Quantum Field Theory II

University of Cambridge Part III Mathematical Tripos

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Abstract: These are the lecture notes for the Advanced Quantum Field Theory course given to students taking Part III Maths in Cambridge during Lent Term of 2015. The main aim is to introduce the Renormalization Group and Effective Field Theories from a path integral perspective.

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Acknowledgments

Nothing in these lecture notes is original. In particular, my treatment is heavily influenced by several of the textbooks listed below, especially Vafa *et al.* and the excellent lecture notes of Neitzke in the early stages, then Schwartz and Weinberg's textbooks and Hollowood's lecture notes later in the course.

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1 Preliminaries

This course is the second course on Quantum Field Theory offered in Part III of the Maths Tripos, so I'll feel free to assume you've already taken the first course in Michaelmas Term (or else an equivalent course elsewhere). You will also find it helpful to know about groups and representation theory, say at the level of the *Symmetries, Fields and Particles* course last term. There may be some overlap between this course and certain other Part III courses this term. In particular, I'd expect the material here to complement the courses on *The Standard Model* and on *Applications of Differential Geometry to Physics* very well. In turn, I'd also expect this course to be useful for courses on *Supersymmetry* and *String Theory*.

1.1 Books & Other Resources

There are many (too many!) textbooks and reference books available on Quantum Field Theory. Different ones emphasize different aspects of the theory, or applications to different branches of physics or mathematics – indeed, QFT is such a huge subject nowadays that it is probably impossible for a single textbook to give an encyclopedic treatment (and absolutely impossible for a course of 24 lectures to do so). Here are some of the ones I've found useful while preparing these notes; you might prefer different ones to me. These lists are alphabetical by author.

This first list contains the main books for the course; you will certainly want to consult (at least) one of **Peskin & Schroeder**, or **Srednicki**, or **Schwartz** repeatedly during the course. They will also be very helpful for people taking the Standard Model course.

- **Peskin, M. and Schroeder, D.,** *An Introduction to Quantum Field Theory*, Addison–Wesley (1996).
  An excellent QFT textbook, containing extensive discussions of both gauge theories and renormalization. Many examples worked through in detail, with a particular emphasis on applications to particle physics.

- **Schwartz, M.,** *Quantum Field Theory and the Standard Model*, CUP (2014).
  The new kid on the block, published just last year after being honed during the author's lecture courses at Harvard. I really like this book – it strikes an excellent balance between formalism and applications (mostly to high energy physics), with fresh and clear explanations throughout.

- **Srednicki, M.,** *Quantum Field Theory*, CUP (2007).
  This is also an excellent, very clearly written and very pedagogical textbook. Along with Peskin & Schroeder and Schwartz, it is perhaps the single most appropriate book for the course, although none of these books emphasize the more geometric aspects of QFT.

- **Zee, A.,** *Quantum Field Theory in a Nutshell*, 2^{nd} edition, PUP (2010).
  An great book if you want to keep the big picture of what QFT is all about firmly

in sight. QFT is notorious for containing many technical details, and its easy to get lost. This book will put you joyfully back on track and remind you why you wanted to learn the subject in the first place. It's not the best place to learn how to do specific calculations, but that's not the point.

There are also a large number of books that are more specialized. Many of these are rather advanced, so I do not recommend you use them as a primary text. However, you may well wish to dip into them occasionally to get a deeper perspective on topics you particularly enjoy. This list is particularly biased towards my (often geometric) interests:

- **Banks, T.** *Modern Quantum Field Theory: A Concise Introduction*, CUP (2008).
  I particularly enjoyed its discussion of the renormalization group and effective field theories. As it says, this book is probably too concise to be a main text.

- **Cardy, J.,** *Scaling and Renormalization in Statistical Physics*, CUP (1996).
  A wonderful treatment of the Renormalization Group in the context in which it was first developed: calculating critical exponents for phase transitions in statistical systems. The presentation is extremely clear, and this book should help to balance the 'high energy' perspective of many of the other textbooks.

- **Coleman, S.,** *Aspects of Symmetry*, CUP (1988).
  Legendary lectures from one of the most insightful masters of QFT. Contains much material that is beyond the scope of this course, but so engagingly written that I couldn't resist including it here!

- **Costello, K.,** *Renormalization and Effective Field Theory*, AMS (2011).
  A pure mathematician's view of QFT. The main aim of this book is to give a rigorous definition of (perturbative) QFT via path integrals and Wilsonian effective field theory. Another major achievement is to implement this for gauge theories by combining BV quantization with the ERG. Repays the hard work you'll need to read it – for serious mathematicians only.

- **Deligne, P.,** *et al.*, *Quantum Fields and Strings: A Course for Mathematicians* vols. 1 & 2, AMS (1999).
  Aimed at professional mathematicians wanting an introduction to QFT. They thus require considerable mathematical maturity to read, but most certainly repay the effort. For this course, I particularly recommend the lectures of Deligne & Freed on *Classical Field Theory* (vol. 1), Gross on the *Renormalization Group* (vol. 1), and especially Witten on *Dynamics of QFT* (vol. 2).

- **Nair, V.P.,** *Quantum Field Theory: A Modern Perspective*, Springer (2005).
  Contains excellent discussions of anomalies, the configuration space of field theories, ambiguities in quantization and QFT at finite temperature.

- **Polyakov, A.,** *Gauge Fields and Strings*, Harwood Academic (1987).
  A very original and often very deep perspective on QFT. Several of the most impor-

tant developments in theoretical physics over the past couple of decades have been (indirectly) inspired by ideas in this book.

- **Schweber, S.,** *QED and the Men Who Made It: Dyson, Feynman, Schwinger and Tomonaga*, Princeton (1994).
  Not a textbook, but a tale of the times in which QFT was born, and the people who made it happen. It doesn't aim to dazzle you with how very great these heroes were[1], but rather shows you how puzzled they were, how human their misunderstandings, and how tenaciously they had to fight to make progress. Inspirational stuff.

- **Vafa, C., and Zaslow, E.,** (eds.), *Mirror Symmetry*, AMS (2003).
  A huge book comprising chapters written by different mathematicians and physicists with the aim of understanding Mirror Symmetry in the context of string theory. Chapters 8 – 11 give an introduction to QFT in low dimensions from a perspective close to the one we will start with in this course. The following chapters could well be useful if you're taking the String Theory Part III course.

- **Weinberg, S.,** *The Quantum Theory of Fields*, vols. 1 & 2, CUP (1996).
  Weinberg's thesis is that QFT is the inevitable consequence of marrying Quantum Mechanics, Relativity and the Cluster Decomposition Principle (that distant experiments yield uncorrelated results). Penetrating insight into everything it covers and packed with many detailed examples. The perspective is always deep, but it requires strong concentration to follow a story that sometimes plays out over several chapters. Particles play a primary role, with fields coming later; for me, this is backwards.

- **Zinn–Justin, J.** *Quantum Field Theory and Critical Phenomena*, 4^{th} edition, OUP (2002).
  Contains a very insightful discussion of the Renormalization Group and also a lot of information on Gauge Theories. Most of its examples are drawn from either Statistical or Condensed Matter Physics.

