7 Identical Particles

One of the most striking facts that helps us understand the Universe is that, apart from details of their motion, all electrons are identical. This fact, though convenient, has no explanation within Quantum Mechanics. In Quantum Field Theory, one learns that particles are simply quantised modes of excitation of a *field*. We thus replace the vast number of electrons our Universe contains with a single quantum field that has hugely many excitations. From this perspective, the fact that all elementary particles of the same species are identical is more or less a tautology, no more surprising than the fact that one factor of x in the monomial x^n is just the same as another.

7.1 Bosons and Fermions

Consider a system consisting of N particles, individually described by some state in a Hilbert space $\mathcal{H}_a$ where $a = 1, \ldots, N$, and suppose $|\alpha_a\rangle \in \mathcal{H}_a$ gives a complete specification of the state of the a^{th} particle, so α_a collectively labels all the quantum numbers describing the particle's energy, orbital angular momentum, spin *etc.*. In the case that the particles are distinguishable – meaning they are each from different *species*, such as an electron, a proton or a neutron – a complete set of states of the full system is then labelled by linear combinations of states

$$|\alpha_1; \alpha_2; \ldots; \alpha_N\rangle = |\alpha_1\rangle|\alpha_2\rangle \cdots |\alpha_N\rangle \in \bigotimes_{a=1}^{N} \mathcal{H}_a \,. \tag{7.1}$$

in accordance with the discussion of section 2.3.1. However, suppose particles 1 and 2 are *indistinguishable*; for example, they might both be electrons. The state (7.1) thus describes a situation where there is one electron with quantum numbers given by α_1 and another whose same properties are labelled by α_2. Since both electrons are identical, it can make no difference which is which. Consequently, there can be no physical difference between this state and the state in which these two particles are exchanged. This does not imply that the two states are identical, but does imply that they can only differ by a phase:

$$|\alpha_2; \alpha_1; \ldots\rangle = \mathrm{e}^{\mathrm{i}\phi}|\alpha_1; \alpha_2; \ldots\rangle \,, \tag{7.2}$$

where ϕ is independent[61] of the particular values of the labels α_a, but may depend on the species of particle we exchange.

If we exchange the pair of particles once again we find

$$|\alpha_1; \alpha_2; \ldots\rangle = \mathrm{e}^{\mathrm{i}\phi}|\alpha_2; \alpha_1; \ldots\rangle = \mathrm{e}^{2\mathrm{i}\phi}|\alpha_1; \alpha_2; \ldots\rangle \tag{7.3}$$

[61] There's a (non-examinable!) exception to this in the case of two spatial dimensions, where the phase may depend on the homotopy class of the path the particles take whilst they are being exchanged. In $2 + 1$ dimensions, these paths can become braided (tangled) and the multi–particle state transforms in a representation of the braid group, first studied in the maths literature by Emile Artin. This leads to the possibility of *anyonic* statistics, which play an important role in some condensed matter systems. Fortunately, we're interested in quantum mechanics on $\mathbb{R}^3$ and in higher dimensions it turns out the paths can always be disentangled, so anyons do not arise.

using the fact that the phase is independent of the particular values of the quantum numbers α_a. Thus $e^{2i\phi} = 1$, so

$$e^{i\phi} = \pm 1 \,. \tag{7.4}$$

The choice of sign depends only on the species of the particle. Identical particles which are symmetric under exchange are called[62] *bosons*, while those that are antisymmetric are said to be *fermions*.

If all N of the particles in our system are identical, then the argument immediately extends to say that the state must be either completely symmetric or completely antisymmetric under exchange of any pair, which case being determined by whether the particles are bosons or fermions. That is, for bosons the state of the system is actually described by a vector

$$|\alpha_1; \alpha_2; \ldots; \alpha_N\rangle \in \mathrm{Sym}^N \mathcal{H} \subset \bigotimes^N \mathcal{H} \,, \tag{7.5}$$

whereas for fermions the state lies in the antisymmetric part

$$|\alpha_1; \alpha_2; \ldots; \alpha_N\rangle \in \bigwedge^N \mathcal{H} \subset \bigotimes^N \mathcal{H} \,. \tag{7.6}$$

As a simple example, a pair of identical fermions with quantum numbers α_1 and α_2 live in the antisymmetric state

$$|\psi\rangle_{\text{fermi}} = \frac{1}{\sqrt{2}}(|\alpha_1; \alpha_2\rangle - |\alpha_2; \alpha_1\rangle) \,, \tag{7.7}$$

whereas the system would have to be described by the symmetric combination

$$|\psi\rangle_{\text{bose}} = \frac{1}{\sqrt{2}}(|\alpha_1; \alpha_2\rangle + |\alpha_2; \alpha_1\rangle) \tag{7.8}$$

if instead these particles were identical bosons.

