Supersymmetry

University of Cambridge, Part III Mathematical Tripos

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Abstract: These are the lecture notes for the Supersymmetry course given to students taking Part III Maths in Cambridge during Lent Term of 2020. They're still very incomplete and will be updated periodically as we go through the course.

Contents

Acknowledgments

Nothing in these lecture notes is original. In particular, my treatment is heavily influenced by several of the textbooks listed below, especially Hori *et al.* and the excellent lecture notes of Neitzke in the early stages.

Preliminaries

These are the lecture notes for the Part III course on Supersymmetry. I'll assume that you're taking the Part III AQFT course alongside this one and that you took the General Relativity course last term. We'll also touch on ideas from the Statistical Field Theory course.

In contrast to previous years, the course takes the perspective that supersymmetry is useful because it provides us with toy models of QFTs that we can study in detail, helping us to understand QFT in general. In the process, we'll see something of the incredibly rich connections between QFT and modern geometry & topology. The course will not emphasize phenomenology.

Books & Other Resources

Here are some of the books & papers I've found useful while preparing these notes; you might prefer different ones to me:

- **Hori, K., Vafa, C.,** *et al.*, *Mirror Symmetry*, AMS (2003).
  This is the main book for (at least the first part of) the course. It's a huge tome comprising chapters written by several different mathematicians and physicists, with the ultimate aim of understanding Mirror Symmetry in the context of string theory. While you might enjoy the mathematical background in the first few chapters, we'll jump in at chapter 8 where the physics starts, and follow the book through to around chapter 16. While the book itself is rather expensive (a new one costs £130 at amazon.co.uk), you can download a free pdf from the publishers here.

- **Neitzke, A.,** *Applications of Quantum Field Theory to Geometry*,
  `https://www.ma.utexas.edu/users/neitzke/teaching/392C-applied-qft/`
  Lectures aimed at introducing mathematicians to Quantum Field Theory techniques that are used in computing Seiberg–Witten invariants. I very much like the perspective of these lectures, and we'll follow Neitzke's notes at various times during the course.

- **Dijkgraaf, R.,** *Les Houches Lectures on Fields, Strings and Duality*,
  `http://arXiv.org/pdf/hep-th/9703136.pdf`
  An modern perspective on what QFT is all about, and its relation to string theory. The emphasis is on topological field theories and dualities rather than supersymmetry *per se*, but we'll meet lots of these ideas during the course.

- **Deligne, P.,** *et al.*, *Quantum Fields and Strings: A Course for Mathematicians*, vols. 1 & 2, AMS (1999).
  Aimed at professional mathematicians wanting an introduction to QFT. They thus require considerable mathematical maturity to read, but most certainly repay the effort. Almost everything here is beyond the level of this course, but you may enjoyed reading the lectures of Deligne & Freed on *Supersolutions* (vol. 1), Seiberg on

Dynamics of $\mathcal{N} = 1$ *Supersymmetric Field Theories in Four Dinmensions* and those of Witten on *The Index of Dirac Operators* (vol. 1) and *Dynamics of Quantum Field Theory* (vol. 2, especially lectures 12 - 19). I'm hoping to cover at least some of the material in the lectures by Seiberg & Witten in the later part of the course.

- **Nakahara, M.,** *Geometry and Topology for Physics*, IOP (2003).
  A good place to look if you're unsure of the background for the mathematical ideas that we meet in lectures. In particular, contains a discussion of the Atiyah-Singer index theorem that is close to the one we'll meet in section 3.96.

I'll add various further suggestions as we go through the course, so please check back here from time to time.

1 Motivation

Before getting started, let's take a minute to see what this course will be about.

1.1 What is supersymmetry?

In any quantum theory containing fermions, the Hilbert space $\mathcal{H}$ splits as

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$$

where $\mathcal{H}_B$ is the bosonic and $\mathcal{H}_F$ the fermionic Hilbert space. That is, states in $\mathcal{H}_B$ contain an even number of fermionic excitations (possibly none), whilst those in $\mathcal{H}_F$ contain an odd number. The theory is *supersymmetric* if it contains an operator Q, acting as

$$Q : \mathcal{H}_B \to \mathcal{H}_F \qquad \text{and} \qquad Q : \mathcal{H}_F \to \mathcal{H}_B \,,$$

and such that[1]

$$Q^2 = 0 \qquad \text{and} \qquad \left\{Q, Q^\dagger\right\} = 2H \,.$$

Here, $Q^\dagger$ is the adjoint (or Hermitian conjugate) of Q *wrt* the inner product on $\mathcal{H}$ while H is the Hamiltonian of the theory.

