Principles of Quantum Mechanics

University of Cambridge Part II Mathematical Tripos

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Abstract: These are the lecture notes for the Principles of Quantum Mechanics course given to students taking Part II Maths in Cambridge during Michaelmas Term of 2021. The main aim is to discuss quantum mechanics in the framework of Hilbert space, following Dirac. Along the way, we talk about transformations and symmetries, angular momentum, composite systems, dynamical symmetries, perturbation theory (both time–independent and time–dependent, degenerate and non–degenerate). We'll finish with a discussion of various interpretations of quantum mechanics, entanglement and decoherence.

Contents

Acknowledgments

Nothing in these lecture notes is original. In particular, my treatment is heavily influenced by the textbooks listed below, especially Weinberg's *Lectures on Quantum Mechanics*, Hall's *Quantum Theory for Mathematicians* and Binney & Skinner's *The Physics of Quantum Mechanics* (though I feel more justified in plagiarising this one). I've also borrowed from previous versions of the lecture notes for this course, by Profs. R. Horgan and A. Davis, which are available online.

I am supported by the European Union under an FP7 Marie Curie Career Integration Grant.

Preliminaries

These are the notes for the Part II course on Principles of Quantum Mechanics offered in Part II of the Maths Tripos, so I'll feel free to assume you took the IB courses on *Quantum Mechanics*, *Methods* and *Linear Algebra* last year. For Part II, I'd expect the material presented here to complement the courses on *Classical Dynamics*, *Statistical Physics* and perhaps *Integrable Systems* very well on the Applied side, in addition to being a prerequisite for *Applications of Quantum Mechanics* next term. On the pure side, the courses *Linear Analysis* or *Functional Analysis* and *Representation Theory* are probably the most closely related to this one.

Books & Other Resources

There are many textbooks and reference books available on Quantum Mechanics. Different ones emphasise different aspects of the theory, or different applications in physics, or give prominence to different mathematical structures. QM is such a huge subject nowadays that it is probably impossible for a single textbook to give an encyclopaedic treatment (and *absolutely* impossible for me to do so in a course of 24 lectures). Here are some of the ones I've found useful while preparing these notes; you might prefer different ones to me.

- **Weinberg, S.** *Lectures on Quantum Mechanics*, CUP (2013).
  This is perhaps the single most appropriate book for the course. Written by one of the great physicists of our times, this book contains a wealth of information. Highly recommended.

- **Binney, J.J. and Skinner, D.** *The Physics of Quantum Mechanics*, OUP (2014).
  The number 1 international bestseller... Contains a much fuller version of these notes (with better diagrams!). Put it on your Christmas list.

- **Dirac, P.** *Principles of Quantum Mechanics*, OUP (1967).
  The notes from an earlier version of this course! Written with exceptional clarity and insight, this is a genuine classic of theoretical physics by one of the founders of Quantum Mechanics.

- **Messiah, A.** *Quantum Mechanics*, Vols 1 & 2, Dover (2014).
  Another classic textbook, originally from 1958. It provides a comprehensive treatment of QM, though the order of the presentation is slightly different to the one we'll follow in this course.

- **Shankar, R.** *Principles of Quantum Mechanics*, Springer (1994).
  A much-loved textbook, with particular emphasis on the physical applications of quantum mechanics.

- **Hall, B.** *Quantum Theory for Mathematicians*, Springer (2013).
  If you're more interested in the functional analysis & representation theory aspects of QM rather than the physical applications, this could be the book for you. Much of it is at a more advanced level than we'll need.

- **Sakurai, J.J. and Napolitano, J.** *Modern Quantum Mechanics*, CUP (3$^{\text{rd}}$ edition, 2020).
  By now, another classic book on QM, presented from a perspective very close to this course. Also contains an excellent introduction to path integrals.

- **Townsend, J.** *A Modern to Approach to Quantum Mechanics*, 2$^{\text{nd}}$ edition, University Science Books (2012).
  Very accessible, but also fairly comprehensive textbook on QM. Starts by considering spin-$\frac{1}{2}$ particle as a simple example of a two–state system.