A further fact that also comes from quantum field theory is that particles with integer spin are always bosons, whilst particles with (odd) half–integer spin are always fermions. Among the elementary particles, examples of bosons thus include the W and Z bosons (hence the name!) and the photon, while examples of fermions include electrons, neutrinos and quarks. You'll learn about how this connection between spin and statistics arises if you take the Part III QFT course.

The above considerations apply to composite particles as well as to elementary ones. When we exchange a pair of identical composite particles, we exchange all of their constituents. Thus, if the composites are made up from an odd number of fermions and any number of bosons, upon exchange we'll acquire an odd number of minus signs, so the whole state must be antisymmetric. For example, a proton consists of three (valence) quarks and many gluons, so is a fermion. So too is a neutron. On the other hand, composites consisting of an even number of fermions and any number of bosons must be symmetric under

[62] They're named after Satyendra Nath Bose and Enrico Fermi, respectively, who first studied their properties.

exchange with an identical composite. For example, a hydrogen atom consists of a proton and an electron, each of which are fermions, so the composite Hydrogen atom is itself a boson.

This behaviour is consistent with addition of angular momentum. When we add the angular momenta of an odd number of particles each with odd half–integer spin, the total angular momentum is also an odd half–integer, whereas when we combine the angular momenta of an even number of particles each with odd half–integer spin, the total angular momentum is an integer. Even without quantum field theory, it would be impossible for all integer spin particles to be fermions, because a composite consisting of an even number of such integer spin particles would also have integer spin, but would be a boson.

7.1.1 Pauli's Exclusion Principle

The fact that the state of identical fermions must be totally antisymmetric has an immediate, striking consequence: no two fermions can be put in exactly the same state simultaneously. This fact is known as *Pauli's exclusion principle*. As an example, since we must always have $\psi_{\sigma,\sigma'}(\mathbf{x},\mathbf{x}') = -\psi_{\sigma',\sigma}(\mathbf{x}',\mathbf{x})$, if both fermions happen to have the same S_z eigenvalue we find $\psi_{\sigma,\sigma}(\mathbf{x},\mathbf{x}') = -\psi_{\sigma,\sigma}(\mathbf{x}',\mathbf{x})$. Hence the spatial wavefunction of the system necessarily vanishes at $\mathbf{x}=\mathbf{x}'$ so there is zero probability that the two particles are located at the same place.

More generally, the state $|\Psi\rangle$ of N identical fermions is represented in terms of the determinants

$$\langle\alpha_1,\alpha_2,\ldots,\alpha_N|\Psi\rangle = \frac{1}{\sqrt{N!}}\begin{vmatrix} \psi_1(\alpha_1) & \psi_1(\alpha_2) & \cdots & \psi_1(\alpha_N) \\ \psi_2(\alpha_1) & \psi_2(\alpha_2) & \cdots & \psi_2(\alpha_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N(\alpha_1) & \psi_N(\alpha_2) & \cdots & \psi_N(\alpha_N) \end{vmatrix} \tag{7.9}$$

that in this context are known as *Slater determinants*. Again, $|\alpha_1,\alpha_2,\ldots,\alpha_n\rangle$ represents a state in which *some* fermion has quantum numbers α_1, another has α_2 and so on, where the labels α_i denote a complete set of quantum numbers for single particle states, including (say) their momentum, spin-z value, *etc.*, while $\psi_1(\alpha_1) = \langle\alpha_1|\psi_1\rangle$ is the amplitude for the 'first' fermion to have quantum numbers α_1. If $\alpha_i = \alpha_j$ for any $i \neq j$, then two columns of this determinant coincide. The amplitude $\langle\ldots,\alpha_i,\ldots,\alpha_i,\ldots|\Psi\rangle$ thus vanishes identically, so there is zero probability of any two fermions being found in an identical state.