Textbooks are expensive. Fortunately, there are lots of excellent resources available freely online. I like these:

- **Dijkgraaf, R.,** *Les Houches Lectures on Fields, Strings and Duality*, `http://arXiv.org/pdf/hep-th/9703136.pdf`
  An modern perspective on what QFT is all about, and its relation to string theory. For the most part, the emphasis is on more mathematical topics (*e.g.* TFT, dualities) than we will cover in the lectures, but the first few sections are good for orientation.

- **Hollowood, T.,** *Six Lectures on QFT, RG and SUSY*, `http://arxiv.org/pdf/0909.0859v1.pdf`
  An excellent mini–series of lectures on QFT, given at a summer school aimed at end–of–first–year graduate students from around the UK. They put renormalization and

[1] I should say 'are'; Freeman Dyson still works at the IAS almost every day.

Wilsonian Effective Theories centre stage. While the final two lectures on SUSY go beyond this course, I found the first three very helpful when preparing the current notes.

- **Neitzke, A.,** *Applications of Quantum Field Theory to Geometry*,
  https://www.ma.utexas.edu/users/neitzke/teaching/392C-applied-qft/
  Lectures aimed at introducing mathematicians to Quantum Field Theory techniques that are used in computing Seiberg–Witten invariants (and a much wider range of related "Theories of class $\mathcal{S}$"). I very much like the perspective of these lectures, and we'll follow a similar path for at least the first part of the course.

- **Osborn, H.,** *Advanced Quantum Field Theory*,
  http://www.damtp.cam.ac.uk/user/ho/Notes.pdf
  The lecture notes for a previous incarnation of this course, delivered by Prof. Hugh Osborn. They cover similar material to the current ones, but from a rather different perspective. If you don't like the way I'm doing things, or for extra practice, take a look here!

- **Polchinski, J.,** *Renormalization and Effective Lagrangians*,
  http://www.sciencedirect.com/science/article/pii/0550321384902876

- **Polchinksi, J.,** *Dualities of Fields and Strings*,
  http://arxiv.org/abs/1412.5704
  The first paper gives a very clear description of the 'exact renormalization group' and its application to scalar field theory. The second is a recent survey of the idea of 'duality' in QFT and beyond. We'll explore this if we get time.

- **Segal, G.,** *Quantum Field Theory* lectures,
  YouTube lectures
  Recorded lectures aiming at an axiomatization of QFT by one of the deepest thinkers around. I particularly recommend the lectures "*What is Quantum Field Theory?*" from Austin, TX, and "*Three Roles of Quantum Field Theory*" from Bonn (though the blackboards are atrocious!).

- **Tong, D.,** *Quantum Field Theory*,
  http://www.damtp.cam.ac.uk/user/tong/qft.html
  The lecture notes for a previous incarnation of the Michaelmas QFT course in Part III. If you feel you're missing some background from last term, this is an excellent place to look. There are also some video lectures from when the course was given at Perimeter Institute.

- **Weinberg, S.,** *What Is Quantum Field Theory, and What Did We Think It Is?*,
  http://arXiv.org/pdf/hep-th/9702027.pdf

- **Weinberg, S.,** *Effective Field Theory, Past and Future*,
  http://arXiv.org/pdf/0908.1964.pdf

These two papers provide a fascinating account of the origins of effective field theories in current algebras for soft pion physics, and how the Wilsonian picture of Renormalization gradually changed our whole perspective of what QFT is about.

- **Wilson, K., and Kogut, J.** *The Renormalization Group and the ϵ-Expansion*, Phys. Rep. **12** 2 (1974),
  http://www.sciencedirect.com/science/article/pii/0370157374900234
  One of the first, and still one of the best, introductions to the renormalization group as it is understood today. Written by somone who changed the way we think about QFT. Contains lots of examples from both statistical physics and field theory.

That's a huge list, and only a real expert in QFT would have mastered everything on it. I provide it here so you can pick and choose to go into more depth on the topics you find most interesting, and in the hope that you can fill in any background you find you are missing.

2 QFT in zero dimensions

Quantum Field Theory is, to begin with, exactly what it says it is: the quantum version of a field theory. But this simple statement hardly does justice to what is the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology, geometry, combinatorics, astrophysics and cosmology. It's also extremely closely related to statistical field theory, probability and from there even to (quasi–)stochastic systems such as finance.

This all–encompassing remit means that QFT is a very powerful tool and you can do yourself damage by trying to operate it without due care. So that we don't get lost in technical minutiæ of particular calculations in specific theories, I want to begin slowly by studying the simplest possible QFT: that of a single, real–valued field in zero dimensional space–time. That's of course a very drastic simplification, and much of the richness of QFT will be absent here. However, you shouldn't sneer. We'll see that even this simple case contains baby versions of ideas we'll study more generally later in the course, and it will provide us with a safe playground in which to check we understand what's going on. Furthermore, it has been seriously conjectured that full, non-perturbative string theory is itself a zero–dimensional QFT (though admittedly with infinitely many fields). I expect that many of the ideas in this chapter will be things you've met before either in last term's QFT course or sometimes even earlier, but my perspective here may well be somewhat different.

2.1 The partition function

If our space–time M is zero–dimensional and connected, then it must be just a single point. In the simplest case, a 'field' on M is then just a map $\phi : \{\mathrm{pt}\} \to \mathbb{R}$, or in other words just a real variable. Notice that in zero dimensions, the Lorentz group is trivial, so in particular there's no notion of the fields' spin. Even more obviously, there are no space–time directions along which we could differentiate our 'field', so there can be no kinetic terms. The action is just a function $S(\phi)$ and the path integral becomes just an ordinary integral. The partition function becomes

$$\mathcal{Z} = \int_{\mathbb{R}} \mathrm{d}\phi \, \mathrm{e}^{-S(\phi)} \tag{2.1}$$

where we'll assume that $S(\phi)$ grows sufficiently rapidly as $|\phi| \to \infty$ so that this integral exists. Typically, we'll take $S(\phi)$ to be a polynomial (with highest term of even degree), such as

$$S(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \cdots \tag{2.2}$$

The partition function then depends on the values of the coupling constants, so

$$\mathcal{Z} = \mathcal{Z}(m^2, \lambda, \cdots) \,. \tag{2.3}$$

As a piece of notation, I'll often write $\mathcal{Z}_0$ for the partition function in the free theory, where the couplings of all but the term quadratic in the field(s) are set to zero.

We should think of the set of couplings as being coordinates on the infinite dimensional 'space of theories' in the sense that, at least for our single field ϕ, the theory is specified once we choose values for all possible monomials ϕ^p.

2.2 Correlation functions

Beyond the partition function, the most important object we wish to compute in any QFT are (normalized) **correlation functions**. These are just weighted integrals

$$\langle f \rangle := \frac{1}{\mathcal{Z}} \int_{\mathbb{R}} \mathrm{d}\phi\, f(\phi)\, \mathrm{e}^{-S(\phi)} \tag{2.4}$$

where along with e^{-S} we've inserted some $f(\phi)$ into the integral. Here, f should be thought of as a function (or perhaps a distribution, such as $\delta(\phi - \phi_0)$ or similar) *on the space of fields.* We'll assume the operators we insert are chosen so as not to disturb the rapid decay of the integrand at large values of the field, and are sufficiently well-behaved at finite ϕ that the integral (2.4) actually exists.

