2 Supersymmetry in Zero Dimensions

In this first chapter, we'll look at supersymmetry in zero dimensions, treating it as a safe playground in which to begin our study. We'll gradually increase the number of dimensions as we go through the course. We start with a discussion of fermionic variables and Berezin integration. We'll then consider 'zero–dimensional QFT'. In practice, this will mean exploring some integrals that are supposed to be toy examples of the partition functions we meet in higher dimensional theories. We'll start by looking at bosonic and fermionic $d = 0$ 'path' integrals separately. Even in this highly over-simplified situation, we can typically only compute these (bosonic) integrals approximately. We'll then see our first examples of how supersymmetry improves things, allowing us to perform the integrals exactly.

2.1 Fermions and super vector spaces

To consider a supersymmetric theory, or in fact any theory involving fermionic fields, the first thing we need is the notion of Grassmann variables. Let's now give a very brief formal introduction to these.

A $\mathbb{Z}_2$-*graded* vector space is a vector space $V = V_0 \oplus V_1$ (over a field that we can take to be $\mathbb{R}$ or $\mathbb{C}$) endowed with a parity operation. Vectors that lie purely in V_0 or purely in V_1 are said to be homogeneous. The *parity* $|v|$ of a homogeneous vector v is given by

$$|v| = \begin{cases} 0 & \text{for } v \in V_0 \\ 1 & \text{for } v \in V_1\,. \end{cases} \tag{2.1}$$

We often call elements $v \in V_0$ *even* whilst elements $w \in V_1$ are called *odd*. Alternatively, looking ahead to physics, we often say such elements are *bosonic* and *fermionic*, respectively. If V_0 and V_1 have dimensions p and q, respectively, the dimension of V is usually denoted $p|q$, keeping track of the bosonic and fermionic dimensions separately. The basic example is $V = \mathbb{R}^{p|q}$.

The usual operations on vector spaces immediately extend to $\mathbb{Z}_2$-graded vector spaces in a way that respects the grading. For example, the *dual* V^* of a (finite dimensional) super vector space, say over $\mathbb{C}$, is again the space of all linear maps $\varphi : V \to \mathbb{C}$, with the even part $(V^*)_0$ being those linear maps that vanish on V_1, and the odd part V_1^* being those maps that annihilate V_0. Similarly, the *direct sum* of two super vector spaces V and W is the super vector space $V \oplus W$ where $(V \oplus W)_0 = V_0 \oplus W_0$ and $(V \oplus W)_1 = V_1 \oplus W_1$, while their *direct product* is the super vector space $V \otimes W$ with even and odd parts

$$\begin{aligned} (V \otimes W)_0 &= (V_0 \otimes W_0) \,\oplus\, (V_1 \otimes W_1) \\ (V \otimes W)_1 &= (V_1 \otimes W_0) \,\oplus\, (V_0 \otimes W_1)\ , \end{aligned}$$

respecting the $\mathbb{Z}_2$ grading.

So far, everything we've defined has been the as it would be for any $\mathbb{Z}_2$ graded vector space. What makes V a super vector space is an unusual choice of symmetry operation.

For 'ordinary' (*i.e.*, purely bosonic) vector spaces A and B, there is an exchange operation s that switches the order in the direct product:

$$s : A \otimes B \to B \otimes A \,, \qquad\qquad s : a \otimes b \mapsto b \otimes a$$

for any $a \in A$ and $b \in B$. This symmetry operator works the same way for $\mathbb{Z}_2$-graded spaces as for standard vector spaces, and can be used to define symmetric and antisymmetric powers. For example,

$$\begin{aligned} A \odot B &= A \otimes B + s(A \otimes B) \\ A \wedge B &= A \otimes B - s(A \otimes B) \,. \end{aligned}$$

However, in a super vector space, we instead define the exchange operator by

$$s : V \otimes W \to W \otimes V \,, \qquad\qquad s : v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \tag{2.2}$$

for any homogeneous elements v and w (and extended by linearity to all of V and W). In other words, interchanging the order of any two fermions gives a minus sign. In consequence, while the symmetric products $\mathrm{Sym}^* V_0$ of the even part of a super vector space are just the usual symmetric products as for any vector space, the 'symmetric' products of an odd vector space is actually the exterior (antisymmetric) algebra on the underlying vector space. That is $\mathrm{Sym}^* V_1 \cong \bigwedge^* V_1$, where on the *rhs* we treat V_1 as a standard vector space 'forgetting' that it is fermionic[6].

Closely related to this is the notion of a *superalgebra*. This is a super vector space $\mathcal{A}$ together with a bilinear multiplication map

$$\cdot \; : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

that respects the grading, in the sense that for any elements $a, b \in \mathcal{A}$ we have $|a \cdot b| = |a| + |b|$. As usual, we'll often drop the multiplication symbol $\cdot$, indicating it just by juxtaposition. A superalgebra $\mathcal{A}$ is said to be *supercommutative* (or often just commutative) if

$$ab = (-)^{|a||b|}\, ba \tag{2.3a}$$

so that again we get a minus sign when any pair of fermionic variables are exchanged. For example, to make $\mathbb{R}^{p|q}$ into a superalgebra, we have the relations

$$x^i x^j = x^j x^i \,, \qquad\qquad x^i \psi^a = \psi^a x^i \,, \qquad\qquad \psi^a \psi^b = -\psi^b \psi^a \,. \tag{2.3b}$$

where $x^i \in \mathbb{R}^{p|0}$ are standard, real variables and $\psi^a \in \mathbb{R}^{0|q}$ are fermionic. Note in particular that $\psi^a \psi^a = -\psi^a \psi^a = 0$ for any fixed a. Thus our superalgebra must contain at least two fermionic variables if it is to have any non-trivial content (beyond that of a usual algebra).

Not all superalgebras are supercommutative. For example, a *Lie superalgebra* is a supervector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where the multiplication

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

[6] More correctly, we let Π denote the parity reversing operator, so $\Pi(V_0 \oplus V_1) = V_1 \oplus V_0$, with V_1 now even. Then $\mathrm{Sym}^* V_1 = \Pi^*(\bigwedge^* \Pi V_1)$; that is, we take antisymmetric powers of the (now bosonic) vector space ΠV_1 and then declare the resulting space to again be fermionic.

obeys the graded antisymmetry rule

$$[X,Y] = -(-)^{|X||Y|}[Y,X]$$

and graded Jacobi identity

$$[X,[Y,Z]] + (-)^{|X|(|Y|+|Z|)}[Y,[Z,X]] + (-)^{|Y|(|Z|+|X|)}[Z,[X,Y]] = 0$$

We often let $\{\ ,\ \}$ denote the Lie bracket restricted to two odd elements in $\mathfrak{g}_1 \subset \mathfrak{g}$; this is the *anticommutator* we saw in the Introduction.

In the same way, we define the ring of polynomials on a super vector space V to be $\mathcal{O}(V) = \mathrm{Sym}^* V^*$, with multiplication taken as in (2.2). Thus, treating V_0 and V_1 as ordinary vector spaces, the ring of polynomials on a super vector space is isomorphic to $\mathrm{Sym}^* V_0^* \otimes \wedge^* V_1^*$. More generally, the space of smooth functions $C^\infty(V)$ on V is

$$C^\infty(V) = C^\infty(V_0) \otimes \wedge^* V_1^*\,,$$

combining smooth functions on V_0 with the exterior algebra of V_1. For example, if $F \in C^\infty(\mathbb{R}^{p|q})$, then we have

$$F(x,\psi) = f(x) + r_a(x)\psi^a + s_{ab}(x)\psi^a\psi^b + \cdots + g(x)\psi^1\psi^2\cdots\psi^q \tag{2.4}$$

where f, r_a, s_{ab}, …, g are smooth functions on $\mathbb{R}^p$. In the context of QFT, such functions (and their generalizations to other ranges and domains) are often called *superfields*, and the functions f, r_a, $s_{ab}\,, \ldots$ are called their *component fields*. Note that the component functions are antisymmetric in their indices, *e.g.* $s_{ab}(x) = -s_{ab}(x)$, inheriting this from the antisymmetry of the fermions.

Parenthetically, let me point out that there should be (at least) two places where you've seen something reminiscent of this before. One is the anticommutation relations

$$\{\Psi^a(x), \Psi^b(y)\} = 0 \tag{2.5}$$

you introduced last term when quantizing fermionic *fields* such as the electron. The relations (2.5) were more involved, because the electrons are spinors, and live in $3+1$ dimensions, whereas our variables ψ^a are just odd elements of some super vector space. Nonetheless, the 'fermionic' nature of the electron field has its mathematical origins in exactly this notion. The second place is in the relation

$$dx^a \wedge dx^b = -dx^b \wedge dx^a$$

of the exterior algebra of forms, that you saw in the General Relativity course last term. Note also that, given a p-form α and a k-form β, their exterior product obeys $\alpha \wedge \beta = (-1)^{kp}\beta \wedge \alpha$. Furthermore, whilst we might write a p-form $\omega \in \Omega^p(N)$ on an n-dimensional manifolds N as $\omega(x) = \omega_{ab\cdots c}(x)dx^a \wedge dx^b \wedge \cdots \wedge dx^c$, a general *polyform* can be written

$$F(x,dx) = f(x) + r_a(x)dx^a + s_{ab}(x)dx^a \wedge dx^b + \cdots + g(x)dx^1 \wedge \cdots \wedge dx^n$$

We'll later understand that the similarity to (2.4) is no accident.

2.1.1 Differentiation and Berezin integration

We'll also need to define differentiation and integration for fermions.

A *derivation* of a commutative superalgebra $\mathcal{A}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ that obeys

$$D(ab) = (Da)b + (-)^{|a||D|} a\,(Db) \tag{2.6}$$

for every $a, b \in \mathcal{A}$. In other words, we have a graded version of the Leibniz rule. For example, on $\mathbb{R}^{p|q}$ we have even derivatives $\partial/\partial x^i$ (the usual derivative on $\mathbb{R}^p$) and also odd derivatives $\partial/\partial\psi^a$, defined by

$$\frac{\partial}{\partial x^i} x^j = \delta_i^j\,, \qquad \frac{\partial}{\partial x^i}\psi^b = 0\,, \qquad \frac{\partial}{\partial \psi^a}\psi^b = \delta_a^b\,, \qquad \frac{\partial}{\partial \psi^a} x^j = 0$$

and the derivation property

$$\frac{\partial}{\partial \psi^a}(\psi^b\psi^c) = \delta_a^b\,\psi^c - \psi^b\,\delta_a^c\ .$$

More generally, a (smooth) *vector field* on $\mathbb{R}^{p|q}$ is a derivation

$$X(x,\psi) = X^i(x,\psi)\frac{\partial}{\partial x^i} + \chi^a(x,\psi)\frac{\partial}{\partial \psi^a}$$

where X^i, $\chi^a \in C^\infty(\mathbb{R}^{p|q})$. The vector field is even if X^i is even and χ^a is odd, whilst it is odd if the X^i are odd and the χ^a even.