It's worth stressing that we've learned only that the state of a pair of identical fermions must be antisymmetric under exchange of *every* one of their quantum numbers. The Hilbert space of a single such spin-s fermion is itself a tensor product $\mathcal{H}_{\text{fermion}} = \mathcal{H}_{\text{spat}} \otimes \mathbb{C}^{2s+1}$, including a factor that describes the fermion's spatial wavefunction (including details of its likely position, orbital angular momentum *etc.*) and a separate factor $\mathcal{H}_{\text{spin}} \cong \mathbb{C}^{2s+1}$ spanned by the possible spin states $\{|s\rangle, |s-1\rangle, \ldots, |-s\rangle\}$ of this spin-s particle. We must exchange *both* the spin and spatial parts of the state in order to find a physically equivalent state.

In the example of our electron pair, $\psi_{\sigma,\sigma'}(\mathbf{x},\mathbf{x}')$ is certainly equivalent to $\psi_{\sigma',\sigma}(\mathbf{x}',\mathbf{x})$, since both correspond to the electron at $\mathbf{x}$ having spin σ and the electron found at $\mathbf{x}'$ having spin σ'. However, antisymmetry of the total state does not imply any relation between the states $\psi_{\sigma,\sigma'}(\mathbf{x},\mathbf{x}')$ and $\psi_{\sigma',\sigma}(\mathbf{x},\mathbf{x}')$: these *are* distinct physical possibilities because the electron at $\mathbf{x}$ has different spin in the two cases. We saw in section 6.1.2 that the spin wavefunctions of the two electrons could be combined into a triplet of spin-1 states

$$|1,1\rangle = |\!\uparrow\rangle|\!\uparrow\rangle\,, \qquad |1,0\rangle = \frac{1}{\sqrt{2}}\left(|\!\uparrow\rangle|\!\downarrow\rangle + |\!\downarrow\rangle|\!\uparrow\rangle\right)\,, \qquad |1,-1\rangle = |\!\downarrow\rangle|\!\downarrow\rangle \tag{7.10}$$

which are symmetric, and a single antisymmetric state

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\!\uparrow\rangle|\!\downarrow\rangle - |\!\downarrow\rangle|\!\uparrow\rangle\right) \tag{7.11}$$

of spin-zero. Because the overall state of the electron pair must be antisymmetric, the spatial wavefunction must be antisymmetric if the electron pair is in any of the spin-1 states, while the spatial wavefunction must be symmetric if the electron pair is in the spin-0 state. Thus, for a given spatial wavefunction $\psi(\mathbf{x},\mathbf{x}')$ the pair has four possible states, and the states

$$\frac{1}{\sqrt{2}}\big(\psi(\mathbf{x},\mathbf{x}') - \psi(\mathbf{x}',\mathbf{x})\big)\otimes\begin{pmatrix}|1,1\rangle\\ |1,0\rangle\\ |1,-1\rangle\end{pmatrix} \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\big(\psi(\mathbf{x},\mathbf{x}') + \psi(\mathbf{x}',\mathbf{x})\big)\otimes|0,0\rangle \tag{7.12}$$

span a basis of the two–electron Hilbert space.

In general, as always, an electron pair can be in a superposition of the four states in (7.12), with different spatial wavefunctions in each term of the superposition. Similarly, an N-fermion system may be in state $|\Psi\rangle$ that is a superposition of different states $|\alpha_1,\alpha_2,\ldots,\alpha_N\rangle$, with different (distinct) values of the α_is, where each of the amplitudes $\langle\alpha_1,\alpha_2,\ldots,\alpha_N|\Psi\rangle$ are given in terms of the single–particle states by Slater determinants (7.9).

7.1.2 The Periodic Table

This section closely follows section 4.5 of Weinberg.

One of the outstanding successes of the early years of quantum mechanics was an understanding of the structure of the Periodic Table of the Elements. The exact dynamics of a heavy atom, with large atomic number Z, is very complicated because each electron not only feels an attractive Coulomb potential $-Ze^2/r$ from the Z protons in the nucleus, but also a repulsive electromagnetic interaction from all the other electrons. However, it turns out that a reasonable approximation – known as the *Hartree approximation* – is to imagine that each electron moves in a (roughly) central potential $V(r)$ created by the net effect of the protons and the other electrons, while neglecting the fact that the response of any given electron to this potential will itself modify the potential. Near the nucleus, $V(r) \approx -Ze^2/r$ as the electron feels the full strength of the attraction of the protons, while at large distances $V(r) \approx -e^2/r$ since the protons' charge is shielded by the $(Z-1)$ other electrons.