Throughout these lectures, we'll be looking in detail at the consequences of the presence of such an operator, understanding why such theories are so special. However, a couple of consequences follow so simply that we may as well state them immediately. Firstly, we have

$$\begin{aligned} [H, Q] &= \frac{1}{2}\left[\left\{Q, Q^\dagger\right\}, Q\right] \\ &= \frac{1}{2}\Big((QQ^\dagger + Q^\dagger Q)Q - Q(QQ^\dagger + Q^\dagger Q)\Big) \\ &= 0 \,, \end{aligned} \tag{1.1}$$

where we've used $Q^2 = 0$ to reach the final line. Provided the Hamiltonian is self-adjoint, we also have $[H, Q^\dagger] = 0$. Thus Q and $Q^\dagger$ are automatically conserved and the transformations they generate will be symmetries of the theory. Since $Q : \mathcal{H}_{B,F} \to \mathcal{H}_{F,B}$, each term in Q must either create or destroy an odd number of fermions and Q itself will be a fermionic operator. Th symmetries it generates therefore mix fermions with bosons and are known as *supersymmetries*, while Q and $Q^\dagger$ are known as *supercharges*.

The second consequence is that, for any state $|\Psi\rangle \in \mathcal{H}$,

$$\begin{aligned} \langle\Psi|H|\Psi\rangle &= \frac{1}{2}\langle\Psi|QQ^\dagger + Q^\dagger Q|\Psi\rangle \\ &= \frac{1}{2}\left\|Q^\dagger|\Psi\rangle\right\|^2 + \frac{1}{2}\|Q|\Psi\rangle\|^2 \geq 0 \,, \end{aligned} \tag{1.2}$$

with equality iff $Q|\Psi\rangle = Q^\dagger|\Psi\rangle = 0$. Thus all states have non-negative energy, and the ground state energy vanishes iff it is invariant under the supersymmetry transformations generated by both Q and its adjoint.

[1] The *anticommutator* is defined for any two operators (whether or not they are fermionic) by $\{A, B\} = AB + BA$. The factor of 2 on the *rhs* of $\{Q, Q^\dagger\} = 2H$ is for later convenience.

This basic structure can be extended in many ways. In a Lorentz invariant quantum field theory, the Hamiltonian H is just one component of the momentum vector P_m so it's natural to expect that the supercharges are also part of a multiplet. The usual (but not the only) way this works is for the Qs and $Q^\dagger$s to transform as spinors under Lorentz transformations. For example, in a Lorentz-invariant supersymmetric QFT in $d = 4$, the basic supersymmetry algebra above is enhanced to

$$\left\{Q_\alpha, Q^\dagger_{\dot\alpha}\right\} = 2\sigma^m_{\alpha\dot\alpha}P_m$$

where $\sigma^m = (1_{2\times 2}, \boldsymbol{\sigma})$ and $\boldsymbol{\sigma}$ are the Pauli matrices[2] whose rows and columns are indexed here by α and $\dot\alpha$. Explicitly,

$$P_{\alpha\dot\alpha} = \sigma^m_{\alpha\dot\alpha}P_m = \begin{pmatrix} H + P_z & P_x - \mathrm{i}P_y \\ P_x + \mathrm{i}P_y & H - P_z \end{pmatrix}$$

so that what we were calling Q before here becomes the linear combination $Q = Q$

A further generalization, known as *extended supersymmetry* is to have several different supercharges Q^a_α, where $a = 1, \ldots, \mathcal{N}$, obeying[3]

$$\left\{Q^a_\alpha, {Q^\dagger}^b_{\dot\alpha}\right\} = 2\delta^{ab}P_{\alpha\dot\alpha} \qquad \text{and} \qquad \left\{Q^a_\alpha, Q^b_\beta\right\} = \epsilon_{\alpha\beta}Z^{ab}\,.$$

where the constants Z^{ab} are known as *central charges*. Note that, from the definition of the anticommutator, the *lhs* of the second equation is certainly symmetric under the simultaneous exchange of both the spinor and 'extended' indices. On the *rhs*, $\epsilon_{\alpha\beta}$ is the $SL(2,\mathbb{C})$ (and hence Lorentz) invariant inner product, so we must also have $Z^{ba} = -Z^{ab}$. The possibility to include central charges reflects the fact that, whilst we still have $(Q^a)^2 = 0$ for each individual value of a, the anticommutator of two different Qs may be non-zero.

We'll explore more of the structure and consequences of this supersymmetry algebra as we go through the course, but for now I want to motivate why we should be interested in such supersymmetric theories at all.

1.2 Why study it?

For much of its development, the main hope for supersymmetry was that it would provide a phenomenologically relevant extension to physics beyond the Standard Model. These hopes were founded on good reasons, ranging from unification of the $SU(3)\times SU(2)\times U(1)$ Standard Model gauge couplings under renormalization group flow, to providing a viable candidate that explains what Dark Matter is, to ensuring that the (Higgs) vacuum of our Universe is stable. In particular, there was great hope that the LHC would provide direct experimental evidence for supersymmetry in Nature.

