3 Supersymmetric Quantum Mechanics

We now turn to consider supersymmetry in d = 1, which is the case of supersymmetric quantum mechanics. Our main aim here will be to use SQM to derive the Atiyah–Singer index theorem, but we'll begin by studying QM from the point of view of path integrals, generalizing the finite–dimensional integrals we looked at in the last chapter.

3.1 Path integrals in Quantum Mechanics

Let's consider the basic case of a (bosonic) quantum particle travelling in \mathbb{R}^n . In the canonical framework, at any time t this particle would be described by a wavefunction $\psi(x) \in \mathcal{H} \cong L^2(\mathbb{R}^n, d^n x)$. The wavefunction evolves in time according to the action of a unitary operator $U(t) : \mathcal{H} \to \mathcal{H}$, with $U(t) = e^{-iHt}$ in the standard case that the Hamiltonian H is time-independent. We'll often Wick rotate $t \to -i\tau$ to Euclidean signature, in which case the time evolution operator becomes $e^{-\tau H}$.

If our particle is initially located at some point $y_0 \in \mathbb{R}^n$, then the amplitude to find it at some $y_1 \in \mathbb{R}^n$ a Euclidean time $\tau = \beta$ later is

$$\langle y_1, \beta | y_0, 0 \rangle = \langle y_1 | e^{-\beta H} | y_0 \rangle = K_\beta(y_0, y_1),$$
 (3.1)

where

$$K: [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

is known as the *heat kernel*. For example, in the simplest case of a free particle of unit mass, the Hamiltonian $H = -\nabla^2/2$ as an operator on \mathcal{H} and the heat kernel is given explicitly by

$$K_{\tau}(y_0, y_1) = \frac{1}{(2\pi\tau)^{n/2}} \exp\left(-\frac{\|y_0 - y_1\|^2}{2\tau}\right)$$
(3.2)

where $||y_0 - y_1||$ is the Euclidean distance between the initial and final points.

Feynman's intuition was that this amplitude could be expressed in terms of the product of the amplitude for it to start at y_0 at $\tau = 0$, then be found at some other location x at an intermediate time $\tau \in (0, \beta)$, before finally being found at y_1 on schedule at $\tau = \beta$. Since we did not measure what the particle was doing at the intermediate time, we should sum (*i.e.* integrate) over all possible intermediate locations x in accordance with the linearity of quantum mechanics. Iterating this procedure, as in figure 4 we break the time interval $[0, \beta]$ into N chunks, each of duration $\Delta \tau = \beta/N$. We then write

$$\langle y_1 | \mathrm{e}^{-\beta H} | y_0 \rangle = \langle y_1 | \mathrm{e}^{-\Delta \tau H} \mathrm{e}^{-\Delta \tau H} \cdots \mathrm{e}^{-\Delta \tau H} | y_0 \rangle$$

$$= \int \langle y_1 | \mathrm{e}^{-\Delta \tau H} | x_{N-1} \rangle \cdots \langle x_2 | \mathrm{e}^{-\Delta \tau H} | x_1 \rangle \langle x_1 | \mathrm{e}^{-\Delta \tau H} | y_0 \rangle d^n x_1 \cdots d^n x_{N-1}$$

$$= \int K_{\Delta \tau} (y_1, x_{N-1}) \cdots K_{\Delta \tau} (x_2, x_1) K_{\Delta \tau} (x_1, y_0) \prod_{i=1}^{N-1} d^n x_i .$$

$$(3.3)$$

In the second line we've inserted identity operators $1_{\mathcal{H}} = \int |x_i\rangle \langle x_i| d^n x_i$ in between each evolution operator; in the present context this can be understood as the concatenation



Figure 4: Feynman's approach to quantum mechanics starts by breaking the time evolution of a particle's state into many chunks, then summing over all possible locations (and any other quantum numbers) of the particle at intermediate times.

identity

$$K_{\tau_1+\tau_2}(x_3, x_1) = \int K_{\tau_2}(x_3, x_2) K_{\tau_1}(x_2, x_1) d^n x_2$$
(3.4)

obeyed by convolutions of the heat kernel. It's a good exercise to check this identity holds using the explicit form (3.2) of K when the potential vanishes, but the above argument makes clear that it must hold in general for any heat kernel.

This more or less takes us to the path integral. From the explicit expression (3.2) for the heat kernel we have

$$\langle y_1 | \mathrm{e}^{-\beta H} | y_0 \rangle = \frac{1}{(2\pi\Delta\tau)^{n/2}} \int \exp\left[-\frac{1}{2} \sum_{i=0}^N \frac{\|x_{i+1} - x_i\|^2}{(\Delta\tau)^2} \,\Delta\tau\right] \prod_{i=1}^{N-1} \frac{d^n x_i}{(2\pi\Delta\tau)^{n/2}} \,. \tag{3.5}$$

We set

$$S_N[x] = \frac{1}{2} \sum_{i=0}^{N} \frac{\|x_{i+1} - x_i\|^2}{(\Delta \tau)^2} \Delta \tau \quad \text{and} \quad \mathcal{D}_N x = \frac{1}{(2\pi \Delta \tau)^{Nn/2}}.$$

At least for smooth paths, taking the limit $N \to \infty$ with β fixed (so $\Delta \tau \to 0$), we recognize $(x_{i+1} - x_i)/\Delta \tau$ as \dot{x} . Thus it's tempting to define the path integral as

$$\int e^{-S[x]} \mathcal{D}x = \int \lim_{N \to \infty} e^{-S_N[x]} \mathcal{D}_N x$$
(3.6)

where $S[x] = \frac{1}{2} \int_0^\beta \|\dot{x}\|^2 d\tau$ is the classical action for a (free) particle on \mathbb{R}^n . The heat kernel can then be written as the path integral

$$\langle y_1 | \mathrm{e}^{-\beta H} | y_0 \rangle = K_\beta(y_0, y_1) = \int_{C_\beta[y_0, y_1]} \mathcal{D}x \, \mathrm{e}^{-S[x]}$$
(3.7)

taken over the value of $x(\tau)$ at each $\tau \in (0, \beta)$. In other words, the path integral is taken over the space of maps $x : [0, \beta] \to \mathbb{R}^n$, with the boundary conditions $x(0) = y_0$ and $x(\beta) = y_1$. In fact, there are many subtleties here. While the limit

$$\lim_{N \to \infty} \left(\mathrm{e}^{-S_N} \, \mathcal{D}_N x \right)$$

does rigorously exist (and is known as the Wiener measure), the limits $\lim_{N\to\infty} e^{-S_N}$ and $\lim_{N\to\infty} \mathcal{D}_N x$ do not exist individually. Furthermore, for the Wiener measure to make sense we must allow arbitrarily jagged, non-differentiable paths. In higher dimensions, QFTs are always defined using some discretization or regularization procedure so as to make the path integral well-defined. Studying the behaviour of such integrals as one refines the discretization, or takes away the regulator, is the mathematical origin of the theory of renormalization that you'll study in the Advanced QFT or Statistical Field Theory courses. We can avoid it in QM because of the rigorous existence of the Wiener measure.

Remarkably²⁰, the Wiener measure can be generalized to cases where the potential $V(x) \neq 0$. Heuristically, this just involves including a potential term in the action in the usual way, but of course all the subtleties are in ensuring that the limit $N \to \infty$ of our discretized version of the path integral really exists. For QM it does, and the rigorous mathematical expression is known as the Feynman-Kac measure. Of course, while the relation (3.7) still holds, the explicit expression (3.2) is no longer valid when $V \neq 0$.

More generally, if $\hat{\mathcal{O}}_i(\hat{x})$ etc. are operators in the canonical picture that depend purely on \hat{x} , then for $\tau_1 < \tau_2 < \cdots < \tau_n < \beta$ we have

$$\begin{aligned} \langle y_{1}, \beta | \mathcal{O}_{n}(\tau_{n}) \cdots \mathcal{O}_{2}(\tau_{2}) \mathcal{O}_{1}(\tau_{1}) | y_{0}, 0 \rangle \\ &= \langle y_{1} | e^{-(\beta - \tau_{n})H} \hat{\mathcal{O}}_{n} \cdots e^{-(\tau_{2} - \tau_{1})H} \hat{\mathcal{O}}_{2} e^{-(\tau_{2} - \tau_{1})H} \hat{\mathcal{O}}_{1} e^{-\tau_{1}H} | y_{0} \rangle \\ &= \int \langle y_{1} | e^{-(\beta - \tau_{n})H} \hat{\mathcal{O}}_{n} | x_{n} \rangle \cdots \langle x_{2} | e^{-(\tau_{2} - \tau_{1})H} \hat{\mathcal{O}}_{1} | x_{1} \rangle \langle x_{1} | e^{-\tau_{1}H} | y_{0} \rangle \prod_{i} \mathrm{d}x_{i} \qquad (3.8) \\ &= \int_{C_{T}[y_{0}, y_{1}]} \mathcal{D}x \ \mathrm{e}^{-S[x]} \prod_{i=1}^{n} \mathcal{O}_{i}(x(\tau_{i})) \end{aligned}$$

where in the final line, the objects \mathcal{O}_i inside the path integral are just ordinary functions, evaluated at the points $x(t_i) \in N^{21}$.

For operators that depend on \hat{p} as well as \hat{x} , given that $p = \delta S/\delta \dot{x} = \dot{x}$, one may think that one should simply replace $\hat{\mathcal{O}}(\hat{x}, \hat{p}) \to \mathcal{O}(x, \dot{x})$ in order to construct a path integral expression for correlation functions of general operators. This is essentially correct (at least for \mathbb{R}^n), but must be done with care: the failure of $[\hat{x}, \hat{p}]$ to vanish is reflected in the path integral by a delicate inequivalence between

$$\lim_{\Delta \tau \to 0} x(\tau) \left[\frac{x(\tau) - x(\tau - \Delta \tau)}{\Delta \tau} \right] \quad \text{and} \quad \lim_{\Delta \tau \to 0} \left[\frac{x(\tau + \Delta \tau) - x(\tau)}{\Delta \tau} \right] x(\tau)$$

as we take away the discretization. In this course, we'll mostly avoid these subtleties by considering only path integrals without insertions, or operators that correspond to functions of x.

