# 6 Symmetries in Quantum Field Theory

Physics of the 20$^{\text{th}}$ Century was largely driven by symmetry principles. Recognition that the behaviour of some physical system was governed by the presence of a symmetry became a key tool that was used to unlock the secrets of physics from hadronic interactions to electroweak physics and the Standard Model, from superconductivity to Bose–Einstein condensates. It's therefore important to understand how symmetry principles arise in QFT, and what their consequences are for the correlation functions and scattering amplitudes we compute.

## 6.1 Symmetries and conserved charges in the classical theory

We'll start from classical field theory. Recall that **Noether's theorem** states that transformations of the fields $\phi \mapsto \phi'$ under which the Lagrangian changes by at most a total derivative[59] correspond to conserved charges. Let's recall how to derive this.

We suppose we have a continuous family of transformations, such that infinitesimally our fields transform as

$$\phi^a(x) \mapsto \phi'^a(x) = \phi^a(x) + \delta\phi^a(x) = \phi^a(x) + \epsilon^r f_r^a(\phi, \partial_\mu \phi) \tag{6.1}$$

where $\epsilon^r$ are constant infinitesimal parameters labelling the transformation, and $f_r^a(\phi, \partial_\mu\phi)$ are some functions of the fields and their derivatives. The transformation is local if each of these functions $f_r^a$ depends on the values of the fields and their derivatives only at the one point $x \in M$, in which case you can think of the transformation as being generated by the vector field

$$\epsilon^r V_r := \int_M \mathrm{d}^d x \sqrt{g}\; \epsilon^r f_r^a(\phi, \partial\phi) \frac{\delta}{\delta\phi^a(x)} \tag{6.2}$$

acting on the infinite dimensional space of fields. The transformation (6.1) is a **symmetry** if the Lagrangian is invariant upto a possible total derivative, *i.e.*

$$\delta\mathcal{L}(\phi, \partial\phi) = \partial_\mu(\epsilon^r K_r^\mu) \tag{6.3}$$

for some spacetime vectors $K_r^\mu$, because in this case the equations of motion will be unaltered.

[59] The Lagrangian changing by at most a total derivative means the classical field equations will be unaffected. It's possible for the field equations to be invariant under further transformations that do not preserve the Lagrangian. For example, consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} m\, \delta_{\mu\nu} \frac{dx^\mu}{dt}\frac{dx^\mu}{dt} \qquad \text{which gives eom} \qquad \frac{d^2 x^\mu}{dt^2} = 0$$

describing a free particle travelling on $\mathbb{R}^n$. The equations of motion are invariant under $x^\mu \mapsto x'^\mu = R^\mu{}_\nu x^\nu$ for any $R \in GL(n, \mathbb{R})$, but only if $R \in O(n)$ is this a symmetry of the Lagrangian. Symmetries of the eom that do not come from symmetries of the Lagrangian are known as **dynamical symmetries** and are often associated with integrable systems. They play an important role in QFT, though we will not discuss them further in this course.

Whatever the transformation, the change in the Lagrangian under (6.1) will be

$$\begin{aligned}\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi^a(x)}\delta\phi^a(x) + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\partial_\mu\delta\phi^a(x) \\ &= \left[\frac{\delta\mathcal{L}}{\delta\phi^a(x)} - \frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\right]\delta\phi^a + \frac{\partial}{\partial x^\mu}\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\delta\phi^a\right].\end{aligned} \tag{6.4}$$

where $\delta\phi^a = \epsilon^r f^a_r(\phi,\partial\phi)$, and so if this transformation is a symmetry we have

$$\left[\frac{\delta\mathcal{L}}{\delta\phi^a(x)} - \frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\right]\delta\phi^a + \frac{\partial}{\partial x^\mu}\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\delta\phi^a - \epsilon^r K^\mu_r\right] = 0\,. \tag{6.5}$$

Defining the **current** $J^\mu_r$ associated to a symmetry transformation by

$$J^\mu_r = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)} f^a_r(\phi,\partial\phi) - K^\mu_r \tag{6.6}$$

equation (6.4) shows that

$$\partial^\mu J_{\mu\,r} = 0 \tag{6.7}$$

along trajectories in field space that obey the classical equations of motion. Note that (6.5) gives the divergence of the current even when the equations of motion are not obeyed.

Given a current, we define the **charge** $Q_r$ corresponding to the transformation with parameter $\epsilon^r$ by

$$Q_r[N] := \int_N *J_r = \int_N J_{\mu\,r}\, n^\mu \sqrt{g}\,\mathrm{d}^{d-1}x\,. \tag{6.8}$$

Here, $N$ is any codimension–one hypersurface in $M$ and $n^\mu$ is a unit normal vector to $N$ (so $g(n,n)=1$ and $g(n,v)=0$ for any $v\in TN$), $\sqrt{g}$ is the square root of the determinant of the metric $g$ on $M$, evaluated along $N$. Thus $\sqrt{g}\,\mathrm{d}^{d-1}x$ is the $(d-1)$–dimensional volume element on $N$. The classical statement that $\partial^\mu J_{\mu\,r}=0$ has the important consequence that the corresponding charge $Q_r$ is conserved. To put this in a general context, suppose that $N_0$ and $N_1$ are two hypersurfaces bounding a region $M'\subset M$ of our space. Then

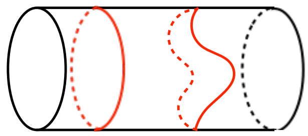
$$Q_r[N_1] - Q_r[N_0] = \int_{N_1} *J_r - \int_{N_0} *J_r = \int_{\partial M'} *J_r = \int_{M'} d*J_r = 0 \tag{6.9}$$

where the third equality is Stokes' theorem and the final equality follows by the conservation equation (6.7). Thus $Q[N]$ depends on the choice of $N$ only through its homology class.

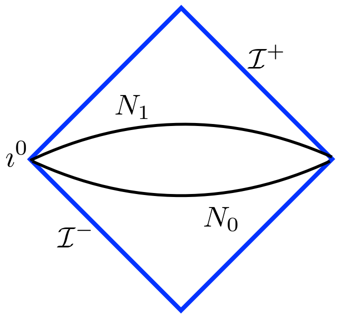
For example, in canonical quantization of the worldsheet CFT in string theory, we often choose $M$ to be a cylinder $S^1\times[0,T]$. The charges are then integrals

$$Q_r = \oint *J_r \tag{6.10}$$

of the currents $J_r$ around any cycle on the cylinder, while the statement that $Q_r[N_1] = Q_r[N_0]$ becomes the statement that the charges are constant whenever we choose two homologous cycles:




A further important example takes the surfaces $N_{1,0}$ to be constant time slices of Minkowski space–time:




shown here in the Penrose diagram of $\mathbb{R}^{1,3}$. The statement that $Q_r[N_1] = Q_r[N_0]$ becomes the statement that the charges $Q_r$ are conserved under time evolution. In this case the constant time slices $N_{0,1}$ are non–compact, so for our derivation to hold we should ensure that the current $j$ decays sufficiently rapidly as we head towards spatial infinity. This also ensures that the integrals defining $Q_r[N_{0,1}]$ converge. You should be familiar with the relation between symmetries, conserved currents and charges from *e.g.* last term's QFT course, if not before.

### 6.2 Symmetries of the effective action

Our treatment of Noether's theorem used the classical equations of motion to deduce that the charge was conserved, and so needs to be re-examined in the quantum theory. Our starting point is to understand how the path integral itself responds to a field transformations. Formally, the path integral measure changes as

$$\mathcal{D}\phi' = \det\left(\frac{\delta\phi'(x)}{\delta\phi(y)}\right) \mathcal{D}\phi \tag{6.11}$$

acquiring a Jacobian from the change of variables, and for an infinitesimal change of the form (6.1) this Jacobian matrix is

$$\frac{\delta}{\delta\phi^b(y)}\left(\phi^a + \epsilon^r f^a_r(\phi,\partial\phi)\right) = \delta^a{}_b\,\delta^d(x-y) + \epsilon^r\frac{\delta f^a_r(\phi,\partial\phi)}{\partial\phi^b(y)} \tag{6.12}$$

so that

$$\det\left(\frac{\delta\phi'(x)}{\delta\phi(y)}\right) = 1 + \mathrm{tr}\left(\epsilon_r\frac{\delta f^a_r(\phi,\partial\phi)}{\delta\phi^b(y)}\right) \tag{6.13}$$

where tr involves a trace over the flavour indices $a, b$ as well as a functional trace over $x$ and $y$.

