Part III Maths **2019**
David Skinner d.b.skinner@damtp.cam.ac.uk

Supersymmetry: Specimen Exam Paper

There will be a 2 hour exam paper, containing three questions. Question 1 is compulsory and carries 20 marks. You then choose to answer EITHER question 2 OR question 3, each of which carry 30 marks. Below are some sample questions.

1. Consider the $d = 0$ action

$$S(z, \bar{z}, \psi_i, \bar{\psi}_i) = |W'(z)|^2 - W''(z)\,\psi_1\psi_2 - \overline{W''(z)}\,\bar{\psi}_2\bar{\psi}_1$$

where $W(z)$ is a polynomial in the bosonic variable z, $W'(z) = \partial W/\partial z$ and the variables ψ_1, ψ_2, $\bar{\psi}_1$, $\bar{\psi}_2$ are fermionic (Grassmann–odd).

Show that the fermionic vector field

$$V_1 = \psi_1 \frac{\partial}{\partial z} - \overline{W'(z)} \frac{\partial}{\partial \psi_2}$$

obeys $\{V_1, V_1\} = 0$ and $V_1(S) = 0$. Find three further, independent fermionic vector fields that also leave the action invariant.

Evaluate:

(a) the partition function $Z = \int \mathrm{e}^{-S(z,\bar{z},\psi_i,\bar{\psi}_i)}\ \mathrm{d}^2 z\, \mathrm{d}^2\psi\, \mathrm{d}^2\bar{\psi}$,

(b) the (unnormalized) expectation value $\langle W'(z)\, g(z)\rangle$, where $g(z)$ is a polynomial,

(c) the (unnormalized) expectation value $\langle f(z)\rangle$, where $f(z)$ is a further polynomial.

[Hint: Consider the effect of rescaling $W(z)$.]

2. In a 1d QFT, let x be a real bosonic scalar field and let $\bar{\psi}$ and ψ be fermionic fields. Consider the action

$$S = \int \mathrm{d}\tau\ \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + \bar{\psi}\frac{d\psi}{d\tau} + \frac{1}{2}\lambda^2 h'(x)^2 + \lambda\, h''(x)\,\bar{\psi}\psi\right]$$

where λ is a coupling constant and $h(x)$ is a smooth function of $x(\tau)$ (and h' the derivative of this function).

(a) Show that this action is invariant under the transformations

$$\begin{aligned}
\delta x &= \epsilon\bar{\psi} - \bar{\epsilon}\psi \\
\delta\psi &= \epsilon(-\dot{x} + \lambda\, h'(x)) \\
\delta\bar{\psi} &= \bar{\epsilon}(\dot{x} + \lambda\, h'(x))
\end{aligned}$$

where ϵ and $\bar{\epsilon}$ are constant fermionic parameters and $\dot{x} = dx/d\tau$. Find the conserved Noether charges Q and $\bar{Q}$ associated to these two symmetries.

(b) Show that $Q\bar{Q} + \bar{Q}Q = 2H$ where H is the Hamiltonian associated to the above Lagrangian. Hence or otherwise show that the energies of the system are non–negative, and that the ground state $|\Psi\rangle$ obeys $Q|\Psi\rangle = \bar{Q}|\Psi\rangle = 0$.

(c) Put this theory on a circle and take $x(\tau)$, $\psi(\tau)$ and $\bar{\psi}(\tau)$ each to be periodic. Show that the partition function $\mathcal{Z} = \int \mathcal{D}x\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\;\mathrm{e}^{-S}$ with these boundary conditions is independent both of the radius of the circle and of λ, provided $\lambda \neq 0$.

(d) What is the operator expression to which this partition function corresponds?

(e) Which bosonic field configurations contribute to $\mathcal{Z}$? By expanding the action in a neighbourhood of these configurations, evaluate $\mathcal{Z}$ in the case that $h(x)$ is a polynomial of degree n with isolated roots.

3. Consider $\mathbb{R}^{2|2}$ with coordinates $(x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm})$. Show that the chiral derivatives

$$D_{\pm} = \frac{\partial}{\partial\theta^{\pm}} - \mathrm{i}\bar{\theta}^{\pm}\frac{\partial}{\partial x^{\pm}}\,, \qquad \bar{D}_{\pm} = -\frac{\partial}{\partial\bar{\theta}^{\pm}} + \mathrm{i}\theta^{\pm}\frac{\partial}{\partial x^{\pm}}$$

obey $\{D_{\pm}, \bar{D}_{\mp}\} = 0$, $\{D_{\pm}, D_{\pm}\} = 0$ and $\{D_{\pm}, \bar{D}_{\pm}\} = 2\mathrm{i}\,\partial/\partial x^{\pm}$.

What is meant by a *chiral superfield*? Show that if $\Phi(x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm})$ is a chiral superfield then it can has a component expansion

$$\Phi(y^{\pm}, \theta^{\pm}) = \phi(y^{\pm}) + \theta^{+}\psi_{+}(y^{\pm}) + \theta^{-}\psi_{-}(y^{\pm}) + \theta^{+}\theta^{-}F(y^{\pm})$$

where $y\pm$ are bosonic coordinates that you should define.

Write down the most general form of supersymmetric action for a theory of a single chiral superfield Φ, together with its complex conjugate. Using arguments based on symmetries and holomorphy, prove that, to all orders in perturbation theory, the superpotential does not receive quantum corrections.