[2]The role of the Pauli matrices here is just as an intertwiner, providing an isomorphism between the tensor product of a left- and a right-Weyl spinor representations of $SO(1,3)$ and the vector representation. In other dimensions, we may modify this to $\{Q_A, Q^\dagger_B\} = 2\Gamma^M_{AB}P_M$ where Γ is some form of Dirac matrix and the range of the spinor indices depends on d.

[3]This equation uses spinors appropriate for $d = 4$, but again extended supersymmetry exists in other dimensions, too.

As you're probably well aware, this has not happened. While the experiment is still ongoing and the full parameter space is still being explored, it now looks as though one of the original phenomenological motivations for supersymmetry — a potential explanation of why the Higgs particle is so much lighter than the Planck scale, known as the *hierarchy problem* — is actually not related to supersymmetry. Worse, although it is still perfectly possible for supersymmetry to exist in Nature at energy scales beyond those accessible to the LHC, it is not at all clear that the political or economic will is there to spend a very large sum of money hunting for it. For these reasons, I no longer think it is reasonable to study supersymmetry with the aim of doing phenomenology. (Others may disagree!)

Instead, the main claim I wish to convince you of is that *supersymmetry is interesting because it allows us to better understand quantum field theory.* This motivation will affect the whole structure and choice of topics in our course.

Let's get a glimpse of what I have in mind. In any Quantum Field Theory, the main objects we wish to understand are *path integrals*. A typical example is[4]

$$\int_{\mathcal{C}} \exp\left(-\frac{1}{\hbar}S[\phi]\right)\mathcal{D}\phi\,.$$

This integral is supposed to be taken over the space $\mathcal{C}$ of all[5] field configurations. That is, every point $\phi \in \mathcal{C}$ corresponds to a configuration of the field – a picture of what the field looks like across our whole Universe M. Since we allow our fields to have arbitrarily small bumps and ripples, $\mathcal{C}$ is typically an infinite dimensional function space. The path integral involves some sort of measure $\mathrm{e}^{-S[\phi]/\hbar}\,\mathcal{D}\phi$ that weights the contribution of each field configuration $\phi \in \mathcal{C}$ by $\mathrm{e}^{-S[\phi]/\hbar}$, where $S[\phi]$ is the classical action. You'll have met something very similar if you took the Statistical Field Theory course last term (or perhaps a Statistical Mechanics course earlier), and you'll learn much more about path integrals in the Advanced Quantum Field Theory course this term.

Unfortunately, in a generic QFT, it's almost impossibly difficult even to make sense of the ingredients in this integral, let alone to perform the integral itself. Typically, the best we can do is perturbation theory. That is, we pick a particular field configuration ϕ_0 that solves the classical field equations (almost always, this will be the 'trivial' configuration $\phi_0 = 0$) and then write down Feynman diagrams describing particles travelling on this background, or vacuum. The particles are really just perturbative excitations $\delta\phi = \phi - \phi_0$ of the quantum field, so you're examining the behaviour of the theory in a neighbourhood of your original field configuration. (Last term, you constructed such Feynman diagrams by acting on the vacuum state with creation & annihilation operators. You'll see how the same diagrams emerge from the path integral in this term's AQFT course.) Even this Feynman diagram expansion is very complicated.

This was not how you learned QM. You didn't jump from considering free particle solutions to Schrödinger's equation straight into perturbation theory. Instead, you first

[4]I've written this path integral assuming that we're in Euclidean signature. For Minkowski signature, we'd instead have $\mathrm{e}^{\mathrm{i}S[\phi]/\hbar}$.

[5]We'll gloss over the subtle and important point of exactly what sort of configurations should be allowed – should they be continuous? differentiable? smooth. The correct answer involves Wilsonian renormalization.

studied a wealth of carefully chosen, exactly solvable examples, almost certainly including particles in a square well, the harmonic oscillator, the pure Coulomb potential and the theory of angular momentum. All of these are idealisations, often containing a large amount of symmetry. Nonetheless, they are often closer to the real world — and so provide a better starting point around which to study perturbation theory — than is free theory. Furthermore, studying them also helps us build our experience and understanding of what QM itself actually is!

Making a QFT supersymmetric allows us to do the same for QFT. While a typical QFT is intractable, in supersymmetric theories we can often obtain *exact* results, at least for some observables. By studying them, we understand QFT itself better, even if our final goal is to do away with supersymmetry.

Better still, these exact results often reveal deep and beautiful connections between QFT and geometry and topology. Examples include many of the highlights of twentieth century mathematics, such as the Atiyah–Singer index theorem, Mirror Symmetry, and Donaldson or Seiberg–Witten invariants of 4-manifolds. The interplay between mathematics and physics in these areas has been one of the most fruitful major lines of research of the past thirty years, with breakthroughs still ongoing. From a physics point of view, they teach us that even the vacuum of a QFT can be a highly subtle, interesting object, and this realisation is now feeding through even to experiments in areas such as Symmetry Protected Topological Phases of certain condensed matter systems. The richness of perspective that supersymmetry brings to our appreciation of QFT is, for me at least, the best reason to study it.