We'll be especially interested in infinitesimal transformation $\phi^a \mapsto \phi'^a = \phi^a + \epsilon^r f_r^a(\phi, \partial\phi)$ that leaves the product of the action and regularized path integral measure invariant, *i.e.*,

$$\mathcal{D}\phi' \, \mathrm{e}^{-S[\phi']/\hbar} = \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]\hbar} . \tag{6.14}$$

(In most cases, the symmetry transformation will actually leave *both* the action and measure invariant separately, but the weaker condition (6.14) is all that is necessary.) Then the partition function in the presence of a source $J_a$ for $\phi^a$ can be written

$$\begin{aligned}
\mathcal{Z}[J] &= \int \mathcal{D}\phi' \; \exp\left[-\frac{1}{\hbar}\left(S[\phi'] + \int_M J_a(x)\, \phi'^a(x)\, \mathrm{d}^d x\right)\right] \\
&\int \mathcal{D}\phi \; \exp\left[-\frac{1}{\hbar}\left(S[\phi] + \int_M J_a(x)\, \phi^a(x)\, \mathrm{d}^d x + \int_M J_a(x)\, \epsilon^r f_r^a(\phi, \partial\phi)\, \mathrm{d}^d x\right)\right] \\
&= \int \mathcal{D}\phi \; \left(1 - \frac{\epsilon^r}{\hbar}\int_M J_a(x)\, f_r^a(\phi, \partial\phi)\, \mathrm{d}^d x + \cdots\right)\; \mathrm{e}^{-(S[\phi] - \int J_a \phi^a \, \mathrm{d}^d x)/\hbar} \\
&= \mathcal{Z}[J]\left(1 - \frac{\epsilon^r}{\hbar}\int_M J_a(x)\, \langle f_r^a(\phi, \partial\phi)\rangle_J \; d^d x + \cdots\right) .
\end{aligned} \tag{6.15}$$

Hence, since all the parameters $\epsilon^r$ may be chosen independently,

$$\int_M J_a(x)\, \langle f_r^a(\phi, \partial\phi)\rangle_J \; d^d x = 0 \tag{6.16}$$

where the expectation value is taken in the presence of the source $J$, normalized so that $\langle 1\rangle_J = 1$.

We'd like to write this result in terms of the 1PI quantum effective action $\Gamma[\Phi^a]$. Recall that this is the Legendre transform of $\mathcal{W}[J] = -\hbar \ln \mathcal{Z}$, with $J_a$ conjugate to a field $\Phi^a$ and where $J_a$ is evaluated at

$$J_{a\Phi}(y) = -\frac{\delta\Gamma[\Phi]}{\delta\Phi^a(y)} \tag{6.17}$$

at which point $\langle \phi^a\rangle_{J_\Phi} = \Phi^a$. At this value of $J$, (6.16) gives

$$0 = \int_M \langle f_r^a(\phi, \partial\phi)\rangle_{J_\Phi} \, \frac{\delta\Gamma[\Phi]}{\delta\Phi^a(x)} \, \mathrm{d}^d x \tag{6.18}$$

which says that the effective action $\Gamma[\Phi^a]$ is invariant under the transformations

$$\Phi^a \mapsto \Phi'^a = \Phi^a + \epsilon^r \langle f_r^a(\phi, \partial\phi)\rangle_{J_\Phi} \tag{6.19}$$

involving the *expectation values* of the transformations of the original, classical action. These symmetries of the 1PI quantum effective action are known as **Slavnov–Taylor identities**.

A very important special case of all this is when the transformations that act *linearly* on the fields, so

$$\begin{aligned} f_r^a(\phi) &= c_r^a(x) + \int_M e_{br}^{(0)a}(x,y)\phi^b(y)\, d^dy + \sum_m \int_M (e_r^{\mu_1\mu_2\cdots\mu_m})^a{}_b(x,y)\, \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_m}\phi^b(y)\, \mathrm{d}^dy \\ &= c_r^a(x) + \int_M d^a_{r\,b}(x,y)\phi^b(y)\, d^dy \end{aligned} \tag{6.20}$$

for some coefficients $\{c_r^a(x), d^a_{r\,b}(x,y)\}$ with[60] $d = e^{(0)} + \sum(-1)^m\partial_{\mu_1}\cdots\partial_{\mu_m}e^{\mu_1\cdots\mu_m}$. Under such a linear transformation, the matrix $\epsilon^r\,\delta f_r^a(x)/\delta\phi^b(y) = \epsilon^r d^a_{r\,b}(x,y)$ appearing in the Jacobian is field independent, at least formally. Thus, even if it does not vanish, it will not affect any normalized correlation functions such as those appearing in the Slavnov-Taylor identities. Furthermore,

$$\begin{aligned} \langle f_r^a(\phi)\rangle_{J_\Phi} &= c_r^a(x) + \int_M d^a_{r\,b}(x,y)\langle\phi^a(y)\rangle_{J_\Phi}\,\mathrm{d}^dy \\ &= c_r^a(x) + \int_M d^a_{r\,b}(x,y)\Phi^b(y)\,\mathrm{d}^dy \\ &= f_r^a(\Phi)\,. \end{aligned} \tag{6.21}$$

Thus, if our symmetry transformation acts linearly on the fields, equation (6.19) becomes

$$0 = \int_M f_r^a(\Phi,\partial\Phi)\,\frac{\delta\Gamma[\Phi]}{\delta\Phi^a(x)}\,\mathrm{d}^dx\,, \tag{6.22}$$

so that these same transformations $\Phi^a \mapsto \Phi^a + \epsilon^r f_r^a(\Phi,\partial\phi)$ are also symmetries of the quantum effective action. This is not generally the case for symmetry transformations which act non-linearly on the classical fields, because the expectation value of a non-linear functional of the fields is not usually the same as the non–linear functional of the expectation values.

The importance of this result is that if we know the quantum effective action is invariant under the same symmetry as was the classical action, then it cannot have been possible to generate terms which violate these symmetries through quantum effects such as loop diagrams. For example, the classical action

$$S[\phi] = \int \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\,\mathrm{d}^dx \tag{6.23}$$

is invariant under the $\mathbb{Z}_2$ transformation $\phi \mapsto -\phi$. Provided we regularize in a way that preserves this symmetry (so our regularization treats $\phi$ and $-\phi$ equally) then we know that $\Gamma[-\phi] = \Gamma[\phi]$. Perturbatively, we can indeed check that every Feynman graph one can draw using the ingredients in (6.23) must have an even number of external legs, so indeed no vertices with odd powers of $\phi$ can be generated in $\Gamma[\phi]$. As a second example, if we have a theory of $n$ scalar fields $\phi^a$ with action

$$S[\phi^a] = \int \frac{1}{2}\delta_{ab}\,\partial^\mu\phi^a\partial_\mu\phi^b + \frac{m^2}{2}\delta_{ab}\,\phi^a\phi^b + \lambda(\delta_{ab}\,\phi^a\phi^b)^2\,\mathrm{d}^dx \tag{6.24}$$

[60] We assume that either $M$ is compact, or that appropriate boundary or asymptotic conditions are placed on the $\phi^a$ and $e_r^{\mu_1\cdots\mu_m\ a}$ so that there is no boundary term here.

that is invariant under $\phi^a \mapsto \phi'^a = R^a{}_b \phi^b$ for $R^a{}_b \in \mathrm{O}(n)$. In this case, provided we choose to regularize the $\phi^a$ in an $\mathrm{O}(n)$ invariant way, the path integral measure acquires a Jacobian

$$\det\left(\frac{\delta\phi'^a(x)}{\delta\phi^b(y)}\right) = \det\left(\delta^d(x-y)\, R^a{}_b(x)\right)$$

which is field independent. The quantum effective action will thus also be $\mathrm{O}(n)$ invariant.

Further examples come from symmetries, such as rotations or translations, whose action on the fields is induced from their action on $M$ itself. For example, the $\mathrm{SO}(d)$ transformation

$$x^\mu \mapsto x'^\mu = L^\mu{}_\nu x^\nu \tag{6.25}$$

of $M$ induce the transformations

$$A_\mu(x) \mapsto A'_\mu(x') = (L^{-1})_\mu{}^\nu A_\nu(L^{-1}x) \qquad \psi^\alpha(x) \mapsto \psi'^\alpha(x) = S^\alpha{}_\beta(L)\,\psi^\beta(L^{-1}x) \tag{6.26}$$

of the photon and electron fields, where $S(L) = \exp(\mathrm{i}L_{\mu\nu}[\gamma^\mu,\gamma^\nu]/4)$ is the spinor representation of the $\mathrm{SO}(d)$. These transformations leave the QED action

$$S_{\mathrm{QED}}[A,\psi] = \int_{\mathbb{R}^d}\left[\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \bar\psi(\not{D}+m)\psi\right]\,\mathrm{d}^d x \tag{6.27}$$

invariant, and are again linear in the fields. Thus we expect that the quantum effective action will also be $\mathrm{SO}(d)$ invariant.