²⁰See e.g. J. Zinn–Justin, Path Integrals in Quantum Mechanics, OUP (2005), or B. Hall, Quantum Theory for Mathematicians, Springer (2013) for a somewhat more mathematical point of view.

²¹A more precise statement would be that they are functions on the space of fields $C_T[y_0, y_1]$ obtained by pullback from a function on N by the evaluation map at time t_i .

Closely related to the heat kernel is the *partition function*. As is familiar from *Statistical Mechanics*, in canonical quantization the partition function of a quantum system is defined to be the trace of the time evolution operator over the Hilbert space:

$$\mathcal{Z}(\beta) = \operatorname{Tr}_{\mathcal{H}}(\mathrm{e}^{-\beta H}), \qquad (3.9)$$

with $1/\beta$ playing the role of a temperature of our system in equilibrium. The partition function also has a natural expression in terms of a path integral. In the case of a single particle moving on \mathbb{R} , take the position eigenstates $|y\rangle$ to be a (somewhat formal) 'basis' of $\mathcal{H} = L^2(\mathbb{R}, dy)$, in which case the partition function becomes

$$\mathcal{Z}(\beta) = \int_{\mathbb{R}} \langle y | \mathrm{e}^{-\beta H} | y \rangle \,\mathrm{d}y = \int_{\mathbb{R}} \left[\int_{C_{\beta}[y,y]} \mathcal{D}x \,\,\mathrm{e}^{-S} \right] \mathrm{d}y \,, \tag{3.10}$$

where the last equality uses out path integral expression (3.7) for the heat kernel. Because we're taking the trace, the path integral here should be taken over maps $x : [0, \beta] \to \mathbb{R}$ such that the endpoints are both mapped to the *same* point²² y. We then integrate over y, erasing the memory of the particular point y. This is just the same thing²³ as integrating over maps $x : S^1 \to \mathbb{R}$ where the worldline has become a circle of circumference β . This shows that

$$\mathcal{Z}(\beta) = \operatorname{Tr}_{\mathcal{H}}(\mathrm{e}^{-\beta H}) = \int_{C_{S^1}} \mathcal{D}x \,\mathrm{e}^{-S}$$
(3.11)

in terms of the path integral.

3.1.1 A free quantum particle on a circle

As an example, let's consider a free particle with action

$$S[x] = \int \frac{1}{2} \dot{x}^2 \,\mathrm{d}\tau \tag{3.12}$$

but where $x \equiv x + 2\pi R$. This model describes quantum mechanics of a bosonic, scalar particle living on a circle of radius R. We'll compute the partition function for this theory first by using the canonical framework you're familiar with from your undergraduate QM courses, and then again using a path integral instead.

In the canonical picture, the Hamiltonian obtained from (3.12) is just $H = p^2/2$ and becomes

$$H = -\frac{1}{2}\frac{d^2}{dx^2}$$

²²The specific heat kernel (3.2) obeys $K_{\beta}(y, y) = K_{\beta}(0, 0)$ so is actually independent of y. Thus on flat space with vanishing potential, this final y integral does not converge. This is an 'infra-red' effect that arises because \mathbb{R} is non-compact. The partition function of a quantum particle does converge if we turn on a potential V(x) with $V \to \infty$ as $|x| \to \infty$. Even without a potential, infra-red effects are absent if our quantum particle lives on a compact space, such as a torus or sphere, rather than on \mathbb{R}^n . We'll see examples of this in the next few sections.

²³To avoid technicalities, I'm being deliberately vague about the (lack of) differentiability of the map; a careful, rigorous treatment leads to the same conclusion using the Wiener measure on S^1 .

under quantization in the position representation. Given the periodicity of x, the Hamiltonian has eigenfunctions

$$\phi_n(x) = \mathrm{e}^{inX/R} \qquad \text{for } n \in \mathbb{Z}$$

with corresponding eigenvalues $E_n = n^2/2R^2$. Thus, the partition function is

$$\mathcal{Z}(\beta) = \operatorname{Tr}_{\mathcal{H}}(\mathrm{e}^{-\beta H}) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\beta n^2/2R^2}$$
(3.13)

as we easily find in the canonical formulation.

Let's now compute the same partition function as a path integral over all maps $x : S^1_{\beta} \to S^1_{2\pi R}$, weighted by the action (3.12). Such maps are classified by their winding number, $m \in \mathbb{Z}$. This winding number is a topological invariant of the map x describing the number of times the worldline S^1_{β} is wrapped around the target $S^1_{2\pi R}$; there is no continuous family of maps that interpolate between a map of winding number m and one of $m' \neq m$. To account for this, we let

$$x_m(\tau) = y(\tau) + \frac{2\pi m\tau R}{\beta}$$
(3.14)

describe maps of fixed winding number m, where $y(\tau + \beta) = y(\tau)$ and the second term accounts for the winding. We then take the path integral to be

$$\mathcal{Z}(\beta) = \sum_{m \in \mathbb{Z}} \int e^{-S[x_m]} Dy$$
(3.15)

including a sum over all topological sectors. At winding number m we have

$$S[x_m] = \frac{2m^2 \pi^2 R^2}{\beta} - \frac{1}{2} \oint_{S^1_\beta} y \frac{d^2 y}{d\tau^2} \,\mathrm{d}\tau \,, \tag{3.16}$$

and, decomposing the periodic function y into its normalized Fourier components

$$y(\tau) = \frac{y_0}{\sqrt{\beta}} + \sum_{n=1}^{\infty} \left[y_n \sqrt{\frac{2}{\beta}} \cos\left(\frac{2\pi n\tau}{\beta}\right) + \tilde{y}_n \sqrt{\frac{2}{\beta}} \sin\left(\frac{2\pi n\tau}{\beta}\right) \right]$$
(3.17)

we take the path integral measure Dy formally to be

$$Dy = \prod_{n=1}^{\infty} \frac{\mathrm{d}y_n \, d\tilde{y}_n}{2\pi} \, .$$

The path integral then gives us the rather formal expression

$$\mathcal{Z}(\beta) = \sum_{m \in \mathbb{Z}} e^{-2\pi^2 m^2 R^2 / \beta} 2\pi R \sqrt{\frac{\beta}{2\pi}} \prod_{n=1}^{\infty} \left(\frac{\beta}{2\pi n}\right)^2$$
(3.18)

for the partition function, where the factor of $2\pi R \sqrt{\beta/2\pi}$ comes from integrating over the constant zero-mode y_0 and the infinite product comes from the non-zero modes. This infinite product clearly requires regularization, and it is convenient to use a ζ -function. Recall that Riemann's ζ -function can be defined for $\operatorname{Re}(s) > 1$ by the sum

$$\zeta(s) = \sum_{n=1}^\infty n^{-s}$$

and extended to all $s \in \mathbb{C}/\{1\}$ by analytic continuation. In particular, we have $\zeta(0) = -1/2$ and $\zeta'(0) = -(1/2) \ln(2\pi)$. To apply this to our case, we consider the related function

$$\tilde{\zeta}(s) = \sum_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2s} = \left(\frac{\beta}{2\pi}\right)^{2s} \zeta(2s)$$

Differentiating term-by-term gives

$$\tilde{\zeta}'(0) = \sum_{n=1}^{\infty} \ln\left(\frac{2\pi n}{\beta}\right)^{-2} = \ln\left[\prod_{n=1}^{\infty}\left(\frac{\beta}{2\pi n}\right)^{2}\right]$$

which we recognize from our divergent path integral, whereas relating the derivative to the derivative of the Riemann ζ -function gives

$$\tilde{\zeta}'(0) = 2\zeta(0) \ln \frac{\beta}{2\pi} + 2\zeta'(0) = -\ln \frac{\beta}{2\pi} - \ln 2\pi = -\ln \beta.$$

Thus the regularized path integral over the non-zero modes gives a contribution of $e^{\tilde{\zeta}'(0)} = 1/\beta$ to the partition function. Altogether, we have

$$\mathcal{Z}(\beta) = R \sqrt{\frac{2\pi}{\beta}} \sum_{m \in \mathbb{Z}} e^{-2\pi^2 m^2 R^2 / \beta} .$$
(3.19)

To see that this agrees with the result (3.13), note first that

$$\sum_{n \in \mathbb{Z}} \delta(x + 2\pi n) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{imx}$$

and hence we have the Poisson resummation identity

$$\sum_{n \in \mathbb{Z}} e^{-(2\pi n)^2 a/2} = \int \sum_{n \in \mathbb{Z}} e^{-x^2 a/2} \delta(x + 2\pi n) dx$$
$$= \frac{1}{2\pi} \int \left(\sum_{m \in \mathbb{Z}} e^{imx} e^{-x^2 a/2} \right) dx = \frac{1}{\sqrt{2\pi a}} \sum_{m \in \mathbb{Z}} e^{-m^2/2a}$$

from which it follows that (3.13) & (3.19) agree.

3.1.2 Path integrals for fermions

Path integrals for fermions may be constructed in just the same way as for bosons, with a few subtleties.

