5 Perturbative Renormalization

In this chapter we'll carry out renormalization of two of the most common theories: $\lambda\phi^4$ and Quantum Electrodynamics, both working to 1–loop accuracy (*i.e.* order $\hbar$ in the quantum effective action). We introduce renormalization schemes as a way of fixing the finite parts of counterterms, and dimensional regularization as a convenient way to control the asymptotic series expansion of a path integral in terms of Feynman graphs.

5.1 One–loop renormalization of $\lambda\phi^4$ theory

Consider the scalar theory

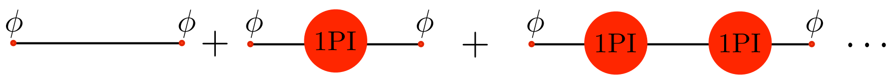
$$S_{\Lambda_0}[\phi] = \int \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right] \mathrm{d}^4x \tag{5.1}$$

with initial couplings m^2 and λ at scale Λ_0. (The mass coupling is dimensionful here.) From the analysis of the previous chapter, we expect that near the Gaussian fixed point, the mass parameter is relevant, while the quartic coupling is marginally irrelevant. Let's see how these expectations are borne out in perturbative calculations.

Firstly, the quadratic terms in ϕ — both the kinetic term $(\partial\phi)^2$ and the mass term — receive corrections from graphs with precisely two external ϕ lines. These are computed by the exact propagator, which in momentum space is the Fourier transform

$$\Delta(k^2) := \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}k\cdot x}\langle\phi(x)\phi(0)\rangle \tag{5.2}$$

of the connected two–point function. If we let $\Pi(k^2)$ denote the sum of all 1PI momentum space Feynman graphs with two external ϕ lines, then we can express the exact propagator as a geometric series

$\Delta(k^2) =$ 

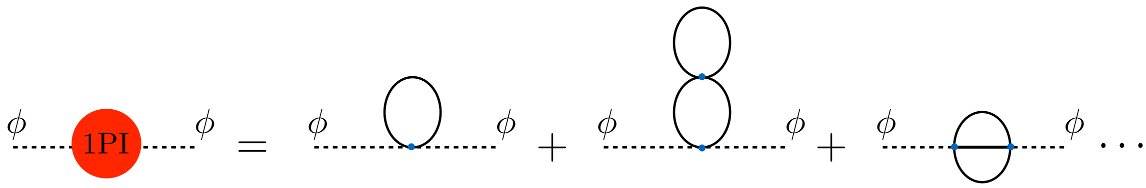
$$= \frac{1}{k^2+m^2} + \frac{1}{k^2+m^2}\Pi(k^2)\frac{1}{k^2+m^2} + \frac{1}{k^2+m^2}\Pi(k^2)\frac{1}{k^2+m^2}\Pi(k^2)\frac{1}{k^2+m^2} + \cdots$$
$$= \frac{1}{k^2+m^2-\Pi(k^2)}\,,$$

where the leading term in this series

$$\Delta^0(k^2) = \frac{1}{k^2+m^2} \tag{5.3}$$

is just the propagator in the classical theory.

The corrections to this classical propagator involve $\Pi(k^2)$, which is often called the **self–energy** of the ϕ field. In perturbation theory, it receives contributions from the loop diagrams



as well as from counterterms that we'll consider below. (The dotted lines are intended to remind us that we amputate the final ϕ propagators in computing the 1PI self-energy diagrams; the two powers of ϕ are being treated as an external source.) In particular, if we're content to work just to 1–loop accuracy, then we only need consider the first of these Feynman graphs, corresponding to the integral

$$-\frac{\lambda}{2}\int_{|p|\leq\Lambda_0}\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2+m^2}$$

over the possible values of the momentum running around the loop. Here, we've regularised our theory by only including modes with momenta $|p| \leq \Lambda_0$ and (as always) are working in Euclidean signature. The factor of $1/2$ is the symmetry factor of the 1–loop graph, while the factor of $-\lambda$ comes from expanding e^{-S} to first order in the coupling. Note also that this particular integral is independent of the momentum k brought in by the external ϕ fields. Using the fact that $\mathrm{Vol}(S^3) = 2\pi^2$, we have

$$\begin{aligned}
-\frac{\lambda}{2}\int_{|p|\leq\Lambda_0}\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2+m^2} &= -\lambda\frac{\mathrm{Vol}(S^3)}{2(2\pi)^4}\int_0^{\Lambda_0}\frac{p^3\,\mathrm{d}p}{p^2+m^2}\\
&= -\frac{\lambda\, m^2}{32\pi^2}\int_0^{\Lambda_0^2/m^2}\frac{u\,\mathrm{d}u}{1+u}\\
&= -\frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2\ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right]\,.
\end{aligned}\tag{5.4}$$

where we introduced the dimensionless variable $u = p^2/m^2$. As expected, this result shows that the mass parameter is relevant: There's a quadratic divergence as we try to take the continuum limit $\Lambda_0 \to \infty$ (as well as a subleading logarithmic one).

If we wish to obtain finite results in the continuum limit, then we must tune the coefficients of each term in our scale-Λ_0 action so as to make (5.4) finite in the limit. Achieving such a tuning by varying (m^2, λ) directly in (5.4) is complicated, but fortunately, as discussed above, we don't need to do this. Instead, we modify the original action by including counterterms:

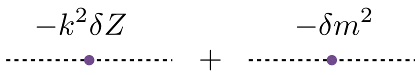
$$S_{\Lambda_0}[\phi] \to S_{\Lambda_0}[\phi] + \hbar S^{\mathrm{CT}}[\phi, \Lambda_0] \tag{5.5}$$

where in this case the counterterm action is

$$S^{\mathrm{CT}}[\phi,\Lambda_0] = \int\left[\frac{1}{2}\delta Z(\partial\phi)^2 + \frac{1}{2}\delta m^2\phi^2 + \frac{1}{4!}\delta\lambda\,\phi^4\right]\mathrm{d}^4x \tag{5.6}$$

with $(\delta Z, \delta m^2, \delta\lambda)$ representing our freedom to adjust the couplings in the original action (including the coupling to the kinetic term — or wavefunction renormalization). These counterterm couplings will depend explicitly on Λ_0, as they represent the tuning that must be performed starting from the Λ_0 cut–off theory.

The fact that the counterterm action is proportional to $\hbar$ means that classical contributions from S^{CT} contribute to the same order in $\hbar$ as 1-loop diagrams from $S_{\Lambda_0}[\phi]$. In particular, the two–point function $\Delta(k^2)$ receives a correction at order $\hbar$ from both the wavefunction renormalization and mass counterterms, contributing to the self–energy $\Pi(k^2)$ as

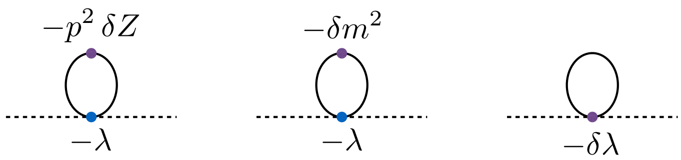


We recall that since the counterterms are being treated as vertices, at first order they contribute with a minus sign coming from the expansion $\mathrm{e}^{-S[\phi]}$. We also note that in momentum space the counterterm $\delta Z(\partial\phi)^2/2$ comes with a factor of k^2 since it couples to the derivatives of the field. The full quantum contribution to the quadratic terms in ϕ arises from the self–energy terms

$$\Pi^{1\,\mathrm{loop}}(k^2) = -k^2\delta Z - \delta m^2 - \frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2\ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right] \tag{5.7}$$

to 1–loop accuracy.

If we required higher accuracy, we should include 1PI graphs built from both higher loops using the propagators and vertices of the original classical action, together with lower loop graphs that also include one or more insertions of the counterterms as vertices. As here, each counterterm vertex counts as a loop when considering the order of the graph. For example, at two loops we have contributions from the 2–loop graphs shown above, together with the graphs



that include counterterms. Note that the wavefunction renormalization counterterm here comes with coefficient p^2, since the momentum carried by the propagator on which it is inserted is p. The result of all these diagrams (which we will not compute) should then be added to (5.7) to obtain $\Pi(k^2)$ to order $\hbar^2$ accuracy.

5.1.1 The on–shell renormalization scheme

The *raison d'etre* of counterterms is to ensure (5.7) has a finite continuum limit, so that they must cancel the part of (5.4) that diverges as $\Lambda_0 \to \infty$. This still leaves us a lot of freedom in choosing how much of the *finite* part of the loop integral can also be absorbed by the counterterms. There's no preferred way to do this, and any such choice is called a **renormalization scheme**. Ultimately, all physically measurable quantities (such as cross–sections, branching ratios, particle lifetimes *etc.*) should be independent of the choice of renormalization scheme.

Our calculation above involved two independent counterterms, δm^2 and δZ, so we'll need two independent renormalization conditions to fix them. One way to proceed is known as the **on–shell scheme**. In this scheme, we fix $(\delta m^2, \delta Z)$ by asking that, once we take the continuum limit, the exact ϕ propagator $\Delta(k^2)$ in (5.2) has a *simple pole* at some experimentally measured value $-k^2 = m^2_{\rm phys}$, and that the *residue* of this pole is unity. These requirements are motivated by the fact that the classical propagator $(k^2+m^2)^{-1}$ has a pole when $-k^2 = m^2$. Here, m^2 is a parameter in the action which would indeed be interpreted as the mass of the particle according to the classical equations of motion. However, we've seen that the value of this coupling (like all others) is shifted by quantum corrections, so the true mass of ϕ could well be very different. Experimentally, we can measure the true mass of a particle by looking for peaks (resonances) in scattering cross–sections[44] where this particle is exchanged. As you learnt when studying scattering theory in Quantum Mechanics, these peaks correspond to poles of the S-matrix in the complex momentum plane.

Since we want $\Delta(k^2)$ to have a simple pole when $k^2 = -m^2_{\rm phys}$, the self–energy contributions must obey

$$\Pi(-m^2_{\rm phys}) = m^2 - m^2_{\rm phys}\,, \tag{5.8a}$$

and since we want this pole to have unit residue, we also require

$$\left.\frac{\partial \Pi}{\partial k^2}\right|_{k^2=-m^2_{\rm phys}} = 0\,. \tag{5.8b}$$

Most often, it's convenient to set up our original action so that the parameter m^2 in the action is indeed the experimentally measured value $m^2_{\rm phys}$, so in this case we'd want

$$\Pi(-m^2_{\rm phys}) = 0\,. \tag{5.8c}$$

Note that this condition doesn't mean the counterterms are chosen to identically cancel quantum corrections to the propagator, just that they cancel at the particular on–shell value $-k^2 = m^2_{\rm phys}$ of the external momentum.

To 1-loop accuracy, taking into account both the loop integral and the counterterm, we found the self–energy (5.7)

$$\Pi^{1\,\rm loop}(k^2) = -k^2\delta Z - \delta m^2 - \frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2\ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right]\,. \tag{5.9}$$

Since the only k^2 dependence on the *rhs* is from the wavefunction renormalization counterterm, condition (5.8b) on the residues gives simply

$$\delta Z = 0 \tag{5.10}$$

at this order. If we wish the Lagrangian parameter m to correspond to the experimentally measured value of the mass, then condition (5.8c) on the location of the pole fixes

$$\delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2\ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right] \tag{5.11}$$

[44] For short–lived particles, it's not feasible to get out your kitchen scales.

in the on–shell renormalization scheme.

Equation (5.10) shows that no wavefunction counterterm is necessary in $\lambda\phi^4$ theory at 1 loop. This is easy to understand: Consider the Feynman diagram contributions to $\Pi(k^2)$ displayed above. In the first two graphs, the momenta flowing around the loops is independent of the external momentum k, so in space–time they will generate corrections to the ϕ^2 term in the effective action, but not to terms involving derivatives of ϕ^2, such as the kinetic term $(\partial\phi)^2$. This is something of a coincidence in $\lambda\phi^4$ theory at 1–loop and in a more generic theory, we would expect the two–point function $\int \mathrm{d}^d x\, \mathrm{e}^{\mathrm{i}k\cdot x}\langle\phi(x)\phi(0)\rangle$ to be sensitive to wavefunction renormalization factors (even at 1–loop) as well as to mass renormalization.

Indeed, even in $\lambda\phi^4$ theory, by the time we get to 2–loops there is non–trivial wavefunction renormalization. We can see this from the final Feynman graph shown above. In this graph, the external momentum k carried by ϕ must flow through at least one of the internal propagators. Expanding this propagator as an infinite power series in k^2, we see that the corresponding loop integral will contribute an infinite series of position space derivative terms acting on two powers of ϕ. The term $\propto k^2$ will correspond to a correction to the kinetic term $(\partial\phi)^2$ and so generates a non–trivial wavefunction renormalization factor. Subleading terms generate new, higher derivative operators in the effective action, such as $(\partial^2\phi)^2$. As we saw in our $d = 1$ QFT, the infinite series of such operators this loop graph generates really shows that the effective theory is non-local. However, just as in $d = 1$, all these higher derivative operators are irrelevant under RG flow of scalar field theory in $d = 4$ and so will vanish when we take $\Lambda_0 \to \infty$. The resulting theory will be local.