The usual way to think about correlation functions comes from probability. So long as the action $S(\phi)$ is $\mathbb{R}$-valued, $\mathrm{e}^{-S} \geq 0$ so we can view $\frac{1}{\mathcal{Z}}\mathrm{e}^{-S}$ as a *probability density* on the space of fields. The correlation function (2.4) is then just the *expectation value* $\langle f \rangle$ of $f(\phi)$ averaged over the space of fields with this measure, with the factor of $1/\mathcal{Z}$ ensuring that the probability measure is normalized. On a higher dimensional space–time M we'll be able to insert functions *at different points in space–time* into the path integral, and the correlation functions will probe whether there is any statistical relation between, say, two functions $f(\phi(x))$ and $g(\phi(y))$ at points $x, y \in M$.

In the context of QFT, we'll often choose the functions we insert to correspond to some quantity of physical interest that we wish to measure; perhaps the energy of the quantum field in some region, or the total angular momentum carried by some electrons, or perhaps temperature fluctuations in the CMB at different angles on the night sky. For reasons that will become apparent, we'll often call these functions 'operators', though the terminology is somewhat inaccurate (particularly in zero dimensions).

Alternatively, recalling that the general partition function $\mathcal{Z}(m^2, \lambda, \cdots)$ depends on the values of all possible couplings, we see that at least for operators that are polynomial in the fields, correlation functions describe the change in this general $\mathcal{Z}$ as we infinitesimally vary some combination of the couplings, evaluated at the point in theory space corresponding to our original model. For example, in the simplest case that $f(\phi) = \phi^p$ is monomial, we have formally

$$\frac{1}{p!}\langle \phi^p \rangle = -\frac{1}{\mathcal{Z}}\, \frac{\partial}{\partial \lambda_p} \mathcal{Z}(m^2, \lambda_i)\bigg|_{*} \tag{2.5}$$

where λ_p is the coupling to $\phi^p/p!$ in the general action, and $*$ is the point in theory space where the couplings are set to their values in the specific action that appears in (2.4).

2.3 The Schwinger–Dyson equations

Let's consider the effect of relabelling $\phi \to \phi + \epsilon$ in the path integral (2.1), where ϵ is a real constant. Since we integrate over the entire real line, trivially

$$\mathcal{Z} = \int_{\mathbb{R}} \mathrm{d}\phi \, \mathrm{e}^{-S(\phi)} = \int_{\mathbb{R}} \mathrm{d}(\phi + \epsilon) \, \mathrm{e}^{-S(\phi+\epsilon)} \tag{2.6}$$

and furthermore $\mathrm{d}(\phi+\epsilon) = \mathrm{d}\phi$ by translation invariance of the measure on $\mathbb{R}$. Taking ϵ to be infinitesimal and expanding the action to first order we find

$$\begin{aligned} \mathcal{Z} = \int_{\mathbb{R}} \mathrm{d}\phi \, \mathrm{e}^{-S(\phi+\epsilon)} &= \int_{\mathbb{R}} \mathrm{d}\phi \, \mathrm{e}^{-S(\phi)} \left[1 - \epsilon \frac{dS}{d\phi} + \cdots \right] \\ &= \mathcal{Z} - \epsilon \int_{\mathbb{R}} \mathrm{d}\phi \, \mathrm{e}^{-S(\phi)} \frac{dS}{d\phi} \end{aligned} \tag{2.7}$$

to first order in ϵ. Comparing both sides of (2.7) we see that inserting $dS/d\phi$ – the first order variation of the action *wrt* the 'field' ϕ – into the path integral for the partition function gives zero. In our zero dimensional context, the statement $dS/d\phi = 0$ in the path integral is the quantum equation of motion for the field ϕ. Alternatively, we can obtain this result directly by integrating by parts:

$$-\int_{\mathbb{R}} \mathrm{d}\phi \, \mathrm{e}^{-S} \frac{dS}{d\phi} = \int_{\mathbb{R}} \mathrm{d}\phi \, \frac{d}{d\phi} \left(\mathrm{e}^{-S} \right) = 0 \tag{2.8}$$

where there is no contribution from $|\phi| \to \infty$ since we assumed $\mathrm{e}^{-S(\phi)}$ decays rapidly. By the way, the derivative $d/d\phi$ here is really best thought of as a derivative *on the space of fields*; it's just that this space of fields is nothing but $\mathbb{R}$ in our zero dimensional example.

For example, suppose we consider the action

$$S = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \tag{2.9}$$

with $m^2 > 0$ and $\lambda \geq 0$. (Again: there can't be any derivative terms because we're in zero dimensions. For the same reason, there's no integral.) The classical equation of motion is

$$m^2 \phi = -\lambda \phi^3 / 6 \,, \tag{2.10}$$

and eq (2.7) says that this relation holds inside the path integral for the partition function.

However, while the classical equations of motion hold in the path integral *for the partition function*, the situation is different for more general correlation functions. We now find

$$\begin{aligned} \langle f \rangle &= \frac{1}{\mathcal{Z}} \int_{\mathbb{R}} \mathrm{d}\phi \, f(\phi+\epsilon) \, \mathrm{e}^{-S(\phi+\epsilon)} \\ &= \langle f \rangle - \frac{\epsilon}{\mathcal{Z}} \int_{\mathbb{R}} \mathrm{d}\phi \, \mathrm{e}^{-S} \left[f(\phi) \frac{dS}{d\phi} - \frac{df}{d\phi} \right] + \cdots \end{aligned} \tag{2.11}$$

to lowest order in ϵ (we'll assume f is at least once differentiable). Therefore we find

$$\left\langle f \, \frac{dS}{d\phi} \right\rangle = \left\langle \frac{df}{d\phi} \right\rangle \tag{2.12}$$

which you can again derive directly by integration by parts. This equation illustrates an important difference between quantum and classical theories. Classically, the equation of motion $dS/d\phi = 0$ *always* holds; the motion always extremizes the action. Noting that if f is some generic smooth function then so too is $df/d\phi$, there is no reason for the *rhs* of (2.12) to vanish. Thus it is *not* true that the classical eom $dS/d\phi = 0$ holds in the quantum system, and we can detect this failure through its effect on correlation functions.

For instance, later on we'll often set $f(\phi) = \prod_{i=1}^{n} p_i(\phi)$ where each of the $p_i(\phi)$ are polynomials. Then we find

$$\left\langle \frac{dS}{d\phi} \prod_{i=1}^{n} p_n(\phi) \right\rangle = \sum_{j=1}^{n} \left\langle \frac{dp_j}{d\phi} \prod_{i\neq j} p_i(\phi) \right\rangle \tag{2.13}$$

so the effect of inserting the classical equation of motion into the path integral is to modify each of the operators $p_i(\phi)$ in our correlation function in turn. This equation (and its cousins in higher dimensional QFT we'll meet later) is known as the **Schwinger–Dyson equation**.