In the first problem set, I ask you to construct the fermionic coherent state $|\eta\rangle = e^{\hat{\psi}\eta}|0\rangle$ and its conjugate $\langle \bar{\eta}| = \langle 0|e^{\hat{\psi}\bar{\eta}}$ and show that they obey the normalization condition $\langle \bar{\eta}|\eta\rangle = e^{\bar{\eta}\eta}$, and provide a resolution of the identity

$$1_{\mathcal{H}} = \int e^{-\bar{\eta}\eta} |\eta\rangle \langle \bar{\eta}| d^2\eta \qquad (3.20)$$

for the fermionic system. Also, using these states one can express the trace of any operator as

$$\operatorname{Tr}_{\mathcal{H}}(\hat{A}) = \int e^{-\bar{\eta}\eta} \langle -\bar{\eta} | \hat{A} | \eta \rangle \,\mathrm{d}^2\eta \tag{3.21}$$

whereas the supertrace – the difference between the trace over states with an even and odd number of fermionic excitations – is represented as

$$\operatorname{STr}_{\mathcal{H}}(\hat{A}) = \operatorname{Tr}_{\mathcal{H}}((-1)^{F}\hat{A}) = \int e^{-\bar{\eta}\eta} \langle \bar{\eta} | \hat{A} | \eta \rangle d^{2}\eta.$$
(3.22)

Note that the usual trace involves a minus sign in the adjoint state $\langle -\eta |$.

We can now use these states to construct the path integral for fermions. The heat kernel for the fermionic system in Euclidean time β is again $\langle \bar{\chi}' | e^{-\beta H} | \chi \rangle$. By taking commutators if necessary, we can always order the Hamiltonian so that in each term, all $\hat{\psi}$ operators appear to the right of all $\hat{\psi}$ operators. With this ordering understood, we write

$$\langle \bar{\chi}' | \mathrm{e}^{-\beta H} | \chi \rangle = \langle \bar{\chi}' | \mathrm{e}^{-\Delta \tau H} \mathrm{e}^{-\Delta \tau H} \cdots \mathrm{e}^{-\Delta \tau H} | \chi \rangle$$

$$= \int \langle \bar{\chi}' | \mathrm{e}^{-\Delta \tau H} | \eta_{N-1} \rangle \cdots \langle \bar{\eta}_2 | \mathrm{e}^{-\Delta \tau H} | \eta_1 \rangle \langle \bar{\eta}_1 | \mathrm{e}^{-\Delta \tau H} | \chi \rangle \prod_{k=1}^{N-1} \mathrm{e}^{-\bar{\eta}_k \eta_k} \mathrm{d}^2 \eta_k$$

$$(3.23)$$

just as for bosons, where we note the presence of the factors of $e^{-\bar{\eta}_k \eta_k}$ coming from the normalization of our fermionic coherent states. Using the fact that $|\eta_k\rangle$ and $\langle \bar{\eta}_k|$ are eigenstates of $\hat{\psi}$ and $\hat{\psi}$ respectively, for an infinitesimal $\Delta \tau$ we have

$$\langle \bar{\eta}_{k+1} | \mathrm{e}^{-\Delta \tau \, H(\hat{\psi}, \hat{\psi})} | \eta_k \rangle = \mathrm{e}^{-\Delta \tau \, H(\bar{\eta}_k, \eta_k)} \, \langle \bar{\eta}_{k+1} | \eta_k \rangle = \mathrm{e}^{-\Delta \tau \, H(\bar{\eta}_{k+1}, \eta_k)} \, \mathrm{e}^{\bar{\eta}_{k+1} \eta_k} \,. \tag{3.24}$$

Then the full heat kernel (3.23) becomes

$$\begin{split} \langle \bar{\chi}' | \mathrm{e}^{-\beta H} | \chi \rangle &= \lim_{N \to \infty} \int \exp\left(\sum_{k=1}^{N} \bar{\eta}_k \eta_{k-1} - \Delta \tau \, H(\bar{\eta}_k, \eta_{k-1})\right) \prod_{k=1}^{N-1} \mathrm{e}^{-\bar{\eta}_k \eta_k} \, \mathrm{d}^2 \eta_k \\ &= \lim_{N \to \infty} \int \exp\left(-\sum_{k=1}^{N} \left[\bar{\eta}_k \frac{\eta_k - \eta_{k-1}}{\Delta \tau} + H(\bar{\eta}_k, \eta_{k-1}) \right] \Delta \tau \right) \, \mathrm{e}^{\bar{\eta}_N \eta_N} \prod_{k=1}^{N-1} \mathrm{d}^2 \eta_k \,, \end{split}$$
(3.25)

where we have set $\eta_0 \equiv \chi$ and $\bar{\eta}_N \equiv \bar{\chi}'$. We recognise the contents of the square brackets as a discretization of the Euclidean action

$$S[\eta,\bar{\eta}] = \int_0^\beta \left[\bar{\eta}\dot{\eta} + H(\bar{\eta},\eta)\right] \mathrm{d}\tau$$

With this action, the heat kernel for the fermionic system can also be written formally as a path integral

$$\langle \bar{\chi}' | e^{-\beta H} | \chi \rangle = \int e^{-S[\psi,\bar{\psi}]} e^{\bar{\psi}(\beta)\psi(\beta)} \mathcal{D}\psi \mathcal{D}\bar{\psi}$$
(3.26)

where we formally integrate over all $\psi(\tau)$ such that $\psi(0) = \chi$ and $\psi(\beta) = \chi'$. Note the presence of the boundary term $\bar{\psi}(\beta)\psi(\beta) = \bar{\chi}'\chi'$ as a remnant of our discretization procedure using these coherent states.

As for the bosons, the partition function of the fermionic system is given by the path integral

$$\mathcal{Z}(\beta) = \operatorname{Tr}(\mathrm{e}^{-\beta H}) = \int \langle -\bar{\chi} | \mathrm{e}^{-\beta H} | \chi \rangle \, \mathrm{e}^{-\bar{\chi}\chi} \, \mathrm{d}^2\chi$$
$$= \int_{\mathrm{AP}} \mathrm{e}^{-S[\psi,\bar{\psi}]} \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi}$$
(3.27)

where again the trace means that the action S is the integral over a circle. The normalization of the coherent states cancels the boundary term in the previous path integral. However, the fact that the trace (3.21) involves $\langle -\bar{\chi}|$ rather than $\langle \chi|$ means that the fermionic fields $\psi(\tau)$ and $\bar{\psi}(\tau)$ should be *antiperiodic* as one goes around the circle. By contrast, the supertrace (3.22) over the Hilbert space of the fermionic system

$$\operatorname{STr}(\mathrm{e}^{-\beta H}) = \operatorname{Tr}((-1)^{F} \mathrm{e}^{-\beta H}) = \int_{\mathrm{P}} \mathrm{e}^{-S[\psi,\bar{\psi}]} \,\mathcal{D}\psi \,\mathcal{D}\bar{\psi}$$
(3.28)

is computed with fermions $\psi(\tau + \beta) = \psi(\tau)$ that are periodic around the circle.

3.2 SQM with a potential

To begin, let's look at a simple extension of the d = 0 model we considered in section 2.3: we'll take a worldline theory of one bosonic field x and a single complex fermion ψ , each of which now depend on the worldline coordinate t. We take the action to be

$$S[x,\psi,\bar{\psi}] = \int \frac{1}{2}\dot{x}^2 + \frac{i}{2}\left(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi\right) - \frac{1}{2}(\partial W)^2 + \bar{\psi}\psi\,\partial^2 W\,\,\mathrm{d}t \tag{3.29}$$

where again W(x) is a (smooth) function of the bosonic field x. Up to boundary terms, this action is invariant under the transformations

$$\delta x = \bar{\epsilon}\psi - \epsilon\bar{\psi}$$

$$\delta \psi = \epsilon(i\dot{x} - \partial W)$$

$$\delta\bar{\psi} = \bar{\epsilon}(-i\dot{x} + \partial W)$$

(3.30)

that generalize the d = 0 supersymmetry transformations (2.22). Checking this is an important exercise!

As before, these transformations are generated by the fermionic vector fields

$$\mathcal{Q} = \int \left[\psi(t) \frac{\delta}{\delta x(t)} - (i\dot{x} - \partial W) \frac{\delta}{\delta \bar{\psi}(t)} \right] dt$$

$$\bar{\mathcal{Q}} = \int \left[-\bar{\psi}(t) \frac{\delta}{\delta x(t)} + (i\dot{x} - \partial W) \frac{\delta}{\delta \psi(t)} \right] dt$$
(3.31)

where the functional derivatives act as

$$\frac{\delta}{\delta x(t')}x(t) = \delta(t - t') \qquad \qquad \frac{\delta}{\delta x(t')}\psi(t) = 0 \qquad \qquad \frac{\delta}{\delta x(t')}\bar{\psi}(t) = 0$$

and similarly for the fermionic derivatives. In this d = 1 case the transformations generated by Q and \overline{Q} obey

$$\{Q, Q\} x = -2i\dot{x} \{Q, \bar{Q}\} \psi = -i\dot{\psi} - \psi \partial^2 h \simeq -2i\dot{\psi} \{Q, \bar{Q}\} \bar{\psi} = -i\dot{\bar{\psi}} + \bar{\psi} \partial^2 h \simeq -2i\dot{\bar{\psi}} ,$$

$$\{Q, \bar{Q}\} \bar{\psi} = -i\dot{\bar{\psi}} + \bar{\psi} \partial^2 h \simeq -2i\dot{\bar{\psi}} ,$$

$$(3.32)$$

where the symbol \simeq here means 'holds on the equations of motion'. Thus we have

$$\left\{\mathcal{Q}, \bar{\mathcal{Q}}\right\} \simeq -2\mathrm{i}\frac{\partial}{\partial t}.$$
 (3.33)

We can find the charges associated to this symmetry by the usual Noether procedure: allowing the parameters $\epsilon, \bar{\epsilon}$ to depend on t, we find the action is no longer invariant, but rather

$$\delta S = \int_M -i\dot{\epsilon}Q - i\dot{\bar{\epsilon}}\bar{Q} \,dt \tag{3.34}$$

where

$$Q = \bar{\psi}(i\dot{x} + \partial h)$$

$$\bar{Q} = \psi(-i\dot{x} + \partial h)$$
(3.35)

are the supercharges.