As for any central potential, $[\mathbf{L}^2, H] = 0$ and $[\mathbf{L}, H] = 0$ so we can label the single–electron states by their values of ℓ and m, together with another integer n labelling the energy levels. However, because $V(r)$ is not precisely $1/r$, unlike the pure Coulomb potential, states with different ℓ do not have identical energy. Because $V(r) \sim 1/r$ near the nucleus, the single–electron wavefunctions behave as r^ℓ for small r, just as in Hydrogen. This mean that states with high values of ℓ are less likely to be found at small r where the potential is deepest, and so typically have slightly larger energy than those of lower ℓ, compared to the pure Coulomb case. Numerical calculations show that the states of roughly equal energy are

$$\begin{aligned}
&1s,\\
&2s,\ 2p,\\
&3s,\ 3p,\\
&4s,\ 3d,\ 4p,\\
&5s,\ 4d,\ 5p,\\
&6s,\ 4f,\ 5d,\ 6p,\\
&7s,\ 5f,\ 7p,\ \ldots
\end{aligned}$$

with energy increasing as one proceeds down the list. Here, the number corresponds to the label n, while $s, p, d, f, \ldots$ stand for *s*harp, *p*rincipal, *d*iffuse, *f*aint, ... and simply correspond to $\ell = 0, 1, 2, 3, \ldots$. This historic notation is standard in atomic spectroscopy, which is still of great importance in the pharmaceutical industry.

Since electrons have spin-$\frac{1}{2}$ they are fermions. Pauli's exclusion principle prevents the electrons in a heavy atom from all burying down into the lowest energy $1s$ state, and forces them to gradually fill up the higher energy levels. There are $2(2\ell+1)$ distinct states with orbital angular momentum ℓ, with the extra factor of 2 coming from the two possible spin states of the electron. Thus, the number of possible states in each line of the above table is 2, $2+6=8$, $2+6=8$, $2+10+6=18$, $2+10+6=18$ and $2+14+10+6=32$, *etc.*

The first two elements, hydrogen and helium, have electrons just in the ground state $1s$. The 8 elements from lithium to neon have electrons in the $n=1$ and $n=2$ states, while the next 8 elements from sodium to argon have electrons in states with $n = 1, 2, 3$. The first excited states of hydrogen and helium are obtained by promoting (one of) their $1s$ electrons up to the $2s$ state, and lie 10.2eV and 19.8eV above the corresponding ground states. These energy differences correspond to the frequency of UV radiation, so this transition cannot be stimulated by optical frequency radiation. By constrast, the first excited state of lithium is obtained by promoting a $2s$ electron to a $2p$ state, and lies a mere 1.85eV above its ground state. This corresponds to the frequency of rather deep red. Elements that lie beyond helium play an important role in astronomical measurements, even though they are present only in trace amounts compared to hydrogen and helium, because their absorption spectra contain lines at easily observed optical frequencies.

The chemical properties of an element are largely determined by the number of electrons in its highest energy level, since these are least tightly bound. Atoms with no electrons outside filled energy levels are particularly stable chemically. These elements are called *noble gases* and include helium ($Z = 2$), neon ($Z = 2 + 8 = 10$), argon ($Z = 2 + 8 + 8 = 18$), krypton ($Z = 2 + 8 + 8 + 18 = 36$), xenon ($Z = 2 + 8 + 8 + 18 + 18 = 54$) and radon ($Z = 2 + 8 + 8 + 18 + 18 + 32 = 86$).