2.4 Perturbation theory

Let's return to the example of $S(\phi) = m^2\phi^2/2 + \lambda\phi^4/4!$. The partition function $\mathcal{Z}(m, \lambda)$ is easy to evaluate at $\lambda = 0$ where we find $\mathcal{Z}(m, 0) = \sqrt{2\pi}/m$. For $\lambda > 0$ it is clear that the integral exists, but it looks hard to evaluate. We can try to treat it perturbatively by expanding in λ:

$$\begin{aligned}
\mathcal{Z}(m,\lambda) &= \int_{\mathbb{R}} \mathrm{d}\phi \sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!}\right)^n \frac{\phi^{4n}}{n!}\, \mathrm{e}^{-\frac{m^2}{2}\phi^2} \\
&\sim \sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!}\right)^n \frac{1}{n!} \int_{\mathbb{R}} \mathrm{d}\phi\, \phi^{4n}\, \mathrm{e}^{-\frac{m^2}{2}\phi^2} \\
&= \frac{\sqrt{2\pi}}{m} \times \left[1 - \frac{1}{8}\frac{\lambda}{m^4} + \frac{35}{384}\frac{\lambda^2}{m^8} + \cdots \right] .
\end{aligned} \tag{2.14}$$

where each term in this result follows from the standard result for a Gaussian integral using integration by parts.

In the second line we've exchanged the order of the integral and sum. This is a very dangerous step: infinite series are convergent iff they converge for some open disc in the complex plane (here for λ). But it's clear that the original integral would have diverged had $\mathrm{Re}(\lambda)$ been negative, so whatever $\mathcal{Z}(m,\lambda)$ actually is, it can't have a convergent power series around $\lambda = 0$. In fact, what we have computed is an **asymptotic series** for $\mathcal{Z}(m,\lambda)$ as $\lambda \to 0^+$. Recall that $\sum_{n=0}^{\infty} a_n\lambda^n$ is an asymptotic series for $\mathcal{Z}(\lambda)$ as $\lambda \to 0^+$ if, for all $N \in \mathbb{N}$

$$\lim_{\lambda\to 0^+} \frac{\left|Z(\lambda) - \sum_{n=0}^{N} a_n\lambda^n\right|}{\lambda^N} = 0 \; .$$

This definition says that for any natural number N, and any $\epsilon > 0$, for sufficiently small $\lambda \in \mathbb{R}_{\geq 0}$ the first N terms of the series differ from the exact answer by less than $\epsilon\lambda^N$.

However, since our series actually diverges, if we instead fix λ and include more and more terms in the sum, we will eventually get worse and worse approximations to the answer.

In fact, we can see this divergence of the perturbation series directly. Writing $\tilde{\phi} = m\phi$ we see that the coefficient of λ^n / m^{4n+1} in $\mathcal{Z}$ is

$$\begin{aligned} a_n &:= \frac{1}{(4!)^n\, n!} \int_{\mathbb{R}} \mathrm{d}\tilde{\phi}\, \mathrm{e}^{-\tilde{\phi}^2/2} \tilde{\phi}^{4n} \\ &= \frac{1}{(4!)^n\, n!} \int_{\mathbb{R}} \mathrm{d}\tilde{\phi}\, \exp\left(-\tilde{\phi}^2/2 + 4n \ln \tilde{\phi}\right) \end{aligned} \tag{2.15}$$

The integrand has maxima when $\tilde{\phi} = \pm\sqrt{4n}$ and as $n \to \infty$ it drops off rapidly away from these stationary points. Thus for large n the integral is dominated from the contribution near these maxima and we have

$$a_n \approx \frac{1}{(4!)^n\, n!} (4n)^{2n} \mathrm{e}^{-2n} \approx \mathrm{e}^{n \ln n} \tag{2.16}$$

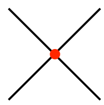
where the second approximation follows from Stirling's approximation $n! \approx \mathrm{e}^{n \ln n}$ for $n \to \infty$. Thus these coefficients asymptotically grow faster than exponentially with n, and the series (2.14) has zero radius of convergence. Perturbation theory thus tells us important, but not complete, information about our QFT.

2.4.1 Graph combinatorics

Working inductively, one can show that the coefficient of the general term $(-\lambda/m^4)^n$ in (2.14) is $\frac{1}{(4!)^n n!} \times \frac{(4n)!}{4^n (2n)!}$. The first factor of this comes straightforwardly from expanding the $\lambda\phi^4/4!$ vertex in the exponential, while the second factor comes from the resulting ϕ integral. Now, $(4n!)/4^n(2n)!$ is the number of ways of joining $4n$ elements into distinct pairs, suggesting that this second numerical factor should have a combinatorial interpretation that saves us having to actually perform the integral. This is what the Feynman rules provide.

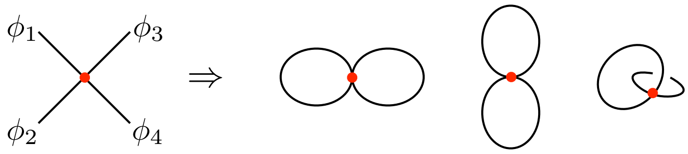
With the action $S(\phi) = m^2\phi^2/2 + \lambda\phi^4/4!$ the Feynman rules are simply

$$\frac{1}{m^2}$$



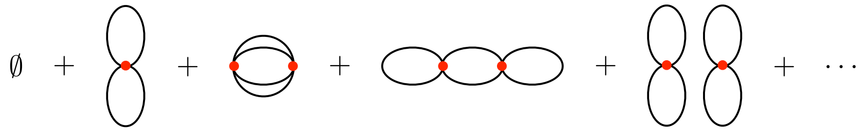
$$-\lambda$$

where the propagator is constant since we are in zero dimensions. The minus sign in the vertex comes from the fact that we are expanding e^{-S}. To compute perturbation series in QFT, Feynman tells us to construct all possible graphs (not necessarily connected) using this propagator and vertex. In the case of the partition function $\mathcal{Z}(m^2, \lambda)$, we want vacuum graphs, *i.e.*, those with no external edges. In constructing all possible such graphs, we imagine the individual vertices carry their own unique 'labels', so that we can tell them apart, and that likewise each of the four ϕ fields present in a given vertex carries its own label. Thus, the term proportional to λ receives contributions from three individual graphs



corresponding to the three possible ways to join up the four ϕ fields into pairs.

The partition function itself is the given by the sum of graphs



$$Z = 1 + \frac{-\lambda}{8m^4} + \frac{\lambda^2}{48m^8} + \frac{\lambda^2}{16m^8} + \frac{\lambda^2}{128m^8} + \cdots$$

where we include both connected and disconnected graphs, with the contribution of a disconnected graph being the product of the contributions of the two connected graphs. Notice that this requires that we assign a factor 1 to the trivial graph $\emptyset$ (no vertices or edges), which is also included as the zeroth–order term in the sum.