Now let's turn to integration. Since any function of a single fermionic variable ψ is of the form $f + \rho\,\psi$, we only have to define $\int \mathrm{d}\psi$ and $\int \mathrm{d}\psi\,\psi$. We ask that our integration measure is translationally invariant, so that if $\psi' = \psi + \eta$ with some fixed $\eta \in \mathbb{R}^{0|1}$ then

$$\int \psi'\,\mathrm{d}\psi' = \int (\psi+\eta)\,\mathrm{d}\psi\ = \int \psi\,\mathrm{d}\psi + \eta\int 1\,\mathrm{d}\psi$$

by linearity of the integral. This implies

$$\int 1\,\mathrm{d}\psi\ = 0\,. \tag{2.7a}$$

We then choose to normalise our integration measure such that

$$\int \psi\,\mathrm{d}\psi = 1\,. \tag{2.7b}$$

These rules are often known as *Berezin integration.* Note that they imply

$$\int \frac{\partial}{\partial\psi}F(\psi)\,\mathrm{d}\psi\ = 0$$

since the derivative removes the single power of ψ that can appear in $F(\psi)$. This allows us to integrate by parts, provided due care is taken of signs.

If we have n fermionic variables ψ^a, repeated application of the above rules shows that the only non-vanishing integral is one whose integrand involves exactly one power of every ψ^a. Specifically, we have

$$\int \psi^1\psi^2\cdots\psi^{n-1}\psi^n\,\mathrm{d}^n\psi = \int \psi^1\psi^2\cdots\psi^n\,\mathrm{d}\psi^n\,\mathrm{d}\psi^{n-1}\,\cdots\,\mathrm{d}\psi^1 \;= 1 \tag{2.8}$$

and, in general

$$\int \psi^{a_1}\psi^{a_2}\cdots\psi^{a_n}\,\mathrm{d}^n\psi = \epsilon^{a_1a_2\cdots a_n} \tag{2.9}$$

with the sign coming from ordering the ψs. Suppose we write $\chi^a = N^a_{\ b}\psi^b$ for some $N \in GL(n)$ and consider integrating the χs against the original measure $\mathrm{d}^n\psi$. Then, by linearity

$$\begin{aligned}\int \chi^{a_1}\chi^{a_2}\cdots\chi^{a_n}\,\mathrm{d}^n\psi &= N^{a_1}_{\ b_1}N^{a_2}_{\ b_2}\cdots N^{a_n}_{\ b_n}\int \psi^{b_1}\psi^{b_2}\cdots\psi^{b_n}\,\mathrm{d}^n\psi \\ &= N^{a_1}_{\ b_1}N^{a_2}_{\ b_2}\cdots N^{a_n}_{\ b_n}\,\epsilon^{b_1b_2\cdots b_n} \\ &= \det(N)\,\epsilon^{a_1a_2\cdots a_n} = \det N\int \chi^{a_1}\chi^{a_2}\cdots\chi^{a_n}\,\mathrm{d}^n\chi\,.\end{aligned} \tag{2.10}$$

Thus we see that for Berezin integration

$$\chi^a = N^a_{\ b}\psi^b \qquad \Rightarrow \qquad \mathrm{d}^n\chi = \frac{1}{\det(N)}\mathrm{d}^n\psi\,, \tag{2.11}$$

where the Jacobian of the change of variables appears upside down (and without a modulus sign) compared to the standard, bosonic rule $\mathrm{d}^n y = |\det N|\,\mathrm{d}^n x$ if $y^a = N^a_{\ b}x^b$.

2.2 QFT in zero dimensions

Let's now get a bit closer to the sort of path integrals we meet in QFT. We'll begin by thinking about purely bosonic and purely fermionic theories, before seeing what we gain by making our integral supersymmetric.

2.2.1 A purely bosonic theory

In our zero-dimensional toy model, the whole Universe M is just a single point:

$$M = \{\mathrm{pt}\}\,.$$

Then, in the simplest case, a 'field' on M is just a map $X : \{\mathrm{pt}\} \to \mathbb{R}$, or in other words just a real variable. The space of all field configurations is also easy to describe: with n such fields, $\mathcal{C} \cong \mathbb{R}^n$ because we completely specify what the fields looks like everywhere on M = pt just by giving their values. The path integral measure $\mathcal{D}X$ becomes just the standard (Lebesgue) measure d^nX on $\mathbb{R}^n$, so the path integral reduces to a standard integral

$$\mathcal{Z} = \int_{\mathbb{R}^n} \mathrm{e}^{-S(X)/\hbar}\,\mathrm{d}^nX\,, \tag{2.12}$$

over $\mathbb{R}^n$.

In zero dimensions, there are no space-time directions along which we could differentiate our 'fields', so the action is just a function $S(X)$ of these real variables, with no 'kinetic terms'. All that really matters is that this function is chosen so that the partition function (2.12) converges, but we'll typically take $S(X)$ to be a polynomial (with highest term of even degree), such as

$$S(X) = \frac{m^2}{2} X^i X^i + \frac{\lambda_{ijkl}}{4!} X^i X^j X^k X^l \,.$$

If the action is purely quadratic, corresponding to a free theory, then the partition function (2.12) is a simple Gaussian integral. However, interesting theories involve interactions, and then exact evaluation of (2.12) may well be beyond us even in this near–trivial zero-dimensional case.

We may hope to approximate (2.12) perturbatively by expanding around the classical limit $\hbar \to 0$. In this limit, the weighting $\mathrm{e}^{-S/\hbar}$ suppresses all contributions to the integral except perhaps those near the minima of S. In particular, if $S(X)$ has a unique, isolated minimum at some point $X_0 \in \mathbb{R}^n$ (which may be the 'trivial' vacuum $X_0 = 0$), the Hessian $\partial_i \partial_j S(X)$ will be positive–definite at X_0. Then, as $\hbar \to 0^+$, we have asymptotically

$$\int_{\mathbb{R}^n} \mathrm{e}^{-S(X)/\hbar}\, d^n X \sim (2\pi\hbar)^{n/2} \frac{\mathrm{e}^{-S(X_0)/\hbar}}{\sqrt{\det\left(\partial_i \partial_j S|_{X_0}\right)}} \left(1 + A_1 \hbar + A_2 \hbar^2 + \cdots\right) \,. \tag{2.13}$$

The proof of this is known as the *method of steepest descent* (or *stationary phase* in the Minkowski case) and should be familiar if you've taken a course on Asymptotic Methods[7].

[7]In case you didn't take such a course, here's an outline of a proof in the case of a single field: Let

$$A(\hbar) = \frac{\mathrm{e}^{+S(X_0)/\hbar}}{\sqrt{\hbar}} \int_a^b \mathrm{e}^{-S(X)/\hbar} f(X)\, \mathrm{d}X$$

and let $\epsilon \in (0, \frac{1}{2})$. Define $B(\hbar)$ in the same way as $A(\hbar)$, but where the integral is taken over the range $[X_0 - \hbar^{\frac{1}{2}-\epsilon}, X_0 + \hbar^{\frac{1}{2}-\epsilon}]$. As $\hbar \to 0$, we have that $A(\hbar) - B(\hbar)$ is smaller than $\hbar^N$ for any $N \in \mathbb{N}$. (We say the difference is *rapidly decaying* in $\hbar$.) Now let $\chi = (X - X_0)/\sqrt{\hbar}$, so

$$B(\hbar) = \int_{-\hbar^\epsilon}^{\hbar^\epsilon} \mathrm{e}^{(S(X_0) - S(X_0 + \chi\sqrt{\hbar}))/\hbar} f(X_0 + \chi\sqrt{\hbar})\, \mathrm{d}\chi \,.$$

Provided the action $S(X)$ and insertion $f(X)$ were smooth, the integrand of this expression is a smooth function of $\sqrt{\hbar}$ when $\hbar \geq 0$. Let $C(\hbar)$ be the same integral as for $B(\hbar)$, but with the integrand replaced by its Taylor expansion around 0 in $\sqrt{\hbar}$, modulo terms of order $\hbar^N$. Then

$$|B(\hbar) - C(\hbar)| \leq K\, \hbar^{N-\epsilon}$$

for some constant $K \geq 0$. Finally, let $D(\hbar)$ be the same as $C(\hbar)$, but where the limits of the integral are $-\infty$ and ∞. Then $D(\hbar)$ is a polynomial in $\sqrt{\hbar}$, while $C(\hbar) - D(\hbar)$ is rapidly decaying in $\hbar$. Since $D(\hbar)$ is a polynomial in $\sqrt{\hbar}$, it admits a Taylor expansion in $\sqrt{\hbar}$ modulo $\hbar^{N-\epsilon}$. Also, the coefficients of odd powers of $\sqrt{\hbar}$ in $D(\hbar)$ are given by integrals of an odd function of χ over all of $\mathbb{R}$, and hence vanish. Finally, we have

$$D(0) = \int_{\mathbb{R}} \mathrm{e}^{-\partial^2 S|_{X_0} \chi^2/2} f(X_0)\, \mathrm{d}\chi = \frac{\sqrt{2\pi}\, f(X_0)}{\sqrt{\partial^2 S|_{X_0}}} \,.$$

Putting all these facts together shows that

$$\int_{\mathbb{R}} \mathrm{e}^{-S(X)/\hbar} f(X)\, \mathrm{d}X = \mathrm{e}^{-S(X_0)/\hbar} \sqrt{\hbar}\, A(\hbar) \sim \sqrt{2\pi\hbar} \frac{\mathrm{e}^{-S(X_0)/\hbar} f(X_0)}{\sqrt{\partial^2 S|_{X_0}}} \sum_{n=0}^{\infty} A_n \hbar^n \,,$$

In particular, expanding X around the classical solution X_0 as $X^i = X_0^i + \delta X^i$, we have

$$S(X) = S(X_0) + \frac{1}{2}\partial_i\partial_j S|_{X_0}\delta X^i\,\delta X^j + \mathcal{O}(\delta X^3) \tag{2.14}$$

so that the leading term

$$\mathcal{Z}_0 = (2\pi\hbar)^{n/2}\frac{\mathrm{e}^{-S(X_0)/\hbar}}{\sqrt{\det(\partial_i\partial_j S|_{X_0})}} \tag{2.15}$$

in the asymptotic series of the partition function is just what we'd obtain as the partition function of a theory a free (purely quadratic) theory.

In the AQFT course, you'll learn that the different terms in this expansion correspond to different types of Feynman diagrams: the classical action $S(X_0)$ evaluated at our critical point gives tree–level vacuum diagrams, the determinant $\det(\partial_i\partial_j S|_{X_0})$ corresponds to 1-loop vacuum diagrams, and the higher-order terms in the series (2.13) correspond to higher-loop quantum corrections.

We don't usually expect this perturbation series to give us exact results about the partition function and in fact, expansions in $\hbar$ typically have zero radius of convergence! If the expansion (2.13) were to converge at any finite $\hbar$, it would have to converge for all $\hbar$ in a disc $D \subset \mathbb{C}$ centered on the origin. But if the action is chosen so that the integral defining the partition function converges whenever $\hbar > 0$, then it surely diverges if we formally attempt continue into the region $\hbar < 0$.