### 6.2.1 Regulators and symmetry

The results above are formal, in the sense that we have not yet said how to define the infinite dimensional determinant

$$\det\left(\frac{\delta\phi'^a(x)}{\delta\phi^b(y)}\right) \qquad \text{or the trace} \qquad \mathrm{tr}\left(\frac{\epsilon^r\delta f^a_r(\phi,\partial\phi)}{\delta\phi^b(y)}\right).$$

To do so, as always we must first regularize to turn our path integral turning into a finite dimensional integral. To what extent the results above carry through depends on how this is achieved. In the simplest case, it may be possible to regularize the theory in a way compatible with the classical symmetry. For example, if we regularize an $\mathrm{SO}(d)$ invariant classical theory either by imposing a cut-off $\Lambda_0$ on the eigenvalues of the $\mathrm{SO}(d)$ invariant Laplacian $-\partial^\mu\partial_\mu$ acting on the fields, or perturbatively by working in $d$ dimensions (perhaps at the cost of gauge invariance), then the quantum effective action is indeed guaranteed to be $\mathrm{SO}(d)$ invariant, even whilst we keep the regulator finite (*e.g.* even at finite $\Lambda_0$ or in $d \neq 4$).

It's also possible that, even though regulators which preserve the classical symmetry exist, for some reason we choose to regularize in a way that violates the symmetry. (This may be because the other regulator is particularly convenient, perhaps because it *does* preserve some other symmetry that we consider to be more 'important', or else because we're simply unaware of the symmetry-preserving regulator.) In this case, the path integral measure will not respect the symmetry of the classical action, and neither the counterterms nor the regularized theory will be invariant. As an example, we may regularize the continuum theory

$$S[\phi] = \int_{T^d} \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi + V(\phi)\,\mathrm{d}^d x \tag{6.28}$$

by replacing $\mathbb{R}^d$ by a lattice $\Lambda \subset \mathbb{R}^d$ with spacing $a$, replacing the action by

$$S_\Lambda[\phi] = \sum_{\mathbf{x}\in\Lambda}\left(\frac{1}{2}\sum_{\mu=1}^{d}\left(\frac{\phi(\mathbf{x}+a\hat{\boldsymbol{\mu}}) - \phi(\mathbf{x})}{a}\right)^2 + V(\phi(\mathbf{x}))\right) \tag{6.29}$$

that samples the field just at lattice points, and integrating over the values of $\phi$ at these lattice points[61]. In this case, with finite lattice spacing $a$ the quantum effective action will be invariant under the discrete subgroup $G \subset \mathrm{SO}(d)$ corresponding to the symmetries of the lattice, but there's no reason to expect it to have full $\mathrm{SO}(d)$ invariance. Nonetheless, as we'll see in more detail below, full $\mathrm{SO}(d)$ invariance is restored in the continuum limit.

Finally, it's possible that *no regularization procedure which preserves the classical symmetry exists*. In this case, the classical symmetry is simply not present in the quantum theory and is said to be **anomalous**. As an example, the conformal transformations

$$\begin{gathered}\delta_{\mu\nu} \mapsto \mathrm{e}^{2\sigma}\delta_{\mu\nu}\,, \qquad \sqrt{\delta}\,\mathrm{d}^4x \mapsto \mathrm{e}^{4\sigma}\sqrt{\delta}\,\mathrm{d}^4x\,, \qquad \gamma^\mu \mapsto \mathrm{e}^{-\sigma}\gamma^\mu \\ A_\mu(x) \mapsto A_\mu(x)\,, \qquad \psi(x) \mapsto \mathrm{e}^{-3\sigma/2}\psi(x)\end{gathered} \tag{6.30}$$

leave invariant the action

$$S[A,\psi] = \int_{\mathbb{R}^4}\left[\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\not{\nabla}\psi\right]\mathrm{d}^4x \tag{6.31}$$

of QED in *four* dimensions, where the charged fermion is taken to be massless. Neither our imposition of a cut-off, nor our analytic continuation to $d$ dimensions, nor the introduction of a lattice preserve this conformal invariance and indeed it is broken in the quantum theory: the $\beta$-function can be understood as measuring the **conformal anomaly**.

While true, the statement that the symmetry is anomalous if we cannot find any regularization compatible with the symmetry is clearly unsatisfactory, as we haven't yet said how we can tell if such a regulator truly doesn't exist, or whether our imagination was simply too limited. A proper answer to this question takes us into considerations of the geometry and topology of the space of fields, and was one of the original places where theoretical physics made contact with differential topology such as the Atiyah–Singer index theorem. Unfortunately, a sensible explanation of this beautiful subject lies beyond the scope of this course. Nonetheless, it's often easy to understand the source of such a failure even from naïve considerations. For example, we might define a metric on the space of fields by

$$ds^2 = \int_M G_{ab}(\phi)\,\delta\phi^a\,\delta\phi^b\,\sqrt{g}\,\mathrm{d}^d x$$

[61] Here $\hat{\boldsymbol{\mu}}$ is a unit basis vector of $\Lambda$. We've taken the original space to be a $d$-dimensional torus – or, equivalently, imposed periodic boundary conditions on $\phi$ – so as to ensure there are only finitely many lattice points.

where $\phi : (M, g) \to (N, G)$ is the field of a (non–)linear sigma model, or by

$$ds^2 = \int_M \mathrm{tr}(\delta A_\mu \, \delta A_\nu) \, g^{\mu\nu} \sqrt{g} \, \mathrm{d}^d x$$

in the case of a gauge theory. In each case, we could construct a path integral measure from some regularization of the associated Riemannian measure on the space of fields. In each case, we see that the metric – and hence the measure – on the space of fields depends on a choice of metric on $M$ and in particular depends on the conformal factor of $g$. Hence there's no reason to expect the quantum theory would be invariant under a change of this conformal factor.

### 6.3 Ward–Takahashi identities

Closely related to the consideration of symmetries of the effective action is to consider symmetries of correlation functions. Consider a class of operators whose only variation under the transformation $\phi \mapsto \phi'$ comes from their dependence on $\phi$ itself (such as scalar operators under rotations). Such operators transform as $\mathcal{O}(\phi) \mapsto \mathcal{O}(\phi')$. At least on a compact manifold $M$ we have

$$\begin{aligned}\int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]/\hbar} \, \mathcal{O}_1(\phi(x_1)) \cdots \mathcal{O}_n(\phi(x_n)) &= \int \mathcal{D}\phi' \, \mathrm{e}^{-S[\phi']/\hbar} \, \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n)) \\ &= \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]/\hbar} \, \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n))\end{aligned} \tag{6.32}$$

The first equality here is a triviality: we've simply relabeled $\phi$ by $\phi'$ as a dummy variable in the path integral. The second equality is non–trivial and uses the assumed symmetry (6.14) under the transformation $\phi \to \phi'$. We see that the correlation function obeys

$$\langle \mathcal{O}_1(\phi(x_1)) \cdots \mathcal{O}_n(\phi(x_n)) \rangle = \langle \mathcal{O}_1(\phi'(x_1)) \cdots \mathcal{O}_n(\phi'(x_n)) \rangle \tag{6.33}$$

so that it is *invariant* under the transformation. This is known as a **Ward–Takahashi identity**, or often (and rather unfairly) just a **Ward identity**.

For example, consider the phase transformation

$$\phi \to \phi' = \mathrm{e}^{\mathrm{i}\alpha} \phi \,, \qquad \bar{\phi} \to \bar{\phi}' = \mathrm{e}^{-\mathrm{i}\alpha} \bar{\phi} \tag{6.34}$$

that leaves the action

$$S[\phi] = \int_M \frac{1}{2} \mathrm{d}\bar{\phi} \wedge * \mathrm{d}\phi + *V(|\phi|^2) \tag{6.35}$$

invariant. The path integral measure will be invariant under this symmetry provided we integrate over as many modes of $\bar{\phi}$ as we do of $\phi$. Correlation functions built from local operators of the form $\mathcal{O}_i = \phi^{r_i} \bar{\phi}^{s_i}$ must obey

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \mathrm{e}^{\mathrm{i}\alpha \sum_{i=1}^n (r_i - s_i)} \, \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \,. \tag{6.36}$$

Considering different (constant) values of $\alpha$ shows that this correlator vanishes unless $\sum_i r_i = \sum_i s_i$. The symmetry thus imposes a **selection rule** on the operators we can insert: only a product of operators that is (in total) invariant under the symmetry can yield a non–zero correlator.

As a second example, suppose $(M, g) = (\mathbb{R}^d, \delta)$ and consider a space–time translation $x \mapsto x' = x - a$ where $a$ is a constant vector. Under this translation, we have

$$\phi(x) \mapsto \phi'(x) = \phi(x - a)\,. \tag{6.37}$$

If the action and path integral measure are translationally invariant and the operators $\mathcal{O}_i$ depend on $x$ only via their dependence on $\phi(x)$, then the Ward identity gives

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \langle \mathcal{O}_1(x - a)\cdots\mathcal{O}_n(x_n - a)\rangle \tag{6.38}$$

for any such vector $a$. Thus, having carried out the path integral, we'll be left with a function $f(x_1, x_2, \ldots, x_n)$ that depends only on the *relative* positions $(x_i - x_j)$. Similarly, if the action & measure are invariant under SO$(d)$ transformations $x \to Lx$ then a correlation function of scalar operators will obey

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \langle \mathcal{O}_1(Lx_1)\cdots\mathcal{O}_n(Lx_n)\rangle\,. \tag{6.39}$$

Combining this with the previous result shows that the correlator can depend only on the rotational (or Lorentz) invariant distances $(x_i - x_j)^2$ between the insertion points.