To work out the numerical factors, let D_n be the set of such graphs that contain precisely n vertices; since each vertex comes with a power of the coupling λ these diagrams will each contribute to the coefficient of λ^n in the expansion of $Z(m^2, \lambda)$. Suppose there are $|D_n|$ graphs in this set. Now, because Feynman instructed us to join up our labelled vertices in *every* possible way, every graph in D_n contains several copies that are identical as topological graphs but differ in the labeling of their vertices. We need to remove this overcounting. D_n is naturally acted on by the group $G_n = (S_4)^n \rtimes S_n$ that permutes each of the four fields in a given vertex (n copies of the permutation group S_4 on 4 elements) and also permutes the labels of each of the n vertices. This group has order $|G_n| = (4!)^n n!$, which is the same factor we saw before from expanding e^{-S} in powers of λ. Thus the asymptotic series may be rewritten as

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} \sim \sum_{n=0}^{\infty} \left(\frac{-\lambda}{m^4}\right)^n \frac{|D_n|}{|G_n|} . \tag{2.17}$$

In detail, the power $(-\lambda)^n$ is the contribution of the coupling constants in each graph, the power of $(1/m^2)^{2n}$ comes from the fact that any vacuum diagram with exactly n 4–valent vertices must have precisely $2n$ edges, each of which contributes a factor of $1/m^2$. Finally, the factor $|D_n|$ is the number of diagrams that contribute at this order and the factor $1/|G_n|$ is the coefficient of this graph in expanding the exponential of the action perturbatively in the interactions.

This way of working out the numerical coefficient requires that we draw all possible graphs obtained by joining up all the fields in all possible ways, as with the single λ vertex above. That's in principle straightforward, but in practice can be very laborious if there are many vertices, or vertices containing many powers of a field. There's another way to think of $|D_n|/|G_n|$ that sometimes makes life easier. An **orbit** Γ of G_n in D_n is a set of labeled graphs in D_n that are identical up to a relabeling of their fields and vertices, so that we can move from one labelled graph to another in the orbit using an element of G_n. Thus an orbit Γ is a topologically distinct graph in D_n. Let O_n be the set of such orbits. The **orbit stabilizer theorem** says that[2]

$$\frac{|D_n|}{|G_n|} = \sum_{\Gamma \in O_n} \frac{1}{|\mathrm{Aut}\,\Gamma|} \tag{2.18}$$

where $\mathrm{Aut}\,\Gamma$ is the stabilizer of any element in Γ in G_n, *i.e.*, the elements of the permutation group G_n that don't alter the labeled graph. For example, if a graph in D_n involves a propagator joining two fields at the same vertex, then exchanging the labeling of those fields won't change the labeled graph. Finally then, we can rewrite our asymptotic series (2.14) as

$$\begin{aligned} \frac{\mathcal{Z}}{\mathcal{Z}_0} &\sim \sum_{n=0}^{\infty} \left[\left(\frac{-\lambda}{m^4} \right)^n \sum_{\Gamma \in O_n} \frac{1}{|\mathrm{Aut}\,\Gamma|} \right] \\ &= \sum_{\Gamma} \frac{1}{|\mathrm{Aut}\,\Gamma|} \frac{(-\lambda)^{|v(\Gamma)|}}{(m^2)^{|e(\Gamma)|}}, \end{aligned} \tag{2.19}$$

where $|v(\Gamma)|$ and $|e(\Gamma)|$ are respectively the number of vertices and edges of the graph Γ.

In zero dimensions, we've rederived the Feynman rule that we should weight each topologically distinct graph by $|v(\Gamma)|$ powers of (minus) the coupling constant $-\lambda$ and $|e(\Gamma)|$ powers of the propagator $1/m^2$, then divide by the symmetry factor $|\mathrm{Aut}\,\Gamma|$ of the graph. More generally, if we have i different types of field, each with propagators $1/P_i$ and interacting via a set of vertice with couplings λ_α, then a graph Γ containing $|e_i(\Gamma)|$ edges of the field of type i and $|v_\alpha(\Gamma)|$ vertices of type α contributes a factor

$$\frac{1}{|\mathrm{Aut}\,\Gamma|} \prod_i \frac{1}{P_i^{|e_i(\Gamma)|}} \prod_\alpha (-\lambda_\alpha)^{|v_\alpha(\Gamma)|} \tag{2.20}$$

to the perturbative series.

To obtain the perturbative series for the partition function we sum this expression over both connected and disconnnected vacuum graphs, including the trivial graph with no vertices. It's often convenient to just include the connected graphs. We then have

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \exp\left[\sum_{\mathrm{conn}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \prod_{i,\alpha} \frac{(-\lambda_\alpha)^{|v_\alpha(\Gamma)|}}{P_i^{|e_i(\Gamma)|}} \right] =: \mathrm{e}^{-W+W_0} \tag{2.21}$$

where the sum in the exponential is only over connected, non–trivial graphs. Particularly in applications to statistical field theory, $W = \ln \mathcal{Z}$ is known as the **free energy**, while $W_0 = \ln \mathcal{Z}_0$. The identity (2.21) easily visualized by writing the power series expansion of the *rhs*, defining the product of two connected graphs to be the disconnected graph whose two connected components are the original graphs.

[2]If you don't know this already, you can find a nicely explained proof on Gowers's Weblog.

In practice, it is often just as quick to think through the possible ways a given topological graph Γ may be obtained by expanding out the vertices in e^{-S} and joining pairs of fields by propagators, as to work out the symmetry factor $|\mathrm{Aut}\,\Gamma|$. I'll leave it to your taste.

2.5 Supersymmetry and localization

Since the perturbative series is only an asymptotic series, it's worth asking if we can ever do better, even for interacting theories. Generically we cannot, but in special circumstances it is possible to evaluate the partition function and even certain correlation functions *exactly*. There are many mechanisms by which this might happen; this section gives a toy model of one of them, known as **localization**, which we'll meet again later (though it'll be in disguise).

Let's take a theory where that in addition to our bosonic field ϕ, we have two fermionic fields ψ_1 and ψ_2. With a zero–dimensional space–time, the space of fields is just $\mathbb{R}^{1|2}$. Given an action $S(\phi, \psi_i)$ the partition function is, as usual,

$$\mathcal{Z} = \int \frac{\mathrm{d}\phi\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2}{\sqrt{2\pi}}\,\mathrm{e}^{-S(\phi,\psi_i)} \tag{2.22}$$

where I've thrown a factor of $1/\sqrt{2\pi}$ into the measure for later convenience. Generically, we'd have to be content with a perturbative evaluation of $\mathcal{Z}$, using Feynman diagrams formed from edges for the ϕ and ψ_i fields, together with vertices from all the different vertices that appear in our action. For a complicated action, even low orders of the perturbative expansion might be difficult to compute in general.

However, let's suppose the action takes the special form

$$S(\phi, \psi_1, \psi_2) = \frac{1}{2}\partial h(\phi)^2 - \psi_1\psi_2\,\partial^2 h(\phi) \tag{2.23}$$

where $h(\phi)$ is some ($\mathbb{R}$-valued) polynomial in ϕ. Note that there can't be any terms in S involving only one of the fermion fields since this term would itself be fermionic. There also can't be higher order terms in the fermion fields since $\psi_i^2 = 0$ for a Grassmann variable, so the only thing special about this action is the relation between the purely bosonic piece and the second term involving $\psi_1\psi_2$.