What (2.13) actually gives is an *asymptotic expansion* of the partition function. Recall that a series $\sum_n a_n\hbar^n$ is asymptotic to a function $I(\hbar)$ if, for all $N \in \mathbb{N}$,

$$\lim_{\hbar\to 0^+}\frac{1}{\hbar^N}\left|I(\hbar) - \sum_{n=0}^{N} a_n\hbar^n\right| = 0\ . \tag{2.16}$$

In other words, with fixed N, for sufficiently small $\hbar \in \mathbb{R}_{\geq 0}$ the first N terms of the series differ from the exact answer by less than $\epsilon\lambda^N$ for *any* $\epsilon > 0$. (The difference is $o(N)$). We write

$$I(\hbar) \sim \sum_{n=0}^{\infty} a_n\hbar^n \qquad \text{as}\ \ \hbar \to 0 \tag{2.17}$$

to mean that the series on the right is an asymptotic expansion of $I(\hbar)$ as $\hbar \to 0$. It's important to remember that the true function may differ from its asymptotic series by transcendental terms; for example, the function $\mathrm{e}^{-1/\hbar^2} \sim 0$ as $\hbar \to 0$, but clearly $\mathrm{e}^{-1/\hbar^2} \neq 0$. Thus, if we instead fix a value of $\hbar$, however small, and include more and more terms in the sum, we will eventually get worse and worse approximations to the answer. Perturbation theory thus tells us important, but not complete, information about our QFT.

2.2.2 A purely fermionic theory

Next let's look at a purely fermionic $d = 0$ QFT. The first step is to compute a Gaussian integral for fermions, modelling the case of a free fermionic theory in higher dimensions.

where $A_0 = 1$. This proves (2.13) in the case of a single field. The generalization to finitely many fields X^a is straightforward. But don't worry, neither of these proofs are central to (nor examinable for) this course.

Since the action must be bosonic, the smallest number of fermions we can consider is two. The action is then

$$S(\psi^1, \psi^2) = A\psi^1\psi^2$$

for some A, and the $d = 0$ partition function is the integral

$$\int e^{-A\psi^1\psi^2/\hbar}\, d\psi^1\, d\psi^2 \,.$$

over these two fermions. Using the fact that $S^2 = 0$, the expansion of $e^{-A\psi^1\psi^2\hbar}$ truncates at the first non-trivial term and we have

$$\int e^{-A\psi^1\psi^2\hbar}\, d\psi^1\, d\psi^2 = \int \left(1 - \frac{A}{\hbar}\psi^1\psi^2\right) d^2\psi = \mathbb{A}/\hbar \,.$$

from the rule (2.8) of Berezin integration.

More generally, for $2m$ fermions and an antisymmetric matrix A_{ab}, the Gaussian integral is

$$\begin{aligned}
\int e^{-A_{ab}\psi^a\psi^b/2\hbar}\, d^{2m}\psi \quad &= \int \sum_{k=0}^{m} \frac{(-)^k}{(2\hbar)^k k!} (A_{ab}\psi^a\psi^b)^k\, d^{2m}\psi \\
&= \frac{(-)^m}{(2\hbar)^m m!} \int A_{a_1a_2}A_{a_3b_4}\cdots A_{a_{2m-1}a_{2m}}\, \psi^{a_1}\psi^{a_2}\cdots\psi^{a_{2m-1}}\psi^{a_{2m}}\, d^{2m}\psi \\
&= \frac{(-)^m}{(2\hbar)^m m!} \epsilon^{a_1a_2\cdots a_{2m-1}a_{2m}} A_{a_1a_2}\cdots A_{a_{2m-1}a_{2m}} \\
&= \left(-\frac{1}{\hbar}\right)^m \mathrm{Pfaff}(A)\,,
\end{aligned} \tag{2.18}$$

where the only term of the expansion that contributes is the one where each fermion appears precisely once. In the final line, we've introduced the *Pfaffian* of a $2m \times 2m$ antisymmetric matrix A by

$$\mathrm{Pfaff}(A) = \frac{1}{2^m m!} \epsilon^{a_1a_2\cdots a_{2m-1}a_{2m}} A_{a_1a_2}\cdots A_{a_{2m-1}a_{2m}} \,. \tag{2.19}$$

For example,

$$\mathrm{Pfaff}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a \,.$$

In the first problem set, I ask you to use fermionic variables to show that $(\mathrm{Pfaff}\, A)^2 = \det A$, so that our Gaussian integral (2.18) can be written as $\pm\sqrt{\det(A)}$. For comparison, recall that the Gaussian integral of the quadratic form $\frac{1}{2}M_{ab}x^a x^b$ of n bosonic variables gives (up to a constant) $\sqrt{1/\det(M)}$, with M a symmetric matrix. Thus, except for a possible numerical factor which we could in any case include in the normalization of the measure, the fermionic result for Gaussian integrals is again just the inverse of the bosonic one.

If our action only contains (finitely many) fermions, then the expansion of e^{-S} will always truncate eventually. The Berezin integral will then just extract the coefficient of whatever terms where all the fermions are present, so the integral will be some polynomial in the coupling constants in the action. For example, suppose we have 4 fermions (so that we can write a non-trivial quartic interaction), with

$$S(\psi^a) = A\left(\psi^1\psi^2 + \psi^3\psi^4\right) + \lambda\psi^1\psi^2\psi^3\psi^4\,.$$

Then $S^3 = 0$ and

$$\begin{aligned}\mathrm{e}^{-S/\hbar} &= 1 - \frac{1}{\hbar}S + \frac{1}{2\hbar^2}S^2 \\ &= 1 - \frac{1}{\hbar}\left[A\left(\psi^1\psi^2 + \psi^3\psi^4\right) - \lambda\psi^1\psi^2\psi^3\psi^4\right] + \frac{A^2}{\hbar^2}\,\psi^1\psi^2\psi^3\psi^4\end{aligned}$$

and the integral $\int \mathrm{e}^{-S/\hbar}\,\mathrm{d}^4\psi = (A/\hbar)^2 - \lambda/\hbar$ analytically. In this sense, in $d = 0$ purely fermionic theories are simpler than purely bosonic ones. Unfortunately, this simplification does not carry over to higher dimensions.

2.3 A supersymmetric theory

We're now ready to consider a $d = 0$ theory containing both a bosonic variable $x \in \mathbb{R}$ and two fermionic variables, ψ^1 and ψ^2. The space of fields is then $\mathcal{C} \cong \mathbb{R}^{1|2}$. A generic action involving these fields takes the form

$$S(x, \psi^1, \psi^2) = V(x) + U(x)\,\psi^1\psi^2 \tag{2.20}$$

for some (smooth) functions U, V of the bosonic fields. Note that (in the absence of fermionic sources) there can't be any terms in S involving only one of the fermion fields since this term would itself be fermionic. There also can't be higher order terms in the fermion fields since $(\psi^a)^2 = 0$ for each fermionic variable.

We'd often take U and V to be polynomials, in which case for the integral of e^{-S} to converge, we just require that $V(x) \to \infty$ as $|x| \to \infty$. Beyond the quadratic terms, the individual monomials in V describe interactions among the bosons and can be represented by vertices in a Feynman diagram, while non-constant terms in U represent couplings between the bosons and fermions. For generic polynomials U and V, this theory is just as intractable as a purely bosonic one: while we can always perform the Berezin integrations exactly (at least for finitely many fermionic variables), doing so leaves us with

$$\frac{1}{\hbar}\int_{\mathbb{R}} U(x)\,\mathrm{e}^{-V(x)/\hbar}\,\mathrm{d}x\,.$$

Typically, this integral is just as difficult as the purely bosonic partition function and again we'd have to be content to evaluate it perturbatively.

Supersymmetry is the magic that will allow us to do better. For later convenience, we'll combine the two real fermions into a single complex one, writing $\psi = \psi^1 + \mathrm{i}\psi^2$ and $\bar{\psi} = \psi^1 - \mathrm{i}\psi^2$. Instead of the generic action (2.20), let's specialise to the case

$$S(x, \psi, \bar{\psi}) = \frac{1}{2}(\partial W)^2 - \bar{\psi}\psi\,\partial^2 W \tag{2.21}$$

where we've imposed a particular relation between U and V, in particular writing $U(x) = (\partial W/\partial x)^2$. This relation has the consequence that $S(x, \psi^1, \psi^2)$ is invariant under the flow generated by the fermionic vector fields

$$\begin{aligned}\mathcal{Q} &= \psi\frac{\partial}{\partial x} + \partial W(x)\frac{\partial}{\partial\bar{\psi}} \\ \mathcal{Q}^\dagger &= \bar{\psi}\frac{\partial}{\partial x} - \partial W(x)\frac{\partial}{\partial\psi}\ .\end{aligned} \tag{2.22}$$

As above, these vector fields are odd derivations of $C^\infty(\mathbb{R}^{1|2})$, acting on the basic variables as

$$\begin{aligned}\mathcal{Q}(x) &= \psi\,, & \mathcal{Q}^\dagger(x) &= \bar{\psi} \\ \mathcal{Q}(\psi) &= 0\,, & \mathcal{Q}^\dagger(\bar{\psi}) &= 0 \\ \mathcal{Q}(\bar{\psi}) &= \partial W(x)\,, & \mathcal{Q}^\dagger(\psi) &= -\partial W\,.\end{aligned}$$

To check that the action is invariant, we compute

$$\begin{aligned}\mathcal{Q}(S) &= \psi\frac{\partial}{\partial x}\left(\frac{1}{2}(\partial W)^2 - \bar{\psi}\psi\,\partial^2 W\right) + \partial W\frac{\partial}{\partial\bar{\psi}}\left(\frac{1}{2}(\partial W)^2 - \bar{\psi}\psi\,\partial^2 W\right) \\ &= \psi\,\partial W\,\partial^2 W - \psi\,\partial W\,\partial^2 W = 0\,,\end{aligned}$$

where we used the fact that $(\psi)^2 = 0$ in computing the first term. A similar calculation shows that $\mathcal{Q}^\dagger(S) = 0$.

Since the action is invariant, we say that $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ generate *supersymmetries* of this zero dimensional theory. Taking the anticommutators of these odd vector fields gives

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\,\partial W\,\psi\frac{\partial}{\partial\bar{\psi}} \qquad \text{and} \qquad \left\{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\right\} = -2\,\partial W\,\bar{\psi}\frac{\partial}{\partial\psi} \tag{2.23a}$$

whilst

$$\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\} = -\partial W\left(\psi\frac{\partial}{\partial\psi} - \bar{\psi}\frac{\partial}{\partial\bar{\psi}}\right) \tag{2.23b}$$

and you can (should!) check that this is a (non-Abelian) Lie superalgebra. The vector fields appearing on the right here are also symmetries of the action as one can check directly: the first two either add some amount of ψ onto $\bar{\psi}$ or vice-versa, but since the action involves these variables only through $\bar{\psi}\psi$, any such change gives ψ^2 or $\bar{\psi}^2$ and hence vanishes. The final vector field rotates the phase of the complex fermion, so obviously leaves the action invariant.