To perform canonical quantization we need to find the Hamiltonian. The momenta of our system are

$$p = \frac{\delta L}{\delta \dot{x}} = \dot{x}, \qquad \pi = \frac{\delta L}{\delta \dot{\psi}} = \mathrm{i}\bar{\psi}$$
 (3.36)

and so the Hamiltonian is

$$H = p\dot{x} + \pi\dot{\psi} - L = \frac{1}{2}p^2 + (\partial h)^2 + \frac{1}{2}\partial^2 h(\bar{\psi}\psi - \psi\bar{\psi}).$$
(3.37)

Classically, we have $\frac{1}{2}\partial^2 h(\bar{\psi}\psi - \psi\bar{\psi}) = \partial^2 h\bar{\psi}\psi = -\partial^2 h\psi\bar{\psi}$, but since $\pi = i\bar{\psi}$ is the momenta conjugate to ψ , quantum mechanically different orderings of the ψ s and $\bar{\psi}$ s are inequivalent. We've chosen a particular ordering in (3.37) for reasons that will soon become apparent.

Upon quantization, we have the usual commutation & anticommutation relations²⁴

$$[\hat{x}, \hat{p}] = i$$
 and $\{\hat{\psi}, \hat{\pi}\} = i.$ (3.38)

Note that, just as for the Dirac field in d = 4, the fermionic operators $\hat{\psi}$ and $\hat{\pi}$ in d = 1 obey anticommutation relations. Just as in d = 0, all bosonic fields commute with all fermionic fields.

²⁴As in any QFT, these are 'equal time' relations, but since in our d = 1 Universe a constant time slice is just a point, there's no need to specify any further arguments for the fields. Thus they reduce to the usual commutation (or anticommutation) relations of QM. Recall also that I've set $\hbar = 1$.

For the bosonic variable, as usual in QM we take the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dx)$ to the usual space of square-integrable wavefunctions on \mathbb{R} . The action of the operators \hat{x} and p on such a wavefunction $\Psi(x)$ is standard:

$$\hat{x}\Psi(x) = x\Psi(x), \qquad \qquad \hat{p}\Psi(x) = -i\frac{d}{dx}\Psi(x). \qquad (3.39)$$

To understand the role of the fermions, we note that since $\pi = i\bar{\psi}$ we can write the fermionic anticommutation relations as $\{\hat{\psi}, \hat{\psi}\} = 1$. This is very reminiscent of the commutation relations $[a, a^{\dagger}] = 1$ of a quantum SHO, except that here we have anticommutators. In the usual SHO, the Hamiltonian $H = \hbar\omega(a^{\dagger}a + \frac{1}{2})$ is determined by the number operator $N = a^{\dagger}a$. Motivated by this, we define the **fermion number operator**

$$F = \bar{\psi}\,\hat{\psi} \tag{3.40}$$

which obeys

$$[F, \hat{\psi}] = -\hat{\psi}, \qquad [F, \hat{\psi}] = +\hat{\psi}$$
 (3.41)

as a consequence of the fundamental relations (3.38). (Note that F itself is a bosonic operator.) This suggests we treat $\hat{\psi}$ as a lowering operator and define a ground state $|0\rangle$ of the fermionic system by $\hat{\psi}|0\rangle = 0$. The first excited state is then $|1\rangle = \hat{\psi}|0\rangle$ just as in the usual SHO, but since $\hat{\psi}^2 = 0$, this 'fermionic oscillator' has no higher excited states.

The Hilbert space of the full quantum system combines these two:

$$\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)|0\rangle \oplus L^2(\mathbb{R}, \mathrm{d}x)|1\rangle = \mathcal{H}_B \oplus \mathcal{H}_F$$
(3.42)

where \mathcal{H}_B and \mathcal{H}_F are the first and second summands, respectively. Note that F acts as 1 on \mathcal{H}_F , and annihilates any state in \mathcal{H}_B .

In the quantum theory, the supercharges Q and \overline{Q} of (3.35) become fermionic operators

$$\hat{Q} = \hat{\psi}(i\hat{p} + \partial h) \qquad \qquad \hat{Q} = \hat{\psi}(-i\hat{p} + \partial h) \qquad (3.43)$$

while the Hamiltonian (3.37) becomes

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}(\partial h)^2 + \frac{1}{2}\partial^2 h\left(\hat{\psi}\hat{\psi} - \hat{\psi}\hat{\psi}\right).$$
(3.44)

We'll drop the hats from operators henceforth: this should cause no confusion so long as one keeps the representation (3.42) in mind. Since $\{\psi, \psi\} = 0 = \{\bar{\psi}, \bar{\psi}\}$ we have immediately

$$\{Q, Q\} = 0$$
 and $\{\bar{Q}, \bar{Q}\} = 0.$ (3.45)

Using the fundamental commutation relations, we also compute

$$\begin{aligned} \{Q,\bar{Q}\} &= \left\{\bar{\psi}(\mathrm{i}p+\partial h),\psi(-\mathrm{i}p+\partial h)\right\} \\ &= \left\{\bar{\psi}(\mathrm{i}p),\psi(-\mathrm{i}p)\right\} + \left\{\bar{\psi}\partial h,\psi\partial h\right\} + \mathrm{i}\left\{\bar{\psi}p,\psi\partial h\right\} - \mathrm{i}\left\{\bar{\psi}\partial h,\psi p\right\} \\ &= p^2 + (\partial h)^2 + \mathrm{i}\left(\left\{\bar{\psi},\psi\right\}p\partial h - \psi\bar{\psi}[p,\partial h]\right) - \mathrm{i}\left(\left\{\bar{\psi},\psi\right\}\partial h p - \psi\bar{\psi}[\partial h,p]\right) \quad (3.46) \\ &= p^2 + (\partial h)^2 + \mathrm{i}(\bar{\psi}\psi - \psi\bar{\psi})[p,\partial h] \\ &= p^2 + (\partial h)^2 + \partial^2 h \left(\bar{\psi}\psi - \psi\bar{\psi}\right). \end{aligned}$$

Thus our supersymmetric Quantum Mechanics carries the algebra

$$\{Q,Q\} = 0, \qquad \{\bar{Q},\bar{Q}\} = 0, \qquad \{Q,\bar{Q}\} = 2H$$
 (3.47)

It was to ensure that $\{Q, \overline{Q}\} = 2H$ that we chose the specific ordering of terms in the Hamiltonian. It follows that

$$[H,Q] = \frac{1}{2} \left[\{Q,\bar{Q}\},Q \right] = \frac{1}{2} (Q\bar{Q} + \bar{Q}Q)Q - \frac{1}{2}Q(Q\bar{Q} + \bar{Q}Q) = 0$$
(3.48)

since $Q^2 = 0$, and similarly $[H, \bar{Q}] = 0$. This says that, as expected the Hamiltonian is invariant under the supersymmetry transformations generated by Q and \bar{Q} . Note also that, since it is proportional to the lowering operator ψ , $\bar{Q} : \mathcal{H}_F \to \mathcal{H}_B$ and annihilates \mathcal{H}_B , while $Q : \mathcal{H}_B \to \mathcal{H}_F$ and annihilates \mathcal{H}_F . Finally, we record that [F, Q] = Q, that $[F, \bar{Q}] = -\bar{Q}$ and that [F, H] = 0.

3.2.1 Supersymmetric ground states and the Witten index

The fact that $H = \frac{1}{2} \{Q, \bar{Q}\}$ where $\bar{Q} = Q^{\dagger}$ has several important consequences. Let $|E\rangle$ be an eigenstate of H with $H|E\rangle = E|E\rangle$. Then, as already mentioned in the Introduction,

$$E = \langle E|H|E \rangle = \frac{1}{2} \langle E|Q\bar{Q} + \bar{Q}Q|E \rangle = \frac{1}{2} \left(\|Q|E \rangle \|^2 + \|\bar{Q}|E \rangle \|^2 \right) \ge 0$$
(3.49)

so that all the eigenvalues of H are non-negative. In particular $H|\Psi\rangle = 0$ iff both $Q|\Psi\rangle = 0$ and $\bar{Q}|\Psi\rangle = 0$, and such a zero-energy state must be a ground state. Thus we see that in SQM *a ground state of zero energy must be invariant under supersymmetry*.

We can explicitly find these ground states as follows. Recall that the fermionic system is spanned by the states $|0\rangle$ and $\bar{\psi}|0\rangle$. Let's represent these as

$$|0
angle
ightarrow \begin{pmatrix} 1\\ 0 \end{pmatrix}$$
 and $\bar{\psi}|0
angle
ightarrow \begin{pmatrix} 0\\ 1 \end{pmatrix}$.

Then Q and \overline{Q} are represented by

$$Q = \bar{\psi}(\mathrm{i}p + \partial h) \rightarrow \begin{pmatrix} 0 & 0 \\ d/dx + \partial h & 0 \end{pmatrix}$$
$$\bar{Q} = \psi(-\mathrm{i}p + \partial h) \rightarrow \begin{pmatrix} 0 & -d/dx + \partial h \\ 0 & 0 \end{pmatrix}$$

A ground state is thus a state $\begin{pmatrix} f \\ g \end{pmatrix}$ where f and g are functions of x that obey

$$\frac{df}{dx} + (\partial h)f = 0$$
, and $-\frac{dg}{dx} + (\partial h)g = 0$. (3.50)

These are solved by $f = Ae^{-h(x)}$ and $g = Be^{+h(x)}$. For a solution in $L^2(\mathbb{R}, dx)$ we must set either A = 0 or B = 0 or both, depending on the behaviour of h(x) as $|x| \to \infty$. If $h(x) \to +\infty$ as $|x| \to \infty$ then the ground state is $(f, 0)^{\mathrm{T}}$, whereas if $h(x) \to -\infty$ as $|x| \to \infty$ then the ground state is $(0, g)^{\mathrm{T}}$. On the other hand, if $h(x) \to \pm \infty$ as $x \to -\infty$ but $h(x) \to \pm \infty$ as $x \to +\infty$ then there is no (normalizable) supersymmetric state of zero-energy.