### 6.3.1 Current conservation in QFT

We can obtain more powerful constraints on correlation functions if our symmetry transformation does not just preserve the action, but the Lagrangian density $\mathcal{L}(x)$ itself. This means that the symmetry holds at each point $x \in M$ separately, not just when integrated over $M$, which will be the case if $K^r_\mu = 0$ in (6.3). Suppose that $\phi \mapsto \phi' = \phi + \epsilon^r f_r(\phi, \partial\phi)$ is a symmetry of $\mathcal{L}(x)$ and the path integral measure when the infinitesimal parameters $\epsilon^r$ are constant. To access the extra information in the fact that $\mathcal{L}(x)$ itself is invariant, we allow the parameters to vary (smoothly) over $M$, so $\epsilon^r \to \epsilon^r(x)$. As in Noether's theorem, the variation of the Lagrangian *and path integral measure* must now be proportional to $\partial_\mu \epsilon^r$, so

$$\mathcal{Z} = \int \mathcal{D}\phi'\, \mathrm{e}^{-S[\phi']/\hbar} = \int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]/\hbar}\left[1 - \int_M *j_r \wedge \mathrm{d}\epsilon^r\right] + O(\epsilon^2) \tag{6.40}$$

to lowest order. Notice that the current here may include possible changes in the path integral measure as well as in the action. The zeroth order term agrees with the partition function on the left, so the first order term must vanish and we have

$$0 = -\int_M *\langle j_r(x)\rangle \wedge \mathrm{d}\epsilon^r = \int_M \epsilon^r(x)\, \mathrm{d} * \langle j_r(x)\rangle\,, \tag{6.41}$$

if either $\partial M = 0$ or the fields decay sufficiently rapidly that there is no boundary contribution. For this to hold for arbitrary $\epsilon$ we must have $\mathrm{d} * \langle j_r(x)\rangle = 0$ so that the *expectation value* of the current obeys a conservation law, just as in classical physics. (In the simple case $(M, g) = (\mathbb{R}^d, \delta)$, this is just the familiar $\partial^\mu\langle j_{r\mu}(x)\rangle = 0$.)

We're now ready to obtain the more powerful constraints on correlation functions. Consider a class of local operators that transform under $\phi(x) \mapsto \phi(x) + \epsilon^r(x) f_r(\phi, \partial\phi)$ as

$$\mathcal{O}(\phi) \mapsto \mathcal{O}(\phi + \epsilon\delta\phi) = \mathcal{O}(\phi) + \epsilon^r \delta_r \mathcal{O} \tag{6.42}$$

to lowest order in $\epsilon^r(x)$, where we've defined $\delta_r\mathcal{O} = \partial\mathcal{O}/\partial\phi\, f_r(\phi, \partial\phi)$. Then, accounting for both the change in the action and measure as well as in the operators,

$$\begin{aligned} &\int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]/\hbar} \prod_{i=1}^{n} \mathcal{O}_i(\phi(x_i)) = \int \mathcal{D}\phi'\, \mathrm{e}^{-S[\phi']/\hbar} \prod_{i=1}^{n} \mathcal{O}_i(\phi'(x_i)) \\ &= \int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]/\hbar} \left[1 - \int_M *j_r \wedge d\epsilon^r\right] \left[\prod_{i=1}^{n} \mathcal{O}_i(x_i) + \sum_{i=1}^{n} \epsilon^r(x_i)\, \delta_r\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j)\right] + O(\epsilon^2)\,. \end{aligned} \tag{6.43}$$

Again, the first equality is a triviality while the second follows upon expanding both $\mathcal{D}\phi'\, \mathrm{e}^{-S[\phi']/\hbar}$ and the operators to first order in the position–dependent parameters $\epsilon^r(x)$. The $\epsilon$-independent term on the rhs exactly matches the lhs, so the remaining terms must cancel. To first order in $\epsilon^r$ this gives

$$\int_M \epsilon^r(x) \wedge d * \left\langle j_r(x) \prod_{i=1}^{n} \mathcal{O}_i(x_i) \right\rangle = -\sum_{i=1}^{n} \left\langle \epsilon^r(x_i)\, \delta_r\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle\,, \tag{6.44}$$

after an integration by parts[62] Note that the derivative on the lhs hits the whole correlation function.

We'd like to strip off the parameters $\epsilon^r(x)$ and obtain a relation purely among the correlation functions themselves. To do this, we must handle the fact that the operator variations on the rhs are located only at the points $\{x_1, \ldots, x_n\} \in M$. Choosing $(M, g) = (\mathbb{R}^d, \delta)$ for simplicity, we write

$$\epsilon^r(x_i)\, \delta_r\mathcal{O}_i(x_i) = \int_M \delta^d(x - x_i)\, \epsilon^r(x)\, \delta_r\mathcal{O}_i(x_i)\, \mathrm{d}^d x$$

as an integral, so that all terms in (6.44) are proportional to $\epsilon^r(x)$. Since these may be chosen arbitrarily, we obtain finally[63]

$$\partial_\mu \left\langle j_r^\mu(x) \prod_{i=1}^{n} \mathcal{O}_i(x_i) \right\rangle = -\sum_{i=1}^{n} \delta^d(x - x_i) \left\langle \delta_r\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle\,. \tag{6.45}$$

stating that the divergence of a correlation function with an insertion of the current $j_r^\mu$ vanishes everywhere except at the locations of other operator insertions. This is the modification of the conservation law $\partial_\mu \langle j_r^\mu(x)\rangle = 0$ for the expectation value of the current

[62] We must either take $M$ compact, or else ask that either $\epsilon^r(x)$ have compact support or impose suitable boundary conditions on the fields so as to avoid any boundary terms in the integration by parts.

[63] The appropriate generalization to a Riemannian manifold $(M, g)$ is

$$\mathrm{d} * \left\langle j_r(x) \prod_{i=1}^{n} \mathcal{O}_i(x_i) \right\rangle = -\ * \sum_{i=1}^{n} \delta^d(x - x_i) \left\langle \delta_r\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle\,,$$

where the exterior derivative on the lhs acts on $x$.

itself. Again, note that the divergence is taken *after* computing the path integral. The relation (6.45) is also known as a Ward–Takahashi identity. In particular, when $M$ is compact, integrating (6.45) over all of $M$ gives

$$\sum_{i=1}^{n}\left\langle \delta_r\mathcal{O}_i(x_i)\prod_{j\neq i}\mathcal{O}_j(x_j)\right\rangle = 0 \tag{6.46}$$

which is just the infinitesimal version of our previous, global form of Ward–Takahashi identity (6.33). However, the local version (6.45) contains far more information.

As an example, let's consider the original use of the local form of the Ward–Takahashi identity, which remains one of the most important. Consider the transformation

$$\psi\mapsto\psi' = \mathrm{e}^{\mathrm{i}\alpha}\psi\,,\qquad \bar\psi\mapsto\bar\psi' = \bar\psi\mathrm{e}^{-\mathrm{i}\alpha}\,,\qquad A_\mu\mapsto A'_\mu = A_\mu \tag{6.47}$$

acting linearly on the electron and photon fields. For constant $\alpha$, these are symmetries of the QED action

$$S_{\mathrm{QED}}[A,\psi] = \int \mathrm{d}^4x\left[\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \bar\psi(\not{\nabla}+m)\psi\right] \tag{6.48}$$

and the regularized path integral measure is also invariant under these transformations,

$$\mathcal{D}\psi\,\mathcal{D}\bar\psi\mapsto\mathcal{D}\psi'\,\mathcal{D}\bar\psi' = \mathcal{D}\psi\,\mathcal{D}\bar\psi\,, \tag{6.49}$$

provided we integrate over an equal number of $\psi$ and $\bar\psi$ modes in the regularized theory. Thus these transformations are indeed symmetries of the path integral.

As above, we promote $\alpha$ to a position–dependent parameter $\alpha(x)$ with

$$\psi(x)\mapsto\mathrm{e}^{\mathrm{i}\alpha(x)}\,\psi(x)\,,\qquad \bar\psi\mapsto\bar\psi(x)\,\mathrm{e}^{-\mathrm{i}\alpha(x)}\,,\qquad A_\mu(x)\mapsto A_\mu(x) \tag{6.50}$$

Although closely related, this is *not* a gauge transformation because the photon field $A_\mu$ itself remains unaffected. The action is of course not invariant under this local transformation, but provided our regularized measure depends only on the fields $(\psi,\bar\psi)$ and not their derivatives, the measure will still be. Thus the only contribution to the current comes from the action and one finds $j^\mu = \mathrm{i}\bar\psi\gamma^\mu\psi$. This is just the charged current to which the photon couples in QED.