Nonetheless, there are apparently a number of differences compared to the supersymmetry algebra we saw in the Introduction. Firstly, although $\{\mathcal{Q}, \mathcal{Q}\}$ annihilates both x and ψ, it does not annihilate $\bar{\psi}$ and so $\mathcal{Q}^2$ is not zero in general. However, we see from (2.21) that $\psi\,\partial^2 W = 0$ is the 'equation of motion' obtained from varying the action *wrt* $\bar{\psi}$. Thus, our supersymmetry algebra $\mathcal{Q}^2 = 0$ (and likewise $\bar{Q}^2 = 0$) hold only 'on-shell'. We'll see how to do better in the first problem set, and also later on in section **??**. Secondly, unlike the algebra $\{Q, Q^\dagger\} = 2H$ we wrote in the introduction, the *rhs* of (2.23b) should not really be interpreted as a 'Hamiltonian'. Indeed, in any $d = 0$ theory there is no 'time' (or any other) direction in our Universe along which to translate, so $H \equiv 0$ identically. We'll study further examples of supersymmetric theories where the supersymmetry transformations close only up to a global (bosonic) symmetry of the action.

Supersymmetric QFTs are drastically simpler than generic ones because (at least in this zero–dimensional example) the method of steepest descent is exact. There are two ways to understand this, and both are important.

Firstly, suppose we rescale $W \to \lambda W$ for some $\lambda \in \mathbb{R}_+$ and let

$$S_\lambda(x, \psi, \bar{\psi}) = \frac{\lambda^2}{2}(\partial W)^2 - \lambda\, \bar{\psi}\psi\, \partial^2 W$$

be the action built from the rescaled W. Because this is nothing more than a renaming of our original, arbitrary $W(x)$, S_λ is invariant under similarly rescaled supersymmetry transformations generated by

$$\mathcal{Q}_\lambda = \psi\frac{\partial}{\partial x} + \lambda\, \partial W \frac{\partial}{\partial \bar{\psi}} \qquad \text{and} \qquad \mathcal{Q}^\dagger_\lambda = \bar{\psi}\frac{\partial}{\partial x} - \lambda\, \partial W \frac{\partial}{\partial \psi}\,. \tag{2.24}$$

Now, the key point is that the 'path' integral[8]

$$\mathcal{I}(\lambda) = \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-S_\lambda}\, \mathrm{d}x\, \mathrm{d}\psi\, \mathrm{d}\bar{\psi}$$

built using this rescaled action is in fact independent of λ. This is because

$$\begin{aligned} \frac{d}{d\lambda}\mathcal{I}(\lambda) &= -\frac{1}{\sqrt{2\pi}} \int \frac{\partial S_\lambda}{\partial \lambda}\, \mathrm{e}^{-S_\lambda}\, \mathrm{d}x\, \mathrm{d}^2\psi \\ &= -\frac{1}{\sqrt{2\pi}} \int \left(\lambda\, (\partial W)^2 - \bar{\psi}\psi\, \partial^2 W\right) \mathrm{e}^{-S_\lambda}\, \mathrm{d}x\, \mathrm{d}^2\psi \end{aligned} \tag{2.25}$$

where we've commuted the derivative through the integral which is allowed since the integral is absolutely convergent. We now observe that $\lambda\, (\partial W)^2 - \bar{\psi}\psi\partial^2 W = -\mathcal{Q}^\dagger_\lambda(\psi\, \partial W)$. Since S_λ is also invariant under $\mathcal{Q}^\dagger_\lambda$, we have

$$\begin{aligned} \frac{d}{d\lambda}\mathcal{I}(\lambda) &= \frac{1}{\sqrt{2\pi}} \int \mathcal{Q}^\dagger_\lambda(\psi\, \partial W)\, \mathrm{e}^{-S_\lambda}\, \mathrm{d}x\, \mathrm{d}^2\psi \\ &= \frac{1}{\sqrt{2\pi}} \int \mathcal{Q}^\dagger_\lambda \left(\psi\, \partial W\, \mathrm{e}^{-S_\lambda(x,\psi,\bar{\psi})}\right) \mathrm{d}x\, \mathrm{d}^2\psi\,. \end{aligned} \tag{2.26}$$

showing that the entire integrand is the $\mathcal{Q}^\dagger_\lambda$ transformation of something. Acting inside (2.26), the derivative $\partial/\partial\psi$ in $\mathcal{Q}^\dagger_\lambda$ strips off the (only) power of ψ. so this term does not survive the Berezin integration. On the other hand, while the term generated by $\bar{\psi}\partial/\partial x$ does survive the Berezin integrals, it's manifestly a total derivative *wrt* x. For any real function $W(x)$ and $\lambda \in \mathbb{R}^+$, we have

$$\partial W\, e^{-\lambda^2(\partial W)^2/2} \;\to\; 0 \quad \text{as } |x| \to \infty\,,$$

[8]Henceforth I'll set $\hbar = 1$. For later convenience, I've chosen to normalise the integration measure over (each) bosonic variable by a factor of $1/\sqrt{2\pi}$.

so there are no boundary terms and this term also vanishes. Thus we have

$$\frac{d}{d\lambda}\mathcal{I}(\lambda) = 0$$

as promised.

Our original problem was to evaluate $\mathcal{I} = \mathcal{I}(1)$. However, since $\mathcal{I}(\lambda)$ is actually independent of λ, we'll get the same result for our integral whatever value of $\lambda \in \mathbb{R}_+$ we use. In particular

$$\mathcal{I}(1) = \lim_{\lambda\to\infty} \mathcal{I}(\lambda)$$

and this observation is useful because it's easy to calculate the integral as $\lambda \to \infty$. For very large λ, the bosonic factor $\mathrm{e}^{-\lambda^2(\partial W)^2/2}$ suppresses all contributions to the integral arbitrarily strongly everywhere except where $\partial W = 0$. Thus, in this limit, the integral receives contributions only from neighbourhoods of the critical points of $W(x)$. This is the key property of *localization.* It's of course very closely related to perturbation theory using steepest descent, because sending $\lambda \to \infty$ here has the same effect as sending $\hbar \to 0$ there. The important difference is that in the supersymmetric case, nothing is lost by taking this limit, as our argument above shows.

The second, slightly more abstract, way to understand the localization is as follows. Let $\mathcal{U}$ be a neighbourhood of $\partial W = 0$ (so $\mathcal{U}$ is perhaps disconnected) and $\mathcal{U}^{\mathrm{c}}$ its complement. On $\mathcal{U}^{\mathrm{c}}$ we can change variables from $(x, \psi, \bar{\psi}) \mapsto (y, \chi, \bar{\chi})$ where

$$y = x - \frac{\bar{\psi}\psi}{\partial W} \qquad \chi = \psi\sqrt{\partial W} \qquad \bar{\chi} = \bar{\psi}\,. \tag{2.27}$$

In the first problem set, you'll show that under this change of variables

$$\mathrm{d}x\, \mathrm{d}^2\psi = \sqrt{\partial W(y)}\, \mathrm{d}y\, \mathrm{d}^2\chi\,, \tag{2.28}$$

where on the *rhs* W is treated as a function of y. The point of this transformation is that $\mathcal{Q}(y) = 0 = \mathcal{Q}^\dagger(y)$, so that y itself is invariant under supersymmetry. Furthermore,

$$S[y, 0, 0] = \frac{1}{2}(\partial W(y))^2 = \frac{1}{2}(\partial W(x))^2 - \partial^2 W(x)\bar{\psi}\psi = S[x, \psi, \bar{\psi}]\,. \tag{2.29}$$

In fact, y is the only independent combination of $(x, \psi, \bar{\psi})$ that is supersymmetrically invariant, so *any* function $h(x, \psi, \bar{\psi})$ obeying $\mathcal{Q}h = 0 = \mathcal{Q}^\dagger h$ may be expressed as $h(y, 0, 0)$.

We immediately see that the contribution to $\mathcal{I}$ from $\mathcal{U}^{\mathrm{c}}$ is

$$\mathcal{I}_{\mathcal{U}^{\mathrm{c}}} = \frac{1}{2\pi}\int_{\mathcal{U}^{\mathrm{c}}} \mathrm{e}^{-S[x,\psi,\bar{\psi}]}\, \mathrm{d}x\, \mathrm{d}^2\psi = \frac{1}{2\pi}\int_{\mathcal{U}^{\mathrm{c}}} \mathrm{e}^{-S[y,0,0]}\, \sqrt{\partial W(y)}\, \mathrm{d}y\, \mathrm{d}^2\chi = 0 \tag{2.30}$$

since the integrand is independent of the fermionic variables χ, $\bar{\chi}$. As before, the only non-vanishing contributions to $\mathcal{I}$ come from a neighbourhood of the points where $\partial W = 0$ at which our coordinate transformation (2.27) breaks down.

To understand what's happening in general, suppose we wish to perform some integral over a space $\mathcal{C}$, but the integrand (including the measure) is actually invariant under a

Figure 1: *Integrals invariant under a group G fermionic transformations localize to an arbitrarily small neighbourhood of the fixed locus of G.*

group G of transformations. If G acts freely, then we can pick coordinates that decompose the integral into an integral over G itself and an integral over the quotient $\mathcal{C}/G$. Since the original integrand was invariant under G, the integral over the group just gives a factor of $\mathrm{vol}(G)$. For example, if $S(x,y)$ and $\mathcal{O}(x,y)$ are each invariant under rotations

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

then the integral

$$\int_{\mathbb{R}^2} \mathrm{e}^{-S(x,y)} \mathcal{O}(x,y)\, \mathrm{d}^2x = 2\pi \int_0^r \mathrm{e}^{-S(r)} \mathcal{O}(r)\, r\mathrm{d}r \tag{2.31}$$

where the factor $2\pi = \mathrm{vol}(SO(2)) = \int_{SO(2)} \mathrm{d}\theta$ comes from integrating over rotations, and the (non-trivial) radial integral is taken over $\mathbb{R}_{\geq 0} \cong \mathbb{R}^2/SO(2)$. In QFT, G may be a group of global transformations which leave the action S and operators $\mathcal{O}$ invariant.

In our case G is a group of *fermionic* symmetries, parametrized by some fermionic coordinate θ. In this case we have $\int_G \mathrm{d}\theta = 0$ by the basic rule of Berezin integration. Consequently, if an integral is invariant under a fermionic symmetry, we expect to get zero. The exception to this is that G may not act freely but may have some fixed locus $\mathcal{C}_0 \subset \mathcal{C}$. In the example above, this was the locus $\partial W = 0$ where the transformations of ψ and $\bar{\psi}$ vanished. As above, let $\mathcal{U}$ be an arbitrarily small, G-invariant open neighbourhood of $\mathcal{C}_0$ and let $\mathcal{U}^{\mathrm{c}}$ be its complement (see figure 1). Then G acts freely on U^{c} and so the integral over $\mathcal{U}^{\mathrm{c}}$ vanishes by the above argument, just as we saw in our example. Thus the integral only receives contributions from an infinitesimal neighbourhood of the fixed locus $\mathcal{C}_0$.[9]

Now we've understood why it localizes, let's finally go ahead and compute our integral. Suppose for simplicity that $W(x)$ is some generic polynomial of degree D with $D-1$ isolated, non-degenerate[10] critical points. Near any such critical point x_* we have

$$W(x) = W(x_*) + \frac{c_*}{2}(x - x_*)^2 + \cdots \tag{2.32}$$

[9]If you're awake, you'll notice that in my bosonic example of rotations, the origin $0 \in \mathbb{R}^2$ was also a fixed point, but we didn't need seem to give it special consideration. This is because in the bosonic case, the integral over G was non-zero ($= 2\pi$) and the point $r = 0$ was a set of measure zero (provided our integral remained appropriately non-singular there). In the fermionic case, by contrast, fixed points give the *only* non-zero contribution.