It's worth emphasising just how remarkable this is. If we simply switch off the fermions, then our Hamiltonian becomes

$$H = \frac{p^2}{2} + V(x)$$

where $V(x) = (\partial h)^2$ is non-negative, but otherwise arbitrary. For generic choices of h we'd have no idea what the ground state wavefunction of this Hamiltonian looks like, though we may be able to approximate it either by a judicious choice of Raleigh–Ritz ansatz, or perhaps perturbing around a nearby potential whose states we do understand. Supersymmetry allows us to take a 'square root' of the Hamiltonian, so that the supersymmetric ground states is determined by a system of first-order equations that we can solve exactly, at least in this case of only one bosonic variable.

Unfortunately, while we can find the exact ground state, it's usually much harder to say anything about excited states. Nonetheless, there are a few important, general features that we can see. First, since H is Hermitian, eigenstates with distinct eigenvalues are orthogonal, so $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ where $H|\Psi\rangle = E_n|\Psi\rangle$ for any $|\Psi\rangle \in \mathcal{H}_n$, and $E_0 = 0$ is the energy of the supersymmetric ground state. Since F, Q and \bar{Q} each commute with H, they preserve each \mathcal{H}_n and in particular we can split $\mathcal{H}_n = \mathcal{H}_{n,B} \oplus \mathcal{H}_{n,F}$ just as for the full Hilbert space. Again we have $Q : \mathcal{H}_{n,B} \to \mathcal{H}_{n,F}$ and annihilates $\mathcal{H}_{n,F}$, whereas $\bar{Q}: \mathcal{H}_{n,F} \to \mathcal{H}_{n,B}$ and annihilates $\mathcal{H}_{n,B}$. In particular, given a state $|b\rangle \in \mathcal{H}_{n,B}$ we have

$$2E_n|b\rangle = (Q\bar{Q} + \bar{Q}Q)|b\rangle = \bar{Q}Q|b\rangle.$$
(3.51)

If $n \neq 0$, so that $|b\rangle$ is not a supersymmetric ground state, then we see that also $|f\rangle = Q|b\rangle \neq 0$. Turning this around, we have that

$$|b\rangle = \bar{Q}\left(\frac{1}{2E_n}|f\rangle\right) \quad \text{for some } |f\rangle \in \mathcal{H}_{n,F}.$$
 (3.52)

Thus any state $|b\rangle \in \mathcal{H}_{n,B}$ with energy $E_n > 0$ is necessarily the \bar{Q} -transformation of some state $|f\rangle \in \mathcal{H}_{n,F}$. A similar argument shows that any state in $\mathcal{H}_{n,F}$ with n > 0 is the Q-transformation of some state in $\mathcal{H}_{n,B}$. Putting these together, we've shown that

$$\mathcal{H}_{n,B} \cong \mathcal{H}_{n,F}$$
 for all $n > 0$.

In particular, bosonic and fermionic states are paired at each energy level. The above argument breaks down for the ground states, because these are annihilated by both Q and \bar{Q} , so we cannot establish an isomorphism between $\mathcal{H}_{0,B}$ and $\mathcal{H}_{0,F}$ and there may be different numbers of bosonic and fermionic ground states.

We define the **Witten index** of a theory to be the difference between the number of bosonic and fermionic ground states. Since the excited states come in pairs, we see that

this can be computed as

$$\dim \mathcal{H}_{0,B} - \dim \mathcal{H}_{0,F} = \operatorname{Tr}_{\mathcal{H}}((-1)^F \mathrm{e}^{-\beta H})$$
(3.53)

with the factor of $(-1)^F$ ensuring that the contribution of excited states cancels out of the sum. For the same reason, this index is independent of β (playing the role of an 'inverse temperature' in statistical mechanics); only the excited ground states are sensitive to the value of β , and their contribution cancels out in pairs.

The importance of the Witten index is that it is insensitive to the details of our Hamiltonian, and in particular insensitive to the detailed form of h(x). This is because as we vary our choice of h, so long as the model remains supersymmetric, the excited states must all move around in pairs. Changing h may cause some excited states to lower their energy until they become ground states, and it may also cause some ground states to acquire positive energy, but in any such effect a bosonic state must always be accompanied by a fermionic state, so the difference in the number of bosonic and fermionic ground states will be unaffected.

We saw above that if $h(x) \to +\infty$ as $|x| \to \infty$ then the unique ground state was of the form $f(x)|0\rangle \in \mathcal{H}_{0,B}$, and hence we would find $\operatorname{Tr}_{\mathcal{H}}(-)^F e^{-\beta H} = +1$. If $h(x) \to -\infty$ as $|x| \to \infty$ then the unique ground state is fermionic and $\operatorname{Tr}_{\mathcal{H}}(-)^F e^{-\beta H} = -1$. Finally, if $h(x) \to \pm \infty$ as $x \to \mp \infty$ then there are no supersymmetric ground states and the Witten index vanishes. The Witten index thus cares about the asymptotic behaviour of h as $|x| \to \infty$, but not to any of its details for finite x.

3.2.2 Path integral computation of the Witten index

We are now ready to use the path integral to compute the Witten index of our basic SQM with potential $\frac{1}{2}(\partial h)^2$. We have

$$\mathcal{I}_{\mathrm{W}} = \mathrm{STr}_{\mathcal{H}}(\mathrm{e}^{-\beta H}) = \int_{\mathrm{P}} \mathrm{e}^{-S_E[x,\psi,\bar{\psi}]} \mathcal{D}x \,\mathcal{D}\psi \,\mathcal{D}\bar{\psi} \,, \qquad (3.54)$$

where the subscript P on the path integral is to remind us that this integral is to be taken with both the bosonic field x and fermionic fields ψ and $\bar{\psi}$ being periodic. The path integral is weighted using the Euclidean action

$$S_E[x,\psi,\bar{\psi}] = \oint \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + \bar{\psi}\frac{d\psi}{d\tau} + \frac{1}{2}(\partial h)^2 + \partial^2 h\,\bar{\psi}\psi\right]\,\mathrm{d}\tau \tag{3.55}$$

which is invariant under the supersymmetry transformations

$$\delta x = \epsilon \bar{\psi} - \bar{\epsilon} \psi, \qquad \delta \psi = \epsilon (-\dot{x} + \partial h), \qquad \delta \bar{\psi} = \bar{\epsilon} (\dot{x} + \partial h) \qquad (3.56)$$

that are the Euclidean continuation of the supersymmetry transformations (3.30). Note that since x is periodic, it's essential that ψ and $\bar{\psi}$ are also periodic, rather than antiperiodic, around the S^1 worldline for these transformations to make sense globally. The partition function $\mathcal{Z}(\beta)$ thus *breaks* supersymmetry globally, and so we cannot (generically) expect to be able to use localization to say anything about it. Localization does work for the Witten index.

Of course, if we allow the parameters ϵ , $\bar{\epsilon}$ to also be antiperiodic, then we can preserve supersymmetry with antiperiodic fermions. However, no *constant* parameter ϵ can be antiperiodic. Requiring the symmetry to hold even for varying parameters $\epsilon(t)$ really means we'd be *gauging* the supersymmetry. This is **supergravity**, here on the worldline. Unfortunately, it is beyond the scope of these notes, though you will encounter it if you're taking a course in String Theory.

We have already argued from the point of view of canonical quantization that the Witten index is independent of β , and insensitive to all but the asymptotic behaviour of h. Let's now see this again from the path integral perspective. Firstly, as in zero dimensions, if we rescale $h(x) \to \lambda h(x)$, then varying $wrt \lambda \in \mathbb{R}_{>}$ gives

$$\frac{d\mathcal{I}_{\mathrm{W}}}{d\lambda} = -\int_{\mathrm{P}} \left[\oint \lambda (\partial h)^2 + \bar{\psi} \psi \partial^2 h \, \mathrm{d}\tau \right] \mathrm{e}^{-S_E[x,\psi,\bar{\psi}]} \, \mathcal{D}x \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \tag{3.57}$$

Just as in zero-dimensions, the λ -rescaled supersymmetry transformations (3.56) show that

$$\oint \lambda (\partial h)^2 + \bar{\psi} \psi \partial^2 h \, \mathrm{d}\tau = Q_\lambda \left(\oint \partial h \, \psi \, \mathrm{d}\tau \right) + \oint \frac{dx}{d\tau} \frac{dh}{dx} \, \mathrm{d}\tau$$

$$= Q_\lambda \left(\oint \partial h \, \psi \, \mathrm{d}\tau \right) + \oint \mathrm{d}h \,. \tag{3.58}$$

The last term vanishes since it is a total derivative integrated over a compact²⁵ S^1 . Thus the insertion in (3.57) is *Q*-exact and the path integral for \mathcal{I}_W is independent of scalings of *h*, as expected. However, if we send $\lambda \to \infty$ then the potential term in the action suppresses all contributions except from maps $x : S^1 \to x_*$, where again x_* is a critical point of *h*. Thus the path integral for the Witten index localizes to a neighbourhood of field configurations obeying

$$\frac{dx}{d\tau} = 0 = \frac{dh}{dx} \tag{3.59}$$

i.e. constant maps to the critical points of h(x).