For infinitesimal $\alpha(x)$ we have

$$\delta\psi(x) = \mathrm{i}\alpha(x)\psi(x)\,,\qquad \delta\bar\psi(x) = -\mathrm{i}\bar\psi(x)\alpha(x)$$

so the local Ward–Takahashi identity (6.45) for the correlation function $\langle\psi(x_1)\bar\psi(x_2)\rangle$ becomes

$$\partial_\mu\langle j^\mu(x)\psi(x_1)\bar\psi(x_2)\rangle = -\mathrm{i}\,\delta^d(x-x_1)\langle\psi(x_1)\bar\psi(x_2)\rangle + \mathrm{i}\,\delta^d(x-x_2)\langle\psi(x_1)\bar\psi(x_2)\rangle \tag{6.51}$$

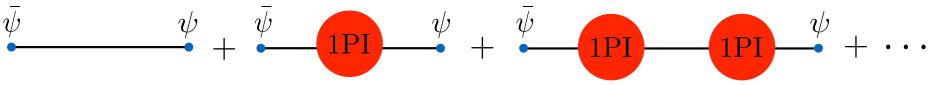
so that the vector $f_\mu(x,x_1,x_2) = \langle j_\mu(x)\psi(x_1)\bar\psi(x_2)\rangle$ is divergence free everywhere except at the insertions of the electron field. This identity is traditionally written in momentum space. We Fourier transform the two-point function of the electron field:

$$
\begin{aligned}
&\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}k_1\cdot x_1}\,\mathrm{e}^{-\mathrm{i}k_2\cdot x_2}\,\langle\psi(x_1)\bar\psi(x_2)\rangle \\
&\qquad = \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}k_1\cdot x_1}\,\mathrm{e}^{-\mathrm{i}k_2\cdot x_2}\,\langle\psi(x_1-x_2)\bar\psi(0)\rangle \\
&\qquad = \int \mathrm{d}^4y\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}k_1\cdot y}\,\mathrm{e}^{\mathrm{i}(k_1-k_2)\cdot x_2}\,\langle\psi(y)\bar\psi(0)\rangle \\
&\qquad = (2\pi)^4\,\delta^4(k_1-k_2)\,S(k_1)
\end{aligned}
\tag{6.52}
$$

where the first equality follows from translational invariance of the correlation function, and where

$$
S(k) = \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}k\cdot x}\langle\psi(x)\bar\psi(0)\rangle
\tag{6.53}
$$

is the exact electron propagator in momentum space. As usual, we can represent this exact propagator in terms of a geometric series

$S(k) =$ 


$$
\begin{aligned}
&= \frac{1}{\mathrm{i}\not{k}+m} + \frac{1}{\mathrm{i}\not{k}+m}\Sigma(\not{k})\frac{1}{\mathrm{i}\not{k}+m} + \frac{1}{\mathrm{i}\not{k}+m}\Sigma(\not{k})\frac{1}{\mathrm{i}\not{k}+m}\Sigma(\not{k})\frac{1}{\mathrm{i}\not{k}+m}\cdots \\
&= \frac{1}{\mathrm{i}\not{k}+m-\Sigma(\not{k})}\,,
\end{aligned}
\tag{6.54}
$$

where the **electron self–energy** $\Sigma(\not{k})$ is the sum of all 1PI Feynman graphs with one external $\psi$ and one external $\bar\psi$ (both amputated).

The remaining term in the Ward identity (6.51) involves the electromagnetic current $j_\mu$. In momentum space this becomes

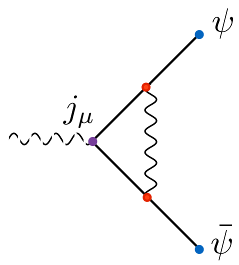
$$
\begin{aligned}
&\int \mathrm{d}^4x\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}p\cdot x}\,\mathrm{e}^{\mathrm{i}k_1\cdot x_1}\,\mathrm{e}^{-\mathrm{i}k_2\cdot x_2}\,\langle j_\mu(x)\psi(x_1)\bar\psi(x_2)\rangle \\
&\quad = \int \mathrm{d}^4x\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}p\cdot(x-x_2)}\,\mathrm{e}^{\mathrm{i}k_1\cdot(x_1-x_2)}\,\mathrm{e}^{\mathrm{i}(p+k_1-k_2)\cdot x_2}\,\langle j_\mu(x-x_2)\psi(x_1-x_2)\bar\psi(0)\rangle \\
&\quad = (2\pi)^4\,\delta^4(p+k_1-k_2)\,S(k_1)\Gamma_\mu(k_1,k_2)S(k_2)\,,
\end{aligned}
\tag{6.55}
$$

where the final line defines the **exact electromagnetic vertex** $\Gamma_\mu(k_1,k_2)$ in terms of the Fourier transform of $\langle j_\mu\psi\bar\psi\rangle$ and the exact electron propagator. To understand this definition, note that $\langle\psi(x_1)j_\mu(x)\bar\psi(x_2)\rangle$ will be given by the sum of all Feynman graphs connecting the electron field insertions at $x_{1,2}$ to the current at $x$. Recalling that $j_\mu = \bar\psi\gamma_\mu\psi$, we see that the leading contribution will simply come from a pair of propagators connecting $\psi(x_1)$ to $\bar\psi(x)$, and $\psi(x)$ to $\bar\psi(x_2)$ respectively. Further contributions will come from diagrams that correct each of these free propagators, turning them into the exact electron propagators on each side; *i.e.*

$$
\langle\psi(x_1)j_\mu(x)\bar\psi(x_2)\rangle \supset \langle\psi(x_1)\bar\psi(x)\rangle\,\gamma_\mu\,\langle\psi(x)\bar\psi(x_2)\rangle\,.
\tag{6.56}
$$

These diagrams tell us nothing new about the *vertex*; they're already part of the exact electron propagator. We thus include factors of $S(k_1)$ and $S(k_2)$ in our definition, accounting for all such diagrams.

The remaining contributions are the ones we care about. They involve graphs that connect the two exact electron propagators together in some way. For example, at leading order, we have the diagram




where the red dots are the usual QED vertex $-\mathrm{i}e\bar{\psi}\not{A}\psi$ while the purple dot denotes the insertion of the composite operator $j_\mu(x)$. Noting the $j_\mu$ is the current to which the photon couples, our picture includes an external photon line joining on to this vertex. This external photon line makes clear that we are computing a correction to the QED vertex, though it is not strictly part of the correlator $\langle j_\mu \psi \bar{\psi} \rangle$ and so should be amputated, as indicated by the dotted line in the diagram. Loop diagrams such as these provide $O(\hbar)$ corrections to the electron–photon vertex function, giving

$$\Gamma_\mu(k_1, k_2) = \gamma_\mu + \text{quantum corrections}\,, \tag{6.57}$$

where the external fermion lines are amputated in $\Gamma_\mu$, as explained above.

Now let's return to the Ward–Takahashi identity (6.51). Taking the Fourier transform of the complete equation, in momentum space this reads

$$(k_1 - k_2)_\mu\, S(k_1)\Gamma^\mu(k_1, k_2)S(k_2) = \mathrm{i}S(k_1) - \mathrm{i}S(k_2) \tag{6.58}$$

or equivalently

$$(k_1 - k_2)_\mu\, \Gamma^\mu(k_1, k_2) = \mathrm{i}S^{-1}(k_2) - \mathrm{i}S^{-1}(k_1) \tag{6.59}$$

by acting with $S^{-1}(k_1)$ on the left and $S^{-1}(k_2)$ on the right. (Recall that these are matrices in spin space.) The significance of this identity is that it relates quantum corrections to the electron kinetic term $\int \bar{\psi}(\not{\partial}+m)\psi\, \mathrm{d}^4x$ to quantum corrections to the electron–photon vertex $-\mathrm{i}e \int \bar{\psi}\not{A}\psi\, \mathrm{d}^4x = -\mathrm{i}e \int j^\mu A_\mu\, \mathrm{d}^4x$. The 'inverse electron propagator' $S^{-1}(k)$ is nothing but the electron kinetic term in the quantum effective action, written in momentum space, and the fact that the rhs of (6.51) involves the difference of these for the electron and positron just cancels the mass term.