[10]That is, $\partial W|_{x_*} = 0$ but $\partial^2 W|_{x_*} \neq 0$.

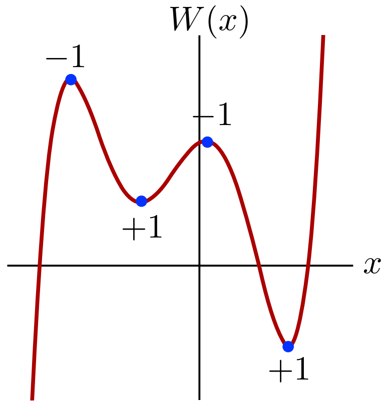

Figure 2: *The supersymmetric path integral receives contributions just from infinitesimal neighbourhoods of the critical points of $W(x)$. These alternately contribute ± 1 according to whether they are minima or maxima.*

where $c_* = \partial^2 W(x_*)$, so the action (2.21) becomes

$$S(x, \psi, \bar{\psi}) = \frac{c_*^2}{2}(x - x_*)^2 - c_* \bar{\psi}\psi + \cdots . \tag{2.33}$$

The higher-order terms will be negligible because, *e.g.* we can always increase λ to focus in on an infinitesimal neighbourhood of x_*. Expanding the exponential in Grassmann variables, the contribution of this critical point to the integral is

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-c_*^2(x-x_*)^2/2} \left[-1 + c_* \bar{\psi}\psi\right] \, \mathrm{d}x \, \mathrm{d}^2\psi &= \frac{c_*}{\sqrt{2\pi}} \int \mathrm{e}^{-c_*^2(x-x_*)^2/2} \, \mathrm{d}x \\ = \frac{c_*}{\sqrt{c_*^2}} = \operatorname{sgn}\left(\partial^2 W|_*\right) \, . \end{aligned} \tag{2.34}$$

Summing over all the critical points, the integral becomes

$$\mathcal{I} \; = \sum_{x_* : \; \partial W|_{x_*} = 0} \operatorname{sgn}\left(\partial^2 W|_{x_*}\right) \tag{2.35}$$

and is thus largely independent of the detailed form of W. In fact, if W is a polynomial of odd degree, then $\partial W = 0$ must have an even number of roots, with $\partial^2 W$ being alternately positive and negative at each successive root. Thus their contributions to (2.35) cancel pairwise and $\mathcal{I}_{\mathrm{odd}} = 0$ identically. On the other hand, if W has even degree then it has an odd number of critical points and we obtain $\mathcal{I}_{\mathrm{ev}} = \pm 1$, with the sign depending on whether $W \to \pm\infty$ as $|x| \to \infty$. (See figure 2.)

The fact that this integral is so simple for arbitrary polynomials $W(x)$ is a really remarkable result! To reiterate, we've found that for any such polynomial, the partition function $\mathcal{I}[W]$ is always either 0 or ± 1. If we imagined trying to compute $\mathcal{I}[W]$ perturbatively, then for a non-quadratic W we'd still have to sum infinitely diagrams using the vertices in the action. In particular, we could certainly draw Feynman diagrams with arbitrarily high numbers of loops involving both x and ψ fields, and these graphs would each contribute to the coefficient of some power of the coupling constants in the perturbative expansion. However, by an apparent miracle, we'd find that these graphs always cancel themselves out; the net coefficient of each such loop graph would be zero with the contributions from graphs where either x or $\bar{\psi}\psi$ run around the loop contributing with opposite sign. The reason for this apparent perturbative miracle is the localization property of the supersymmetric integral.

2.4 Landau–Ginzburg theories and the chiral ring

For a further example, consider a theory of a complex boson z and *two* complex fermions ψ_1, ψ_2. We pick a holomorphic function $W(z)$ and choose the action

$$S[z,\psi_i] = |W'(z)|^2 + W''(z)\psi_1\psi_2 - \overline{W}''(\bar{z})\bar{\psi}_1\bar{\psi}_2 \tag{2.36}$$

This action is invariant under two sets of supersymmetry transformations, generated by

$$\begin{aligned} \mathcal{Q}_1 &= \psi_1\frac{\partial}{\partial z} + \overline{\partial W}\frac{\partial}{\partial \psi_2} & \qquad \mathcal{Q}_1^\dagger &= \bar{\psi}_1\frac{\partial}{\partial \bar{z}} + \partial W\frac{\partial}{\partial \bar{\psi}_2} \\ \mathcal{Q}_2 &= \psi_2\frac{\partial}{\partial z} - \overline{\partial W}\frac{\partial}{\partial \psi_1} & \qquad \mathcal{Q}_2^\dagger &= \bar{\psi}_2\frac{\partial}{\partial \bar{z}} - \partial W\frac{\partial}{\partial \bar{\psi}_1}\ . \end{aligned} \tag{2.37}$$

Note that since W is holomorphic, these generators now obey

$$\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0\,, \qquad \left\{\mathcal{Q}_i^\dagger, \mathcal{Q}_j^\dagger\right\} = 0 \tag{2.38}$$

identically.

The same localization principle as before implies that

$$\mathcal{I} = \int \mathrm{e}^{-S[z,\psi_i]}\,\mathrm{d}^2 z\,\mathrm{d}^4\psi$$

only receives contributions from a neighbourhood of the critical points of W, which are again fixed points of the supersymmetry transformations. For example, away from this neighbourhood, we can change variables $(z,\psi_i) \to (y,\chi_i)$ as

$$y = z - \frac{\psi_1\psi_2}{\partial W}\,, \qquad \chi_i = \sqrt{\partial W}\,\psi_i$$

and write the action as $S[y,0]$. After changing variables, the integrand is independent of the fermions and so vanishes under the Berezin integration.

In this case, the result of the contribution from the critical points will be slightly different. Let z_* be an isolated, non-degenerate critical point of W so that nearby

$$W(z) = W(z_*) + \frac{\alpha}{2}(z-z_*)^2 + \cdots\ . \tag{2.39}$$

Around such a critical point, only the quadratic pieces

$$S^{(2)}[z,\psi_i] = |\alpha(z-z_*)|^2 + \alpha\psi_1\psi_2 - \bar{\alpha}\bar{\psi}_1\bar{\psi}_2 \tag{2.40}$$

are important. Thus the integral in this case becomes

$$
\begin{aligned}
\mathcal{I} &= \frac{1}{2\pi}\int \mathrm{e}^{-S}\,\mathrm{d}^2z\,\mathrm{d}^4\psi \\
&= \frac{1}{2\pi}\sum_{z_*\,:\,\partial W|_{z_*}=0}\int \mathrm{e}^{-|\alpha(z-z_*)|^2}\,|\alpha|^2\psi_1\psi_2\bar\psi_1\bar\psi_2\,\mathrm{d}^2z\,\mathrm{d}^4\psi \\
&= \frac{1}{2\pi}\sum_{z_*\,:\,\partial W|_{z_*}=0}|\alpha|^2\int \mathrm{e}^{-|\alpha(z-z_*)|^2}\,dz\,d\bar z \\
&= \sum_{z_*\,:\,\partial W|_{z_*}=0} 1 = \#\,(\text{critical points of } W)\,,
\end{aligned}
\tag{2.41}
$$

just counting (no longer with sign) the number of critical points.

It's important to realise that localization can be used to calculate much more than just the partition function. Suppose $\mathcal{O}$ is any function of the fields that is invariant under (say) the antiholomorphic supersymmetry transformations generated by $\mathcal{Q}^\dagger = \mathcal{Q}_1^\dagger + \mathcal{Q}_2^\dagger$. We will allow $\mathcal{O}$ to have arbitrary behaviour under the holomorphic supersymmetry transformations, so $\mathcal{Q}_i(\mathcal{O}) \neq 0$ necessarily. Then $\mathcal{O}$ may have arbitrary dependence on (z,ψ_i) but, away from $\overline{\partial W}=0$, it can depend on $\bar z$ and $\bar\psi_i$ only through $\bar y$ and $\bar\psi_1+\bar\psi_2$. Then the correlation function[11]

$$\langle\mathcal{O}\rangle = \frac{1}{2\pi}\int \mathrm{e}^{-S}\,\mathcal{O}\,\mathrm{d}^2z\,\mathrm{d}^4\psi\,,$$

will again localize to a neighbourhood of the fixed–point locus $\overline{\partial W}=0$ as the integrand is independent of $\bar\psi_1-\bar\psi_2$ on $\mathcal{U}^{\mathrm{c}}$. As an example, any holomorphic function $f(z)$ of z alone is certainly $\bar\delta$-invariant, so

$$
\begin{aligned}
\langle f\rangle &= \frac{1}{2\pi}\int f(z)\,|\partial^2 W|^2\,\mathrm{e}^{-\frac12|\partial W|^2}\,\mathrm{d}z\,\mathrm{d}\bar z \\
&= \sum_{z_*\,:\,\partial W|_{z_*}=0}\frac{1}{2\pi}\int f(z_*)\,|\partial^2 W|^2\,\mathrm{e}^{-\frac12|\partial W|^2} \\
&= \sum_{z_*\,:\,\partial W|_{z_*}=0} f(z_*)\,.
\end{aligned}
$$

localizing again to the critical points of W.

Since $(\mathcal{Q}^\dagger)^2=0$, a simple way to construct such an operator is to let $\Lambda(z,\bar z,\psi_i,\bar\psi_i)$ be *any* operator and take $\mathcal{O}=\mathcal{Q}^\dagger(\Lambda)$. However, such operators have vanishing correlation functions, because

$$\langle\mathcal{Q}^\dagger\Lambda\rangle = \frac{1}{2\pi}\int(\mathcal{Q}^\dagger\Lambda)\,\mathrm{e}^{-S}\,\mathrm{d}^2z\,\mathrm{d}^4\psi = \frac{1}{2\pi}\int\mathcal{Q}^\dagger\left(\Lambda\,\mathrm{e}^{-S}\right)\,\mathrm{d}^2z\,\mathrm{d}^4\psi\,, \tag{2.42}$$

which vanishes for the same reasons as above. Thus the interesting operators are those that are in the kernel of $\mathcal{Q}^\dagger$ (so $\mathcal{Q}^\dagger\mathcal{O}=0$), but not in its image (so $\mathcal{O}\neq\mathcal{Q}^\dagger(\Lambda)$ for any Λ).