Textbooks are expensive, so check what's available in your college library before shelling out – any reasonably modern textbook on QM will be adequate for this course. There are also lots of excellent resources available freely online, including these:

- Here are the lecture notes from another recent version of this course.

- Here are the lecture notes from the Part II Further Quantum Mechanics course in the Cavendish.

- These are the lecture notes from Prof. W. Taylor's graduate course on QM at MIT. The first half of his course covers material that is relevant here, while scattering theory will be covered in next term's Applications of Quantum Mechanics Part II course.

- This is the lecture course on Quantum Mechanics from Prof. R. Fitzpatrick at the University of Texas, Austin.

1 Introduction

In classical mechanics, a particle's motion is governed by Newton's Laws. These are second order o.d.e.s, so to determine the fate of our particle we must specify two initial conditions. We could take these to be the particle's initial position $\mathbf{x}(t_0)$ and velocity $\mathbf{v}(t_0)$, or it's initial position $\mathbf{x}(t_0)$ and momentum $\mathbf{p}(t_0)$. Once these are specified the motion is determined (provided of course we understand how to describe the forces that are acting). This means that we can solve Newton's Second Law to find the values $(\mathbf{x}(t), \mathbf{p}(t))$ for $t > t_0$[1]. So as time passes, our classical particle traces out a trajectory in the space M of possible positions and momenta, sketched in figure 1. The space M is known as *phase space* and in our case, for motion in three dimensions, M is just $\mathbb{R}^6$. In general M comes with a rich geometry known as a *Poisson structure*; you'll study this structure in detail if you're taking the Part II courses on Classical Dynamics or Integrable Systems, and we'll touch on it later in this course, too.

Classical observables are represented by functions

$$f : M \to \mathbb{R} \qquad\qquad f : (\mathbf{x}, \mathbf{p}) \mapsto f(\mathbf{x}, \mathbf{p})\,.$$

For example, we may be interested in the kinetic energy $T = \mathbf{p}^2/2m$, potential energy $V(\mathbf{x})$, angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$, or a host of other possible quantities. A priori, these functions are defined everywhere over M, but if we want to know about the energy or angular momentum of *our specific particle* then we should evaluate them not at some random point $(\mathbf{x}, \mathbf{p}) \in M$, but *along the particle's phase space trajectory.* For example, if at time t the particle has position $\mathbf{x}(t)$ and momentum $\mathbf{p}(t)$, then its angular momentum is $\mathbf{x}(t) \times \mathbf{p}(t)$. Thus the values of the particle's energy, angular momentum *etc.* may depend on time, though of course our *definition* of these quantities does not[2]. In this way,

[1] We can certainly do this numerically, at least for t in a sufficiently small open subset around t_0.

[2] Technically, we take the pullback of f by the embedding map $\imath : [t_0, \infty) \to M$.

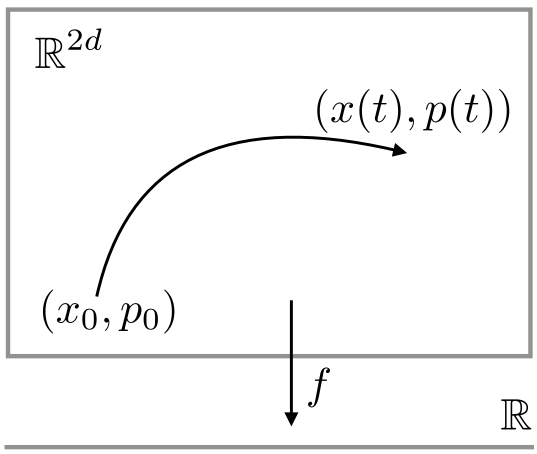

Figure 1: *A particle's trajectory in phase space. Observables are represented by functions $f : \mathbb{R}^{2d} \to \mathbb{R}$, evaluated along a given particle's trajectory.*

everything we could possibly want to know about a single, pointlike particle is encoded in its phase space trajectory.