Differentiating (6.51) wrt $k_1$ and then taking the limit $k_{1,2} \to k$ gives

$$\Gamma_\mu(k, k) = -\mathrm{i}\frac{\partial}{\partial k^\mu}S^{-1}(k) = \gamma_\mu + \mathrm{i}\frac{\partial}{\partial k^\mu}\Sigma(\not{k}) \tag{6.60}$$

showing indeed that the quantum corrections to the vertex are completely determined by the quantum corrections to the fermion propagator. Evaluating this at $k = 0$ shows that the quantum effective action renormalizes the complete covariant derivative term $\int \bar{\psi} \not{D} \psi \, \mathrm{d}^4x$ together, whilst the $k$ dependence in $\Sigma(\not{k})$ amounts to saying that higher derivative corrections must always be of the form $\sim \bar{\psi} D^p \psi$, again involving *covariant* derivatives. (You'll explore this further in the problem sets.)

### 6.4 Emergent symmetries

As we saw in chapter 4, irrespective of the details of its microscopic origin, at low energies a QFT is governed by the values of a relatively small number of relevant or marginal couplings. These couplings correspond to relevant and marginal operators that (typically) involve only a small number of powers of the fields or their derivatives. It's often the case that these few relevant and marginal operators are invariant under a wider range of field transformations than a generic, irrelevant operator would be. The effects of irrelevant operators are strongly suppressed at low energies, making it appear as though the theory has the larger symmetry group. Thus, symmetry can be **emergent** in the low–energy theory, even if not it is present in the microscopic theory.

As an example, consider a theory of electromagnetism coupled to several generations of charged fermions, denoted $\psi_i$, each with the same charge $-e$. We might imagine that $\psi_i$ describe the three generations of charged leptons in the Standard Model. The most general Lorentz– and gauge–invariant Lagrangian we can write down for these fields that contains only relevant and marginal operators is

$$\begin{aligned}\mathcal{L}[A, \psi_i] = &\frac{1}{4e^2} Z_3 \, F^{\mu\nu} F_{\mu\nu} \\ &+ \sum_{i,j} \left[ (Z_L)_{ij} \, \bar{\psi}_{Lj} \not{D} \psi_{Li} + (Z_R)_{ij} \, \bar{\psi}_{Rj} \not{D} \psi_{Ri} + M_{ij} \bar{\psi}_{Li} \psi_{Rj} + \bar{M}_{ij} \bar{\psi}_{Ri} \psi_{Lj} \right] ,\end{aligned} \tag{6.61}$$

where

$$\psi_{Li} = \frac{1}{2}(1 + \gamma_5)\psi_i \,, \qquad \psi_{Ri} = \frac{1}{2}(1 - \gamma_5)\psi_i$$

are the left– and right–handed parts of the fermions, where $Z_3$ and $Z_{L,R}$ are possible wavefunction renormalization factors for the photon[64] for and leptons, and where $M_{L,R}$ are lepton mass terms. For the Lagrangian (6.61) to be real, the matrices $Z_{L,R}$ must be Hermitian, while their eigenvalues must be positive if we are to have the correct sign kinetic terms.

If the wavefunction renormalization matrices $(Z_{L,R})_{ij}$ are non–diagonal then the form of (6.61) suggests that processes such as $\psi_2 \to \psi_1 + \gamma$ are allowed, so that the absence of such a process in the Standard Model would seem to indicate an important new symmetry. However, this is a mirage. We introduce renormalized fields $\psi'_{L,R}$ defined by $\psi_L = S_L \psi'_L$ and $\psi_R = S_R \psi_R$. The Lagrangian for the new fields takes the same form, but with new matrices

$$Z'_L = S_L^\dagger Z_L S_L \,, \qquad Z'_R = S_R^\dagger Z_R S_R \,, \qquad M' = S_L^\dagger M S_R \,.$$

[64] It is conventional to denote the photon wavefunction renormalization factor by $Z_3$.

Now take $S_L$ to have the form $S_L = U_L D_L$, where $U_L$ is the unitary matrix that diagonalizes the positive–definite Hermitian matrix $Z_L$, and $D_L$ is a diagonal matrix whose entries and the inverses of the eigenvalues of $Z_L$. Such an $S_L$ ensures that $Z'_L = 1$, and we can arrange $Z_R = 1$ similarly. This condition does not completely fix the unitary matrix $U_L$, because if $Z'_L = 1$ then it is unchanged by conjugation by a further unitary matrix. We can use this remaining freedom to diagonalize the mass matrix $M$. The polar decomposition theorem implies that any complex square matrix $M$ can be written as $M = VH$ where $V$ is unitary and $H$ is a positive semi–definite Hermitian matrix. Thus, we perform a further field redefinition $\psi'_L = S'_L\psi''_L$ and $\psi'_R = S'_R\psi''_R$ with $S'_L = (S'_R)^\dagger V^\dagger$ and choose $S'_R$ to be the unitary matrix that diagonalizes $H$.

In terms of the new fields the Lagrangian (6.61) becomes finally (dropping all the primes)

$$\begin{aligned}\mathcal{L}[A,\psi] &= \frac{1}{4e^2} Z_3\, F^{\mu\nu}F_{\mu\nu} + \sum_i \left[\bar{\psi}_{Li}\not{D}\psi_{Li} + \bar{\psi}_{Ri}\not{D}\psi_{Ri} + m_i\bar{\psi}_{Li}\psi_{Ri} + m_i\bar{\psi}_{Ri}\psi_{Li}\right] \\ &= \frac{1}{4e^2} Z_3\, F^{\mu\nu}F_{\mu\nu} + \sum_i \bar{\psi}_i(\not{D} + m_i)\psi_i\ .\end{aligned} \tag{6.62}$$

This form of the Lagrangian manifestly shows that the 'new' fields $\psi_i$ have conserved individual lepton numbers. It's easy to write down an interaction that would violate these individual lepton numbers, such as $Y_{ijkl}\bar{\psi}_i\gamma^\mu\psi_j\,\bar{\psi}_k\gamma_\mu\psi_l$. However, all such operators have mass dimension $> 4$ and so are suppressed in the low–energy effective action. Lepton number conservation is merely an accidental property of the Standard Model, valid[65] only at low–energies.

Higher dimension operators can lead to processes such as proton decay that are impossible according to the dimension $\leq 4$ operators that dominate the low–energy behaviour. Thus, although such processes are highly suppressed, they are very distinctive signatures of the presence of higher dimension operators. Experimental searches for proton decay put important limits on the scale at which the new physics responsible for generating these interactions comes into play. Sorting out the details in various different possible extensions of the Standard Model is one of the main occupations of particle phenomenologists.

In fact, there are arguments to suggest that there *are no* continuous global symmetries in a quantum theory of gravity. Certainly; there are no such continuous global symmetries in string theory (though discrete global symmetries do exist). From this perspective, *all* the continuous symmetries that guided the development of so much of 20$^{\text{th}}$ Century physics may be low–energy accidents.

### 6.5 Low energy effective field theory

The concept of a symmetry being emergent under renormalization group flow is tremendously powerful. In trying to construct low energy effective actions, we should simply

[65]In fact, certain non–perturbative processes known as sphalerons lead to a very small violation of lepton number even in with dimension 4 operators. However the difference $B-L$ between baryon number and lepton number is precisely conserved in the Standard Model, yet is believed to be just an accidental symmetry.

identify the relevant degrees of freedom for the system we wish to study and then write down all possible interactions that are compatible with the expected symmetries. At low energies, the most important terms in this action will be those that are least suppressed by powers of the scale $\Lambda$. Thus, to describe some particular low–energy phenomenon, we simply write down the lowest dimension operators that are capable of causing this effect. Let's illustrate this by looking at several examples.

### 6.5.1 Why is the sky blue?

As a first example, we'll use effective field theory to understand how light is scattered by the atmosphere. Visible light has a wavelength between around 400nm and 700nm, while atmospheric $N_2$ has a typical size of $\sim 7 \times 10^{-6}$nm, nearly a million times smaller. Thus, when sunlight travels through the atmosphere we do not expect to have to understand all the details of the microscopic $N_2$ molecules, so we neglect all its internal degrees of freedom and model the $N_2$ by a complex scalar field $\phi$ so that excitations of $\phi$ correspond to creation of an $N_2$ molecule (with excitations of $\bar{\phi}$ creating anti–Nitrogen). Importantly, because the $N_2$ molecules are electrically neutral, $\phi$ is uncharged so $D_\mu\phi = \partial_\mu\phi$ and Nitrogen does not couple to light via a covariant derivative.