[11] Often, one normalizes correlation functions by the partition function; that is, one takes $\langle\mathcal{O}\rangle = \int \mathrm{e}^{-S}\mathcal{O}\,\mathcal{D}X \,/\, \int \mathrm{e}^{-S}\,\mathcal{D}X$. In this course our correlation functions will be unnormalized. The conventions in AQFT or Statistical Field Theory may be different

We call the equivalence class

$$H_{\mathcal{Q}^\dagger} = \frac{\{\mathcal{O} : \mathcal{Q}^\dagger \mathcal{O} = 0\}}{\{\mathcal{O} = \mathcal{Q}^\dagger \Lambda\}} \tag{2.43}$$

the $\mathcal{Q}^\dagger$-*cohomology.* Because $\langle \mathcal{O} + \mathcal{Q}^\dagger \Lambda' \rangle = \langle \mathcal{O} \rangle + \langle \mathcal{Q}^\dagger \Lambda' \rangle = \langle \mathcal{O} \rangle$, correlation functions depend only on the cohomology equivalence class of an operator.

Suppose for $r = 1, \dots n$ the operators $\mathcal{O}_r \in H_{\mathcal{Q}^\dagger}$. Then $\bar{Q}(\prod_r \mathcal{O}_r) = 0$ follows by the Leibniz rule, so the product is $\mathcal{Q}^\dagger$-closed. Furthermore, if we change one of the $\mathcal{O}_r$s by a $\mathcal{Q}^\dagger$-exact piece, say $\mathcal{O}_1 \to \mathcal{O}_1 + \mathcal{Q}^\dagger \Lambda_1$, then

$$\prod_{r=1}^{n} \mathcal{O}_a \to \left(\mathcal{O}_1 + \mathcal{Q}^\dagger \Lambda_1\right) \prod_{r=2}^{n} \mathcal{O}_r = \prod_{r=1}^{n} \mathcal{O}_r + \mathcal{Q}^\dagger \left(\Lambda_1 \prod_{r=2}^{n} \mathcal{O}_a\right)$$

and so the cohomology class of the product remains unchanged. Correlation functions of products of $\mathcal{Q}^\dagger$-invariant operators are thus only sensitive to the $\mathcal{Q}^\dagger$-cohomology. In the context of supersymmetry, the $\mathcal{Q}^\dagger$-cohomology is often called the *chiral ring*: we can add, subtract and multiply such cohomology classes, but not (in general) divide them.

In particular, to compute the chiral ring for bosonic fields, let $f(z)$ be a holomorphic function so that $\mathcal{Q}^\dagger f = 0$ as before. Since $\mathcal{Q}^\dagger(f\bar{\psi}_1) = f(z)\,\partial W$ and ∂W is also holomorphic, we see that any holomorphic function that has ∂W as a factor is equivalent to zero in the $\bar{Q}$-cohomology. Thus the chiral ring $\mathcal{R} = \mathbb{C}[z]/\{\mathcal{I}\}$ where $\mathcal{I}$ is the ideal generated by ∂W. For example, if

$$W = \frac{z^{n+1}}{n+1} - az$$

where a is a constant, then $\partial W = z^n - a$. The chiral ring is then the ring of polynomials in z, modulo the relation $z^n = a$. The non-trivial elements in this ring are thus $\{1, z, z^2, \dots, z^{n-1}\}$.

The fact that correlation functions of operators that are invariant under supersymmetry depend on the cohomology class represented by the operator is the starting–point for much of the mathematical interest in QFT: we design our supersymmetric QFT so that this cohomology is the cohomology of an interesting space. For example, Donaldson's theory of invariants of 4-manifolds that are homeomorphic but not diffeomorphic, and the Gromov–Witten generalization of intersection theory can both be understood as examples of (higher–dimensional) supersymmetric QFTs where the localization / cancellation is precise. We'll see this much more in the following chapters.

It's also important to mention that we can, of course, try to compute correlation functions of operators that are *not* $\bar{Q}$-closed. However, there's no reason to expect such correlation functions will be amenable to techniques of localization, so we'll typically have to resort to a perturbative treatment as for any QFT.

2.5 The Duistermaat–Heckmann localization formula

For a final example in zero dimensions, let's look at something with a little more geometry.

Let (M, ω) be a symplectic manifold. That is, M is a (smooth) $2n$-dimensional manifold on which we have a 2-form ω that obeys

$$\begin{aligned} d\omega = 0 &\qquad \text{so } \omega \text{ is closed} \\ \omega(X, Y) = 0 &\qquad \text{for all vector fields } Y \text{ iff } X = 0 \text{ identically} \end{aligned}$$

The second condition is a non-degeneracy condition. We can equivalently write it as the condition that the Liouville measure $\omega^n \neq 0$. Concretely, if x^a are local coordinates[12] on M then we can write $\omega = \omega_{ab}(x)\, dx^a \wedge dx^b$ and the Liouville measure

$$\omega^n = \det(\omega_{ab})\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{2n} \ .$$

Since it is non-degenerate, ω is invertible and

$$\omega^{-1} = (\omega^{-1})^{ab} \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b}$$

defines a Poisson bracket on M. This Poisson bracket obeys the Jacobi identity as a consequence of ω being closed.

Now suppose X is a vector field on M and that ω is invariant along the flow generated by X. This means the Lie derivative of ω along X vanishes, so

$$0 = \mathcal{L}_X \omega = (\imath_X d + d\imath_X)\, \omega = d(\imath_X \omega)$$

where the second equality uses Cartan's homotopy formula for the Lie derivative of a form[13], and the final equality uses the fact that ω is closed. We thus see that the contraction $\imath_X \omega$ is itself a closed 1-form. Provided $b_1(M) = 0$, all closed 1-forms on M must be exact, so we can find a function h such that[14] $dh = -\imath_X \omega$. Then X is said to be a *Hamiltonian vector field*, with h the corresponding Hamiltonian. Equivalently, given a function h, the corresponding Hamiltonian vector field is $X = -\omega^{-1}(dh, \cdot\,) = -\{h, \cdot\}$ obtained by inserting h into the first slot of the Poisson bracket. We often write X as X_h if we wish to emphasize its relation to h. You should be familiar with this from Classical Dynamics (although perhaps in a less abstract language) where, in the most standard case, $M = \mathbb{R}^{2n}$ is the space of positions and momenta and the Hamiltonian $h : M \to \mathbb{R}$ is interpreted as the energy. Hamilton's equations say that dynamical trajectories are determined by the equations $\dot{f} = \{f, h\} = X_h(f)$, so that the function $f : M \to \mathbb{R}$ changes in time by flowing along X_h.

We'll be concerned with symplectic manifolds M that are compact and have no boundary. We'll also require that our Hamiltonian vector field $X = \omega^{-1}(dh)$ generates a $U(1)$ action on M. That is, the generic orbit of X is a circle, but there may be fixed points $x_* \in M$ where the vector field vanishes and the circle shrinks to zero. We'll also assume for simplicity that any such fixed points are isolated. The key example to keep in mind is

[12] The x^as are *all* the coordinates here, including both the qs and ps, so $x^a = (q^i, p_j)$! In particular, $a = 1, \ldots, 2n$, whereas $i, j = 1, \ldots, n$.

[13] If you're not familiar with this formula, you can prove it *e.g.* by comparing the component form of the Lie derivative to the component form of the *rhs*. I leave the proof as an exercise.

[14] The minus sign here is a convention, chosen to agree with standard definitions in classical mechanics.

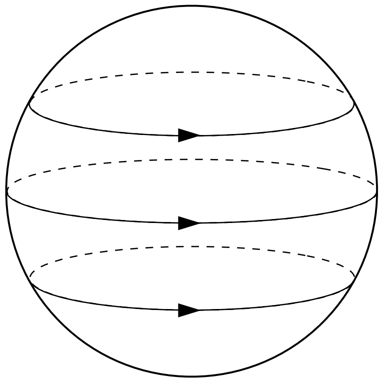

Figure 3: S^2 *is a symplectic manifold. The circle action rotates the* S^2 *around an axis, leaving the poles fixed.*

$M = S^2$ with symplectic form $\omega = \sin\theta\, d\theta \wedge d\phi$. The vector field $X = \partial/\partial\phi$ is Hamiltonian with $h = \cos\theta$, and generates a circle action as in figure 3. The fixed points x_* of the circle action are the North pole ($\theta = 0$) and South pole ($\theta = \pi$), where[15] $X(x_*) = 0$.

In this situation, the Duistermaat–Heckmann theorem states that, for $\alpha \in \mathbb{R}$, the integral

$$\int_M \frac{\omega^n}{n!}\, \mathrm{e}^{\mathrm{i}\alpha h}$$

reduces to a sum of contributions over the fixed points x_*. In the example where $M = S^2$ with the above symplectic form and Hamiltonian, this integral becomes elementary and we find

$$\int_M \frac{\omega^n}{n!}\, \mathrm{e}^{\mathrm{i}\alpha h} = \int_{S^2} \mathrm{e}^{\mathrm{i}\alpha\cos\theta}\, \sin\theta\, d\theta \wedge d\phi = 2\pi \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\alpha z}\, dz = \frac{2\pi}{\mathrm{i}\alpha}\left(\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{-\mathrm{i}\alpha}\right) .$$

As promised, the answer is a sum of contributions of the *integrand* $\mathrm{e}^{\mathrm{i}\alpha h(x_*)}$ itself, evaluated at the North and South poles. Aside from the universal factor $2\pi/\mathrm{i}\alpha$, the contribution of each fixed point is weighted by a factor of ± 1 whose role we will understand momentarily. The Duistermaat–Heckmann theorem states that thee same will be true for *any* compact, symplectic M and Hamiltonian h.

We can give a simple derivation of this theorem using supersymmetry. The 'fields' of our model will be x^a, local coordinates on M, and $2n$ fermions ψ^a which transform as tangent vectors to M. To sound fancy, we can say that the space of fields is $\mathcal{C} = \Pi TM$, the parity–reversed tangent bundle to M. That is, $\mathcal{C}$ looks just like the total space TM of the tangent bundle to M, except that we're treating the fibre directions as fermionic. A generic smooth function of our fields looks like

$$F(x,\psi) = f(x) + \rho_{a_1}(x)\,\psi^{a_1} + g_{a_1a_2}(x)\,\psi^a\psi^b + \cdots + r_{a_1a_2\ldots a_{2n}}(x)\,\psi^{a_1}\psi^{a_2}\cdots\psi^{a_{2n}} \ ,$$

[15]Figure 3 makes it clear that $X = 0$ at the North and South poles, but this is not easy to see from the expression $X = \partial/\partial\phi$ because the coordinates (θ, ϕ) break down at the poles. Recall (*e.g.* from the theory of angular momentum in QM) that $\partial/\partial\phi = x\partial/\partial y - y\partial/\partial x$ where (x, y, z) are Cartesian coordinates on a copy of $\mathbb{R}^3$ in which our S^2 is embedded as the unit sphere. The Cartesian expression clearly has a zero along the z-axis.

and so we identify $C^\infty(\Pi TM) \cong \Omega^*(M)$, the space of forms on M of arbitrary degree.