Now consider the transformations

$$\delta\phi = \epsilon_1\psi_1 + \epsilon_2\psi_2\,, \qquad \delta\psi_1 = \epsilon_2\partial h\,, \qquad \delta\psi_2 = -\epsilon_1\partial h \tag{2.24}$$

where ϵ_i are fermionic parameters. These are supersymmetry transformations in this zero–dimensional context; take the Part III Supersymmetry course to meet supersymmetry in higher dimensions. The most important property of these transformations is that they are *nilpotent*[3]. Under (2.24) the action (2.23) transforms as

$$\delta S = \partial h\,\partial^2 h(\epsilon_1\psi_1 + \epsilon_2\psi_2) - (\epsilon_2\partial h)\psi_2\,\partial^2 h - \psi_1(-\epsilon_1\partial h)\partial^2 h = 0 \tag{2.25}$$

[3] That is, $\delta_1^2 = 0$, $\delta_2^2 = 0$ and $[\delta_1, \delta_2] = 0$, where δ_1 is the transformation with parameter $\epsilon_2 = 0$, *etc.*. You should check this from (2.24) as an exercise!

and is thus invariant — this is what the special relation between the bosonic and fermionic terms in S buys us. (To obtain this result we used the fact that Grassmann variables anticommute.) It's also true that the integral measure $\mathrm{d}\phi\,\mathrm{d}^2\psi$ is likewise invariant; I'll leave this too as an exercise.

The action being supersymmetric will drastically simplify this QFT. Let $\delta\mathcal{O}$ be the supersymmetry variation of some operator $\mathcal{O}(\phi,\psi_i)$ and consider the correlation function $\langle\delta\mathcal{O}\rangle$. Since $\delta S = 0$ we have

$$\langle\delta\mathcal{O}\rangle = \frac{1}{Z_0}\int \mathrm{d}\phi\,\mathrm{d}^2\psi\,\mathrm{e}^{-S}\,\delta\mathcal{O} = \frac{1}{\mathcal{Z}_0}\int \mathrm{d}\phi\,\mathrm{d}^2\psi\,\delta\left(\mathrm{e}^{-S}\mathcal{O}\right) . \tag{2.26}$$

The supersymmetry variation here acts on both ϕ and the fermions ψ_i in $\mathrm{e}^{-S}\mathcal{O}$. But if it acts on a fermion ψ_i then the resulting term does not contain that ψ_i and hence cannot contribute to the integral because $\int \mathrm{d}\psi\,1 = 0$ for Grassmann variables. On the other hand, if it acts on ϕ then while the resulting term may survive the Grassmann integral, it is a total derivative in the ϕ field space. Thus, provided $\mathcal{O}$ does not disturb the decay of e^{-S} as $|\phi|\to\infty$, any such correlation function must vanish, $\langle\delta\mathcal{O}\rangle = 0$.

In particular, if we choose $\mathcal{O}_g = \partial g\,\psi_1$ for some $g(\phi)$, then setting the parameters $\epsilon_1 = -\epsilon_2 = \epsilon$ we have

$$0 = \langle\delta\mathcal{O}_g\rangle = \epsilon\langle\partial g\,\partial h - \partial^2 g\,\psi_1\psi_2\rangle . \tag{2.27}$$

The significance of this is that the quantity $\partial g\,\partial h - \partial^2 g\,\psi_1\psi_2$ is the first–order change in the action under the *deformation* $h \to h+g$, again so long as g does not alter the behaviour of h as $|\phi|\to\infty$. The fact that $\langle\delta\mathcal{O}_g\rangle = 0$ tells that the partition function $\mathcal{Z}[h]$, which we might think depends on all the couplings in the vertices in the polynomial h, is in fact largely insensitive to the detailed form of h because we can deform it by any other polynomial of the same degree or lower. The most important case is if we choose g to be proportional to h, then our deformation just rescales $h \to (1+\lambda)h$ and so we see that $\mathcal{Z}[h]$ is independent of the overall scale of h. By iterating this procedure, we can imagine rescaling h by a large factor so that the bosonic part of the action $(\partial h)^2/2 \to \Lambda^2(\partial h)^2/2$. As $\Lambda\to\infty$, the factor e^{-S} exponentially suppresses any contribution to Z except from an infinitesimal neighbourhood of the critical points of h where $\partial h = 0$. This phenomenon is known as **localization** of the path integral.

It's now straightforward to work out the partition function. Near any such critical point ϕ_* we have

$$h(\phi) = h(\phi_*) + \frac{c_*}{2}(\phi-\phi_*)^2 + \cdots \tag{2.28}$$

where $c_* = \partial^2 h(\phi_*)$, so the action (2.23) becomes

$$S(\phi,\psi_i) = \frac{c_*^2}{2}(\phi-\phi_*)^2 + c_*\psi_1\psi_2 + \cdots . \tag{2.29}$$

The higher order terms will be negligible as we focus on an infinitesimal neighbourhood of ϕ_*. Expanding the exponential in Grassmann variables the contribution of this critical

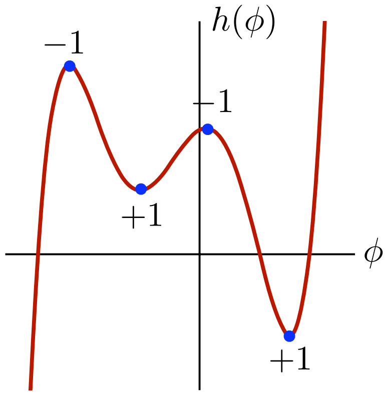


Figure 1: *The supersymmetric path integral receives contributions just from infinitesimal neighbourhoods of the critical points of $h(\phi)$. These alternately contribute ± 1 according to whether they are minima or maxima.*

point to the partition function is

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int \mathrm{d}\phi\,\mathrm{d}^2\psi\,\mathrm{e}^{-c_*(\phi-\phi_*)^2/2}\,[1-c_*\psi_1\psi_2] &= \frac{c_*}{\sqrt{2\pi}}\int \mathrm{d}\phi\,\mathrm{e}^{-c_*(\phi-\phi_*)^2} \\ &= \frac{c_*}{\sqrt{c_*^2}} = \mathrm{sgn}\left(\partial^2 h|_*\right)\ .\end{aligned} \tag{2.30}$$

Summing over all the critical points, the full partition function thus becomes

$$\mathcal{Z}[h] \;=\; \sum_{\phi_*\,:\,\partial h|_{\phi_*}=0} \mathrm{sgn}\left(\partial^2 h|_{\phi_*}\right) \tag{2.31}$$

and, as expected, is largely independent of the detailed form of h. In fact, if h is a polynomial of odd degree, then $\partial h = 0$ must have an even number of roots with $\partial^2 h$ being alternately > 0 and < 0 at each. Thus their contributions to (2.31) cancel pairwise and $\mathcal{Z}[h_{\mathrm{odd}}] = 0$ identically. On the other hand, if h has even degree then it has an odd number of critical points and we obtain $\mathcal{Z}[h_{\mathrm{ev}}] = \pm 1$, with the sign depending on whether $h \to \pm\infty$ as $|\phi| \to \infty$. (See figure 1.)