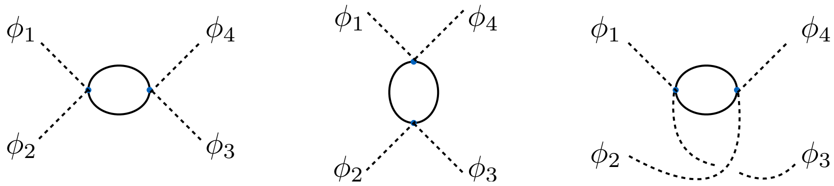
Notice that we cannot sensibly take the limit $\Lambda_0 \to \infty$ in either the 1-loop correction (5.4) or the counterterm (5.11) separately. However, the combined contribution

$$\Pi_{1-\text{loop}}(k^2) = -\delta m^2 - \frac{\lambda}{32\pi^2}\left[\Lambda_0^2 - m^2 \ln\left(1 + \frac{\Lambda_0^2}{m^2}\right)\right] = 0 \tag{5.12}$$

is perfectly well–behaved in the continuum limit. This reflects the fact that neither the continuum action nor the path integral measure $\mathcal{D}\phi$ over *all* modes exists: It does *not* make sense to take the limit $\Lambda_0 \to \infty$ *before* computing the path integral. However, if we hold Λ_0 fixed at some finite value and compute the path integral to any desired accuracy in $\hbar$, then we can make sense of the final result as $\Lambda_0 \to \infty$. While our tuning (5.11) is good to 1–loop accuracy, if we computed the path integral to higher order in perturbation theory, including the contribution of higher–loop Feynman diagrams, we would have to make further tunings in $(\delta m^2, \delta Z)$, proportional to higher powers of the coupling λ, so as to still retain a finite limiting result. In particular, the fact that the combined contribution to $\Pi(k^2)$ vanishes identically (not just when $-k^2 = m^2$) is again a 1–loop accident in $\lambda\phi^4$ theory.

5.1.2 Renormalization of the quartic coupling

The 1-loop correction to the quartic vertex is given by the three Feynman graphs



each with four external ϕ fields. In conventions where each external momentum k_i is flowing into the diagram, these correspond to the integrals

$$\begin{aligned}\frac{\lambda^2}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2+m^2}\,\frac{1}{(p+k_1+k_2)^2+m^2}&\\ +\;\frac{\lambda^2}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2+m^2}\,\frac{1}{(p+k_1+k_4)^2+m^2}&\\ +\;\frac{\lambda^2}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2+m^2}\,\frac{1}{(p+k_1+k_3)^2+m^2}&\end{aligned}\tag{5.13}$$

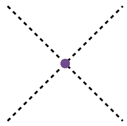
in momentum space, taken over the region $|p|\leq\Lambda_0$. If the loop momentum has magnitude much greater than the physical mass m or the momenta k_i carried in by the external fields, then we see that each integral behaves as $\sim\int^{\Lambda_0}p^3\mathrm{d}p/p^4$ and so diverges logarithmically if we naïvely try to send $\Lambda_0\to\infty$.

Because the external momentum flows through the propagators that make up the loop, the Feynman graphs above also generate non–local contributions to the effective action. This is just a momentum space reflection of the fact that, in the position space Feynman graphs, the two vertices in each graph are at different points x, $x'\in M$. Expanding (5.13) in powers of the external momenta generates an infinite series of derivative interactions such as $\sim\partial^k\phi^2\,\partial^k\phi^2$, each containing an even number of derivatives acting in some way on the four external ϕ fields. In the UV region, the appropriate expansion is in powers of $|k_i|/|p|$. Thus, for every addition power of the external momenta, the loop integral is suppressed by a further power of $1/|p|$. You should check that, just as we expected, all such derivative terms are *irrelevant* — although turning on the coupling λ in our initial scale Λ_0 theory generates such non–local terms, they all vanish in the continuum limit $\Lambda_0\to\infty$ as we are driven to focus on the renormalized trajectory.

The pure correction to the ϕ^4 coupling comes just from the k_i-independent part of each of the three loop diagrams, all of which are equal. The ϕ^4 coupling thus receives a 1–loop correction

$$\begin{aligned}3\frac{\lambda^2}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^2}&=\frac{3\lambda^2}{16\pi^2}\int_0^{\Lambda_0}\frac{p^3\mathrm{d}p}{(p^2+m^2)^2}=\frac{3\lambda^2}{32\pi^2}\int_0^{\Lambda_0^2/m^2}\frac{u\,\mathrm{d}u}{(1+u)^2}\\ &=\frac{3\lambda^2}{32\pi^2}\left[\ln\left(1+\frac{\Lambda_0^2}{m^2}\right)-\frac{\Lambda_0^2}{\Lambda_0^2+m^2}\right]\end{aligned}\tag{5.14}$$

which is indeed logarithmically divergent. To obtain a finite continuum limit we tune our initial coupling using the counterterm contribution



Again, the counterterm must remove the logarithmic divergence in (5.21), and a choice of renormalization scheme is a choice of how to fix the finite parts $\delta\lambda$. A standard choice would be to take

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left[\ln\frac{\Lambda_0^2}{m^2} - 1\right] \tag{5.15}$$

in which case the pure ϕ^4 coupling in the quantum effective action is

$$\lambda_{\text{eff}} = \lambda - \frac{3\hbar\lambda^2}{32\pi^2}\left[\ln\left(1+\frac{m^2}{\Lambda_0^2}\right) + \frac{m^2}{m^2+\Lambda_0^2}\right] \tag{5.16}$$

which is again perfectly finite as $\Lambda_0 \to \infty$. In particular, with this renormalization scheme, in the continuum limit we find $\lambda_{\text{eff}} = \lambda$ so that the pure ϕ^4 coupling in the quantum effective action again coincides with the parameter λ in the classical action.

5.1.3 Irrelevant interactions and the quantum effective action

To uncover something more interesting, let's reconsider the effect of the external momenta in the previous calculation. As we said above, these will mean that the quantum effective action contains an infinite number of quartic interactions including ever more derivatives of the fields. To compute them, we must evaluate the loop integrals in (5.13) at $k_i \neq 0$. Let's now do this, concentrating on the first integral.

We first note Feynman's trick

$$\int_0^1 \frac{\mathrm{d}x}{[xA+(1-x)B]^2} = \frac{1}{B-A}\left[\frac{1}{xA+(1-x)B}\right]_0^1 = \frac{1}{AB} \tag{5.17}$$

which allows us to combine the two propagators as *e.g.*

$$\begin{aligned}\frac{1}{(p+k_{12})^2+m^2}\,\frac{1}{p^2+m^2} &= \int_0^1 \frac{\mathrm{d}x}{[x((p+k_{12})^2+m^2)+(1-x)(p^2+m^2)]^2}\\ &= \int_0^1 \frac{\mathrm{d}x}{[p^2+m^2-2xp\cdot k_{12}+xk_{12}^2]^2}\\ &= \int_0^1 \frac{\mathrm{d}x}{[(p-xk_{12})^2+m^2+x(1-x)k_{12}^2]^2}\end{aligned} \tag{5.18}$$

in terms of $k_{12}^2 = (k_1+k_2)^2$. Defining $\ell = p - xk_{12}$ we have the loop integral

$$\frac{\lambda^2}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2+m^2}\,\frac{1}{(p+k_1+k_2)^2+m^2} = \frac{\lambda^2}{2}\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\int_0^1\frac{\mathrm{d}x}{[\ell^2+m^2+x(1-x)k_{12}^2]^2}\,. \tag{5.19}$$

We're really supposed to integrate this over the region $|p| \leq \Lambda_0$. However, since our interest is ultimately in the $\Lambda_0 \to \infty$ continuum limit, we can instead integrate over the region

$|\ell| \le \Lambda_0$ which differs from the original region by terms of order $|k_{12}|/\Lambda_0$.[45] Interchanging the order of the integrals, which is allowed as they are both absolutely convergent whenever Λ_0 is finite, we have

$$\begin{aligned}\int \mathrm{d}^4\ell \int_0^1 \frac{\mathrm{d}x}{[\ell^2+m^2+x(1-x)k_{12}^2]^2} &= \mathrm{Vol}(S^3)\int_0^1 \mathrm{d}x \int_0^{\Lambda_0} \frac{\ell^3\mathrm{d}\ell}{[\ell^2+m^2+x(1-x)k_{12}^2]^2} \\ &= \pi^2 \int_0^1 \mathrm{d}x \int_0^{\Lambda_0^2} \frac{\ell^2\,\mathrm{d}(\ell^2)}{[\ell^2+m^2+x(1-x)k_{12}^2]^2} \\ &= \pi^2 \int_0^1 \left[\ln\left(\frac{\Lambda_0^2+m^2+x(1-x)k_{12}^2}{m^2+x(1-x)k_{12}^2}\right) + \frac{m^2+x(1-x)k_{12}^2}{\Lambda_0^2+m^2+x(1-x)k_{12}^2} - 1\right]\mathrm{d}x\,. \end{aligned} \tag{5.20}$$

Including the contributions from the remaining two diagrams and keeping only terms which do not vanish as $\Lambda_0 \to \infty$ (since these are the only parts we have reliably computed after our change of integration region), these three loop diagrams actually yield

$$\frac{\lambda^2}{32\pi^2}\int_0^1 \left[\ln\frac{\Lambda_0^2}{m^2-x(1-x)s} + \ln\frac{\Lambda_0^2}{m^2-x(1-x)t} + \ln\frac{\Lambda_0^2}{m^2-x(1-x)u} - 3\right]\mathrm{d}x\,, \tag{5.21}$$

where we've written the external momenta in terms of the **Mandelstam variables**

$$s = -(k_1+k_2)^2\,, \qquad t = -(k_1+k_4)^2\,, \qquad u = -(k_1+k_3)^2\,. \tag{5.22}$$

These terms are present in the quantum effective action, written in momentum space, with the factors of s, t and u indicating the presence of derivatives acting on the fields in the position space quantum effective action.

Combined with the classical action and counterterm as before, the total coefficient of ϕ^4 in the effective action is thus

$$\lambda + \hbar\,\delta\lambda - \frac{\hbar\lambda^2}{32\pi^2}\int_0^1 \left[\ln\frac{\Lambda_0^2}{m^2-x(1-x)s} + \ln\frac{\Lambda_0^2}{m^2-x(1-x)t} + \ln\frac{\Lambda_0^2}{m^2-x(1-x)u} - 3\right]\mathrm{d}x$$

to order $\hbar$. With our earlier choice

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left[\ln\frac{\Lambda_0^2}{m^2} - 1\right] \tag{5.23}$$

for the counterterm, this is

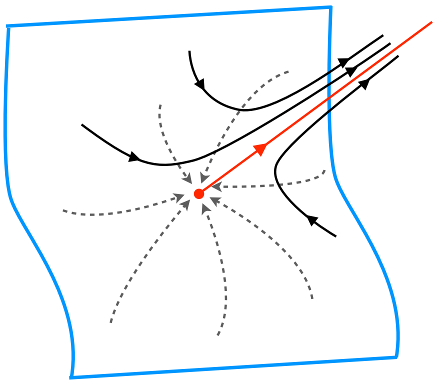
$$\begin{aligned}\mathcal{A}(k_i) &= \lambda - \frac{\hbar\lambda^2}{32\pi^2}\int_0^1 \left[\ln\frac{m^2}{m^2-x(1-x)s} + \ln\frac{m^2}{m^2-x(1-x)t} + \ln\frac{m^2}{m^2-x(1-x)u}\right]\mathrm{d}x \\ &= \lambda_{\text{eff}} + \frac{\hbar\lambda_{\text{eff}}^2}{32\pi^2}\int_0^1 \left[\ln\left(1-\frac{x(1-x)s}{m_{\text{phys}}^2}\right) + \ln\left(1-\frac{x(1-x)t}{m_{\text{phys}}^2}\right) + \ln\left(1-\frac{x(1-x)u}{m_{\text{phys}}^2}\right)\right]\mathrm{d}x\,. \end{aligned} \tag{5.24}$$

where we can replace $(m^2,\lambda) \mapsto (m_{\text{phys}}^2, \lambda_{\text{eff}})$ to this order in $\hbar$. (For the second term in (5.24), which has an explicit factor of $\hbar$ in front, at order $\hbar$ this is true no matter what

[45]Note that the two regions are certainly the same if the integral was over all of $\mathbb{R}^4$ as the measure $\mathrm{d}^4p = \mathrm{d}^4\ell$ is translationally invariant. Thus the difference between the two regions vanishes as $\Lambda_0 \to \infty$.

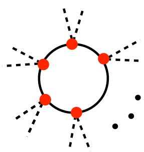
renormalization scheme we use, since any scheme dependence would have to come with a further factor of $\hbar$.)

The first thing to notice about (5.24) is that it is completely finite in the continuum limit. We only needed to tune the initial values of m^2 and λ – the relevant and (classically) marginal couplings – in order to obtain this result, even though this represents an infinite series of higher–derivative interactions in the quantum action. Put differently, although quantum effects do indeed turn on this infinite series of irrelevant higher derivative operators (together with others we'll consider momentarily), the coefficients of these operators are completely fixed in terms of $(m^2_{\rm phys}, \lambda_{\rm eff})$. We can understand this result in terms of our previous picture[46]



of RG flow as follows. In the continuum limit, we've focussed in on some renormalized trajectory, but by tuning our initial couplings (m^2, λ) we're only finitely far along this trajectory even in the continuum limit. There's no claim that the trajectory always stays 'perpendicular' or 'normal' to the critical surface; travelling along the trajectory may well turn on other, irrelevant operators. However, the presence of these operators is simply a consequence of being on the renormalized trajectory, so they do not come with independent coefficients. At least in principle, the relation between the coefficients of such irrelevant operators and $(m^2_{\rm phys}, \lambda_{\rm eff})$ can indeed be understood as an equation describing our renormalized trajectory in the space of all possible couplings.

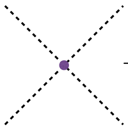
The quantum effective action also contains further irrelevant operators with more powers of the field (and again arbitrarily many derivatives), coming from 1PI loop diagrams of the general form



[46]It took me ages to construct this figure, so you can bet I'm going to use it as often as possible!

These loop diagrams, containing $e > 2$ propagators, behave in the UV like $\int^{\Lambda_0} \mathrm{d}^4 p / p^{2e}$ and so are finite in the continuum limit.[47] Again, the coefficients of these operators in $\Gamma[\phi]$ are completely fixed in terms of our location along the renormalized trajectory, parametrized by $(m^2_{\text{phys}}, \lambda_{\text{eff}})$.