Now we choose our action. The first terms we'll need are

$$S_0(x,\psi) = -\mathrm{i}\alpha(h(x) + \omega_{ab}(x)\psi^a\psi^b)\,.$$

These terms are invariant under the supersymmetry transformations generated by

$$\mathcal{Q} = \psi^a\frac{\partial}{\partial x^a} + X^a(x)\frac{\partial}{\partial \psi^a}$$

as one can easily verify using the fact that X is Hamiltonian, with associated Hamiltonian function h. This $\mathcal{Q}$ has a simple geometric meaning: since functions of (x,ψ) are identified with forms on M, whenever $\mathcal{Q}$ acts on any superfield (such as the action) we can equivalently view it as the operator

$$\mathcal{Q} = d + \imath_X\,.$$

In particular, the first term $\psi^a\partial/\partial x^a$ in $\mathcal{Q}$ corresponds to the exterior derivative d, while the second term strips off a ψ^a replacing it with the vector field X^a and so represents the operation of contraction $\imath_X$ acting on forms. Furthermore,

$$\frac{1}{2}\{\mathcal{Q},\mathcal{Q}\} = d\imath_X + \imath_X d = \mathcal{L}_X\,, \tag{2.44}$$

so our $\mathcal{Q}$ squares to the Lie derivative along X. In particular, $\mathcal{Q}^2 = 0$ when acting on forms that are constant along the orbits of X.

To aid with the localization, we need to add another term to the action. In the simplest case, we do this by first picking a postive-definite metric g on M. We then deform the action to

$$S_\lambda = S_0 + \lambda\mathcal{Q}\left(g(\psi,X)\right) = S + \lambda\left(g(X,X) - \psi^a\psi^c\,\partial_c X_a\right)\,,$$

where $X_a = g_{ab}X^b$ and $\lambda \in \mathbb{R}_\geq$ is a constant. Provided the metric is invariant under the $U(1)$ action, from (2.44) we have $\mathcal{Q}^2(g(\psi,X)) = \mathcal{L}_X(g_{ab}X^b\,dx^a) = 0$. Therefore $\mathcal{Q}(S_\lambda) = 0$ and our deformed actions remain supersymmetric for all λ.

Now let's consider the partition function of our theory. As usual, this is the integral

$$\mathcal{Z} = \int_{\Pi TM} \mathrm{e}^{-S_\lambda(x,\psi)}\,\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi \tag{2.45}$$

over the space of fields. It may seem surprising that, even though M may be curved, we're integrating using the naïve measure $\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi$. In fact, this measure on ΠTM is invariant under orientation–preserving diffeomorphisms $f: M \to M$. To see this, suppose that x^a are coordinates on a neighbourhood of $m \in M$ and y^a are coordinates on a neighbourhood of $f(m)$. Then the pushforward of any vector X at m is the vector

$$f_*(X) = X^a(f(x))\frac{\partial y^b}{\partial x^a}\frac{\partial}{\partial x^b} = Y^b(y)\frac{\partial}{\partial y^b}\,.$$

at $f(m)$. In particular, since the ψ^a are fermionic elements of the tangent bundle, under a diffeomorphism they pushforward to $\chi^b = \psi^a\,(\partial y^b/\partial x^a)$. Therefore the measure transforms under pullback as[16]

$$f^*(d^{2n}y\,d^{2n}\chi) = \det\left(\frac{\partial y^b}{\partial x^a}\right) d^{2n}x\;d^{2n}\!\left(\frac{\partial y^d}{\partial x^c}\psi^c\right) = d^{2n}x\,d^{2n}\psi$$

where we have used the transformation law (2.7b) for the Berezin measure. We see that $d^{2n}x\,d^{2n}\psi$ is a canonically defined measure on ΠTM, because the bosonic and fermionic measures transform oppositely[17].

As in previous sections, the most important feature of the partition function (2.45) is that it is actually independent of the value of $\lambda \in \mathbb{R}_{\geq}$. Again, this is because

$$-\frac{\partial}{\partial\lambda}\int \mathrm{e}^{-S_\lambda}\,\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi = \int \mathcal{Q}\left(g(\psi,X)\right)\mathrm{e}^{-S_\lambda}\,\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi = \int \mathcal{Q}\left(g(\psi,X)\,\mathrm{e}^{-S_\lambda}\right)\,\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi\,,$$

which vanishes because after acting with $\mathcal{Q}$, each term in the integrand is either missing some fermion or else is a total derivative on M. As a special case, setting $\lambda = 0$ and integrating out the fermions we have

$$\mathcal{Z} = \int_{\Pi TM} \mathrm{e}^{-S_0(x,\psi)}\,\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi = \frac{(\mathrm{i}\alpha)^n}{n!}\int_M \omega^n\,\mathrm{e}^{\mathrm{i}\alpha h} \tag{2.46}$$

so, up to the harmless factor $(\mathrm{i}\alpha)^n$, the partition function is equal to the integral in the Duistermaat–Heckmann theorem.

To actually evaluate $\mathcal{Z}$, we instead consider the limit $\lambda \to \infty$. Because g is positive-definite, at large λ the term $\lambda\,g(X,X)$ in S_λ suppresses all contributions to the integral except those near the zeros of X, *i.e.*, the fixed points of the $U(1)$ action. Near each fixed point, the method of steepest descent (2.13) gives

$$\mathcal{Z} \sim \frac{(2\pi)^n}{n!}\sum_{x_*\in M\,:\,X(x_*)=0} \mathrm{e}^{\mathrm{i}\alpha h(x_*)}\;\left.\frac{\epsilon^{a_1b_1a_2b_2\cdots a_nb_n}\,(\partial_{a_1}X_{b_1})(\partial_{a_2}X_{b_2})\cdots(\partial_{a_n}X_{b_n})}{\sqrt{\det(\partial_a\partial_b g(X,X))}}\right|_{x_*}\,, \tag{2.47}$$

as the leading–order term when $\lambda \to \infty$. The numerator here comes from integrating out the dominant fermion term $\psi^a\psi^c\partial_c X_a$, while the denominator from the Gaussian approximation to the bosonic integral near $x = x_*$. Generically, we'd expect the leading–order term (2.47) to receive corrections from a power series in $1/\lambda$, as in (2.13). However, supersymmetry ensures the answer is actually λ-independent, so in particular it agrees with the $\lambda \to \infty$ limit. Thus (2.47) is in fact the exact answer. (This implies that each term in the sub–leading power series is actually zero even at finite λ. This is true, but hard to see directly.)

We can simplify the ratio in (2.47) as follows. First, let's suppose $\dim(M) = 2$. Then near each fixed point, we choose (Darboux) coordinates (q,p) on the tangent space

[16] We assume the coordinate transformation is orientation–preserving so that $\det(\partial y/\partial x) > 0$ and there is no need for a modulus sign in the bosonic measure.

[17] This is closely related to the fact that the (bosonic) cotangent bundle T^*N to any manifold N has a canonically defined measure.

$T_{x_*}M \cong \mathbb{R}^2$ such that the fixed point x_* is the origin $(q,p)=(0,0)$ and the symplectic form $\omega = dq \wedge dp$. In these coordinates, X rotates us around the origin on circles of constant radius, so

$$X = k\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right)$$

for some $k \in \mathbb{Z}$. The Hamiltonian function associated to X is $h = \frac{1}{2}k(q^2+p^2)$. The obvious $U(1)$-invariant metric is just the Euclidean metric $g = dq^2 + dp^2$, in which case

$$\epsilon^{ab}\partial_a(g_{bc}X^c) = \partial_q X_p - \partial_p X_q = 2k\,, \qquad \text{whilst} \qquad \sqrt{\det(\partial_a\partial_b(X^cX_c))} = 2k^2$$

Hence the ratio in (2.47) is just $1/k$. In particular, for the previous example $M = S^2$, we have $k=+1$ at one fixed point and $k=-1$ at the other fixed point, because if we orient the tangent planes the same way then the circle action X rotates us in an opposite sense (anticlockwise/anticlockwise) at the North and South poles.

The higher dimensional case works similarly. Near each fixed point x_*, we choose Darboux coordinates (q^i, p_j) on the tangent space $T_{x_*}M \cong \mathbb{R}^{2n}$ such that $\omega = dq^i \wedge dp_i$ and X acts as

$$X = \sum_{i=1}^{n} k_i\left(q^i\frac{\partial}{\partial p_i} - p_i\frac{\partial}{\partial q^i}\right).$$

Thus X rotates us around circles in each $\mathbb{R}^2$ factor, with the $k_i \in \mathbb{Z}$ allowing for the possibility that we rotate around at different rates in each $\mathbb{R}^2$ factor. We then find

$$\left.\frac{\epsilon^{a_1b_1a_2b_2\cdots a_nb_n}\,(\partial_{a_1}X_{b_1})(\partial_{a_2}X_{b_2})\cdots(\partial_{a_n}X_{b_n})}{\sqrt{\det(\partial_a\partial_b g(X,X))}}\right|_{x_*} = \prod_i \frac{1}{k_i}$$

so the ratio in the steepest descent formula is just the product of weights of the $U(1)$ action. Comparing (2.47) to (2.46) gives the final form of the Duistermaat-Heckmann formula

$$\int_M \frac{\omega^n}{n!}\,\mathrm{e}^{\mathrm{i}\alpha h} = \left(\frac{2\pi}{\mathrm{i}\alpha}\right)^n \sum_{x_*\in M\,:\,X(x_*)=0} \frac{\mathrm{e}^{\mathrm{i}\alpha h(x_*)}}{\prod_i k_i(x_*)}\,. \tag{2.48}$$

We emphasize that this is valid for any compact (M,ω) and any Hamiltonian $h: M \to \mathbb{R}$, provided the fixed points are isolated and non-degenerate.

2.5.1 A non-Abelian generlization

There are many ways in which the Duistermaat–Heckmann formula (and its supersymmetric derivation) can be generalized. One example is where the symplectic manifold carries the action of a Lie group $\mathcal{G}$, no longer required to be $U(1)$. That is, we have $\dim(\mathcal{G})$ vector fields $X_i = X_i^a(x)\,\partial/\partial x^a$ on M which obey $[X_i, X_j] = f_{ij}^k X_k$ where f_{ij}^k are the structure constants of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. We require that the symplectic form is preserved by the $\mathcal{G}$-action, so $\mathcal{L}_{X_i}(\omega) = 0$ for all i.

We make a choice[18] of $\phi^i \in \mathfrak{g}^*$ and define our supersymmetry operator $\mathcal{Q}$ by

$$\mathcal{Q} = \psi^a\frac{\partial}{\partial x^a} + \phi^i X_i^a(x)\frac{\partial}{\partial \psi^a}$$

[18] As a simple example, suppose $M = \mathbb{R}^{2n}$ with $x^a = (q^i, p_j)$ the usual Darboux coordinates. Then the momenta p_j are Hamiltonians whose corresponding vectors $\partial/\partial q^j$ generate translations. Then the ϕ^i correspond to a choice of direction in $\mathbb{R}^n$ along which we may wish to translate.

so that the ϕ^i parametrize the particular $\mathfrak{g}$-transformations we can perform. As before, we have $\mathcal{Q}^2 = \mathcal{L}_{\phi^i X_i}$ and so $\mathcal{Q}^2 = 0$ when acting on functions $F(x,\psi,\phi)$ that are $\mathfrak{g}$-invariant. In particular, the basic action S_0 naturally generalizes to

$$S_0 = -\mathrm{i}\left(\phi^i h_i(x) + \omega_{ab}(x)\,\psi^a\psi^b\right)$$

and is $\mathcal{Q}$-invariant.