For a nearby map, we set $x(\tau) = x_* + \delta x(\tau)$ and expand the action to quadratic order in $\delta x(\tau)$. This gives

$$S_E^{(2)} = \int_0^\beta \left[\frac{1}{2} \delta x \left(-\frac{d^2}{d\tau^2} + (h''(x_*))^2 \right) \delta x + \bar{\psi} \left(\frac{d}{d\tau} + h''(x_*) \right) \psi \right] d\tau$$
(3.60)

as the action for field configurations in the neighbourhood of our localization point. It's straightforward to compute the Witten index using this quadratic theory: Since both the bosonic field and fermionic fields must be periodic, we can expand them in Fourier modes as

$$\delta x(\tau) = \sum_{n \in \mathbb{Z}} \delta x_n \, \mathrm{e}^{2n\pi \mathrm{i}\tau/\beta} \,, \qquad \qquad \psi(\tau) = \sum_{n \in \mathbb{Z}} \psi_n \, \mathrm{e}^{2n\pi \mathrm{i}\tau/\beta} \tag{3.61}$$

²⁵If you prefer, the integral $\int_0^\beta (dx/d\tau)(dh/dx) d\tau = h(x(\beta)) - h(x(0))$ vanishes since $x(\tau)$ is periodic.

with $\delta x_{-n} = \delta x_n^*$ as $x(\tau)$ is real, and where $\bar{\psi}(\tau) = \overline{\psi(\tau)}$. The path integral becomes an integral over all these Fourier modes, giving

$$\int e^{-S_E^{(2)}} \mathcal{D}\delta x \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} = \frac{\det(\partial_\tau + h''(x_*))}{\sqrt{\det(-\partial_\tau^2 + (h''(x_*))^2)}} = \frac{\prod_{n \in \mathbb{Z}} (2\pi i n/\beta + h''(x_*))}{\sqrt{\prod_{n \in \mathbb{Z}} ((2\pi n/\beta)^2 + (h''(x_*))^2)}} = \frac{h''(x_*)}{|h''(x_*)|}$$
(3.62)

so that, summing over all critical points

$$\mathcal{I}_{W} = STr(e^{-\beta H}) = \sum_{x_{*}: h'(x_{*})=0} \frac{h''(x_{*})}{|h''(x_{*})|}.$$
(3.63)

Note that all the terms with $|n| \neq 0$ have cancelled out between the numerator and denominator. This result is in agreement with what we obtained (more straightforwardly) from the canonical perspective: if h is a polynomial of even degree, then $\mathcal{I}_{W} = \pm 1$ according to whether the coefficient of the leading term in h is positive or negative, whereas if the leading term in h has odd degree, $\mathcal{I}_{W} = 0$. This also coincides with what we obtained from our zero-dimensional example, which is not surprising since the path integral here reduced to constant maps.

3.3 Nonlinear Sigma Models

To get something more interesting, we need to study a quantum system with closer connections to geometry. Consider a particle travelling freely on a general Riemannian manifold (N,g) which we'll take to have dimension n. Our worldline fields describe a map $x: M \to N$; that is, for each point τ on the worldline $M, x(\tau)$ is a point in N. It's often convenient to describe N using coordinates. If an open patch $U \subset N$ has local co-ordinates x^a for $a = 1, \ldots, n$, then we let $x^a(\tau)$ denote²⁶ the coordinates of the image point $x(\tau)$. We can interpret $x(\tau)$ as a possible trajectory a particle might take as it travels through the space N. (See figure 5.) In this context, N is called the **target space** of the theory.

We choose the **non-linear sigma model** action

$$S[x] = \frac{1}{2} \int_M \left[g_{ab}(x) \dot{x}^a \dot{x}^b \right] \mathrm{d}\tau , \qquad (3.64)$$

where $g_{ab}(x)$ is the pullback to the worldline M of the Riemannian metric on N, τ is worldline (Euclidean) time, and $\dot{x}^a = dx^a/d\tau$. Under a small variation δx of x the change in the action is

$$\delta S[x] = \int_{M} \left[g_{ab}(x) \dot{x}^{a} \dot{\delta x}^{b} + \frac{1}{2} \frac{\partial g_{ab}(x)}{\partial x^{c}} \delta x^{c} \dot{x}^{a} \dot{x}^{b} \right] d\tau$$

$$= \int_{M} \left[-\frac{d}{d\tau} \left(g_{ac}(x) \dot{x}^{a} \right) + \frac{1}{2} \frac{\partial g_{ab}(x)}{\partial x^{c}} \dot{x}^{a} \dot{x}^{b} \right] \delta x^{c} d\tau + g_{ab}(x) \dot{x}^{b} \delta x^{a} \Big|_{\partial M} .$$
(3.65)

²⁶More precisely, $x^{a}(\tau)$ are the pullbacks to M of coordinates on U by the map x.



Figure 5: The theory (3.64) describes a map from an abstract worldline into the Riemannian target space (N, g), interpreted as single particle Quantum Mechanics on N.

Requiring that the bulk term vanishes for arbitrary variations $\delta x^a(\tau)$ gives the Euler– Lagrange equations

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0$$
(3.66)

where $\Gamma_{bc}^{a} = \frac{1}{2}g^{ad} \left(\partial_{b}g_{cd} + \partial_{c}g_{bd} - \partial_{d}g_{bc}\right)$ is the Levi–Civita connection on N, again pulled back to the worldline. Thus the field equation (3.66) says that, classically, the particle travels along a geodesic in (N, g).

We can try to quantize this NLSM in the usual way. Firstly, the momentum conjugate to $x^{a}(\tau)$ is

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab} \dot{x}^b$$

and so we obtain the usual quantum commutation relations²⁷

$$[\hat{x}^a, \hat{p}_b] = \mathrm{i}\,\delta^a_{\ b}$$

We can take the Hilbert space to be $\mathcal{H} = L^2(N, \sqrt{g} d^n x)$, the space of complex-valued functions that are square-integrable *wrt* the obvious measure $\sqrt{g} d^n x$. Then commutation relations show that, as usual, \hat{p}_a acts on Schrödinger wavefunctions as $\hat{p}_a = -i\partial_a$.

However, the Hamiltonian is ambiguous. Classically, we have

$$H = p_a \dot{x}^a - L = \frac{1}{2}g^{ab}(x)p_a p_b$$

and the quantum version of this will certainly be some form of Laplacian acting on our wavefunctions. The problem is that, because the metric g depends on x, we have to decide how to order the \hat{x} s and \hat{p} s in the quantum Hamiltonian operator \hat{H} . It's reasonable to ask that our ordering choice respects general coordinate invariance of the target space and, since H contains at most two ps, \hat{H} should be a second order differential operator

²⁷As confirmation of these relations, the boundary term in (3.65) shows that the symplectic potential on the space of maps is $\Theta = p_a \, \delta x^a = g_{ab}(x) \dot{x}^b \, \delta x^a$ and the canonical commutation relations follow from the associated symplectic form $\Omega = \delta \Theta = \delta p_a \wedge \delta x^a$.

acting on wavefunctions $\Psi(x)$, involving no more than two derivatives of the metric. These restrictions do not fix \hat{H} completely, but give

$$\hat{H}\Psi(x) = -\frac{1}{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^a}\left(\sqrt{g}\,g^{ab}\frac{\partial}{\partial x^b}\Psi\right) + \alpha R\,\Psi = -\frac{1}{2}\nabla^a\nabla_a\Psi + \alpha R\Psi$$

where the first term is the covariant Laplacian acting on scalar functions, and the second term is some arbitrary multiple of the Ricci scalar of g. This ambiguity will be cured in the supersymmetric model.

To construct the supersymmetric extension of this, we introduce dim N complex fermions ψ^a (with $\bar{\psi}^a = \overline{\psi}^a$) and consider the rather intimidating action

$$S[x,\psi,\bar{\psi}] = \int_{M} \frac{1}{2} g_{ab} \dot{x}^{a} \dot{x}^{b} + i g_{ab} \bar{\psi}^{a} (\nabla_{\tau}\psi)^{b} - \frac{1}{2} R_{abcd} \psi^{a} \bar{\psi}^{b} \psi^{c} \bar{\psi}^{d} d\tau , \qquad (3.67)$$

where the fermion kinetic term uses the pullback to M of the covariant derivative acting on sections of TN:

$$(\nabla_{\tau}\psi)^a = \frac{d}{d\tau}\psi^a + \Gamma^a_{bc}\frac{dx^b}{d\tau}\psi^c$$

and where R_{abcd} is the Riemann curvature of the target space metric. Note that, for a generic choice of target space (N, g), the (pullback of the) metric components $g_{ab}(x)$, connection $\Gamma^a_{bc}(x)$ and Riemann curvature will all depend on the worldline field $x^a(\tau)$. We'll understand the origin of this action better when we consider superspace in the following chapters; you'll also explore it further in the first problem set.

The most important fact about this action is that, up to boundary terms, it is invariant under the supersymmetry transformations

$$\delta x^{a} = \epsilon \psi^{a} - \bar{\epsilon} \psi^{a}$$

$$\delta \psi^{a} = \epsilon \left(i\dot{x}^{a} - \Gamma^{a}_{bc} \bar{\psi}^{b} \psi^{c} \right)$$

$$\delta \bar{\psi}^{a} = \bar{\epsilon} \left(-i\dot{x}^{a} - \Gamma^{a}_{bc} \bar{\psi}^{b} \psi^{c} \right) ,$$
(3.68)

and it's very good practice to check you can show this. The conserved Noether charges corresponding to these transformations are

$$Q = i\bar{\psi}^{a} \left(g_{ab}\dot{x}^{b} + ig_{bc}\bar{\psi}^{b}\Gamma^{c}_{ad}\psi^{d} \right)$$

$$\bar{Q} = -i\psi^{a} \left(g_{ab}\dot{x}^{b} + ig_{bc}\bar{\psi}^{b}\Gamma^{c}_{ad}\psi^{d} \right) ,$$
(3.69)

respectively. Much easier to see is that the action is also invariant under the transformations

$$\psi^a \mapsto e^{-i\alpha}\psi^a, \qquad \bar{\psi}^a \mapsto e^{+i\alpha}\bar{\psi}^a$$
(3.70)

generated by the Noether charge

$$F = g_{ab}\bar{\psi}^a\psi^b. \tag{3.71}$$

Conservation of F means that evolution in the quantum theory does not create or destroy excitations of the ψ s.

Let's now quantize this theory. The momenta conjugate to (x^a, ψ^a) are

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab} \left(\dot{x}^b + \mathrm{i} g_{bc} \bar{\psi}^b \Gamma^c_{ad} \psi^d \right)$$

with the extra piece coming from the covariant part of the fermion kinetic term, and

$$\pi_a = \frac{\delta L}{\delta \dot{\psi}^a} = \mathrm{i} g_{ab} \bar{\psi}^b$$

for the fermions. We again have the basic commutation & anticommutation relations

$$[\hat{x}^a, \hat{p}_b] = \mathrm{i}\delta^a_{\ b}, \qquad \left\{\hat{\psi}^a, \hat{\bar{\psi}}_b\right\} = g^{ab} \qquad (3.72)$$

with all other commutators & anticommutators being trivial.