Elements with either a few more or few less electrons than required to fill a shell have their chemical properties determined by this number, known as the *valence* and counted positive for extra electrons and negative for fewer. If there is just one electron in the highest energy level, then this electron is easily stripped away, so the element is chemically reactive. These are the *alkali metals* and include lithium ($Z = 2+1 = 3$), sodium ($Z = 2+8+1 = 11$), potassium ($Z = 2 + 8 + 8 + 1 = 19$) *etc.* If a large number of such atoms combine to form a solid crystal, then it is energetically favourable for the valence electrons to be shared throughout the solid rather than clinging to their original atom. These electrons thus form a sort of fluid that is free to flow within the crystal when stimulated by a small external force. This gives the crystal high electrical and high thermal conductivity, making it a *metal*. (You'll study this in much more detail if you take the Applications of Quantum Mechanics course next term.) Atoms with two electrons more than the noble gases are also chemically reactive, though not as reactive as the alkalis. They are known as the *alkaline metals* and include beryllium ($Z = 2 + 2 = 4$), magnesium ($Z = 2 + 8 + 2 = 12$), calcium ($Z = 2 + 8 + 8 + 2 = 20$) *etc.*

On the other hand, atoms with one electron missing from their highest energy level tend to strongly attract other electrons and so are chemically reactive non–metals, often reacting violently when brought into contact with metals. These elements are called *halogens* and include fluorine ($Z = 2 + 8 - 1 = 9$), chlorine ($Z = 2 + 8 + 8 - 1 = 17$), bromine ($Z = 2+8+8+18-1 = 35$), iodine ($Z = 2+8+8+18+18-1 = 53$) *etc.* Elements with two electrons missing from their highest energy level include oxygen ($Z = 2 + 8 - 2 = 8$), sulfur ($Z = 2 + 8 + 8 - 2 = 16$), *etc.* These elements are again chemically reactive, though less so than the halogens.

The inclusion of the $4f$ and $5f$ states in the sixth and seventh energy levels, respectively, are responsible for the long sequence of *rare earths* in the middle of the periodic table. Numerical calculations show that wavefunctions of the $2(2 \cdot 3 + 1) = 14$ different $4f$ states have small probability to lie outside the wavefunctions of the two $6s$ states, despite having slightly higher energy. Consequently, these elements are all rather similar chemically. The same is true of the 14 different $5f$ states compared to the wavefunctions of the two $7s$ states. The 14 elements in which the $4f$ states are being filled run from lanthanum ($Z = 2 + 8 + 8 + 18 + 18 + 2 + 1 = 57$) to ytterbium ($Z = 70$) and are known as *lanthanides*, while the next chemically similar sequence are the *actinides*, running from actinium ($Z = 2 + 8 + 8 + 18 + 18 + 32 + 2 + 1 = 89$) to nobelium ($Z = 102$). Beyond this point the nuclei themselves become so unstable that the element tends to undergo radioactive decay before it has chance to participate in any chemical reaction.