The whole point of the 1PI quantum effective action $\Gamma[\phi]$ is that its vertices incorporate all possible quantum effects already. Thus, if we want to compute a scattering amplitude it is sufficient to compute with *tree–level* diagrams built from the vertices and propagator of the quantum effective action. Beginning with just a $\lambda\phi^4$ interaction in the classical action, $\Gamma[\phi]$ contains no interactions with fewer than four fields[48], so if in particular we wish to compute a four–particle scattering amplitude, the only tree diagram we can consider is



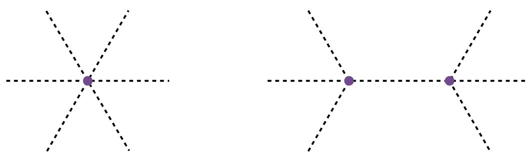
$$- \left.\frac{\delta^4 \Gamma(\phi)}{\delta \phi^4}\right|_{\phi=0}$$

where the vertex corresponds to all possible terms in $\Gamma[\phi]$ with exactly four fields, including all the irrelevant, higher derivative terms. We already know the answer: if the fields we're scattering correspond to momentum eigenstates with momenta (k_1, k_2, k_3, k_4), it's

$$\delta^4\left(\sum_{i=1}^{4} k_i\right) \mathcal{A}(k_i)$$

where $\mathcal{A}(k_i)$ is given in equation (5.24).

Of course, the work in computing all the quartic terms in the quantum effective action is precisely the same work we have to do to compute this one–loop amplitude, and indeed the calculation is usually presented without mentioning $\Gamma[\phi]$ just as a way of getting a result for this amplitude, correct to order $\hbar$. Exactly as we saw earlier in low dimensions, the virtue of knowing the effective action is that we can put the same result to work in computing higher point scattering amplitudes. For example, the six particle amplitude is given by the tree diagrams



using the vertex $\delta^6\Gamma[\phi]/\delta\phi^6|_{\phi=0}$ that incorporates the effect of the 1-loop diagram above, together with tree diagrams built from the vertex $\delta^4\Gamma[\phi]/\delta\phi^4|_{\phi=0}$ and propagator $\left(\delta^2\Gamma[\phi]/\delta\phi^2\right)^{-1}$.

[47]The expression $\int^{\Lambda_0} \mathrm{d}^4 p/p^{2e}$ is only a good approximation to our integral in the UV region where $p^2 \gg m^2$, k_i^2. Thus, while both the integral and the actual loop integral receive vanishing contributions from this UV region, the true loop integral has a finite contribution from the low energy region where the masses and external momenta are non–negligible.

[48]This result is obvious at 1-loop, as there are no 1PI 1–loop diagrams with 3 external legs. We'll prove it more generally in the next chapter.

It's also interesting to consider what happens if we start from a more general classical action, including couplings to higher powers of ϕ and its derivatives to begin with. According to our picture of RG flow, we expecting these higher couplings to be irrelevant and thus have negligible effect at any finite energy if we start with fixed numerical values for the dimensionless couplings g_i and take the continuum limit. Let's briefly see how this works, choosing an action

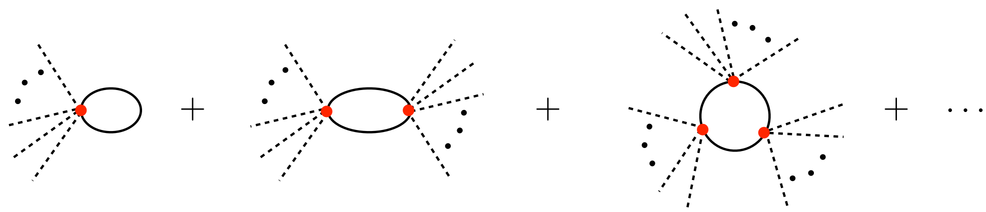
$$S_{\Lambda_0}[\phi] = \int \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi)\right] \mathrm{d}^4x \,, \tag{5.25}$$

where for simplicity we assume the potential

$$V(\phi) = \sum_{k\geq 2} g_{2k}\,\Lambda_0^{4-2k}\,\phi^{2k} \tag{5.26}$$

includes only even powers of ϕ and no derivatives of ϕ.

To compute the coupling to ϕ^{2m} in the quantum effective action, we need to consider all 1PI Feynman diagrams we can build using these vertices. To 1-loop accuracy, the only such graphs are of the generic type



where each graph has a total of $2m$ (amputated) external legs. These graphs are of course just a four dimensional version of the graphs we already considered in our effective theories in zero and one dimension, and indeed they can be understood as the contribution to the quantum effective action from the determinant $\det(-\nabla^2 + m^2 + V''(\phi))^{-1/2}$ obtained by integrating out quadratic fluctuations around a background value of ϕ. We saw that the order $\hbar$ contribution to the asymptotic series of the quantum effective action can always be understood in terms of such determinants.

A 1-loop graph with a total of e propagators contributes an amount

$$\propto \int \frac{\mathrm{d}^4p}{(2\pi)^4} \prod_{j=1}^{e} \frac{1}{(p+K_j)^2 + m^2}$$

to the ϕ^{2m} coupling in the momentum space quantum effective action[49], where $p + K_j$ is the total momentum carried by the j^{th} propagator. If the cut–off Λ_0 is much larger than the mass or typical scale of the incoming momenta, then this behaves as

$$\int^{\Lambda_0} \frac{p^3\,\mathrm{d}p}{p^{2v}} \sim \begin{cases} \Lambda_0^2 & \text{when } e = 1 \\ \ln\Lambda_0 & \text{when } e = 2 \end{cases} \tag{5.27}$$

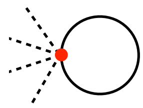
[49]As is by now familiar, diagrams with more than one vertex are non–local and so also generate and infinite series of derivative interactions. These terms are suppressed in the UV by powers of $|K_j|/\Lambda_0$ compared to the non–derivative terms.

in the UV region, and is UV finite if there are more than two propagators. On the other hand, for every vertex ϕ^{2k+2} in $V(\phi)$ that is involved in such a graph, as well as the dimensionless coupling g_{2k+2} we have a factor of $\Lambda_0^{4-2k-2} = \Lambda_0^{2(1-k)}$, which is a suppression at large Λ_0 if $k > 1$.

Since the above loop integrals with more than two propagators ($e > 2$) are all UV finite, every such diagram that involves one or more ϕ^6 or higher vertex is suppressed by a positive power of Λ_0 and so vanishes in the continuum limit. This also applies to the logarithmically divergent loop integrals with $e = 2$, and any quadratically divergent loop integral with $e = 1$ provided if its vertex is ϕ^8 or higher.[50]

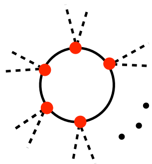
The only divergent loop diagrams are thus the mass and quartic vertex diagrams

we've already considered. There is also a finite contribution to the quartic coupling coming from the diagram



involving a single propagator and a ϕ^6 vertex. However, since we in any case have to tune the initial coupling of ϕ^4 using a counterterm, we can always incorporate the effects of this finite contribution in a choice of renormalisation scheme. These terms are thus nothing beyond what we've already seen.

Thus, even starting from our generic potential $V(\phi)$, the only contributions to ϕ^{2m} with $m > 2$ that do not vanish in the continuum limit come from 1-loop integrals that are built exclusively using the ϕ^4 vertex



as before. There must be exactly m propagators in this loop, so these contributions to the quantum effective action are finite. Thus, even allowing for arbitrary irrelevant operators in the original classical action, when we take the continuum limit *the strength of the interactions in the quantum effective action* $\Gamma[\phi]$ *is completely fixed by our location along the renormalized trajectory.* The irrelevant couplings in the classical action play no role.

[50] I caution you that the extension of this argument to higher loops is much more involved. Nonetheless, the WIlsonian picture of couplings evolving with scale assures us it *must* hold.

5.1.4 Dimensional regularization and the $\overline{\text{MS}}$ scheme

While the idea of integrating out momenta only up to a cut–off Λ_0 is very intuitive, in more complicated examples it becomes very cumbersome to perform the loop integrals over $|p|$ with a finite upper limit. More seriously, in a gauge theory, simply imposing a cut–off in momentum space is incompatible with gauge invariance

We saw that whether a coupling was relevant, marginal or irrelevant is largely determined just by dimensional analysis, and that the the dimension of an operator or coupling depends on the dimension of the space on which the QFT lives. This suggests that we can regularize our loop integrals by computing in some generic number of dimensions d. Since couplings may be relevant or irrelevant as d changes, they *can't* diverge if we keep d arbitrary. Having computed our loop integrals in a generic number of dimensions, we include counterterms to tune the initial couplings in our d–dimensional theory so as to obtain a finite limit as we approach the physically relevant dimension (usually, $d = 4$).

Unlike a lattice regularization, or choosing to integrate only finitely many momentum modes in the path integral, dimensional regularization is only a *perturbative* regularization scheme: whilst it does allow us to regulate individual loop integrals over the full range $|p| \in [0, \infty)$ (as we shall see), it does *not* provide any definition of a finite–dimensional path integral measure. Furthermore, in order to 'approach' the physical dimension, we need to analytically continue the results of our generic d–dimensional theory through non–integer values of d. I stress that this is purely a convenient device for regularizing loop integrals — there is no suggestion that Nature 'really' lives in non–integer dimensions. These conceptual shortcomings are compensated by its practical convenience: dimensional regularization is very easy to implement, whilst integrating complicated integrals over a finite range $|p| \in [0, \Lambda_0)$ is often prohibitively difficult.

To see how this works, let's repeat our previous calculation of the 1–loop corrections to $\Pi(k^2)$ in $\lambda\phi^4$ theory, now working in some generic space–time dimension d. In d dimensions, the quartic coupling λ has non–zero mass dimension $[\lambda] = 4 - d$, so we write

$$\lambda = \mu^{4-d} g(\mu) \tag{5.28}$$

where the new coupling $g(\mu) = \lambda/\mu^{4-d}$ is dimensionless. We see that μ is an *arbitrary* mass scale, such as a kilogram, and $g(\mu)$ is the value of our quartic coupling in units of this scale. I stress that μ is not a cut–off; it's just an arbitrary scale introduced so to allow us to use dimensionless couplings. We will usually choose μ to be of the typical scale of the experiment we're interested in performing: if you're working with subatomic particles, it's not likely that you'll want to express your answers in terms of kilotonnes, or inverse lightyears.

In terms of this scale, we obtain the 1-loop correction to the mass

$$-\frac{1}{2}g(\mu)\,\mu^{4-d}\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{1}{p^2+m^2} = -g(\mu)\,\mu^{4-d}\frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d}\int_0^\infty\frac{p^{d-1}\,\mathrm{d}p}{p^2+m^2} \tag{5.29}$$

in dimensional regularization. To compute the area of a unit sphere in d dimensions, we use the following trick. For any $d \in \mathbb{N}$, comparing Cartesian and polar coordinates gives us

$$(\sqrt{\pi})^d = \int_{\mathbb{R}^d} \prod_{i=1}^{d} \mathrm{e}^{-x_i^2}\, \mathrm{d}x_i = \mathrm{Vol}(S^{d-1}) \int_0^\infty \mathrm{e}^{-r^2}\, r^{d-1}\, \mathrm{d}r = \frac{1}{2}\mathrm{Vol}(S^{d-1})\, \Gamma(d/2)\,, \tag{5.30}$$

where we recognize the radial integral as (half) a gamma function. Thus, when $d \in \mathbb{N}$ we have

$$\mathrm{Vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(d/2\right)}\,. \tag{5.31}$$

We now *define* this to be $\mathrm{Vol}(S^{d-1})$ for any $d \in \mathbb{C}$ by analytic continuation.

The remaining integral in (5.29) is

$$\begin{aligned} \mu^{4-d} \int_0^\infty \frac{p^{d-1}\, \mathrm{d}p}{p^2+m^2} &= \frac{1}{2}\mu^{4-d} \int_0^\infty \frac{(p^2)^{d/2-1}\, \mathrm{d}(p^2)}{p^2+m^2} \\ &= \frac{m^2}{2}\left(\frac{\mu}{m}\right)^{4-d} \int_0^1 (1-u)^{\frac{d}{2}-1}\, u^{-\frac{d}{2}}\, \mathrm{d}u \\ &= \frac{m^2}{2}\left(\frac{\mu}{m}\right)^{4-d} \frac{\Gamma(\frac{d}{2})\,\Gamma(1-\frac{d}{2})}{\Gamma(1)}\,, \end{aligned} \tag{5.32}$$

where $u = m^2/(p^2+m^2)$, and in the last line we recognized the integral form of the Euler beta–function

$$B(s,t) = \int_0^1 u^{s-1}(1-u)^{t-1}\, \mathrm{d}u = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}\,, \tag{5.33}$$

again writing this in terms of gamma functions. We'll see that the presence of such gamma functions is a ubiquitous feature of dimensional regularization.

Combining the pieces, the 1-loop contribution to $\Pi(k^2)$ is

$$\Pi_{1\,\mathrm{loop}}(k^2) = -\frac{g(\mu)\, m^2}{2(2\pi)^d}\, \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \left(\frac{\mu}{m}\right)^{4-d} \frac{\Gamma(\frac{d}{2})\,\Gamma(1-\frac{d}{2})}{\Gamma(1)} = -\frac{g(\mu)\, m^2}{2(4\pi)^{d/2}} \left(\frac{\mu}{m}\right)^{4-d} \Gamma\left(1-\frac{d}{2}\right) \tag{5.34}$$

in dimensional regularization. $\Gamma(z)$ has a pole whenever z is a non–positive integer, so our loop integral diverges in $d = 4$ as expected. More precisely, the Γ–function the asymptotic expansion

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \qquad \text{as } \ \epsilon \to 0^+\,, \tag{5.35}$$

where $\gamma \approx 0.577$ is known as the Euler–Mascheroni constant. Thus, if we set $d = 4-\epsilon$, our 1–loop diagram is asymptotic to

$$\frac{g(\mu)\, m^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + 1\right] + \mathcal{O}(\epsilon) \tag{5.36}$$

as $\epsilon \to 0^+$. The divergence we saw as $\Lambda_0 \to \infty$ in the cut–off regularization has become a *pole* in $d = 4$ in dimensional regularization. (The logarithm of μ^2/m^2 is perfectly finite, because μ is not a cut–off so we have no wish to send $\mu \to \infty$.)