The fact that the partition function is so simple in this class of theories is a really remarkable result! To reiterate, we've found that for any form of polynomial $h(\phi)$, the partition function $\mathcal{Z}[h]$ is always either 0 or ± 1. If we imagined trying to compute $\mathcal{Z}[h]$ perturbatively, then for a non–quadratic h we'd still have to sum infinitely diagrams using the vertices in the action. In particular, we could certainly draw Feynman graphs Γ with arbitrarily high numbers of loops involving both ϕ and ψ_i fields, and these graphs would each contribute to the coefficient of some power of the coupling constants in the perturbative expansion. However, by an apparent miracle, we'd find that these graphs always cancel themselves out; the net coefficient of each such loop graph would be zero with the contributions from graphs where either ϕ or $\psi_1\psi_2$ run around the loop contributing with opposite sign. The reason for this apparent miracle is the localization property of the supersymmetric integral.

In supersymmetric theories in higher dimensions, complications such as spin mean the cancellation can be less powerful, but it is nonetheless still present and is responsible for making supersymmetric quantum theories 'tamer' than non–supersymmetric ones. As an important example, diagrams where the Higgs particle of the Standard Model runs around a loop can have the effect of destabilizing the mass of the Higgs, sending it up to a very high scale. (We'll understand this later on.) This Spring, CERN will resume its search for a hypothesized supersymmetric partner to the Higgs that many people postulated should be present so as to cancel these dangerous loop diagrams; the ultimate mechanism is the one we've seen above, though it's power is filtered through the lens of a much more complicated theory.

I also want to point out that localization is useful for calculating much more than just the partition function. For each $a \in \{1, 2, 3, \ldots\}$ suppose that $\mathcal{O}_a(\phi, \psi_i)$ is an operator that obeys $\delta\mathcal{O}_a = 0$, *i.e.* each operator is invariant under supersymmetry transformations (2.24). Then the (unnormalized) correlation function

$$\left\langle \prod_a \mathcal{O}_a \right\rangle = \int \frac{\mathrm{d}\phi\, \mathrm{d}^2\psi}{\sqrt{2\pi}}\, \mathrm{e}^{-S} \prod_a \mathcal{O}_a \tag{2.32}$$

again localizes to the critical points of h. Once again, this is because deforming $h \to h + g$ leaves the correlator invariant since the deformation affects the correlation function as

$$\left\langle \prod_a \mathcal{O}_a \right\rangle \stackrel{h\to h+g}{\longrightarrow} \left\langle \delta\mathcal{O}_g \prod_a \mathcal{O}_a \right\rangle = \left\langle \delta \left(\mathcal{O}_g \prod_a \mathcal{O}_a \right) \right\rangle = 0 \tag{2.33}$$

which vanishes by the same arguments as before. Here, we used the fact that $\delta\mathcal{O}_a = 0$ to write the operator on the *rhs* as a total derivative.

Of course, if any of the $\mathcal{O}_a$ are already of the form $\delta\mathcal{O}'$, so that this $\mathcal{O}_a$ is itself the supersymmetry transformation of some $\mathcal{O}'$, then $\langle \prod \mathcal{O}_a \rangle = 0$ which is not very interesting. The interesting operators are those which are δ-closed ($\delta\mathcal{O} = 0$) but not δ-exact ($\mathcal{O} \neq \delta\mathcal{O}'$). These operators describe the **cohomology** of the nilpotent operator δ. This is the starting–point for much of the mathematical interest in QFT: we can build supersymmetric QFTs that compute the cohomology of interesting spaces. For example, Donaldson's theory of invariants of 4–manifolds that are homeomorphic but not diffeomorphic, and the Gromov–Witten generalization of intersection theory can both be understood as examples of (higher–dimensional) supersymmetric QFTs where the localization / cancellation is precise.

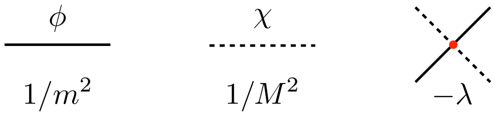
We'll also meet essentially the same idea again in a slightly different context later in this course when we study BRST quantization of gauge theories.

2.6 Effective theories: a toy model

Now I want to introduce a very important idea which will be central to our understanding of QFT in higher dimensions. Suppose we have two real–valued fields ϕ and χ, so that the space of fields is $\mathbb{R}^2$, and let the action be

$$S(\phi, \chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2 \tag{2.34}$$

so that λ provides a coupling between the fields. We have the Feynman rules

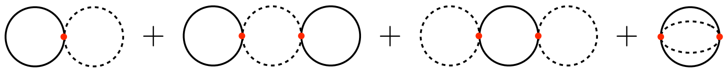


which may be used to compute perturbative expressions for correlation functions such as

$$\langle f \rangle = \frac{1}{\mathcal{Z}} \int_{\mathbb{R}^2} \mathrm{d}\phi \, \mathrm{d}\chi \, \mathrm{e}^{-S(\phi,\chi)} f(\phi,\chi)$$

in the usual way. For example, we have

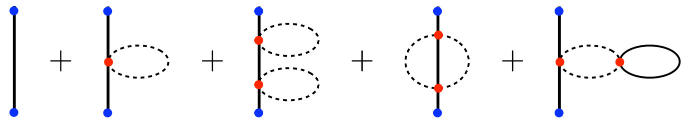
$$\ln\left[\frac{Z}{Z_0}\right] =$$



$$= -\frac{\lambda}{4m^2M^2} \quad + \frac{\lambda^2}{16m^4M^4} \quad + \frac{\lambda^2}{16m^4M^4} \quad + \frac{\lambda^2}{8m^4M^4}$$

as the sum of connected vacuum diagrams, and also

$$\frac{1}{2}\langle \phi^2 \rangle =$$



$$= \frac{1}{m^2} \quad - \frac{\lambda}{2m^4M^2} \quad + \frac{\lambda^2}{4m^6M^4} \quad + \frac{\lambda^2}{2m^6M^4} \quad + \frac{\lambda^2}{4m^6M^4}$$

where the blue dots represent the insertions of the two powers of ϕ.

I want to arrive at this result in a different way. Suppose we're interested in correlation functions of operators that depend only on ϕ; for example, we might imagine that the field χ has a very high mass such that our experiment isn't powerful enough to observe real χ production — we can only measure properties of the ϕ field. Having no idea what χ is doing suggests that we should perform its path integral first, *i.e.*, we average over χ configurations at each fixed ϕ.