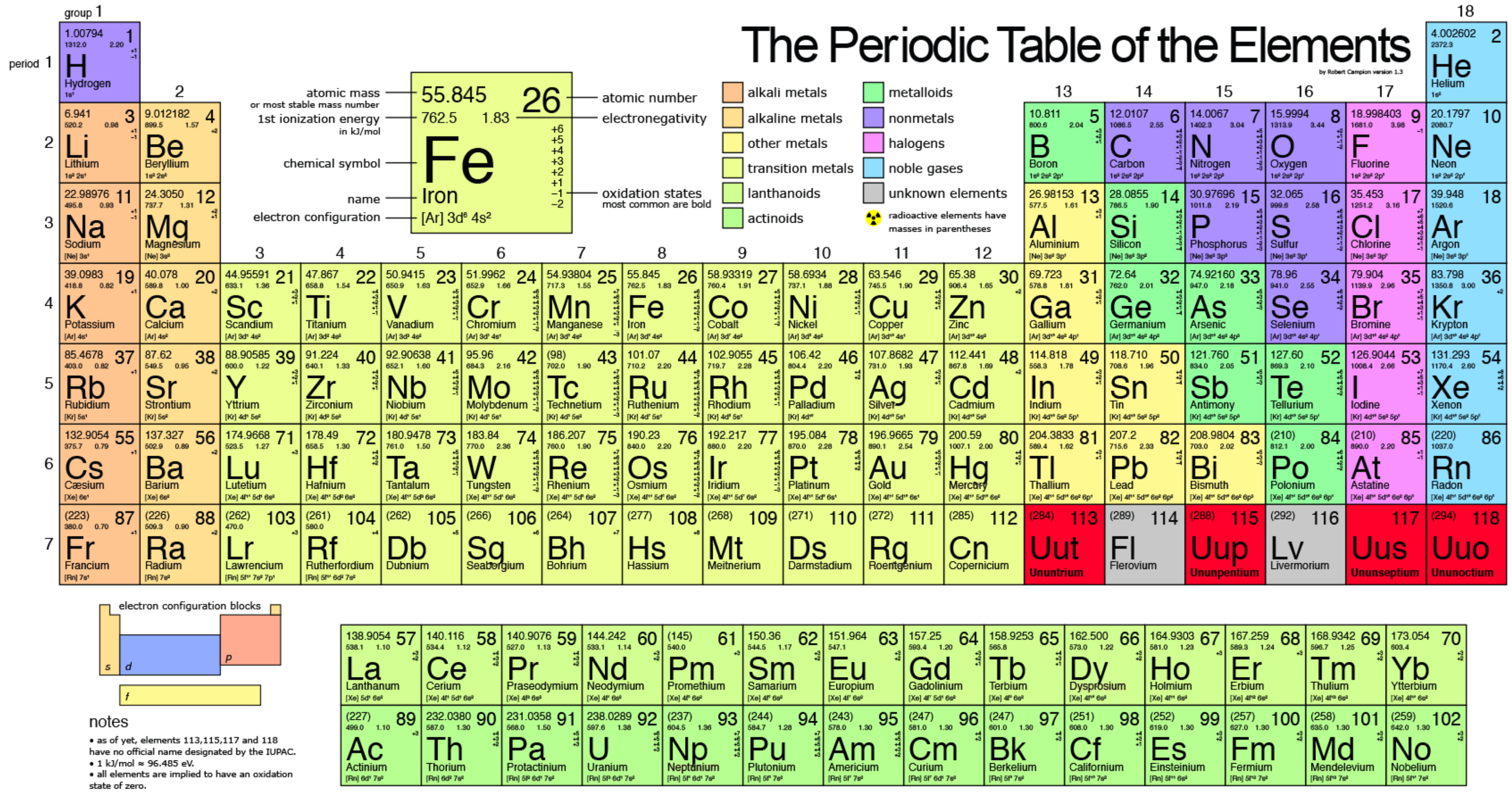

Figure 16: *The Periodic Table of the Elements. For formatting reasons, the lanthanides and actinides are traditionally shown below the rest of the table. (Figure from Wikimedia.)*

7.1.3 White Dwarfs, Neutron Stars and Supernovae

7.2 Exchange and Parity in the Centre of Momentum Frame

Let's now consider the effects of exchanging two identical particles on their spatial wavefunction. Letting $\mathbf{X}_1$, $\mathbf{P}_1$ and $\mathbf{X}_2$, $\mathbf{P}_2$ denote the position and momentum operators of the two particles, exchanging $1 \leftrightarrow 2$ implies that

$$\begin{aligned} \mathbf{X}_{\mathrm{com}} = \frac{\mathbf{X}_1 + \mathbf{X}_2}{2} \qquad &\mapsto \qquad \frac{\mathbf{X}_2 + \mathbf{X}_1}{2} = \mathbf{X}_{\mathrm{com}} \\ \mathbf{P}_{\mathrm{com}} = \mathbf{P}_1 + \mathbf{P}_2 \qquad &\mapsto \qquad \mathbf{P}_2 + \mathbf{P}_1 = \mathbf{P}_{\mathrm{com}}\,, \end{aligned} \tag{7.13}$$

whilst

$$\begin{aligned} \mathbf{X}_{\mathrm{rel}} = \mathbf{X}_1 - \mathbf{X}_2 \qquad &\mapsto \qquad \mathbf{X}_2 - \mathbf{X}_1 = -\mathbf{X}_{\mathrm{rel}} \\ \mathbf{P}_{\mathrm{rel}} = \frac{\mathbf{P}_1 - \mathbf{P}_2}{2} \qquad &\mapsto \qquad \frac{\mathbf{P}_2 - \mathbf{P}_1}{2} = -\mathbf{P}_{\mathrm{rel}}\,. \end{aligned} \tag{7.14}$$

Thus, exchange acts trivially on the centre-of-momentum coordinates, but *acts on the relative coordinates just like a parity transformation.* Since $\mathrm{Y}_\ell^m(-\mathbf{x}) = (-1)^\ell \mathrm{Y}_\ell^m(\mathbf{x})$ under parity, we see that if two identical particles have relative orbital angular momentum ℓ, the spatial part of the wavefunction will be either symmetric or antisymmetric under exchange according to whether ℓ is odd or even.