We now use counterterms in the d–dimensional theory to tune our couplings so as to obtain a finite limit as $d \to 4$. This is the analogue of choosing the counterterms to give a finite limit as $\Lambda_0 \to \infty$ in the cut–off regularization above. Once again, there's a lot of arbitrariness in how we accomplish this. It's of course perfectly possible to use the same, on–shell renormalization scheme as we did when regularizing our integrals with a cut–off. However, in dimensional regularization other choices of scheme turn out to be more convenient.

The simplest choice of renormalization scheme available in conjunction with dimensional regularization is called **minimal subtraction** (MS). Here, one simply chooses the counterterm

$$\delta m^2 = -\frac{g(\mu)\, m^2}{16\pi^2 \epsilon} \qquad \text{(MS scheme)} \tag{5.37}$$

so as to remove the purely divergent part of the loop diagram (5.36). However, it's often even more convenient to use **modified minimal subtraction** ($\overline{\text{MS}}$), where the counterterm is chosen to also remove the Euler–Mascheroni constant and the $\log 4\pi$ term,

$$\delta m^2 = -\frac{g(\mu)\, m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) \qquad (\overline{\text{MS}}\text{ scheme}). \tag{5.38}$$

In this scheme, we'd find

$$\Pi_{1\,\text{loop}}(k^2) = \frac{g(\mu)\, m^2}{32\pi^2}\left[\ln\frac{\mu^2}{m^2} - 1\right], \tag{5.39}$$

which is perfectly finite in $d = 4$. Note also that in $d = 4$, the dimensionless coupling g coincides with the original coupling λ.

Let's also take a look at the renormalization of the quartic coupling in dimensional regularization. Like λ itself, the loop integrals (5.13) have mass dimension

$$2[\lambda] + [\mathrm{d}^d p/p^4] = 2(4-d) + d - 4 = 4 - d$$

in d-dimensions. In dimensional regularization it'll be more convenient to study instead the correction to the dimensionless coupling $g(\mu) = \lambda\mu^{d-4}$, which is given by the dimensionless integrals

$$\frac{g^2\mu^{4-d}}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^2+m^2}\frac{1}{(p+k_1+k_2)^2+m^2} + \text{ other channels}, \tag{5.40}$$

corresponding to the 1-loop Feynman graphs we saw earlier. As before, setting $k_i = 0$ to extract the correction to the pure ϕ^4 coupling in the quantum effective action, we have

$$\begin{aligned}\mu^{4-d}\int_0^\infty \frac{p^{d-1}\,\mathrm{d}p}{(p^2+m^2)^2} &= \frac{\mu^{4-d}}{2}\int_0^\infty \frac{(p^2)^{(d-2)/2}}{(p^2+m^2)^2}\,\mathrm{d}(p^2)\\ &= \frac{1}{2}\left(\frac{\mu}{m}\right)^{4-d}\int_0^1 u^{1-\frac{d}{2}}(1-u)^{\frac{d}{2}-1}\,\mathrm{d}u\\ &= \frac{1}{2}\left(\frac{\mu}{m}\right)^{4-d}\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(2)}.\end{aligned} \tag{5.41}$$

Thus, using our result (5.31) for $\mathrm{Vol}(S^{d-1})$, we have a total 1-loop contribution

$$\begin{aligned}3g^2\frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d}\,\frac{1}{2}\left(\frac{\mu}{m}\right)^{4-d}\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(2)} &= \frac{3g^2}{2(4\pi)^{d/2}}\left(\frac{\mu}{m}\right)^{4-d}\Gamma\left(2-\frac{d}{2}\right)\\ &\sim \frac{3g^2}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\log\frac{4\pi\mu^2}{m^2}\right)+\mathcal{O}(\epsilon)\,,\end{aligned} \tag{5.42}$$

as $\epsilon = 4-d \to 0^+$. Just like the mass correction (5.36), we've found a pole when $d=4$, reflecting the divergence of the original $d=4$ loop integral.

Yet again, to obtain a finite limit we tune our d–dimensional coupling using the counterterm $\delta\lambda$. In the $\overline{\mathrm{MS}}$ scheme we choose this to be

$$\delta\lambda = \frac{3g^2}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\log 4\pi\right) \tag{5.43}$$

removing both the pole as $\epsilon\to 0$ and the $\gamma - \log 4\pi$ terms. Altogether, restoring $\hbar$, in the $\overline{\mathrm{MS}}$ scheme we've found that in the quantum effective action the scalar field has a quartic coupling with strength

$$g_{\mathrm{eff}}(\mu) = g(\mu) - \frac{3\hbar g^2(\mu)}{32\pi^2}\log\frac{\mu^2}{m^2} + O(\hbar^2)\,. \tag{5.44}$$

Here, the leading term is the classical coupling (which may be thought of as a 1PI tree diagram with four amputated external legs), while the order $\hbar$ term is the result of our 1–loop calculation combined with the contribution of the $\overline{\mathrm{MS}}$ counterterm. Once again, this result is perfectly finite in $d=4$.

In exactly four dimensions, the ϕ^4 coupling is naturally dimensionless, *i.e.* it's just a number. The coupling we actually measure is g_{eff} in the quantum effective action, since this is the one that incorporates all the quantum effects of the original theory, which are certainly present in our experiment. The numerical value of g_{eff} cannot depend on the choice of scale μ, which was nothing more than a choice of units in which to write the d dimensional coupling $g(\mu)$ in the classical action. For this to be compatible with the result (5.44), we must have

$$0 = \mu\frac{\partial}{\partial\mu}g_{\mathrm{eff}}(\mu) = \mu\frac{\partial}{\partial\mu}\left[g(\mu) - \frac{3\hbar g^2(\mu)}{32\pi^2}\log\frac{\mu^2}{m^2} + O(\hbar)\right]\,, \tag{5.45}$$

at least to 1-loop accuracy. Consequently, the coupling $g(\mu)$ in the original action must vary with μ, having β-function

$$\beta(g) = \mu\frac{\partial g}{\partial\mu} = \frac{3\hbar g^2}{16\pi^2} + O(\hbar^2)\,. \tag{5.46}$$

Notice that this β–function is proportional to $\hbar$: at the classical level we had

$$g(\mu) \overset{\text{classical}}{=} \lambda\mu^{d-4}\,,$$

so classically $g(\mu)$ would have been independent of μ in $d=4$, which just says that the quartic coupling was classically marginal in four dimensions. The fact that the β-function is positive even when $d = 4$ shows that the classical assessment of g as marginal was incorrect and in fact the quartic coupling is actually marginally *irrelevant* in $d = 4$, at least in a neighbourhood of the Gaussian critical point.

Solving (5.46) for the coupling gives

$$\frac{1}{g(\mu')} = \frac{1}{g(\mu)} + \frac{3\hbar}{16\pi^2} \ln \frac{\mu}{\mu'} + O(\hbar^2) \,, \tag{5.47}$$

relating the coupling $g(\mu)$ at scale μ to the coupling $g(\mu')$ at a different scale μ'. How should we understand this running coupling? For example, since any real calculation using the original action will inevitably have to be carried out using perturbation theory, should we do perturbation theory in $g(\mu)$ or in

$$g(\mu') = \frac{g(\mu)}{1 + \frac{3\hbar g(\mu)}{16\pi^2} \ln \frac{\mu}{\mu'}} \approx g(\mu) - \frac{3\hbar g^2(\mu)}{16\pi^2} \ln \frac{\mu}{\mu'} + \cdots \qquad ?$$

If we were able to calculate the path integral exactly then, by definition, the result would be independent of μ. However, in real life we can only compute finitely many Feynman diagrams and then perturbation theory in $g(\mu)$ will certainly be different from perturbation theory in $g(\mu')$. Can we choose the coupling $g(\mu)$ to be arbitrarily small in the hope of achieving excellent perturbative results?

Remarkably, the answer is *no*! In fact, we have no freedom at all to choose the coupling because it is totally fixed in terms of an energy scale Λ_{ϕ^4} that is inherently present in the theory as the scale at which the coupling diverges. That is $1/g(\mu') = 0$ at $\mu' = \Lambda_{\phi^4}$. Using this in (5.47), at any scale μ (such as the scale of our experiments) the coupling is determined in terms of Λ_{ϕ^4} by

$$g(\mu) = \frac{16\pi^2}{3\hbar} \frac{1}{\ln(\Lambda_{\phi^4}/\mu)} \,, \tag{5.48}$$

(at least as long as out one–loop result for the running may be considered accurate). Note that the scale Λ_{ϕ^4} can be given invariant meaning by measuring it in terms of the physical mass m_{phys} of the particle. We simply have to hope that $g(\mu) \ll 1$ at the scales we are interested in so that perturbation theory can stand a chance of being reliable, and (5.48) shows that this will be the case only for experimental energies far below Λ_{ϕ^4}. The fact that we've traded a dimensionless coupling g for an energy scale Λ_{ϕ^4} is an example of dimensional transmutation at work.

However, something is not quite right here. To *define* a continuum QFT with finite values of relevant and marginal couplings at our experimental scale μ we tuned our initial couplings to be closer and closer to the critical surface, so as to keep the scale at which the RG trajectory passed closest to the (Gaussian) critical point finite. Our treatment of ϕ^4 theory – including a counterterm $\delta\lambda$ – was based on the classical picture that this quartic coupling was marginal. We now realize that in fact it's (marginally) irrelevant. Like the irrelevant ϕ^6 and higher couplings we considered earlier, from a Wilsonian point of view, it is in principle possible to generate an irrelevant quartic coupling in the quantum effective action but it's value should be fixed in terms of the genuinely relevant couplings which give coordinates along the RG trajectory. However, in this scalar theory, the only such relevant 'couplings' are the mass and kinetic terms. We can solve the theory of a massive free scalar exactly, and we know that no non–trivial quartic coupling is generated. Thus the only value of the quartic coupling that really exists in the continuum is $\lambda = 0$ and we say the theory is **trivial**.

Our perturbative treatment is not powerful enough to say what really occurs as we head into the deep UV, where the coupling becomes large — perturbation theory certainly breaks down in this region. More sophisticated treatments indeed show that $\lambda\phi^4$ does not exist as a continuum QFT.

5.2 One–loop renormalization of QED

We next turn to Quantum Electrodynamics, the theory of a (massive) charged Dirac spinor coupled to the electromagnetic field. The classical action for this theory is

$$S_{\rm QED}[A,\psi] = \int \mathrm{d}^d x \left[\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(\not{\nabla} + m)\psi\right] \tag{5.49}$$

where the covariant derivative in the fermion kinetic term is $\not{\nabla}\psi = \gamma^\mu(\partial_\mu - \mathrm{i}A_\mu)\psi$, and the Dirac matrices γ_μ obey $\{\gamma_\mu, \gamma_\nu\} = +2\delta_{\mu\nu}$. I'm working in conventions where these γ matrices are each *anti*-Hermitian $(\gamma^\mu)^\dagger = -\gamma^\mu$, and then the generators $S^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$ of the compact rotation group SO(d) are all Hermitian (unlike the boost generators for the non–compact group SO($d-1,1$).) In Euclidean signature, it's also natural to define $\bar{\psi} = (\psi)^\dagger$ without the factor of γ^0, which plays no distinguished role. You should check that the action (5.49) is then real. In order for the covariant derivative $\nabla_\mu = \partial_\mu - \mathrm{i}A_\mu$ to make sense, the gauge field must have mass dimension 1 even in d dimensions. Thus, the electric charge e has mass dimension $(4-d)/2$, so at the classical level we expect that e is relevant when $d < 4$, irrelevant in $d > 4$ and marginal in $d = 4$.

To do perturbation theory, we'd like the kinetic terms to be canonically normalized, so we introduce a rescaled photon field $A_\mu^{\rm new} = e^{-1}A_\mu^{\rm old}$. In terms of this rescaled field the action becomes

$$S_{\rm QED}[A^{\rm new},\psi] = \int \mathrm{d}^d x \left[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(\not{\partial} + m)\psi - \mathrm{i}e\bar{\psi}\not{A}\psi\right] \tag{5.50}$$

with e appearing only in the electron–photon vertex, as befits a coupling. Notice that the rescaled photon field has mass dimension

$$[A^{\rm new}] = [A^{\rm old}] - [e] = \frac{d-2}{2}\,, \tag{5.51}$$

just like a scalar field in d dimensions. To obtain the classical photon propagator, we write the Maxwell term in this canonically normalized action as

$$\begin{aligned}\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}\,\mathrm{d}^d x &= \frac{1}{4}\int -\mathrm{i}\left(k^\mu\tilde{A}^\nu(-k) - k^\nu\tilde{A}^\mu(-k)\right)\mathrm{i}\left(k_\mu\tilde{A}_\nu(k) - k_\nu\tilde{A}_\mu(k)\right)\mathrm{d}^d k \\ &= \frac{1}{2}\int k^2\left(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\tilde{A}_\mu(-k)\tilde{A}_\nu(k)\,\mathrm{d}^d k\,,\end{aligned} \tag{5.52}$$

in terms of an integral over momentum space. Therefore, as you learned last term, in Lorenz (or Landau) gauge the tree–level photon propagator is

$$\Delta^0_{\mu\nu}(k) = \frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{5.53}$$

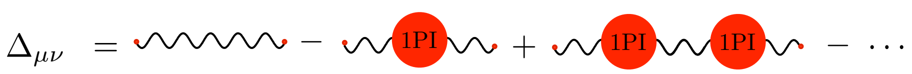
in momentum space. The Lorenz gauge condition $\partial^\mu A_\mu(x) = 0$ is reflected in the fact that the propagator obeys $k^\mu \Delta^0_{\mu\nu}(k) = 0$, a condition you'll also meet when studying Ward identities in the problem sheets. This condition ensures that only transverse polarizations propagate.