The natural Hilbert space on which to represent these commutation & anticommutation relations can be constructed as follows. Firstly, as in the non-supersymmetric NLSM, we take the Hilbert space of the bosonic system to be $L^2(N, \sqrt{g} \, d^n x)$, so that as usual \hat{x}^a and \hat{p}_a correspond to multiplication by the coordinate x^a and differentiation $-i\partial/\partial x^a$. Next, as in section 3.2, we take the vacuum of the fermionic system to be defined by $\psi^a|0\rangle = 0$ for all $a = 1, \ldots, n$. Then the Hilbert space of the $(\psi, \bar{\psi})$ -system is spanned by the states obtained by acting on this $|0\rangle$ with any of the 'raising' operators²⁸ $\bar{\psi}^b$, with each component of $\bar{\psi}^b$ acting at most once. The anticommutation relations $\bar{\psi}^a \bar{\psi}^b = -\bar{\psi}^b \bar{\psi}^a$ mean that we can interpret these states as given a basis of all *p*-forms on *N*, with the correspondence

In other words, $\bar{\psi}^a$ corresponds to taking the exterior product $dx^a \wedge$. On the other hand, the anticommutation relations $\{\psi^a, \bar{\psi}^b\} = g^{ab}$ and vacuum condition $\psi^a |0\rangle = 0$ show that

$$\begin{split} \psi^{e} \bar{\psi}^{a} \bar{\psi}^{b} \bar{\psi}^{c} \cdots \bar{\psi}^{d} |0\rangle &= \{\psi^{e}, \bar{\psi}^{a} \bar{\psi}^{b} \bar{\psi}^{c} \cdots \bar{\psi}^{d}\} |0\rangle \\ &= \left(g^{ae} \bar{\psi}^{b} \bar{\psi}^{c} \cdots \bar{\psi}^{d} - \bar{\psi}^{a} g^{be} \bar{\psi}^{c} \cdots \bar{\psi}^{d} + \bar{\psi}^{a} \bar{\psi}^{b} g^{ce} \cdots \bar{\psi}^{d} + \cdots\right) |0\rangle, \end{split}$$

$$(3.73)$$

and hence we should interpret the action of ψ^a as *contraction* by the vector field $g^{ab} \frac{\partial}{\partial x^b}$.

Combining the bosonic and fermionic systems together, in total we have

$$\mathcal{H} = \Omega^*(N) \otimes \mathbb{C} = \bigoplus_{p=0}^n \Omega^p(N) \otimes \mathbb{C}$$
(3.74)

the space of all complex valued polyforms on N, square-integrable with respect to the inner product

$$\langle \alpha | \beta \rangle = \int_{N} \overline{\alpha} \wedge \ast \beta \tag{3.75}$$

²⁸Again, henceforth we drop the 'hats'.

where * is the Hodge star. Notice that if we choose a wavefunction $\Psi(x)|0\rangle \leftrightarrow \Psi(x) \in \Omega^0(N)$ (*i.e.* a function rather than a higher form), this inner product reduces to $\langle \Psi | \Psi \rangle = \int_N \sqrt{g} |\Psi(x)|^2 d^n x$, just as we had in the purely bosonic model above, whilst repeated use of (3.73) connects the indices of two forms together²⁹. We can decompose $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$, where \mathcal{H}_B is the space of *even* forms whilst \mathcal{H}_F is the space of *odd* forms.

One of the beautiful features of this model is that the supercharges Q and \bar{Q} also have natural geometric interpretations. In the canonical framework, the supercharges (3.69) are

$$\hat{Q} = i\hat{\psi}^a \hat{p}_a$$
 and $\hat{Q} = -i\hat{\psi}^a \hat{p}_a$. (3.76)

Thus, acting on $\Omega^*(N) \otimes \mathbb{C}$, we have

$$Q = i\bar{\psi}^a p_a \leftrightarrow dx^a \wedge \frac{\partial}{\partial x^a} = d, \qquad (3.77)$$

and so is just the exterior derivative. Similarly,

$$\bar{Q} = -\mathrm{i}\psi^a p_a \leftrightarrow d^{\dagger} = (-1)^{n(p+1)+1} * d *$$
(3.78)

is the adjoint of Q wrt the inner product (3.75). Note that $d^{\dagger} : \Omega^p(N) \to \Omega^{p-1}(N)$ for $p \ge 1$ and annihilates functions, just as we expect for ψ acting on a state with $p \ \overline{\psi}s$.

We fix the ordering ambiguity in the quantum Hamiltonian \hat{H} by requiring that the d = 1 supersymmetry algebra $\{Q, \bar{Q}\} = 2H$ holds. Notice that since [F, Q] = Q and $[F, \bar{Q}] = -\bar{Q}$, this definition of H also ensures that [F, H] = 0 so that fermion number is also conserved in the quantum theory. Geometrically, our Hamiltonian is the operator

$$H = \frac{1}{2} \{Q, \bar{Q}\} \qquad \leftrightarrow \qquad \frac{1}{2} \Delta = \frac{1}{2} \left(dd^{\dagger} + d^{\dagger}d \right)$$
(3.79)

where Δ is the natural generalization of the scalar Laplacian when acting on forms. To understand this, note for example that when acting on a function f(x), since $d^{\dagger}f = 0$ we have

$$\begin{split} \Delta f &= (dd^{\dagger} + d^{\dagger}d)f = d^{\dagger}df = *d * (\partial_{a}f \, dx^{a}) \\ &= *d\left(\frac{\sqrt{g}}{(n-1)!}g^{ab}\,\partial_{a}f \,\epsilon_{bcd\dots e}dx^{c} \wedge dx^{d} \wedge \dots \wedge dx^{e}\right) \\ &= *\frac{1}{(n-1)!}\partial_{m}\left(\sqrt{g}\,g^{ab}\,\partial_{a}f \,\epsilon_{bcd\dots e}dx^{m} \wedge dx^{c} \dots \wedge dx^{e}\right) \\ &= *\partial_{a}\left(\sqrt{g}\,g^{ab}\,\partial_{b}f\right)\frac{1}{g}\,dx^{1} \wedge dx^{2} \dots \wedge dx^{n} \\ &= \frac{1}{\sqrt{g}}\,\partial_{a}(\sqrt{g}\,g^{ab}\partial_{b}f)\,, \end{split}$$

which is the usual curved space Laplacian for functions. Also, since d^{\dagger} is the adjoint of dwrt the inner product (3.75), this Laplacian obeys

$$(\omega, \Delta\omega) = (\omega, dd^{\dagger}\omega) + (\omega, d^{\dagger}d\omega) = \|d^{\dagger}\omega\|^2 + \|d\omega\|^2 \ge 0$$
(3.80)

²⁹Since $* : \Omega^p(N) \to \Omega^{n-p}(N)$ for each p, α and $*\beta$ here are each polyforms in general. In the inner product $\langle \alpha | \beta \rangle$, we take all the pieces of $\alpha \wedge *\beta$ that combine to form an *n*-form and integrate this over *N*. All other terms are zero – by antisymmetry if we try to make a form of degree > n, and by definition if we try to integrate a form of degree < n over *N*.

and so is positive, as expected for our supersymmetric quantum mechanics.

A form ω is said to be **harmonic** if $\Delta \omega = 0$. A fundamental theorem of Riemannian geometry, known as Hodge's theorem, states that harmonic forms are in 1:1 correspondence with de Rham cohomology classes. That is,

$$\operatorname{Harm}^{p}(N) \cong H^{p}_{\mathrm{dR}}(N) \,,$$

where

$$H^p_{\mathrm{dR}}(N) = \frac{\ker(d:\Omega^p(N) \to \Omega^{p+1}(N))}{\operatorname{im}(d:\Omega^{p-1}(N) \to \Omega^p(N))} \,.$$

It is clear from (3.80) that harmonic forms must be closed. The role of the extra condition $d^{\dagger}\omega = 0$ is to select a representative of the equivalence class $\omega \sim \omega + d\beta$ in cohomology. It thus plays the same role as fixing the gauge freedom $A \sim A + d\lambda$ in Maxwell theory by requiring the photon to obey the Lorenz gauge condition $d^{\dagger}A = 0$. Putting all this together, we see that supersymmetric ground states of the quantum NLSM (3.67) with p fermionic excitations correspond to the de Rham cohomology $H^p(N)$ of the target space.

Homology and Cohomology

Let's give a small introduction to de Rham cohomology and its duality to homology. Since $\partial C = 0$ the pairing depends only on the cohomology class of ω , since

$$(C, \omega + d\beta) = \int_C \omega + d\beta = \int_C \omega + \int_{\partial C} \beta = (C, \omega).$$

Likewise, (C, ω) depends only on the homology class of C, for if the cycles C_1 and C_2 can be smoothly deformed into one another, then we can find a (p + 1)-cycle D whose boundary $\partial D = C_1 - C_2$. Then

$$(C_1, \omega) - (C_2, \omega) = \int_{C_1} \omega - \int_{C_2} \omega = \int_{\partial D} \omega = \int_D d\omega = 0$$

3.3.1 The Witten Index of a NLSM

On a generic compact Riemannian manifold N, it can be hard to solve the equations $d\omega = 0$, $d^{\dagger}\omega = 0$ to find explicit expressions for harmonic forms, so unlike the simple potential case considered in section 3.2.1, we cannot usually obtain an exact description of the ground states. But because the space of harmonic forms is given by a purely (differential) topological object, $H^*_{dR}(N)$, this space — and in particular its dimension — is often much easier to compute. In particular,

$$\operatorname{Tr}((-1)^{F} e^{-\beta H}) = \sum_{p=0}^{n} (-1)^{p} \dim \operatorname{Harm}^{p}(N) = \sum_{p=0}^{n} (-1)^{p} \dim H^{p}_{\mathrm{dR}}(N) = \chi(N), \quad (3.81)$$

and so the Witten index of SQM on a compact Riemannian manifold N is just the **Euler** characteristic $\chi(N)$ of N. For example, using the duality between cohomology and

homology outlined in Box 3.3, we see that a sphere S^n has $\chi(S^n) = 1 + (-1)^n$ as it is a connected manifold with the only non-trivial cycle being in dimension n, whereas a compact Riemann surface Σ_g of genus g has $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$, as it is connected manifold with 2g independent non-contractible 1-cycles and Σ_g itself forming a non-trivial 2-cycle.