The behaviour of the entire state under exchange is determined by whether the particles in question are bosons or fermions. In the case of identical bosons, the spins must be combined to form a symmetric overall spin state when ℓ is even, or an antisymmetric overall spin state when ℓ is odd so as to ensure the overall state is always symmetric under exchange. For fermions, the opposite holds so as to ensure overall antisymmetry.

if the neutrons are produced in a state where their relative orbital angular momentum is ℓ in the centre of momentum frame, their spatial wavefunction acquires a factor of $(-1)^\ell$ under exchange of the two neutrons.

7.2.1 Identical Particles and Inelastic Collisions

The requirement that the state describing N identical bosons/fermions be totally symmetric/antisymmetric under exchange of any pair of identical particles has important consequences in collision processes where such particles are created or destroyed.

For example, consider an exotic type of 'atom' consisting of a spin-0 pion (denoted by π^-) bound electromagnetically to a deuterium nucleus (denoted by D^+ – itself consisting of a proton and a neutron, bound together by the strong nuclear force). The bound state energy levels of this atom due to the electromagnetic Coulomb attraction between the π^- and D^+ have the same form as in Hydrogen and, in particular, the ground state is $|1,0,0\rangle$, an s-wave having $\ell = 0$. Because the pion is much heavier than the electron, the Bohr radius of the $D^+\pi^-$ 'atom' is much smaller than that of Hydrogen and the π^- wavefunction closely hugs the D^+ nucleus. The D^+ is known to have spin-1 whilst the π^- is spin zero, so the $1s$ ground state of the $D^+\pi^-$ 'atom' has total angular momentum $j = 1$.

Now, as well as their electromagnetic interactions, the π^- and D^+ also interact via the short-range strong nuclear force. This causes the 'atom' to be unstable, with the π^- rapidly being absorbed by the D^+, causing the system to disintegrate into a pair of neutrons (each denoted N):

$$\pi^- + D^+ \longrightarrow N + N\,. \tag{7.15}$$

(In particle physics terminology, processes such as this, where different types of particles appear in the initial and final states, are known as *inelastic*. An *elastic* process is one in which the initial & final states contain the same particles.)

Neutrons are fermions with spin-$\frac{1}{2}$, so the final state must be antisymmetric under their exchange. One possibility is for their spins to combine into one of the triplet

$$|\uparrow\rangle|\uparrow\rangle\,, \qquad \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right)\,, \qquad |\downarrow\rangle\downarrow\rangle$$

of symmetric spin-1 states, combined with a spatial wavefunction that is antisymmetric under exchange. From above, this will be the case if the relative orbital angular momentum of the two neutrons is odd. The other possibility is for the spins to form the antisymmetric spin-0 state

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

combined with a relative spatial wavefunction of even ℓ.

Total angular momentum is conserved, so the spin and orbital angular momentum of the final state must combine to give $j = 1$ as for the initial ground state of the $\pi^- D^+$. In section 6 we learnt that when we combine systems with angular momenta ℓ and s, the combined system could have total angular momentum $j \in \{\ell + s, \ell + s - 1, \ldots |\ell - s|\}$ depending on the relative alignment of the two subsystems. Thus, in the case that the neutrons spins combine to the antisymmetric spin-0 state, then $j = \ell$. However, since this case requires ℓ even for fermionic statistics, we see it is ruled out. The remaining case is that the neutrons spins combine to give net spin-1, so that

$$j \in \{\ell + 1, \ell, \ell - 1\}\,.$$

Since fermionic statistics here require that ℓ is odd, the only possibility to get $j = 1$ is if $\ell = 1$ and the spin and orbital angular momenta of the neutron pair are neither perfectly aligned nor perfectly anti-aligned. Thus the final state has $j = \ell = s = 1$.