5.2.1 Vacuum polarization: loop calculation

In quantum theory, the exact Lorenz gauge photon propagator

$$\Delta_{\mu\nu}(k) := \int \mathrm{d}^d x \; \mathrm{e}^{\mathrm{i}k\cdot x}\langle A_\mu(x) A_\nu(0)\rangle \tag{5.54}$$

differs from the classical expression (5.53) because of the photon's interaction with the charged electron field; the exact propagator accounts for the effect quantum fluctuations of the electron field have on the photon's propagation. In exactly the same way as we did for scalar field theory earlier (or even earlier in $d = 0$ and $d = 1$), to incorporate these effects we write



$$= \Delta^0_{\mu\nu}(k) + \Delta^0_{\mu\rho}(k)\Pi^\rho{}_\sigma(k)\Delta^{0\sigma}{}_\nu(k) + \Delta^0_{\mu\rho}(k)\Pi^\rho{}_\sigma(k)\Delta^{0\sigma}{}_\kappa(k)\Pi^\kappa{}_\lambda(k)\Delta^{0\lambda}{}_\nu(k) + \cdots$$

where the **photon self–energy** $\Pi^\rho{}_\sigma(k)$ is the sum of all 1PI graphs with two external photon lines. This is again similar to what we saw for the scalar, with the difference that because the photon field A_μ carries a target space index, the photon self–energy is a tensor.

Below, we'll show that $\Pi^\rho{}_\sigma(k)$ takes the form

$$\Pi^\rho{}_\sigma(k) = k^2\left(\delta^\rho{}_\sigma - \frac{k^\rho k_\sigma}{k^2}\right)\pi(k^2) \tag{5.55}$$

in terms of a dimensionless scalar function $\pi(k^2)$. Accepting this for the moment, the factor

$$P^\rho{}_\sigma = \delta^\rho{}_\sigma - \frac{k^\rho k_\sigma}{k^2}$$

is the same factor as we saw in the tree–level photon propagator and projects onto polarization states transverse to k^σ. As for any projection operator, this factor is idempotent:

$$P^\rho{}_\sigma P^\sigma{}_\kappa = P^\rho{}_\kappa\,. \tag{5.56}$$

The fact that the self–energy diagrams and tree–level propagator both involve the same projection operator allows us to simplify our expression for the exact photon propagator to

$$\begin{aligned}\Delta_{\mu\nu}(k) &= \Delta^0_{\mu\nu} + \Delta^0_{\mu\rho}\,\Pi^\rho{}_\sigma\,\Delta^{0\sigma}{}_\nu + \Delta^0_{\mu\rho}\,\Pi^\rho{}_\sigma\,\Delta^{0\sigma}{}_\kappa\,\Pi^\kappa{}_\lambda\,\Delta^{0\lambda}{}_\nu + \cdots \\ &= \Delta^0_{\mu\nu}\left(1+\pi(k^2)+\pi^2(k^2)+\pi^3(k^2)\cdots\right) \\ &= \frac{\Delta^0_{\mu\nu}(k)}{1-\pi(k^2)}\,,\end{aligned} \tag{5.57}$$

again summing the geometric series.

Just as the classical propagator $\Delta^0_{\mu\nu}(k)$ was the inverse of the Maxwell kinetic term

$$\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}\,\mathrm{d}^dx = \frac{1}{2}\int k^2\left(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\tilde{A}_\mu(-k)\tilde{A}_\nu(k)\,\mathrm{d}^dk \tag{5.58}$$

in momentum space, so too the exact photon propagator $\Delta_{\mu\nu}(k)$ can be thought of as resulting from the term

$$S^{(2)}_{\text{eff}}[\tilde{A}] = \frac{1}{2}\int [1-\pi(k^2)]\,k^2\left(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\tilde{A}_\mu(-k)\tilde{A}_\nu(k)\;\mathrm{d}^dk \tag{5.59}$$

in the momentum space quantum effective action that is quadratic in the photon field. We interpret the factor of $[1-\pi(k^2)]$ as representing the effects of quantum fluctuations in the electron and photon fields on the photon's propagation. In particular, if we expand $\pi(k^2)$ as a power series in k^2, the part $\pi(0)$ that is independent of k^2 just provides an overall factor multiplying the classical Maxwell action. In other words, in position space this term is

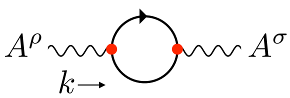
$$S^{(2)}_{\text{eff}}[A] = \frac{1-\pi(0)}{4}\int F^{\mu\nu}(x)F_{\mu\nu}(x)\,\mathrm{d}^dx \tag{5.60}$$

and so contributes to wavefunction renormalization for the photon. As always, the remaining, k^2-dependent terms in $\pi(k^2)$ correspond to an infinite series of higher derivative interactions of the schematic form $\partial^n F^{\mu\nu}\,\partial^n F_{\mu\nu}$. This infinite series again means that the effective theory is non-local, but the higher derivative couplings are irrelevant in $d=4$ so at low energies non–locality is strongly suppressed at length scales larger than the inverse electron mass.

After these general considerations, let's now compute the leading contribution to $\Pi^{\rho\sigma}(k)$. We'll make use of dimensional regularization, so it will be convenient to work with a dimensionless coupling $g(\mu)$, introduced into the classical action via

$$e^2 = \mu^{4-d}\,g^2(\mu) \tag{5.61}$$

in terms of an arbitrary mass scale μ. Thus the electron–photon vertex is written as $-\mathrm{i}\mu^{2-d/2}g(\mu)\int\bar{\psi}\not{A}\psi\;\mathrm{d}^dx$. At sub–leading order in $\hbar$, the unique loop graph we need to consider is



where a virtual e^+e^- pair is formed and then reabsorbed. (There is no symmetry factor here because, unlike the propagator for a real scalar field, the electron–positron propagator is *oriented.*) As always, in computing the self–energy contribution the external photon propagators are amputated. The Feynman rules following from (5.50) give

$$\begin{aligned}\Pi^{\rho\sigma}_{1\,\text{loop}}(k) &= -\mu^{4-d}(\mathrm{i}g)^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{tr}\left(\frac{1}{\mathrm{i}\not{p}+m}\,\gamma^\rho\,\frac{1}{\mathrm{i}(\not{p}-\not{k})+m}\,\gamma^\sigma\right)\\ &= \mu^{4-d}g^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{\mathrm{tr}\left((-\mathrm{i}\not{p}+m)\gamma^\rho(-\mathrm{i}(\not{p}-\not{k})+m)\gamma^\sigma\right)}{(p^2+m^2)((p-k)^2+m^2)}\,.\end{aligned} \tag{5.62}$$

Note the overall sign: we obtain a factor of $(-)^2(-\mathrm{i}g)^2$ expanding $\mathrm{e}^{-S_{\text{QED}}/\hbar}$ to second order in the electron–photon vertex, and an additional factor of -1 from the fermionic loop. To understand this additional sign, suppose the electron propagator is

$$S^\beta{}_\alpha(x,y) = \langle \psi^\beta(x)\bar{\psi}_\alpha(y)\rangle \tag{5.63}$$

in position space, where α, β are Dirac spinor indices. Expanding $\mathrm{e}^{-S_{\text{QED}}[A,\psi]/\hbar}$ to second order in the electron–photon vertex, we meet a term

$$\begin{aligned}&(-\mathrm{i}g)^2\mu^{4-d}\left\langle \int \bar{\psi}\not{A}\psi(x)\,\mathrm{d}^d x \int \bar{\psi}\not{A}\psi(y)\,\mathrm{d}^d y\right\rangle\\ &\qquad = (-\mathrm{i}g)^2\mu^{4-d}\int d^d x\,\mathrm{d}^d y \not{A}^\alpha{}_\beta(x)\,\not{A}^\gamma{}_\delta(y)\,\langle \bar{\psi}_\alpha(x)\psi^\beta(x)\,\bar{\psi}_\gamma(y)\,\psi^\delta(y)\rangle\\ &\qquad \supset (-\mathrm{i}g)^2\mu^{4-d}\int d^d x\,\mathrm{d}^d y \not{A}^\alpha{}_\beta(x)\,\not{A}^\gamma{}_\delta(y)\,\langle \psi^\beta(x)\bar{\psi}_\gamma(y)\rangle\,\langle \bar{\psi}_\alpha(x)\psi^\delta(y)\rangle\,,\end{aligned} \tag{5.64}$$

where in the last line we've paired up the electron fields to give two propagators, being careful to move an even number of fermions through one another. The term $\langle\psi^\beta(x)\bar{\psi}_\gamma(y)\rangle$ is precisely the propagator $S^\beta{}_\gamma(x,y)$ representing the electron travelling from one vertex to the next. However, the term $\langle\bar{\psi}_\alpha(x)\psi^\delta(y)\rangle = -\langle\psi^\delta(y)\bar{\psi}_\alpha(x)\rangle$ is *minus* the propagator $S^\delta{}_\alpha(y,x)$ representing propagation back to the first vertex, because the two fermions naturally appeared in the opposite order.

More generally, in pairing a string of $\bar{\psi}\psi$'s from the action into propagators, we note

$$\begin{aligned}&\langle \bar{\psi}_{\alpha_1}\psi^{\beta_1}(x_1)\,\bar{\psi}_{\alpha_2}\psi^{\beta_2}(x_2)\,\cdots\,\bar{\psi}_{\alpha_n}\psi^{\beta_n}(x_n)\rangle\\ &\qquad\supset \langle\psi^{\beta_1}(x_1)\bar{\psi}_{\alpha_2}(x_2)\rangle\,\langle\psi^{\beta_2}(x_2)\bar{\psi}_{\alpha_3}(x_3)\rangle\,\cdots\,\langle\psi^{\beta_{n-1}}(x_{n-1})\bar{\psi}_{\alpha_n}\rangle\,\langle\bar{\psi}_{\alpha_1}(x_1)\psi^{\beta_n}(x_n)\rangle\\ &\qquad = -S^{\beta_1}_{\alpha_2}(x_1,x_2)\,S^{\beta_2}_{\alpha_3}(x_2,x_3)\,\cdots\,S^{\beta_{n-1}}_{\alpha_n}(x_{n-1},x_n)\,S^{\beta_n}_{\alpha_1}(x_n,x_1)\end{aligned} \tag{5.65}$$

with all but one propagator naturally in the correct order. You should check as an exercise that the same result holds no matter which order we choose to join up the fermions. Thus, we always obtain an additional minus sign, beyond those in the individual propagators and vertices, for every fermion loop. At one loop, this minus sign can equivalently be understood as coming from the fact that the path integral over the Grassmann valued electron field gives $\det(\not{\nabla}+m)$ in the *numerator* of the remaining path integral, whereas had the electron been a (complex) boson we would have obtained this factor in the denominator. Exponentiating this determinant into the effective action $\mathrm{e}^{-\Gamma[A]/\hbar}$ also shows we get an extra minus sign for the fermion loop. (We'll consider this determinant further below.)

Having understood its constituents, we're now ready to evaluate $\Pi^{\rho\sigma}_{1\,\text{loop}}(k)$ in (5.62). The loop integral is somewhat complicated, and we'll need to work hard to evaluate it.[51] We begin by recalling the Feynman trick (5.17) that allows us to can combine the two propagators in (5.62) as

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{[(p^2+m^2)(1-x)+((p-k)^2+m^2)x]^2} &= \int_0^1 \frac{\mathrm{d}x}{[p^2+m^2-2xp\cdot k+k^2x]^2}\\ &= \int_0^1 \frac{\mathrm{d}x}{[(p-kx)^2+m^2+k^2x(1-x)]^2}\,.\end{aligned} \tag{5.66}$$

We change variables $p \to p' = p + kx$, whereupon (5.62) becomes (dropping the prime)

$$\Pi^{\rho\sigma}_{1\,\text{loop}}(k) = \mu^{4-d}g^2\int\frac{\mathrm{d}^dp}{(2\pi)^d}\int_0^1\mathrm{d}x\;\frac{\mathrm{tr}\left((-\mathrm{i}(\not{p}+\not{k}x)+m)\gamma^\rho(-\mathrm{i}(\not{p}-\not{k}(1-x))+m)\gamma^\sigma\right)}{[p^2+\Delta]^2}\,, \tag{5.67}$$

where we've introduced $\Delta = m^2 + k^2x(1-x)$ as shorthand.

The next step is to perform the traces over the Dirac matrices. We'll do this treating the Dirac spinors as having 4 components[52] as appropriate for our final goal of $d = 4$. Thus

$$\begin{aligned}\mathrm{tr}(\gamma^\rho\gamma^\sigma) &= 4\,\delta^{\rho\sigma}\\ \mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) &= 4(\delta^{\mu\rho}\delta^{\nu\sigma}-\delta^{\mu\nu}\delta^{\rho\sigma}+\delta^{\mu\sigma}\delta^{\nu\rho})\end{aligned} \tag{5.68}$$

in Euclidean signature, so that

$$\begin{aligned}&\mathrm{tr}\left((-\mathrm{i}(\not{p}+\not{k}x)+m)\gamma^\rho(-\mathrm{i}(\not{p}-\not{k}(1-x))+m)\gamma^\sigma\right)\\ &\quad= 4\left[-(p+kx)^\rho(p-k(1-x))^\sigma+(p+kx)\cdot(p-k(1-x))\delta^{\rho\sigma}\right.\\ &\qquad\left.-(p+kx)^\sigma(p-k(1-x))^\rho+m^2\delta^{\rho\sigma}\right]\,.\end{aligned} \tag{5.69}$$

Using this in the above loop integral, we obtain

$$\begin{aligned}\Pi^{\rho\sigma}_{1\,\text{loop}}(k) &= 4\mu^{4-d}g^2\int\frac{\mathrm{d}^dp}{(2\pi)^d}\int_0^1\mathrm{d}x\;\frac{1}{[p^2+\Delta]^2}\\ &\quad\times\left[-(p+kx)^\rho(p-k(1-x))^\sigma+(p+kx)\cdot(p-k(1-x))\delta^{\rho\sigma}\right.\\ &\qquad\left.-(p+kx)^\sigma(p-k(1-x))^\rho+m^2\delta^{\rho\sigma}\right]\,,\end{aligned} \tag{5.70}$$

which would be quadratically divergent in $d = 4$.