The presence of the atmosphere explicitly breaks Lorentz invariance, defining a preferred rest frame with 4-velocity $u^\mu = (1, 0, 0, 0)$, so the kinetic term of the $\phi$ field is $\int \mathrm{d}^4x\, \frac{1}{2}\bar{\phi}u^\mu\partial_\mu\phi$ showing that the field $\phi$ has mass dimension 3/2. The lowest dimension gauge invariant couplings between $\phi$ and $A_\mu$ we can write down are

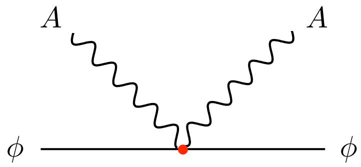
$$|\phi|^2 F^{\mu\nu}F_{\mu\nu} \qquad \text{and} \qquad |\phi|^2 u^\mu u^\nu F_{\mu\kappa}F_\nu{}^\kappa\,,$$

where we've allowed ourselves to use the preferred 4-velocity $u^\mu$. In $d = 4$, each of these interactions has mass dimension 7 so, schematically, the effective interactions responsible for this scattering must take the form

$$S^{\mathrm{int}}_\Lambda[A, \phi] = \int \mathrm{d}^4x \left[\frac{g_1}{8\Lambda^3}\phi^2F^2 + \frac{g_2}{8\Lambda^3}\phi^2(u\cdot F)^2 + \cdots\right] \tag{6.63}$$

where the couplings $g_{1,2}(\Lambda)$ are dimensionless and $\Lambda$ is the cut–off scale. In the case at hand, the obvious cut–off scale is the inverse size of the $N_2$ molecule whose orbital electrons are ultimately responsible for the scattering. We expect our effective theory really contains infinitely many further terms involving higher powers of $\phi$, $F$ and their derivatives. However, on dimensional grounds these will all be suppressed by higher powers of $\Lambda$ and so will be negligible at energies $\ll \Lambda$. The $\phi^2F^2$ terms themselves must be retained if we want to understand how light can be scattered by $\phi$ at all.

Now let's consider computing a scattering amplitude $\phi + \gamma \to \phi + \gamma$ using the theory (6.63). The vertices $\phi^2F^2$ and $\phi^2(u \cdot F)^2$ each involve two copies of $\phi$ and two copies of the photon, so can both contribute to this scattering at tree–level. In particular, for $g_2$ we find the diagram



$$\sim \frac{|\mathbf{k}|^2 g_2}{\Lambda^3}$$

Because the interaction proceeds via the fieldstrength $F$ rather than $A$ directly, it involves a derivative of each photon, bringing down a power of the photon's momentum. For massless particles such as the photon, $|\mathbf{k}| = \omega$, so the amplitude $\mathcal{A} \propto g_2\omega^2/\Lambda^3$ and the tree–level scattering cross–section takes the form

$$\sigma_{\text{Rayleigh}} = |\mathcal{A}|^2 \propto g_2^2\,\frac{\omega^4}{\Lambda^6} \tag{6.64}$$

characteristic of **Rayleigh scattering**. Loop corrections to this cross–section will involve higher powers of the coupling $g_2/\Lambda^3$ and so will be suppressed for photon energies $\ll \Lambda$. Since the cross–section increases rapidly with frequency, blue light is scattered much more than red light, so the daylight sky in a direction away from the sun appears blue.

Our treatment of the scattering using the simple effective action (6.63) is only justified if $\omega \ll \Lambda$, where $\Lambda$ was the inverse size of the $N_2$ molecules responsible for the scattering at a microscopic level. It will thus fail for photons of too high energy, or if the visible light enters a region where the scattering is done by larger particles. In particular, as water droplets coalesce in the atmosphere they can easily reach sizes in excess of the wavelength of visible light. In this case the relevant scale $\Lambda$ is the inverse size of the water droplet, and (6.63) will be unreliable for light in the visible spectrum; there are infinitely many higher order terms $|\phi|^{2r}F^{2s}/M^{2s+3r-4}$ (and also further derivative interactions) that will be just as important. Thus there's no reason to expect that higher frequencies will be scattered more. Clouds are white.

### 6.5.2 Why does light bend in glass?

In vacuum, the lowest–order gauge, Lorentz and parity invariant action we can write for the photon is of course the Maxwell action

$$S_{\text{M}}[A] = \frac{1}{4\mu_0}\int_{\mathbb{R}^{3,1}} \mathrm{d}^4x\, F^{\mu\nu}F_{\mu\nu} = \frac{1}{4}\int_{\mathbb{R}^{3,1}} \mathrm{d}t\,\mathrm{d}^3x\,\left(\epsilon_0\mathbf{E}\cdot\mathbf{E} - \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right) \tag{6.65}$$

where $\mu_0$ is the magentic permeability of free space. For later convenience, we've written this term out in non–relativistic notation, using $c^2 = 1/\mu_0\epsilon_0$ to introduce the electric permittivity $\epsilon_0$ in the electric term.

In the presence of other sources, we should add a new term to this action that describe interactions between the photons and the sources. Low-energy effective field theory provides a powerful way to think about these new interaction terms. We suppose the degrees of freedom we expect to be important at some scale $\Lambda$ can be described by some field(s) $\Phi$, which may have arbitrary spin, charge *etc.*. Then in general we'd expect the new action to be

$$S = S_{\text{M}}[A] + S_{\Lambda}^{\text{int}}[A,\Phi]\,. \tag{6.66}$$

The Euler–Lagrange equations become

$$\partial^\mu F_{\mu\nu} = \mu_0 J_\nu \,, \tag{6.67}$$

where the current $J_\mu(x) := \delta S^{\rm int}_\Lambda / \delta A^\nu(x)$ is defined to be the variation of the new interaction terms. Together with the Bianchi identity $\partial_{[\kappa} F_{\mu\nu]} = 0$, these give Maxwell's equations.

For example, consider a piece of glass. Glass is an insulator, so the Fermi surface lies in a band gap. Thus, so long as the light with which we illuminate our glass has sufficiently low–frequency, the electrons will be unable to move and the insulator has *no* relevant degrees of freedom. In this case, we must have $S_{\rm int}[A,\Phi] = S_{\rm int}[A]$, so the interaction Lagrangian must be a sum of gauge invariant terms built from $A$. However, while the local structure of glass is invariant under rotations and reflections, a lump of glass is certainly not a Lorentz invariant as it has a defines a preferred rest frame. So in writing our effective interactions, there's no reason to impose Lorentz invariance. Thus for glass we should add a term

$$S^{\rm int}_\Lambda[A] = \int_{\rm glass} {\rm d}t\, {\rm d}^3x\, \frac{1}{2} \left(\chi_{\rm e}\, \mathbf{E}\cdot\mathbf{E} - \chi_{\rm m} \mathbf{B}\cdot\mathbf{B} + \cdots\right) \tag{6.68}$$

where the dimensionless couplings $\chi_{\rm e}(\Lambda)$ and $\chi_{\rm m}(\Lambda)$ are respectively the **electric** and **magnetic susceptibilities** of the glass. (Invariance under reflections rules out any $\mathbf{E}\cdot\mathbf{B}$ term.) Higher–order polynomials in $\mathbf{E}$, $\mathbf{B}$ and their derivatives are certainly allowed, but by dimensional analysis must come suppressed by a power of the electron bang–gap energy $\Lambda$. The field equations obtained from $S_{\rm M} + S^{\rm int}_\Lambda$ show that light travels through the glass with reduced speed given by

$$c^2_{\rm glass} = \frac{1}{(\epsilon_0 + \chi_{\rm e})} \left(\frac{1}{\mu_0} + \chi_{\rm m}\right) ,$$

This leads to Snell's Law at an interface and the appearance of bending.

Notice that our EFT argument doesn't tell us anything about the *values* of $\chi_{\rm e}$ or $\chi_{\rm m}$; for that we'd need to know more about the microphysics of the silicon in the glass. However, it does predict that integrating out the high energy degrees of freedom (the electrons in the glass) will lead to an effective Lagrangian that at low–energies must look like (6.68). Since $\mathbf{E}$ and $\mathbf{B}$ each have mass dimension $+1$, any higher powers of these fields would be suppressed by powers of the energy scale required to excite electrons.

Similarly, a crystal such as calcium carbonate or quartz has a lattice structure that breaks rotational symmetry. For such materials there is no reason for the effective Lagrangian to be rotationally symmetric, so we should expect different permeabilities and permittivities for the different components of $\mathbf{E}$ and $\mathbf{B}$:

$$S_{\rm crystal} = \int {\rm d}t\, {\rm d}^3x\, \frac{1}{2} \left((\chi_{\rm e})_{ij} {\rm E}_i {\rm E}_j - (\chi_{\rm m})_{ij} {\rm B}_i {\rm B}_j\right) \,. \tag{6.69}$$

This leads to different speeds of propagation for the different polarization states of light, resulting in the phenomenon of **birefringence**. (See figure 9.)