These considerations also help us to determine the intrinsic parity of the pion. Recall from section 4.6 that the parity operator Π acts on a state $|\mathbf{x}\rangle$ as

$$\Pi|\mathbf{x}\rangle = \eta|\mathbf{x}\rangle \tag{7.16}$$

where the value of $\eta \in \{+1, -1\}$ is known as the *intrinsic parity* that, like spin, depends on the type of particle that the state $|\mathbf{x}\rangle$ represents. More generally, if $|\mathbf{x}_1, \kappa_1; \mathbf{x}_2, \kappa_2; \cdots; \mathbf{x}_N, \kappa_N\rangle$ describes an N-particle state in which a particle of type κ_a is definitely located at $\mathbf{x}_a$, then from (**??**) the parity operator acts as

$$\Pi|\mathbf{x}_1, \kappa_1; \mathbf{x}_2, \kappa_2; \cdots; \mathbf{x}_N, \kappa_N\rangle = \eta_1\eta_2\cdots\eta_N\,|-\mathbf{x}_1, \kappa_1; -\mathbf{x}_2, \kappa_2; \cdots; -\mathbf{x}_N, \kappa_N\rangle\,. \tag{7.17}$$

where the $\eta_k = \pm 1$ is the *intrinsic parity* of the k^{th} particle. (This holds whether or not the particles are distinguishable.)

Like the spin of fundamental particles, these intrinsic parities are independent of any details of the spatial wavefunction. Intrinsic parities thus have no effect in elastic processes, where the same particles are present in both the initial and final states. The intrinsic parity of a particle can often be determined by examining inelastic processes in which the particle participates.

In the example $\pi^- D^+ \to NN$ above, equating the parities of the initial and final states gives

$$\eta_\pi \, \eta_D = (-1)^\ell \, \eta_N^2 = -1 \,, \tag{7.18}$$

since the initial atomic state $|1,0,0\rangle$ has $\ell = 0$ and hence no parity other than the intrinsic parities of the pion and deuteron. Provided parity is conserved by the strong nuclear interactions causing this decay, we conclude that the π^- and D^+ must have opposite intrinsic parity. Now, the D^+ is predominantly an s-wave bound state of a proton and neutron, so $\eta_D = (-1)^\ell \, \eta_P \, \eta_N = \eta_P \, \eta_N$, and furthermore the proton and neutron can always be chosen to have the same intrinsic parity since they are related by an 'isospin' symmetry[63]. Thus $\eta_D = +1$ and hence the pion must have intrinsic parity $\eta_\pi = -1$.

As we mentioned in section 4.5.1, one of the great surprises of particle physics came in the 1950s. Cosmic rays were found to contain various types of particles, including two similar of similar mass called the τ^+ and the θ^+. Both particles decay quickly, the predominant channels being

$$\theta^+ \longrightarrow \pi^+ + \pi^0 \qquad \text{and} \qquad \tau^+ \longrightarrow \pi^+ + \pi^+ + \pi^- \,. \tag{7.19}$$

The angular distribution of the pions in the final states could be observed in a cloud chamber or bubble chamber, and studying these patterns showed that the pions were produced in an $\ell = 0$ state in both cases. Since $\eta_\pi = -1$ (irrespective of the electric charge of the pion, for the same isospin reason as above), the θ^+ should have intrinsic parity $\eta_\theta = \eta_\pi^2 = +1$, while the τ^+ should have $\eta_\tau = -1$. However, as measurements improved the masses and lifetimes of the two particles became indistinguishable. The puzzle was resolved in 1956 when Lee and Yang proposed that the two particles were in fact one and the same, but that parity was not conserved in their decay process. Their proposal was largely ignored until Lee persuaded his colleague, Madame Chien–Shiung Wu, to test it using the β–decay of a certain isotope of cobalt. Wu's experiments showed that parity is indeed violated in the weak interactions. The fact that $\mathbf{x} \to -\mathbf{x}$ is *not* a symmetry of Nature can be accommodated in QFT, though the deep reason for it remains mysterious[64].

[63]The Standard Model has an (approximate) 'internal' $SU(2)$ symmetry known as *isospin* that rotates protons into neutrons; they are different states of the same *nucleon*, somewhat like the two different spin states $|\uparrow\rangle$, $|\downarrow\rangle$ of the same electron.

[64]There are various natural ways for parity violation to originate in string theory, but needless to say, none of them have been verified experimentally.