We're now ready to perform the loop integral. Observing that whenever $d \in \mathbb{N}$, any term involving an odd number of powers of momentum would vanish, we drop these terms. For the same reason, we replace

$$p^\mu p^\nu \to \frac{1}{d}\delta^{\mu\nu}p^2 \qquad\text{and}\qquad p^\mu p^\nu p^\rho p^\sigma \to \frac{(p^2)^2}{d(d+2)}\left[\delta^{\mu\nu}\,\delta^{\rho\sigma}+\delta^{\mu\rho}\,\delta^{\nu\sigma}+\delta^{\mu\sigma}\,\delta^{\nu\rho}\right]$$

[51] Don't worry: asking you to reproduce this laborious calculation is not the sort of thing I'm keen on for exams.

[52] In certain supersymmetric theories, it is often convenient to work instead with d–dimensional spinors, which is known as **dimensional reduction**, rather than dimensional regularization. Here, we'll stick with the simpler idea of naively working with a spinor representation of SO(4), rather than SO(d).

where the tensor structure is fixed by SO(d) invariance and permutation symmetry, and the numerical factors are determined by contracting both sides with d-dimensional metrics. After these replacements the integrand depends only on p^2, so the angular integrals may be performed trivially to obtain

$$\frac{\mathrm{d}^d p}{(2\pi)^d} = \mathrm{Vol}(S^{d-1})\,\frac{p^{d-1}\,\mathrm{d}p}{(2\pi)^d} = \frac{1}{(4\pi)^{d/2}\,\Gamma(d/2)}\,(p^2)^{\frac{d}{2}-1}\,\mathrm{d}(p^2) \tag{5.71}$$

as in section 5.1.4. Thus (5.70) becomes

$$\begin{aligned}\Pi^{\rho\sigma}_{1\,\mathrm{loop}}(k) &= 4\mu^{4-d}\,\frac{g^2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})} \\ &\times \int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}(p^2)\,(p^2)^{\frac{d}{2}-1}\left[\frac{p^2(1-\frac{2}{d})\,\delta^{\rho\sigma} + (2k^\rho k^\sigma - k^2\delta^{\rho\sigma})x(1-x) + m^2\delta^{\rho\sigma}}{(p^2+\Delta)^2}\right].\end{aligned} \tag{5.72}$$

To go further, we use the integrals

$$\begin{aligned}\int_0^\infty \mathrm{d}(p^2)\frac{(p^2)^{\frac{d}{2}-1}}{(p^2+\Delta)^2} &= \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\frac{\Gamma(2-\frac{d}{2})\,\Gamma(\frac{d}{2})}{\Gamma(2)} \\ \int_0^\infty \mathrm{d}(p)^2\,\frac{(p^2)^{\frac{d}{2}}}{(p^2+\Delta)^2} &= \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}}\frac{\Gamma(1+\frac{d}{2})\,\Gamma(1-\frac{d}{2})}{\Gamma(2)}\end{aligned} \tag{5.73}$$

that can be evaluated using the substitution $u = \Delta/(p^2+\Delta)$ and the definition of the Euler B–function.

Putting everything together, one finds that the 1-loop contribution to vacuum polarization is given by

$$\begin{aligned}\Pi^{\rho\sigma}_{1\,\mathrm{loop}}(k) &= \frac{4g^2\mu^{4-d}}{(4\pi)^{d/2}}\,\Gamma\!\left(2-\frac{d}{2}\right) \\ &\times \int_0^1 \mathrm{d}x\left[\frac{\delta^{\rho\sigma}(m^2 - x(1-x)k^2) - \delta^{\rho\sigma}(m^2 + x(1-x)k^2) + 2x(1-x)k^\rho k^\sigma}{\Delta^{2-\frac{d}{2}}}\right] \\ &=: (k^2\,\delta^{\rho\sigma} - k^\rho k^\sigma)\,\pi_{1\,\mathrm{loop}}(q^2)\,,\end{aligned} \tag{5.74a}$$

where in the last line we have defined $\pi_{1\,\mathrm{loop}}(k^2)$ to be the dimensionless quantity

$$\pi_{1\,\mathrm{loop}}(k^2) = -\frac{8g^2(\mu)\,\Gamma\big(2-\frac{d}{2}\big)}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\,x(1-x)\left(\frac{\mu^2}{\Delta}\right)^{2-d/2}. \tag{5.74b}$$

and we recall that $\Delta = m^2 + k^2 x(1-x)$.

As promised, the loop integral has come out proportional to the operator $\delta^{\rho\sigma} - k^\rho k^\sigma/k^2$ projecting onto polarizations transverse to k. Gauge invariance is also maintained in lattice regularization, but would fail if one simply imposed a cut–off Λ_0, because the requirement that fields only contain Fourier modes with $|p| \leq \Lambda_0$ is not preserved under the gauge transformation $\psi \to e^{i\chi(x)}\psi$, even if it is true of ψ and χ separately.[53] The desire to maintain manifest gauge invariance was one of the main motivation to use dimensional regularization in the first place, and our result (5.74a) vindicates this decision: we see that the 1-loop correction $\Pi^{\rho\sigma}_{1\,\text{loop}}(k^2)$ is proportional to $(k^2\delta^{\rho\sigma} - k^\rho k^\sigma)$, so

$$k_\rho\,\Pi^{\rho\sigma}_{1\,\text{loop}}(k) = 0\,. \tag{5.75}$$

This signifies that the quantum effective action is also gauge invariant (at least to order $\hbar$ accuracy, but in fact it holds in general). We'll explore this important point further in the following chapters.

5.2.2 Counterterms in QED

The first thing to notice about our result (5.74a) is that it, if all couplings remain constant, it will diverge in the physically interesting dimension $d = 4$ where $\Gamma(2 - \frac{d}{2})$ has a pole. To obtain a finite result in $d = 4$ we must tune the initial couplings in the action, which as always we do by introducing counterterms. For QED the counterterms are

$$S^{\text{CT}}[A, \psi, \epsilon] = \int \mathrm{d}^d x \left[\delta Z_3 \frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \delta Z_2\, \bar{\psi}\not{\nabla}\psi + \delta m\, \bar{\psi}\psi\right]. \tag{5.76}$$

Adding these to the classical action (5.50) allows us to tune the initial values of the photon and electron wavefunction kinetic terms, and the electron mass. The labels $(\delta Z_3, \delta Z_2)$ for the photon and electron wavefunction renormalization counterterms are conventional (and the wavefunction renormalization factors themselves are likewise called (Z_3, Z_2), respectively). The fact that the entire kinetic term for the electron, including the gauge covariant derivative operator $\not{\nabla} = \not{\partial} - ie\not{A}$, receives only one counterterm assumes that the regularized path integral preserves gauge invariance: our dimensionally regularized loop integral is indeed gauge invariant, so $\not{\partial}\psi$ and $i\not{A}\psi$ cannot appear independently. We'll study this further in section 6.3.1.

To fix the counterterm δZ_3 we set $d = 4 - \epsilon$ and find the asymptotic expression

$$\pi_{1\,\text{loop}}(k^2) \;\sim\; -\frac{g^2(\mu)}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{\Delta}\right) + O(\epsilon) \tag{5.77}$$

as $\epsilon \to 0^+$, where again γ is the Euler–Mascheroni constant. The contribution

$-\delta Z_3$

from the counterterm $\frac{1}{4}\delta Z_3\, F^{\rho\sigma}F_{\rho\sigma}$ must remove this pole, and in the $\overline{\text{MS}}$ scheme we'd set

$$\delta Z_3 = -\frac{g^2(\mu)}{12\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) \tag{5.78}$$

[53] In fact, the conceptually simple idea of integrating over modes only up to a cut–off *can* be done in gauge theory, but requires the introduction of a fair amount of technology beyond the scope of this course; see *e.g.* K. Costello's book cited in the introduction.

so as also to remove the contribution $\propto (-\gamma + \ln 4\pi)$. (To check that this counterterm does indeed cancel the pole, note that $\int_0^1 \mathrm{d}x\, x(1-x) = \frac{1}{6}$.) Thus the total contribution to the effective photon self–energy at $\mathcal{O}(\hbar)$ is

$$\Pi^{\rho\sigma}_{1\,\mathrm{loop}}(k) = \left(k^2 \delta^{\rho\sigma} - k^\rho k^\sigma\right)\, \pi_{1\,\mathrm{loop}}(k^2) \tag{5.79}$$

where

$$\pi_{1\,\mathrm{loop}}(k^2) = \frac{g^2(\mu)}{2\pi^2} \int_0^1 \mathrm{d}x\, x(1-x)\, \ln\left[\frac{m^2 + x(1-x)k^2}{\mu^2}\right] \tag{5.80}$$

in the $\overline{\mathrm{MS}}$ scheme.

Strikingly, the loop correction to the photon propagator has created the logarithm[54]

$$\ln\left[m^2 + x(1-x)\,k^2\right]$$

in momentum space. This is quite unlike anything you've seen at tree–level, where Feynman diagrams are always rational functions of momenta, but it is very similar to the logarithms we obtained from integrating out fields in lower dimensional examples. In the present case, the logarithm has a branch cut in the region $m^2 + x(1-x)k^2 \leq 0$. For $x \in [0,1]$ we have $0 \leq x(1-x) \leq 1/4$, so the branch point is inaccessible for real Euclidean q. However, in Lorentz signature with $k_0 = \mathrm{i}E$ we have

$$x(1-x)\,(E^2 - \mathbf{k}^2) \geq m^2\,. \tag{5.81}$$

in Lorentzian signature, so the smallest value energy at which the branch point can be reached is

$$E^2 = (2m)^2\,. \tag{5.82}$$

This is precisely the threshold energy for the creation of a *real* (as opposed to virtual) electron–positron pair.

5.2.3 The β-function of QED

Knowing the effective action allows us to read off the β-function for the electric charge. To relate the photon kinetic term – involving wavefunction renormalization – to the β function for the electromagnetic coupling e, we first undo our rescaling $A^{\mathrm{old}}_\mu = eA^{\mathrm{new}}_\mu$ and work back in terms of the original gauge field A^{old}_μ. Then (strictly in $d = 4$) the quadratic term (5.60) in the effective action becomes

$$\begin{aligned} S^{(2)}_{\mathrm{eff}}[A^{\mathrm{old}}] &= \frac{1}{4g^2_{\mathrm{eff}}} \int F^{\mu\nu}F_{\mu\nu}\, \mathrm{d}^4z \\ &= \frac{1-\pi(0)}{4g^2(\mu)} \int F^{\mu\nu}F_{\mu\nu}\, \mathrm{d}^4z \\ &= \frac{1}{4}\left[\frac{1}{g^2(\mu)} - \frac{\hbar}{12\pi^2}\ln\frac{m^2}{\mu^2} + O(\hbar^2)\right] \int F^{\mu\nu}F_{\mu\nu}\, \mathrm{d}^4z\,. \end{aligned} \tag{5.83}$$

[54] Actually, it has produced a certain integral of a logarithm involving $x(1-x)$. This integral can be explicitly computed in terms of dilogarithms, but we won't need to know the result.

where the final expression uses our 1-loop result (5.80). In this way, we can view the vacuum polarization as a quantum correction to the value of the coupling $1/g^2(\mu)$ in front of the classical kinetic term.

Arguing again that the numerical value of the dimensionless coupling g_{eff} in the quantum effective action quantity cannot depend on our choice of scale μ, we have

$$0 = \mu\frac{\partial}{\partial\mu}\left[\frac{1}{g^2(\mu)} - \frac{\hbar}{12\pi^2}\ln\frac{m^2}{\mu^2}\right] = -\frac{2}{g^3(\mu)}\beta(g) + \frac{\hbar}{6\pi^2} \tag{5.84}$$

so that the β function for $g(\mu)$ is

$$\beta(g) = \frac{\hbar g^3(\mu)}{12\pi^2} + O(\hbar^2)\,. \tag{5.85}$$

Thus the coupling at scale μ' is related to that at scale μ by

$$\frac{1}{g^2(\mu')} = \frac{1}{g^2(\mu)} + \frac{\hbar}{6\pi^2}\ln\frac{\mu}{\mu'} + O(\hbar^2)\,, \tag{5.86}$$

to this order.

Just as for ϕ^4 theory, the running coupling defines a natural scale inherent in our theory. We define Λ_{QED} to be the scale at which the coupling diverges. Then at any other scale μ the value of the coupling cannot be chosen freely, but is fixed in terms of Λ_{QED} by

$$g^2(\mu) = \frac{6\pi^2}{\hbar}\frac{1}{\ln(\Lambda_{\text{QED}}/\mu)}\,. \tag{5.87}$$

Since the positive β-function (5.85) shows that (like the quartic scalar coupling) the QED coupling is in fact marginally irrelevant, we expect its value in the continuum theory should be fixed by our location along the renormalized trajectory parametrized purely in terms of the relevant and truly marginal couplings. But for QED the only such couplings are the electron mass and electron and photon kinetic terms. Once again, for any value of these masses and kinetic coefficients, we have a free theory and no interaction terms are generated at any point along the RG trajectory. Consequently, it is believed that pure QED *does not exist* as a continuum QFT in four dimensions![55]

In practical terms, as for the quartic scalar theory, this result is of no great importance. We can treat QED as a low–energy effective theory, valid provided the energy scale μ of our experiments remains well below Λ_{QED}. Experimentalists tell us that at the scale $\mu = m_{\text{e}} \approx 511\,\text{keV}$ of the electron mass, the dimensionless coupling $\alpha = g^2(m_{\text{e}})/4\pi \approx 1/137$. From this data point, equation (5.87) gives

$$\Lambda_{\text{QED}} \approx 10^{286}\ \text{GeV} \qquad (!)$$

which is an extremely high energy, and certainly well beyond any point at which we claim to even vaguely trust QFT as a description of Nature. In Nature, QED unifies with the weak interactions at around ~ 100 GeV, where the physics of non–Abelian gauge theories comes into play, modifying our conclusions here.

[55] Once again, to be sure one should really go further than this 1–loop calculation, but what we can be sure of already is that any continuum version of interacting QED would necessarily be strongly coupled. More sophisticated treatments support the view that in fact, no such continuum interacting theory exists.