It seems undemocratic to single out a particular choice of ϕ^i, and we can avoid this by also integrating over $\mathfrak{g}$ in our partition function. To do this, rather than thinking of ϕ^i as fixed, we let it represent a Euclidean coordinate on $\mathfrak{g}$, chosen so that the Euclidean measure $\mathrm{d}^{\dim\mathcal{G}}\phi$ coincides with the evaluation at $\mathfrak{g} = T_e\mathcal{G}$ of the Haar measure on $\mathcal{G}$. Then we integrate over $\mathfrak{g}$ using the measure

$$\mathrm{e}^{-\epsilon(\phi,\phi)/2}\,\frac{d^{\dim\mathcal{G}}\phi}{\mathrm{vol}(\mathcal{G})}\,,$$

where $\mathrm{vol}(\mathcal{G})$ is the volume of $\mathcal{G}$ computed using our Haar measure. The Gaussian factor is constructed using the Killing form $(\ ,\ )$ on $\mathfrak{g}$ and is inserted to ensure the integral over $\mathfrak{g}$ converges. At $\lambda = 0$ our partition function then becomes

$$\begin{aligned}\mathcal{Z}_{\mathcal{G}}(\epsilon) &= \int\limits_{\mathfrak{g}\times\Pi TM} \exp\left[-\frac{\epsilon}{2}(\phi,\phi) + \mathrm{i}\phi^i h_i(x) + \mathrm{i}\,\omega_{ab}(x)\psi^a\psi^b\right]\frac{d^{\dim\mathcal{G}}\phi\, d^{2n}x\, d^{2n}\psi}{\mathrm{vol}(\mathcal{G})}\\ &= \frac{1}{\mathrm{vol}(\mathcal{G})}\left(\frac{2\pi}{\epsilon}\right)^{\dim\mathcal{G}}\int\limits_{\Pi TM}\exp\left[-\frac{1}{2\epsilon}(h,h) + \mathrm{i}\,\omega_{ab}(x)\psi^a\psi^b\right]d^{2n}x\,d^{2n}\psi\,,\end{aligned} \tag{2.49}$$

where in the second line we have performed the Gaussian integral over the ϕ^is. Integrating out the fermions we arrive at

$$\mathcal{Z}_{\mathcal{G}}(\epsilon) = \frac{1}{\mathrm{vol}(\mathcal{G})}\left(\frac{2\pi}{\epsilon}\right)^{\dim\mathcal{G}}\int\limits_M \exp\left[-\frac{1}{2\epsilon}(h,h)\right]\frac{\omega^n}{n!} \tag{2.50}$$

as the underlying bosonic integral. Alternatively, adding a judiciously chosen $\mathcal{Q}$-exact term to the action before integrating out the fermions and scaling $\lambda\to\infty$, $\mathcal{Z}_{\mathcal{G}}$ can again be evaluated by localization.

2.5.2 A glimpse of two-dimensional Yang-Mills theory

Now let me convince you that this hasn't all just been about playing around with some pretty integrals. Let's consider Yang-Mills theory, with gauge group G. On $\mathbb{R}^{3,1}$, Yang-Mills theory is one of the main ingredients of the Standard Model, but we'll restrict our considerations to a two-dimensional version that lives on some closed, compact Riemann surface Σ.

As usual, the basic field is a connection ∇. At least locally on Σ, we can write $\nabla = d + A$ where the gauge field $A = A^\alpha_\mu(x)\,t_\alpha\,dx^\mu$ (at least locally on Σ), and $\{t_\alpha\}$ are a basis of the Lie algebra of G. Unlike the exterior derivative, the connection ∇ is not nilpotent, but obeys $\nabla^2 = F$, where F is the curvature or Yang-Mills fieldstrength. In terms of the gauge field, we have

$$F = \nabla^2 = dA + A \wedge A = \frac{1}{2}\left(\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + f^\alpha_{\beta\gamma} A^\beta_\mu A^\gamma_\nu\right) t_\alpha \, dx^\mu \wedge dx^\nu \, .$$

locally. The space $\mathcal{A}$ of connections is an infinite dimensional affine space whose tangent space $T_\nabla \mathcal{A} \cong \Omega^1(\Sigma, \mathfrak{g})$ at any point $\nabla \in \mathcal{A}$, because the difference between two connections is a 1-form on Σ that transforms in the adjoint.

Gauge transformations are maps $g : \Sigma \to G$ that act on the connection as $g : \nabla \mapsto g\nabla g^{-1}$ and therefore

$$F \mapsto gFg^{-1} \qquad\qquad A \mapsto gAg^{-1} - dg\, g^{-1} \, . \tag{2.51}$$

The space $\mathcal{G}$ of all gauge transformations can be identified with the space $\mathrm{Maps}(\Sigma, G)$ of maps to the gauge group, and the corresponding (infinite dimensional) Lie algebra is

$$\mathrm{Lie}(\mathcal{G}) = T_e\mathcal{G} \cong \Omega^0(\Sigma, \mathfrak{g}) \, .$$

In particular, writing $g = \mathrm{e}^\lambda$ and taking λ infinitesimal, (2.51) reduces to

$$F \mapsto F - [F, \lambda] \qquad\qquad A \mapsto A - \nabla\lambda$$

where $\nabla\lambda = dA + [A, \lambda]$. Geometrically, we can view this gauge transformation as generated by the vector field

$$X_\lambda = -\int_\Sigma (\nabla\lambda(\sigma))^\alpha \frac{\delta}{\delta A^\alpha(\sigma)}$$

on $\mathcal{A}$. (The variational derivative here acts as $(\delta/\delta A^\alpha(\sigma))A^\beta(\sigma') = \delta^\beta_\alpha \, \delta^2(\sigma - \sigma')$. The integral over $\sigma \in \Sigma$ in the definition of X can be viewed as the continuous part of a 'sum over components'. We let δ denote the exterior derivative on $\mathcal{A}$.)

One of the reasons that Yang-Mills theory is special in two dimensions is that $\mathcal{A}$ is naturally an infinite–dimensional symplectic manifold with symplectic form

$$\omega = \frac{1}{2}\int \mathrm{Tr}(\delta A \wedge \delta A)$$

Remarkably, the vector field X_λ that generates gauge transformations is Hamiltonian, and its associated Hamiltonian function is nothing but the curvature F. More specifically, let

$$h(\lambda) = \int_\Sigma \mathrm{Tr}(\lambda F) \, .$$

Then taking the exterior derivative on $\mathcal{A}$, we have

$$\delta h(\lambda) = \int_\Sigma \mathrm{Tr}(\lambda \, \nabla\delta A) = -\int_\Sigma \mathrm{Tr}(\nabla\lambda \wedge \delta A) = -\imath_{X_\lambda}\omega$$

where the first equality uses the fact that $\delta F = \nabla\delta A = d(\delta A) + [A, \delta A]$.

The action for Yang-Mills theory is

$$S_{\mathrm{YM}}[\nabla] = \frac{1}{2\mathrm{g}^2}\int_\Sigma \mathrm{Tr}(F\wedge *F) \tag{2.52}$$

as usual. Here, g^2 a coupling and Tr is a G-invariant quadratic form on the Lie algebra of G, normalized so that $\mathrm{Tr}(t_\alpha t_\beta) = -\frac{1}{2}\delta_{\alpha\beta}$. The Yang-Mills partition function is then an integral of $\mathrm{e}^{-S_{\mathrm{YM}}[\nabla]}$ over the space $\mathcal{A}/\mathcal{G}$ of equivalence classes of connections up to gauge transformations. Somewhat heuristically, we can perform this by instead integrating over $\mathcal{A}$ and dividing by $\mathrm{vol}(\mathcal{G})$ to compensate for overcounting gauge-equivalent field configurations. That is,

$$\mathcal{Z}_{\mathrm{YM}} = \frac{1}{\mathrm{vol}(\mathcal{G})}\int_{\mathcal{A}} \mathrm{e}^{-S_{\mathrm{YM}}[\nabla]}\, DA$$

where DA is a formal Euclidean measure on $\mathcal{A}$. In fact, since the symplectic form ω has constant (or δ-function) coefficients, in two dimensions we can equally interpret this as a formal Liouville measure on the infinite–dimensional symplectic space $\mathcal{A}$.

It should be clear that the partition function of YM_2 is a infinite–dimensional analogue of the non-Abelian localization story we outlined above. To make the correspondence even clearer, we introduce a scalar field ϕ and a fermionic field ψ, where

$$\phi \in \Omega^0(\Sigma,\mathfrak{g}) \cong \mathrm{Lie}(\mathcal{G}) \qquad \text{and} \qquad \psi \in \Omega^1(\Sigma,\mathfrak{g}) \cong \Pi T_A\mathcal{A}\,.$$

Thus the scalar lives in the Lie algebra of $\mathcal{G}$ whilst ψ transforms as a fermionic tangent vector to the space of gauge fields. Note that, for our application, we've chosen the fermion to be a 1-form on Σ rather than a spinor. We can then rewrite rewrite the Yang-Mills path integral in first-order form as

$$\mathcal{Z}_{\mathrm{YM}} = \int_{\mathrm{Lie}(\mathcal{G})\times\Pi T\mathcal{A}} \exp\left[\mathrm{i}\int_\Sigma \mathrm{Tr}(\phi F) + \frac{1}{2}\int_\Sigma \mathrm{Tr}(\psi\wedge\psi) - \frac{\mathrm{g}^2}{2}\int_\Sigma *\,\mathrm{Tr}(\phi^2)\right]\frac{DA\,D\psi\,D\phi}{\mathrm{vol}(\mathcal{G})}\,.$$

Note that because we haven't given the fermion any kinetic term, its path integral is trivial (reflecting the fact that the Euclidean measure on $\mathcal{A}$ is already the Liouville measure). Completing the square in ϕ and then performing the Gaussian integrals over both ϕ and ψ returns us to the standard form (2.52) of the Yang-Mills action. Note also that the extended action is invariant under the supersymmetry transformations[19]

$$\delta A = \mathrm{i}\psi \qquad\qquad \delta\psi = -\nabla\phi \qquad\qquad \delta\phi = 0$$

as one can readily check using the fact that Σ has no boundary. These transformations square to a gauge transformation of A along ϕ, or in other words to the Lie derivative along the Hamiltonian vector field X_ϕ. As above, the supersymmetric form of the action can be used to localize the Yang-Mills partition function, reducing it to an exactly calculable integral over the moduli space of flat connections ($\nabla \in \mathcal{A}$ s.t. $F = 0$). Unfortunately, it would take us too far afield to explain this here, but you can find the story in the paper Two Dimensional Gauge Theories Revisited, J. Geom. Phys. **9**, 4 (1992) by E. Witten, where the non-Abelian localization story was also first introduced.

[19] I caution you that these are not (quite) the usual supersymmetry transformations in Yang-Mills theory, though they are closely related. See section **??** for the standard case.