**Figure 9**: *Ancient Viking texts describe a sólarstei that could be used to determine the direction of the Sun even on a cloudy day, and was an important navigational aid. It's believed that this sunstone was a form of calcium carbonate ($CaCO_3$), or calcite. Calcite is birefringent, so different polarizations of light travel through it at different speeds, leading to multiple imaging. Near the Arctic, sunlight is quite strongly polarized, with the polarization reduced in directions away from the sun due to random scattering from the atmosphere. Thus, by moving a calcite crystal around one can detect the direction of the Sun even when obscured by clouds.*

As we've emphasized above, the effective actions (6.68)-(6.69) are only the lowest–order terms in the infinite series we obtain from integrating out high–energy modes corresponding to the atoms in the glass or crystal. The next order terms take the schematic form[66]

$$\sim \int \mathrm{d}t\, \mathrm{d}^3x \left[\frac{\alpha}{\Lambda^2}\mathrm{E}^4 + \frac{\beta}{\Lambda^2}\mathrm{B}^4 + \frac{\gamma}{\Lambda^2}\mathrm{E}^2\mathrm{B}^2 + \frac{\delta}{\Lambda^2}(\partial \mathrm{E})^2 + \frac{\varepsilon}{\Lambda^2}(\partial \mathrm{B})^2\right]$$

and on dimensional grounds will be suppressed by an energy scale $\Lambda$ of order the excitation energy of the insulating material. Such higher order terms lead to Euler–Lagrange equations that are nonlinear in the electric and magnetic fields. We'd expect them to become important as the energy density of the electric or magnetic fields grows to become appreciable on the scale $\Lambda$. Indeed, we've seen an example of this in the Euler–Heisenberg effective action for QED at energy scales much lower than the mass of the lightest charged particle. Aside from breaking Lorentz invariance, the only rôle of the insulating material here is to reduce the scale $\Lambda$ to the *effective* mass of the lightest charge carrier – the band structure of crystals means that this can be very much lower than the 511 keV electron mass.

Such non-linear terms mean that a powerful laser will not pass directly through an insulating material in a straight line, but rather will be scattered in a very complicated non–linear way. Indeed this is observed (see figure 10) and is the starting–point for the whole field of **Nonlinear Optics**.

[66] Alternatively, we can take $\epsilon = \epsilon(\mathbf{E}, \mathbf{B})$ and $\mu = \mu(\mathbf{E}, \mathbf{B})$ so that the permittivity and permeability are themselves functions of the fields.

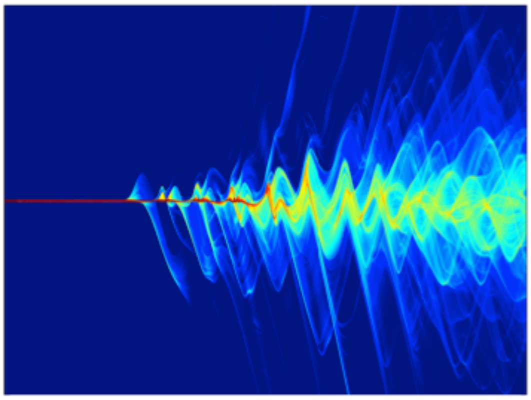


**Figure 10**: *Higher–order terms in the effective action lead to nonlinear optical effects such as the generation of these harmonics from the incoming laserbeam on the left. Notice that the scattered light has a higher frequency than the incoming red light. (Figure taken from the Nonlinear Optics group at Universiteit Twente, Netherlands.)*

### 6.5.3 Quantum gravity as an EFT

The final example of a low-energy effective field theory I'd like to discuss is General Relativity. Including (as we should) a cosmological constant $\lambda$, the Einstein–Hilbert action for General Relativity is

$$S_{\mathrm{EH}}[g] = \int \mathrm{d}^4x\sqrt{-g}\left[\lambda + \frac{1}{16\pi G_{\mathrm{N}}}\mathrm{R}(g)\right] \tag{6.70}$$

where $R(g)$ is the Ricci scalar. The cosmological constant has mass dimension 4 and so is relevant (in fact, this is the most relevant operator of all). The Riemann tensor involves the second derivative of the metric and so has mass dimension +2, showing that the Newton constant $G_{\mathrm{N}}$ must have mass dimension $-2$ in four dimensions, as is well–known. If we work perturbatively around flat space, writing $g = \delta + \sqrt{G_{\mathrm{N}}}h$ so that the metric fluctuation $h$ has canonically normalized kinetic terms appropriate for a spin-2 field, then we will obtain a positive power of the Newton constant $G_{\mathrm{N}}$ in front of all interactions $\sim h^3$ and higher. These interactions are thus all *irrelevant*.

Clearly, we require such metric interactions in order to account for experimental phenomena (things fall down!). How is this compatible with our understanding of QFT? One possibility is as follows. We know that it is perfectly possible (and indeed generic) for the quantum effective action representing a continuum QFT to contain a (perhaps infinite) number of irrelevant interactions, provided their coefficients are fixed in terms of the parameters governing the relevant and marginal interactions — in other words, we generate irrelevant interactions as we move away from a critical point along a renormalized trajectory. In the case of gravity, near the Gaussian fixed point (free, massless spin-2 field) there are no relevant or marginal interactions, so the renormalized trajectory would just stay as free theory. Thus, if this is the case, gravity must be controlled by some other, strongly coupled critical point in the UV. This scenario goes by the name of **asymptotic safety**. It's an active topic of research, but I think it's safe to say it's a long shot. Another, in my opinion more likely, possibility is that Nature is telling us that GR simply isn't a fundamental, UV complete theory. New degrees of freedom are needed to make sense of Planck scale physics — perhaps strings.

Nonetheless, GR makes perfect sense as a low–energy effective field theory. In this case, we should expect that the Einstein–Hilbert action is just the first term in an infinite series of possible higher–order couplings. Diffeomorphism invariance restricts these higher–order terms to be products of the metric and covariant derivatives of the Riemann tensor, and the first few terms are

$$S_\Lambda^{\text{eff}}[g] = \int \mathrm{d}^4x\sqrt{-g}\left[c_0\Lambda^4 + c_1\,\Lambda^2 R + c_2\,R^2 + c_3\,R^{\mu\nu}R_{\mu\nu} + c_4\,R^{\mu\nu\rho\kappa}R_{\mu\nu\rho\kappa} + \cdots\right] \tag{6.71}$$

where the couplings $c_i$ are dimensionless. In fact, it can be shown that a linear combination of the couplings $c_2$, $c_3$ and $c_4$ is proportional to the four–dimensional Gauss–Bonnet term which is topological and does not affect perturbation theory.

### 6.6 Charges, quantum states and representations

Let's integrate the Ward identity over some region $M' \subseteq M$ with boundary $\partial M' = N_1 - N_0$, just as we studied classically. We'll first choose $M'$ to contain all the points $\{x_1, \ldots, x_n\}$ so that the integral receives contributions from all of the terms on the *rhs* of (6.45). Then

$$\langle Q[N_1] \prod_i \mathcal{O}_i(x_i)\rangle - \langle Q[N_0] \prod_i \mathcal{O}_i(x_i)\rangle = -\sum_{i=1}^{n} \langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j)\rangle \tag{6.72}$$

where the charge $Q[N] = \int_N *j$ just as in the classical case. In particular, if $M' = M$ and $M$ is closed (*i.e.*, compact without boundary) then we obtain

$$0 = \sum_{i=1}^{n} \langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j)\rangle \tag{6.73}$$

telling us that if we perform the symmetry transform throughout space–time then the correlation function is simply invariant, $\delta\langle \prod_i \mathcal{O}_i\rangle = 0$. This is just the infinitesimal version of the result we had before in (6.33).

On the other hand, if only one some of the $x_i$ lie inside $M'$, then only some of the $\delta$-functions will contribute. In particular, if $I \subseteq \{1, 2, \ldots, n\}$ then we obtain

$$\langle Q[N_1] \prod_{i=1}^{n} \mathcal{O}_i(x_i)\rangle - \langle Q[N_0] \prod_{i=1}^{n} \mathcal{O}_i(x_i)\rangle = -\sum_{i\in I} \langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j)\rangle\,. \tag{6.74}$$

whenever $x_i \in M'$ for $i \in I$. Only those operators enclosed in $M'$ contribute to the changes on the *rhs*.

Note that the condition that $M$ be closed cannot be relaxed lightly. On a manifold with boundary, to define the path integral we must specify some boundary conditions for the fields. The transformation $\phi \mapsto \phi'$ may now affect the boundary conditions, which lead to further contributions to the *rhs* of the Ward identity. For a relatively trivial example, the condition that the net charges of the operators we insert must be zero becomes modified to the condition that the *difference* between the charges of the incoming and outgoing states (boundary conditions on the fields) must be balanced by the charges of the operator insertions.

A much more subtle example arises when the space–time is non–compact and has infinite volume. In this case, the required boundary conditions as $|x| \to \infty$ are that our fields take some constant value $\phi_0$ which lies at the minimum of the effective potential. Because of the suppression factor $\mathrm{e}^{-S[\phi]}$, such field configurations will dominate the path integral on an infinite volume space–time. However, it may be that the (global) minimum of the potential is not unique; if $V(\phi)$ is minimized for any $\phi \in \mathcal{M}$ and our symmetry transformations move $\phi$ around in $\mathcal{M}$ the symmetry will be **spontaneously broken**. You'll learn much more about this story if you're taking the Part III Standard Model course.