac{d^n p}{(2\pi)^n} \frac{d^n \ell}{(2\pi)^n} \frac{e^{ip \cdot (x-y)}}{(p^2 + m^2)^2 (\ell^2 + m^2)}. \end{aligned} \quad (3.61)$$

This is the same order λ contribution to the 2-point function $\langle \Phi(x)\Phi(y) \rangle$ that we'd obtain from (Fourier transforming) the momentum space Feynman rules for $\lambda\Phi^4$ theory, with the graph treated as a Feynman graph in \mathbb{R}^n rather than a one-dimensional Universe.

To obtain the full perturbative expansion of $\langle \Phi(x_1)\Phi(x_2) \cdots \Phi(x_n) \rangle$ we now sum over all graph topologies appropriate to our 4-valent interaction. Thus

$$\langle \Phi(x_1)\Phi(x_2) \cdots \Phi(x_n) \rangle = \sum_{\Gamma} \frac{(-\lambda)^{|v(\Gamma)|}}{|\text{Aut } \Gamma|} \int_0^\infty d^{|e(\Gamma)|} T \int_{C_{\Gamma}[x_1, x_2, \dots, x_n]} \mathcal{D}\phi e^{-S_{\Gamma}[\phi]}, \quad (3.62)$$

where $|e(\Gamma)|$ and $|v(\Gamma)|$ are respectively the number of edges and vertices of Γ .

Thus, the integral over the lengths of all the edges of our graph in (3.62) is best thought of as an integral over the space of all possible Riemannian metrics on Γ , up to diffeomorphism invariance. Furthermore, in summing over graphs Γ we were really summing over the topological type of our one dimensional Universe. Notice that the vertices of our graphs are singularities of the one-dimensional Riemannian manifold, so we're allowing our Universe to have such wild (even non-Hausdorff) behaviour. So for fixed lengths T_e the path integral over $x(t)$ is the 'matter' QFT on a fixed background space Γ , while the integral over the lengths of edges in Γ together with the sum over graph topologies is Quantum Gravity.

This worldline approach to perturbative QFT is close to the way Feynman originally thought about the subject, presenting his diagrams at the Pocono Conference of 1948. The relation of this approach to higher (four) dimensional QFT as we usually think about it was worked out by Dyson a year later, long before people used path integrals to compute anything in higher dimensions. Above, we've described just the simplest version of this picture, relevant for a scalar theory on the target space. There are more elaborate $d = 1$ QFTs that would allow us to obtain target space Quantum Mechanics for particles with

spin, and we could also allow for more interesting things to happen at the interaction vertices of our worldgraphs. In this way, one can build up worldline approaches to many perturbative QFTs. This way of thinking can still be useful in practical calculations today, and still occasionally throws up conceptual surprises, but we won't pursue it further in this course.

This picture is also very close to perturbative String Theory. There, as you'll learn if you're taking the Part III String Theory course, the worldgraph Γ is replaced by a two dimensional worldsheet (Riemann surface) Σ , the $d = 1$ worldline QFT replaced by a $d = 2$ worldsheet CFT³⁵. Likewise, the integral over the moduli space of Riemannian metrics on Γ becomes an integral over the moduli space of Riemann surfaces, and finally the sum over graphs is replaced by a sum over the topology of the Riemann surface. We know that the worldgraph approach to QFT only captures some aspects of perturbation theory, and in the following chapters we'll see that deeper insight is provided by QFT proper. Asking whether there's a similarly deeper approach to String Theory will take you to the mystic shores of String Field Theory, about which very little is known.

³⁵CFT = Conformal Field Theory.