We define the **effective action** $S_{\text{eff}}(\phi)$ for the ϕ field to be the result of carrying out this χ integral. Thus

$$S_{\text{eff}}(\phi) := -\hbar \log\left[\int_{\mathbb{R}} \mathrm{d}\chi \, \mathrm{e}^{-S(\phi,\chi)/\hbar}\right] \tag{2.35}$$

where I've restored the powers of $\hbar$. Once we have found this effective action, we can use it to compute $\langle f \rangle$ for any observable that depends only on ϕ. Of course there's nothing mysterious here, we're simply choosing in which order to do our integrals.

In general computing S_{eff} can be difficult, but in our toy example it's straightforward because χ appears only quadratically in $S(\phi,\chi)$. We have

$$\int_{\mathbb{R}} \mathrm{d}\chi \, \mathrm{e}^{-S(\phi,\chi)/\hbar} = \mathrm{e}^{-m^2\phi^2/2\hbar} \sqrt{\frac{2\pi\hbar}{M^2 + \lambda\phi^2/2}} \tag{2.36}$$

where the first factor is the χ-independent part of the original action and the square root comes from the Gaussian integral over χ. Hence the effective action (2.37) is

$$\begin{aligned} S_{\text{eff}}(\phi) &= \frac{m^2}{2}\phi^2 + \frac{\hbar}{2}\ln\left[1+\frac{\lambda}{2M^2}\phi^2\right] + \frac{\hbar}{2}\ln\frac{M^2}{2\pi\hbar} \\ &= \left(\frac{m^2}{2}+\frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \cdots \\ &=: \frac{m^2_{\text{eff}}}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \cdots . \end{aligned} \tag{2.37}$$

The important point is that the effect of integrating out the 'high energy field' χ has *changed* the structure of the action. In particular, the mass term of the ϕ field has been shifted

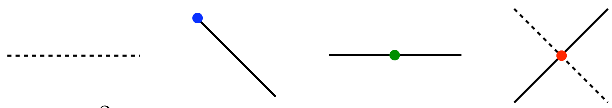
$$m^2 \to m^2_{\text{eff}} = m^2 + \frac{\hbar\lambda}{2M^2} . \tag{2.38}$$

Even more strikingly, we've generated an infinite series of new coupling terms

$$\lambda_4 = -\frac{3\hbar}{2}\frac{\lambda^2}{M^4}, \qquad \lambda_6 = 15\hbar\frac{\lambda^3}{M^6}, \qquad \lambda_{2k} = (-1)^{k+1}\hbar\,\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}} \tag{2.39}$$

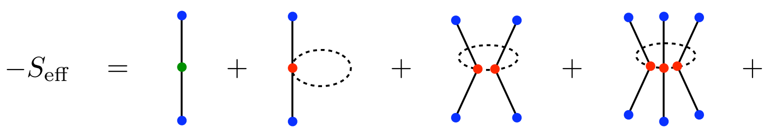
describing self–interactions of ϕ.

It's important to observe that the ϕ mass shift and new ϕ self–interactions all vanish as $\hbar \to 0$; they are *quantum* effects. Notice also that they're each suppressed by powers of the (high) mass M. It's useful to think a little more detail about how these new couplings arise. We can perform the χ path integral using Feynman graphs, provided we remember that the ϕ field is not propagating, but may appear on an external leg. The χ propagator and vertices are



$1/M^2 \qquad \phi \qquad -m^2 \qquad -\lambda$

where the blue 1–valent vertex represents an insertion of the ϕ field, coming from expanding the action in powers of the vertices. These ingredients lead to the following perturbative construction of S_{eff}:



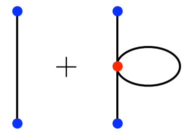
$$= -\frac{m^2}{2}\phi^2 \quad -\frac{\lambda}{4M^2}\phi^2 \quad +\frac{\lambda^2}{16M^4}\phi^4 \quad -\frac{\lambda^3}{48M^6}\phi^6$$

where we note that since $-S_{\text{eff}}$ is the *logarithm* of the χ integral, only connected diagrams appear.

The diagrammatic expansion shows that the new interactions of ϕ have actually been generated by loops of the χ field. In our effective description that knows only about the behaviour of the ϕ field, we can no longer 'see' the χ field 'circulating' around the loop. Instead, we perceive this just as a new interaction vertex for ϕ. (The fact that the new terms in $S_{\rm eff}$ come just from *single* loops of χ, and hence come with just a single power of $\hbar$, is special to the fact that χ appears in the original action (2.34) only quadratically. Generically there would be higher–order corrections.)

Using this effective action, we now find

$$\frac{1}{2}\langle\phi^2\rangle \quad = \quad \cdots \quad + \quad \cdots \quad +$$



$$= \frac{1}{m_{\rm eff}^2} \quad - \frac{\lambda_4}{2m_{\rm eff}^6} \quad +$$

where the propagator and vertices here are the ones appropriate for the effective action $S_{\rm eff}$. Using the definition of the new couplings in terms of the original λ and M, this unsurprisingly agrees with our answer before, correct to order λ^2. However, once we had the effective action, we arrived at this answer using just two diagrams, whereas above it required five. If we only care about a single correlation function then the work involved in first computing $S_{\rm eff}$ and then using the new set of Feynman rules to compute the low–energy correlator is roughly the same as just using the original action to compute this correlator directly. On the other hand, if we wish to compute many low–energy correlators then we're clearly better off investing a little time to work out $S_{\rm eff}$ first.

However, the real point I wish to make is this: the way we experience the world is *always* through $S_{\rm eff}$. Naively at least, we have no idea what new physics may be lurking just out of reach of our most powerful accelerators; there may be any number of new, hitherto undiscovered species of particle, or new dimensions of space–time, or even wilder new phenomena. More importantly, when describing low–energy physics you should only seek to describe the behaviour of the degrees of freedom (fields) that are relevant and accessible at the energy scale at which you're conducting your experiments, *even when you know what the more fundamental description is.* We know that a glass of water consists of very many H_2O molecules, that these molecules are bound states of atoms which each consist of many electrons orbiting around a central nucleus, that this nucleus comprises of protons and neutrons stuck together by a strong force mediated by pions, and that all these hadrons are themselves seething masses of quarks and gluons. But it would be very foolish to imagine we should describe the properties of water that are relevant in everyday life by starting from the Lagrangian for QCD.

Let me make one final comment. In the example above, we started from a very simple action in equation (2.34) and obtained a more complicated effective action (2.37) after integrating out the unobserved degree of freedom χ. A more generic case would start from a general action (invariant under $\phi \to -\phi$ and $\chi \to -\chi$ for simplicity)

$$S'(\phi, \chi) = \sum_{i,j} \frac{\lambda_{i,j}}{(2i)!\,(2j)!} \phi^{2i} \chi^{2j} \tag{2.40}$$

in which all possible even monomials in ϕ and χ are allowed. For example, we may have arrived at this action by integrating out some other field that was unknown in our above considerations. In this generic case, the effect of integrating out χ will not generate *new* interactions for ϕ — all possible even self–interactions are included anyway — but rather the values of the coupling constants $\lambda_{i,0}$ will get shifted, just as for the mass shift we saw above. The lesson to remember is that the effect of integrating out degrees of freedom is to change the values of the coupling constants in the effective action.