But that is not our World. To the best of our current experimental knowledge, our World is a quantum, not classical, one. Initially, these experiments were based on careful studies of atomic spectroscopy and blackbody radiation, but nowadays I'd prefer to say that the best evidence of quantum mechanics is simply that we use it constantly in our everyday lives. Each time you listen to music on your stereo, post a photo on Instagram or make a call on your phone you're relying on technology that's only become possible due to our understanding of the quantum structure of matter. Whenever you plug something into the mains, you're using electricity that's in part generated by nuclear reactions in which quantum mechanics is essential, while much of modern medicine relies on new drugs designed with the benefit of the improved understanding of chemistry that quantum mechanics provides.

In such a quantum world, instead of a phase space trajectory, everything we could want to know about a particle is encoded in a vector ψ in Hilbert space $\mathcal{H}$. As you met in IB Quantum Mechanics, this *state vector* evolves in time according to Schrödinger's equation. In the quantum world, observables are represented by certain operators $\mathcal{O}$. The operators you saw in IB QM had the same sort of form as observables classical mechanics, such as the kinetic energy operator $\hat{T} = \hat{\mathbf{p}}^2/2m$ or angular momentum operator $\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$. However, rather than being functions, these operators are (roughly) linear maps

$$\mathcal{O} : \mathcal{H} \to \mathcal{H} \,.$$

Again, in the first instance these operators are defined throughout $\mathcal{H}$, but if we're interested in knowing about the energy or angular momentum of our particular quantum particle, then we should find out what happens when they act on the specific $\psi \in \mathcal{H}$ that describes the state of our particle at time t. (See figure 2.)

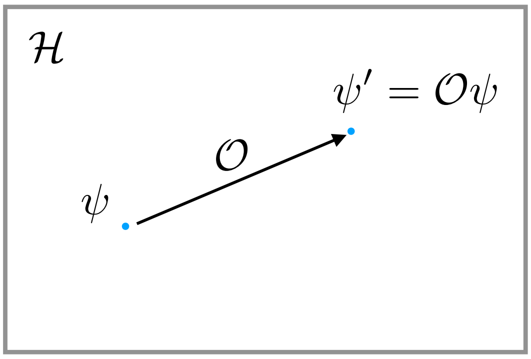

Figure 2: *In Quantum Mechanics, complete knowledge of a particle's states is determined by a vector in Hilbert space. Observables are represented by Hermitian linear operators* $\mathcal{O} : \mathcal{H} \to \mathcal{H}$.

In the following chapters we'll study what Hilbert space is and what it's operators do in a more general framework than you saw last year, building your insight into the mathematical structure of quantum mechanics. Much of this is just linear algebra, but the Hilbert spaces we'll care about in QM are often infinite-dimensional, so we also make contact with Functional Analysis. Furthermore, although last year you 'guessed' the form of quantum operators by analogy with their classical counterparts, we'll see that at a deeper level many of them can be understood to have their origins in symmetries of space and time; the operators just reflect the way these symmetry transformations act on Hilbert space, rather than on (non-relativistic) space-time. In this way, Quantum Mechanics makes contact with Representation Theory.

So, mathematically, much of Quantum Mechanics boils down to a mix of Functional Analysis and Representation Theory. It's even true that it provides a particularly interesting example of these subjects. *But this is not the reason we study it.* We study Quantum Mechanics in an effort to understand and appreciate *our* World, not some abstract mathematical one. You're all intimately familiar with vector spaces, and you're (hopefully!) also very good at solving Sturm–Liouville type eigenfunction / eigenvalue problems. But the real skill is in understanding how this formalism relates to the world we see around us.

It's not obvious. Newton's laws are (at least generically) non-linear differential equations and we can't usually superpose solutions. General Relativity teaches us that spacetime is not flat. So it's not at all clear that our particle should in fact be described by a point in a vector space, any more than it was obvious to Aristotle that bodies actually stay in uniform motion unless acted on by a force, or clear to the Ancients that the arrival of solar eclipses, changes of the weather, or any other natural phenomenon are actually governed by calculable Laws, rather than the whims of various gods. For this reason, instead of emphasising how weird and different Quantum Mechanics is, I'd prefer to make you appreciate how it actually underpins the physics you're already familiar with. No matter how good you are at solving eigenvalue problems, if you don't see how these relate to your everyday physical intuition, knowledge you've built up since first opening your eyes and learning to crawl, then you haven't really understood the subject.

Let's begin.