5.2.4 Decoupling: on-shell schemes *vs* $\overline{\rm MS}$

Before moving on, I wish to point out a small peculiarity inherent in the $\overline{\rm MS}$ renormalization scheme. If we perform experiments at low energies, we do not expect to have large corrections coming from particles whose masses are much higher than the energy scale of our experiments: for example, you did not worry about corrections from the muon or tau particle when studying classical electrodynamics.

Indeed, this is just what we'd find if, instead of the $\overline{\rm MS}$ scheme, we'd used a renormalization scheme that fixes the value of the counterterm δZ_3 in terms of $\pi_{1\,\rm loop}(k^2)$ at some definite scale. For example, in the on–shell scheme we fix the counterterm by requiring that (just like the classical propagator) the exact photon propagator $\Delta^0_{\mu\nu}(k)/[1-\pi(k^2)]$ has a simple pole with unit residue when the photon's momentum obeys $k^2=0$. Thus, in the on–shell scheme, we'd fix δZ_3 by demanding $\pi(0)=0$ so that the quantum corrections (including both the loop integral and counterm) vanish at the on–shell point $k^2=0$.

For the purposes of extracting the β–function, it's more convenient to ask instead that the quantum corrections vanish at our arbitrary scale $k^2=\mu^2$. That is, we set

$$\delta Z_3 = -\frac{\hbar g^2(\mu)}{2\pi^2}\int_0^1 {\rm d}x\, x(1-x)\left(\frac{2}{\epsilon}-\gamma+\ln\frac{4\pi\mu^2}{m^2+x(1-x)\mu^2}\right) \tag{5.88}$$

cancelling the value of $\pi_{1\,\rm loop}(\mu^2)$. The total contribution to $\pi(k^2)$ in such a scheme is then

$$\pi(k^2) = \frac{\hbar g^2(\mu)}{2\pi^2}\int_0^1 {\rm d}x\, x(1-x)\,\ln\left(\frac{m^2+x(1-x)k^2}{m^2+x(1-x)\mu^2}\right)+O(\hbar^2)\,. \tag{5.89}$$

As in equation (5.84), asking that the total contribution to the effective action is independent of the scale μ means that the coupling must run, and in this scheme we find the β-function

$$\beta(g) = \frac{g^3(\mu)}{2\pi^2}\int_0^1 {\rm d}x\,\frac{x^2(1-x)^2\mu^2}{m^2+x(1-x)\mu^2}\,, \tag{5.90}$$

as you should check. This does indeed approach zero when $\mu\ll m$, so that the coupling automatically stops running at scales far below the electron mass.

However, in the $\overline{\rm MS}$ scheme, the β-function

$$\beta_{\overline{\rm MS}} = \frac{g^3(\mu)}{12\pi^2}+O(\hbar^2)$$

of (5.85) shows no suppression at any scale and the coupling (5.86) still runs even at scales $\mu\ll m_e$ much lower than the mass of the lightest (indeed the *only*) charged particle. There's nothing wrong with this in principle: as $\mu\to 0$ there's a balance in the effective action (5.83) between $g^2(\mu)\to 0$ and the growing effect of the loop contribution $\propto \ln(1/\mu^2)\to\infty$ so that the actual *observable physics* does remain constant. Nonetheless, it's strange to have loop effects dominating the classical ones, so for some purposes it's better to find a way to mimic the decoupling of the electron in $\overline{\rm MS}$.

To do so, we consider *two* different theories. One includes the electron and is valid at scales $\mu>m_e$, while the second contains no electron field and is valid at scales $\mu<m_e$.

Physical quantities in the two theories are matched at $\mu = m_e$. The effects of the electron (or other heavy particle) are then manually frozen out as we continue on to our other theory at lower scales. In particular, since pure Maxwell theory is free, for any $\mu < m_e$ the fine structure constant $\alpha(\mu) = g^2(\mu)/4\pi$ will remain frozen at its value $\approx 1/137$ at the electron mass.

This matching procedure, discarding the effects particles whose masses are $\gg \mu$ by hand, is certainly rather cumbersome. However, for most purposes (particularly in more complicated theories such as Yang–Mills theory, or the full Standard Model) the $\overline{\text{MS}}$ scheme is so convenient that most physicists view this as a price worth paying.

5.3 The Euler–Heisenberg effective action

The electron field appears purely quadratically in the QED action (5.50)[56]

$$S_{\text{QED}}[A,\psi] = \int \left[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(\nabla\!\!\!/ + m)\psi\right] \mathrm{d}^4x\,, \tag{5.91}$$

so we can perform its path integral exactly. The electron is fermionic, so its path integral yields $\det(\nabla\!\!\!/ + m)$ in the numerator. The remaining photon path integral thus involves the effective action

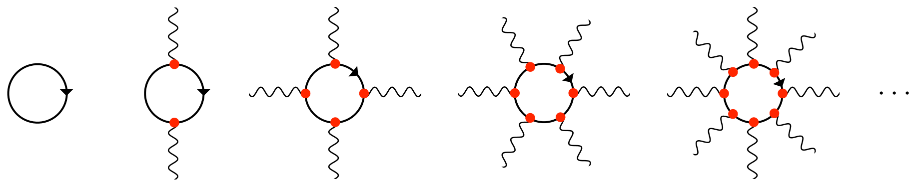
$$\Gamma_{\text{eff}}[A] = \frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}\,\mathrm{d}^4x \;-\; \hbar\,\ln\det(\nabla\!\!\!/ + m) \tag{5.92}$$

for the photon that takes into account quantum effects due to the electron. (As always, this expression is somewhat formal: if we don't regulate the path integral measure then the effective action will diverge. We'll consider a convenient regularization below.)

The functional determinant is non–polynomial in the photon field A_μ. Expanding it as an inifinite series around $A_\mu = 0$ gives

$$\begin{aligned}\ln\det(\nabla\!\!\!/ + m) &= \ln\det(\partial\!\!\!/ + m) + \mathrm{tr}\ln(1 - \mathrm{i}e(\partial\!\!\!/ + m)^{-1}A\!\!\!/)\\ &= \ln\det(\partial\!\!\!/ + m) + \sum_{n=1}^{\infty}\frac{(-\mathrm{i}e)^n}{n}\int\prod_{i=1}^{n}\mathrm{d}^4x_i\;\mathrm{Tr}(S(x_n,x_1)A\!\!\!/(x_1)S(x_1,x_2)A\!\!\!/(x_2)\cdots S(x_{n-1},x_n)A\!\!\!/(x_n))\end{aligned} \tag{5.93}$$

where $S(x_i, x_{i+1})$ is the Dirac propagator $(\partial\!\!\!/ + m)^{-1}$ and Tr indicates a trace over the Dirac gamma matrices. The terms on the rhs correspond to the 1PI Feynman diagrams[57]



in which the electron runs around a loop, radiating external photons as it goes. We recognise the 1-loop vacuum polarization graph as a member of this infinite series, with two external photons.

[56] Here, we've rescaled the photon to have canonically normalized kinetic terms, so $\nabla\!\!\!/\psi = \gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)\psi$ and e is dimensionless as we're in exactly four Euclidean dimensions.

[57] In the problem sheets, you'll show that the terms involving an odd number of external photons necessarily vanish, as a consequence of Furry's theorem.

Quite remarkably, in the case of a constant electromagnetic field, Euler and Heisenberg were able to find a closed form expression for $\ln\det(\nabla\!\!\!/ + m)$ that sums up the effects of *all* of these diagrams. They did this in 1936, well before QFT was on a firm footing and over a decade before Feynman first presented his diagrams! To obtain their result, we first simplify the effective action slightly using the following observation: $\ln\det(\nabla\!\!\!/ + m) = \operatorname{tr}\ln(\nabla\!\!\!/ + m)$ includes both a trace over the Dirac spinor indices as well as a functional trace. Since the trace of any odd number of γ matrices vanishes, we have

$$\operatorname{tr}\ln(\nabla\!\!\!/ + m) = \operatorname{tr}\ln(-\nabla\!\!\!/ + m) \tag{5.94}$$

and therefore

$$\begin{aligned}\operatorname{tr}\ln(\nabla\!\!\!/ + m) &= \frac{1}{2}\left[\ln\det(\nabla\!\!\!/ + m) + \ln\det(-\nabla\!\!\!/ + m)\right] \\ &= \frac{1}{2}\ln\det(-\nabla\!\!\!/^{2} + m^2)\,.\end{aligned} \tag{5.95}$$

We also have

$$\begin{aligned}\nabla\!\!\!/^{2} = \gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu &= \left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right)\nabla_\mu\nabla_\nu \\ &= \nabla^\mu\nabla_\mu - eS^{\mu\nu}F_{\mu\nu}\,,\end{aligned} \tag{5.96}$$

where $S^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]$ are the (Hermitian) generators of $\mathrm{SO}(4) \cong SU(2)\times SU(2)/\mathbb{Z}_2$ in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation appropriate for a Dirac spinor. This final term involving $S^{\mu\nu}$ is the origin of the electron's magnetic moment in the Dirac equation.

Our task is to understand the functional[58]

$$\begin{aligned}&\frac{1}{2}\operatorname{tr}\ln(-\nabla^2 + eS^{\mu\nu}F_{\mu\nu} + m^2) \\ &\quad= \frac{1}{2}\operatorname{tr}\ln\left[(-(\partial - \mathrm{i}eA)^2 + m^2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + e\begin{pmatrix}(\mathbf{B}+\mathrm{i}\mathbf{E})\cdot\boldsymbol{\sigma} & 0\\ 0 & (\mathbf{B}-\mathrm{i}\mathbf{E})\cdot\boldsymbol{\sigma}\end{pmatrix}\right],\end{aligned} \tag{5.97}$$

where $\boldsymbol{\sigma}$ are the usual Pauli matrices and the trace is a sum over the eigenvalues of the operator, acting on spinor–valued functions on $\mathbb{R}^4$, as well as a Dirac trace. Since the trace is basis independent, we're free to evaluate this in any basis, and we'll choose to work with the position basis. We'll do this by making use of the connection between QFT and worldline quantum mechanics that we saw in section 3.4, so

$$\Gamma_{\text{eff}}[A] = \int\left[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left\langle x\,\middle|\,\ln(-\nabla^2 + eS^{\mu\nu}F_{\mu\nu} + m^2)\middle|\,x\right\rangle\right]d^4x \tag{5.98}$$

where $\{|x\rangle\}$ is a basis of position eigenstates (we temporarily suppress the spinor indices). Next, we use the asymptotic relation

$$\int_{s_0}^{\infty}\mathrm{e}^{-sX}\,\frac{\mathrm{d}s}{s} \sim -\ln X - \ln s_0 + \text{finite} \tag{5.99}$$

[58]Our treatment here follows closely chapter 33 of M. Schwartz, *Quantum Field Theory and the Standard Model.*

as $s_0 \to 0^+$ to rewrite the logarithm of the operator as

$$\frac{1}{2}\left\langle x \,\middle|\, \ln(-\nabla^2 + eS^{\mu\nu}F_{\mu\nu} + m^2) \,\middle|\, x\right\rangle = \lim_{s_0 \to 0^+} \frac{1}{2}\int_{s_0}^{\infty} \mathrm{e}^{-sm^2/2} \left\langle x \,\middle|\, \mathrm{e}^{-s(-\nabla^2 + eS^{\mu\nu}F_{\mu\nu})} \,\middle|\, x\right\rangle \frac{\mathrm{d}s}{s}\,, \tag{5.100}$$

up to a divergent term that is independent of A_μ. Noting that, in the position space representation of Euclidean quantum mechanics, $\nabla^2 = (\partial - \mathrm{i}eA(x))^2 = (\hat{p} - \mathrm{i}eA(\hat{x}))^2$ in terms of the momentum operator $\hat{p} = -\partial/\partial x$, our problem has reduced to evaluating the quantum mechanical expectation value $\langle x|\mathrm{e}^{-s\hat{H}}|x\rangle$ with Hamiltonian $\hat{H} = (\hat{p} + eA(\hat{x}))^2 + \frac{e}{2}S^{\mu\nu}F_{\mu\nu}(\hat{x})$. Indeed, if $\{|\psi_n\rangle\}$ form a basis of eigenstates of $\hat{H}$, then the new term in the effective action involves

$$\begin{aligned}\int \mathrm{d}^4x \langle x|\mathrm{e}^{-s\hat{H}}|x\rangle &= \int \mathrm{d}^4x \sum_n \langle x|\psi_n\rangle \langle \psi_n|\mathrm{e}^{-s\hat{H}}|x\rangle \\ &= \int \mathrm{d}^4x \sum_n |\psi_n(x)|^2 \mathrm{e}^{-sE_n}\,.\end{aligned} \tag{5.101}$$

We'll be able to do this in the case of constant electromagnetic fields, so that $\partial_\mu F_{\alpha\beta} = 0$.

Let's begin with just a constant magnetic field $\mathbf{B}$, and choose to align the z-axis with the direction of $\mathbf{B}$. Then we can pick the gauge $A_y = Bx$ with $A_\mu = 0$ for $\mu \neq y$. In this gauge the Hamiltonian becomes

$$\hat{H} = \hat{p}_0^2 + \hat{p}_x^2 + \hat{p}_z^2 + (\hat{p}_y - eBx)^2 - eB\sigma_z \tag{5.102}$$

where $eB\sigma_z$ arises from the magnetic moment $S^{\mu\nu}F_{\mu\nu}$ and σ_z is the usual Pauli matrix. You should have met this system in undergraduate quantum mechanics: it's just a harmonic oscillator in the disguise of **Landau levels** for an electron in a constant magnetic field. This should come as no surprise: the Feynman diagrams we've drawn show an electron moving around a circle in the presence of a background magnetic field. The energy of the n^{th} level is

$$E_n(p_t, p_y, p_z, \pm) = p_t^2 + p_z^2 + 2eB\left(n + \frac{1}{2}\right) \mp eB \tag{5.103}$$

where p_t, p_z are the eigenvalues of the momenta and the choice of sign $\pm$ corresponds to the electron having its spin aligned or anti-aligned with the magnetic field. (Note that the energy eigenvalues are independent of p_y, so there is a large degeneracy in this system.) The corresponding energy eigenstate is

$$\langle x|\psi_n\rangle = \chi_n^{\pm}\left(x - \frac{p_y}{eB}\right) \mathrm{e}^{-\mathrm{i}(p_t t + p_y y + p_z z)} \tag{5.104}$$

where $\chi_n^{\pm}$ is the n^{th} excited state of a harmonic oscillator, tensored with a spin-up or spin-down state.

FILL IN DETAILS OF DENSITY OF STATES CALCULATION

Thus we obtain

$$\begin{aligned}\int \mathrm{d}^4x \langle x|\mathrm{e}^{-\hat{H}s}|x\rangle &= \frac{eBL^4}{(2\pi)^3} \sum_{\pm} \mathrm{e}^{\pm eB} \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}(p_t^2 + p_z^2)s} \mathrm{d}p_t\, \mathrm{d}p_z \sum_{n=0}^{\infty} \mathrm{e}^{-se(2n+1)B} \\ &= \frac{2L^4 eB}{8\pi^2} \frac{1}{s} \frac{\cosh(esB)}{\sinh(esB)}\,.\end{aligned} \tag{5.105}$$

Had we started with a constant electric field, from (5.97) we would have the same result, but with $\mathbf{B} \to \mathrm{i}\mathbf{E}$. The resulting effective Lagrangian for general constant $F_{\mu\nu}$ can be written as

$$\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \lim_{s_0\to 0^+} \frac{e^2}{32\pi^2}\int_{s_0}^{\infty} \frac{\mathrm{d}s}{s}\,\mathrm{e}^{-sm^2}\frac{\mathrm{Re}\cosh(esX)}{\mathrm{Im}\cosh(esX)}\,\tilde{F}^{\mu\nu}F_{\rho\sigma}\,, \tag{5.106}$$

where

$$X^2 = \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\tilde{F}^{\mu\nu}F_{\mu\nu} = (\mathbf{B}+\mathrm{i}\mathbf{E})^2 \tag{5.107}$$

and $\tilde{F}^{\mu\nu} = \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

This expression is singular as $s_0 \to 0$, as we anticipated in (5.99). Physically, s represents worldline proper time – a Schwinger parameter – along the electron's S^1 worldline, so the divergence at the lower limit corresponds to the UV region where the electron **EXPLAIN**.

We can regularize the answer by simply imposing a cut–off at s_0, but then we need to renormalize (tune our initial coupling) so as to obtain a finite effective action as $s_0 \to 0$. Expanding (5.106) as a series in the coupling e we have

$$\frac{\mathrm{Re}\cosh(esX)}{\mathrm{Im}\cosh(esX)}\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{4}{e^2s^2} + \frac{2}{3}F^{\mu\nu}F_{\mu\nu} - \frac{e^2s^2}{45}\left[(F^{\mu\nu}F_{\mu\nu})^2 + \frac{7}{4}(\tilde{F}^{\mu\nu}F_{\mu\nu})^2\right] + O(e^4) \tag{5.108}$$

and the first two terms cause divergences in the $s \to 0$ region of the integral $\mathrm{d}s/s$. The leading divergence is independent of the fields and can be removed by tuning the initial vacuum energy density. In fact, this term comes from the field independent term $\ln\det(\not{\partial}+m)$ corresponding to the 1-loop vacuum Feynman graph. It can also be removed by dividing by the partition function of a free (uncharged) electron. The subleading term leads to a logarithmic divergence in the coefficient of the usual photon kinetic term $F^{\mu\nu}F_{\mu\nu}$. It can thus be removed by including a wavefunction renormalization – a counterterm for this kinetic term. The simplest choice of renormalization scheme is just minimal subtraction, where we just remove these divergent pieces by hand. Altogether, we're left with the effective action

$$\Gamma[A] = \int\left\{\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{e^2}{32\pi^2}\int_0^{\infty}\mathrm{e}^{-sm^2}\left[\frac{\mathrm{Re}\cosh(esX)}{\mathrm{Im}\cosh(esX)}\,\tilde{F}^{\mu\nu}F_{\mu\nu} - \frac{4}{e^2s^2} - \frac{2}{3}F^{\mu\nu}F_{\mu\nu}\right]\frac{\mathrm{d}s}{s}\right\}\mathrm{d}^4x \tag{5.109}$$

as found by Euler & Heisenberg.

This effective action for the photon takes into account the quantum effects of the electron. In our derivation, we needed to assume the electromagnetic field was constant. We thus expect that the true effective action contains further terms such as $(\partial F)^2$ that involve derivatives of the Maxwell fieldstrength. On dimensional grounds, every derivative must be suppressed by some mass scale, and the only scale in the problem is the mass of the electron. Thus we expect the Euler-Heisenberg action (5.109) to be an accurate description of very high intensity light (where the non-linear F^{2k} terms are significant) but of frequency much lower than the electron mass. For example, expanding (5.109) we find a term

$$\frac{\alpha^2}{90}\frac{1}{m^4}\left[(F^{\mu\nu}F_{\mu\nu})^2 + \frac{7}{4}(\tilde{F}^{\mu\nu}F_{\mu\nu})^2\right]$$

which describes a four-photon interaction. This is the leading term in light-by-light scattering, leading to an amplitude for this process $\propto \alpha^2(\omega/m)^4$, valid in the regime where the photon energy transfer $\omega \ll m$.

5.3.1 Physical interpretation of vacuum polarization

When light propagates through a region containing an insulating medium with no relevant degrees of freedom, on general grounds we expect the low–energy effective field theory to be governed by an action that modifies the coefficients of the electric and magnetic fields in the usual Maxwell action by terms that respect the microscopic symmetries of the medium. In the present case, the medium is simply *the vacuum itself*! Since the vacuum is Lorentz invariant, these modifications must be proportional to the Lorentz invariant combination $F^{\mu\nu}F_{\mu\nu} = 2(\mathbf{E}^2 - \mathbf{B}^2)$. In (5.83) we see explicitly that this is true. If we place a medium such as water in the presence of an electric field, it will become polarized due to the large dipole moment of the H_2O molecules. Likewise, at distances $\gtrsim 1/m$ the vacuum itself becomes a dielectric medium in which virtual electron–positron pairs form dipoles, polarizing the vacuum.

The first effect is that vacuum polarization leads to a measureable change in the Coulomb potential. Recall that in the non-relativistic limit (in Lorentzian signature), the Fourier transform of the (Feynman gauge) photon propagator $\delta^{\rho\sigma}/k^2$ is the Coulomb potential $V(r) = e^2/4\pi r$, as I hope is familiar from Rutherford scattering. Let's compute the 1-loop quantum corrections to this result. We consider a scattering process in which two spin-$\frac{1}{2}$ charged particles interact electromagnetically. Using the exact photon propagator, we have

$$\begin{aligned} S(1,2\to 1',2') &= -\frac{e_1e_2}{4\pi^2}\,\delta^4(k_1+k_2-k_{1'}-k_{2'})\,\bar{u}_{1'}\gamma^\mu u_1\,\Delta_{\mu\nu}(p)\,\bar{u}_{2'}\gamma^\nu u_2 \\ &= -\frac{e_1e_2}{4\pi^2}\,\delta^4(k_1+k_2-k_{1'}-k_{2'})\,\bar{u}_{1'}\gamma^\mu u_1\,\frac{\Delta^0_{\mu\nu}(p)}{1-\pi(p^2)}\,\bar{u}_{2'}\gamma^\nu u_2 \end{aligned} \tag{5.110}$$

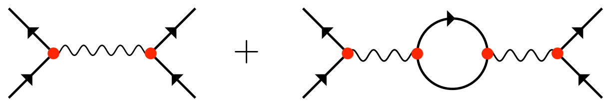
where $p = k_1 + k_2 = k_{1'} + k_{2'}$ is the total incoming (or outgoing) momenta, $u_{1,2}$ and $\bar{u}_{1',2'}$ are the momentum space on–shell Dirac wavefunctions for the incoming and outgoing particles, respectively. Using the fact that these initial and final Dirac wavefunctions obey $(\mathrm{i}\not{k}_{1,2} + m)u_{1,2} = 0$ and $(-\mathrm{i}\not{k}_{1',2'} + m)\bar{u}_{1',2'} = 0$ we have

$$\bar{u}_{1'}\gamma^\mu u_1\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\bar{u}_{2'}\gamma_\mu u_2 = \bar{u}_{1'}\gamma^\mu u_1\,\frac{1}{p^2}\,\bar{u}_{2'}\gamma_\mu u_2\,. \tag{5.111}$$

The photon self–energy term $\pi(p^2)$ comes from quantum corrections and so is $O(\hbar)$. Thus, to leading order in $\hbar$, the scattering amplitude is given by

$$S(1,2\to 1',2') = -\frac{e_1e_2}{4\pi^2p^2}\,\delta^4(k_1+k_2-k_{1'}-k_{2'})\,\bar{u}_{1'}\gamma^\mu u_1\,[1+\pi(p^2)]\,\bar{u}_{2'}\gamma_\mu u_2\,. \tag{5.112}$$

Of course, we can think of these terms as coming from the Feynman diagrams



so the 1–loop diagram modifies the classical answer by the factor $[1 + \pi(p^2)]$.

Let's now take the non–relativistic limit, as appropriate for Rutherford scattering. In this limit, the energy transfer $p^0 \ll |\mathbf{p}|$ and

$$\bar{u}_{1'}\gamma^\mu u_1 \approx \begin{pmatrix} -\mathrm{i}\delta_{\sigma_1,\sigma_{1'}} \\ \mathbf{0} \end{pmatrix} ,$$

where σ_i, is the spin angular momentum quantum number of the i^{th} external particle. The factor of $\delta_{\sigma\sigma'}$ enforces that the spins of the two particles should be aligned. Thus, in the non–relativistic limit we have

$$S(1,2 \to 1',2') \approx \frac{e_1 e_2}{4\pi^2 \mathbf{p}^2}\, \delta^4(k_1 + k_2 - k_{1'} - k_{2'})\, \left[1 + \pi(\mathbf{p}^2)\right]\, \delta_{\sigma_1\sigma_{1'}}\, \delta_{\sigma_2,\sigma_{2'}}\,. \tag{5.113}$$

By comparison, in non–relativistic quantum mechanics, the Born approximation for scattering two charged particles off a scalar potential $V(\mathbf{r})$ is given by

$$S_{\text{Born}}(1,2 \to 1',2') = \frac{e_1 e_2}{4\pi^2}\, \delta^4(k_1 + k_2 - k_{1'} - k_{2'})\, \delta_{\sigma_1,\sigma_{1'}}\, \delta_{\sigma_2,\sigma_{2'}} \int \mathrm{d}^3\mathbf{r}\, V(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,. \tag{5.114}$$

This shows that the 1-loop corrected amplitude (5.113) looks just like the amplitude we would find from Born level scattering off a modified classical potential whose Fourier transform is $e_1 e_2[1 + \pi(\mathbf{p})^2]/\mathbf{p}^2$, or in other words

$$V_1(\mathbf{r}) = e_1 e_2 \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}} \left[\frac{1 + \pi(\mathbf{p}^2)}{\mathbf{p}^2}\right] . \tag{5.115}$$

In the region where $\mathbf{p}^2 \ll m^2$ (momentum transfer is very small compared to the electron mass), our result (5.80) for $\pi(p^2)$ gives

$$\begin{aligned} \pi(\mathbf{p}^2) &= \pi(0) + \frac{g^2(\mu)}{2\pi^2} \int_0^1 \mathrm{d}x\, x(1-x)\, \ln\left[1 + x(1-x)\frac{\mathbf{p}^2}{m^2}\right] \\ &= \pi(0) + \frac{g^2(\mu)}{2\pi^2}\frac{\mathbf{p}^2}{m^2} \int_0^1 \mathrm{d}x\, x^2(1-x)^2 + O((|\mathbf{p}|/m)^4) \\ &= \pi(0) + \frac{g^2(\mu)}{60\pi^2}\frac{\mathbf{p}^2}{m^2} + O((|\mathbf{p}|/m)^4)\,. \end{aligned} \tag{5.116}$$

The term $\pi(0)$ modifies the strength of the classical Coulomb interaction, while the subleading term $\propto \mathbf{p}^2$ cancels the $\mathbf{p}^2$ coming from the photon propagator. Thus in this regime we find

$$V_1(\mathbf{r}) \approx -e_1 e_2 \frac{(1 + \pi(0))}{4\pi r} - \frac{e_1 e_2\, g^2(\mu)}{60\pi^2 m^2}\, \delta^{(3)}(\mathbf{r})\,. \tag{5.117}$$

Treating the term proportional to $\delta^{(3)}(\mathbf{r})$ as a perturbation, we see that the $\ell = 0$ bound states of Hydrogen (which are the only ones to have non–zero support at $\mathbf{r} = 0$) receive a shift in their energy. For example, the $|n, \ell, m\rangle = |2, 0, 0\rangle$ state is lowered by the measurable amount of ~ 27 MHz by this Uehling contribution, splitting its degeneracy with the $|2, 1, 0\rangle$ state. This is part of an effect known as the **Lamb shift**, measured shortly after the end of the Second World War. It's discovery posed a challenge to theoretical physicists to come to terms with loop corrections, which had previously been found to diverge.

More accurately, in the on–shell renormalization scheme where $\pi(0) = 0$, the 1–loop correction yields a potential

$$V(r) = -\frac{e_1 e_2}{4\pi r}\left(1 + \frac{g^2}{6\pi^2}\int_1^\infty \mathrm{e}^{-2mrx}\,\frac{2x^2+1}{2x^4}\,\sqrt{x^2-1}\,\mathrm{d}x\right) \tag{5.118}$$

known as the **Uehling potential**. The interpretation is that the external point charge of strength e_1 sitting at $\mathbf{r} = 0$ polarizes the vacuum, attracting virtual particles of opposite charge towards it and repelling their antiparticles as they circulate around the loop.