We can get a further, very different expression for the Euler characteristic by considering the path integral expression

$$\operatorname{Tr}((-1)^{F} \mathrm{e}^{-\beta H}) = \int \mathrm{e}^{-S[x,\psi,\bar{\psi}]} \mathcal{D}x \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi}$$
(3.82)

for the Witten index, where we recall that this should be taken over fields living on a circle of circumference β , where the fermions (as well as $x^a(\tau)$) are periodic around the S^1 worldline. From the canonical point of view, we understood that since only ground states contributed to the Witten index, $\text{Tr}((-1)^F e^{-\beta H})$ is independent of β . To see this from the path integral perspective, note that the whole action

$$S[x,\psi,\bar{\psi}] = \bar{Q} \left[\oint g_{ab} \bar{\psi}^a \left(\mathrm{i}\dot{x}^b - \Gamma^b_{cd} \bar{\psi}^c \psi^d \right) \,\mathrm{d}\tau \right]$$
(3.83)

and hence is \overline{Q} -exact.

if we rescale $\tau \mapsto \tau' = \tau/\beta$, then the action becomes (dropping the primes)

$$S = \int_0^1 \frac{1}{2\beta} g_{ab} \dot{x}^a \dot{x}^b + \mathrm{i}g_{ab} \bar{\psi}^a \nabla_\tau \psi^b + \frac{\beta}{2} R_{abcd} \psi^a \bar{\psi}^b \psi^c \bar{\psi}^d \,\mathrm{d}\tau$$

$$= (3.84)$$

To understand this, it will be helpful to introduce an auxiliary bosonic field $B_a(\tau)$, writing the action as

$$S[x,\psi,\bar{\psi},B] = \int_0^\beta B_a \dot{x}^a + \frac{1}{2} g^{ab} B_a B_b + i g_{ab} \bar{\psi}^a \nabla_\tau \psi^b + \frac{1}{2} R_{abcd} \psi^a \bar{\psi}^b \psi^c \bar{\psi}^d \, \mathrm{d}\tau \,. \tag{3.85}$$

Eliminating B_a using its algebraic equation of motion $B_a = -g_{ab}\dot{x}^b$ leads back to our original action. In the presence of B_a , the $\bar{\epsilon}$ -supersymmetry transformations are modified to become

$$\delta x^{a} = -\bar{\epsilon}\psi^{a}, \qquad \qquad \delta\psi^{a} = 0$$

$$\delta B_{a} = \Gamma^{c}_{ab}B_{c}\psi^{b} - \frac{1}{2}R_{abcd}\bar{\psi}^{b}\psi^{c}\psi^{d}, \qquad \qquad \delta\bar{\psi}^{a} = g^{ab}\left(B_{b} - g_{de}\bar{\psi}^{e}\Gamma^{d}_{bc}\psi^{c}\right). \qquad (3.86)$$

and one can check that the action (3.85) is invariant under these transformations, and that they are nilpotent.

note that not only is the action invariant under these transformations, but is exact:

$$S = \bar{Q} \left[\int_0^\beta g_{ab} \bar{\psi}^a \left(\mathrm{i}\dot{x}^b + \frac{1}{2} g^{bc} B_c \right) \,\mathrm{d}\tau \right] \,. \tag{3.87}$$

As usual, this path integral localizes to the fixed point locus of the supersymmetry transformations

$$\delta\psi^a = \epsilon \left(i\dot{x}^a - \Gamma^a_{bc} \bar{\psi}^b \psi^c \right) , \qquad \qquad \delta\bar{\psi}^a = \bar{\epsilon} \left(-i\dot{x}^a - \Gamma^a_{bc} \bar{\psi}^b \psi^c \right) ,$$

or in other words to

3.3.2 The Atiyah–Singer Index Theorem

In this section, we assume that the target space N is even dimensional. If we restrict the previous theory to the case $\bar{\psi} = \psi$, then the action simplifies to

$$S[x,\psi] = \oint \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b + \frac{1}{2} g_{ab} \psi^a \nabla_\tau \psi^b \,\mathrm{d}\tau \,, \qquad (3.88)$$

with no curvature term since $R_{a[bcd]} = 0$ is an algebraic Bianchi identity. For this action, as the Levi-Civita connection Γ is torsion-free, the supersymmetry transformations become simply

$$\delta x^a = \epsilon \psi^a$$
 and $\delta \psi^a = -\epsilon \dot{x}^a$. (3.89)

This corresponds to taking $\epsilon = -\bar{\epsilon}$ in (3.68). This is sometimes known as $\mathcal{N} = \frac{1}{2}$ supersymmetry in d = 1. The Noether charge corresponding to these transformations is

$$Q = \frac{1}{2} g_{ab} \psi^a \dot{x}^b \,, \tag{3.90}$$

while the momenta conjugate to x^a and ψ^a are

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab} \dot{x}^a + \frac{1}{2} \psi_c \Gamma^c_{ab} \psi^b \qquad \text{and} \qquad \pi_a = \frac{\delta L}{\delta \dot{\psi}^a} = \frac{1}{2} g_{ab} \psi^b \,, \tag{3.91}$$

where in the expression for p_a we have lowered the index on ψ using the metric.

Upon canonical quantization, the commutation relations among the fields become

$$[x^{a}, p_{b}] = \mathrm{i}\delta^{a}_{\ b} \qquad \text{and} \qquad \left\{\psi^{a}, \psi^{b}\right\} = 2g^{ab} \qquad (3.92)$$

As usual, the Hilbert space of the bosonic fields can be taken to be $L^2(N, \sqrt{g} d^n x)$, and (as explained in the Box) the natural quantization of the fermionic fields ψ is to take $\mathcal{H}_{\psi} = S$, the space of Dirac spinors, with the fermions acting as the Dirac matrix γ^a . Mixing the two constructions, the Hilbert space of this system is the space $L^2(S(N), \sqrt{g} d^n x)$ of squareintegrable sections of the spin bundle on N.

The supercharge $Q = \psi^a p_a$ is then the Dirac operator $i \nabla$, and the Hamiltonian $H = Q^2 = -\nabla^2$.

Spinors in n = 2m dimensions

The Dirac γ -matrices on an even dimensional Riemannian manifold obey

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}$$

where i, j, \ldots are tangent space indices. We will take the γ -matrices to be Hermitian. Over \mathbb{C} , we can construct m = n/2 raising and lowering operators from these by introducing $\gamma_{\pm}^{I} = \frac{1}{2}(\gamma^{2I} \pm i\gamma^{2I+1})$ where $I = 1, \ldots, m$. These obey

$$\{\gamma^{I}_{+},\gamma^{J}_{-}\}=\delta^{IJ}\,,\qquad\qquad \{\gamma^{I}_{+},\gamma^{J}_{+}\}=0\,,\qquad\qquad \{\gamma^{I}_{-},\gamma^{J}_{-}\}=0$$

In particular, starting from a spinor χ that obeys $\gamma_{-}^{I}\chi = 0$ for all I, we obtain the Dirac representation of Spin(n) by acting on χ with any combination of the raising operators γ_{+}^{I} , where each γ_{+}^{I} acts at most once. This representation has dimensions $2^{n/2}$, with the generators of Spin(n) acting as $\Sigma^{ij} = -\frac{i}{4}[\gamma^{i}, \gamma^{j}]$. One may check that these Σ^{ij} indeed obey the algebra

$$\left[\Sigma^{ij}, \Sigma^{kl}\right] = i\left(\delta^{ik} \Sigma^{jl} + \delta^{jl} \Sigma^{ik} - \delta^{jk} \Sigma^{il} - \delta^{il} \Sigma^{jk}\right)$$

defining $\mathfrak{spin}_n \cong \mathfrak{so}_n$.

The Dirac representation is reducible. Because the generators Σ^{ij} are quadratic in the γ s, under a Spin(n) transformation, spinors constructed from the vacuum χ by applying an even number of γ_+ operators do not mix with those constructed by applying an odd number of γ_+ s. We define the Hermitian matrix (generalizing γ^5 in n = 4 dimensions)

$$\gamma^{n+1} = \mathbf{i}^{n/2} \gamma^1 \gamma^2 \cdots \gamma^n$$

which obeys

$$(\gamma^{n+1})^2 = \mathbf{1}\,, \qquad \quad \{\gamma^{n+1}, \gamma^i\} = 0 \qquad \text{ and } \qquad \left[\gamma^{n+1}, \Sigma^{ij}\right] = 0$$

We say a spinor has *even chirality* if it is in the +1 eigenspace of γ^{n+1} , and *odd chirality* if it is in the -1 eigenspace. The space S of spinors thus splits as $S = S^+ \oplus S^-$.

In curved space, the vielbein $e^i{}_a$ defined up to SO(n) transformations by $g_{ab} = \delta_{ij} e^i{}_a e^j{}_b$ gives an orthonormal frame at each $p \in N$. The inverse vielbein $e^a{}_i$ obeys $e^a{}_i e^i{}_b = \delta^a{}_b$ and $e^a{}_i e^j{}_a = \delta^j{}_i$. We introduce a spin connection 1-form $\omega^i{}_j$ by demanding that the vierbeins are covariantly constant

$$\nabla_a e^i{}_b = \partial_a e^i{}_b - \Gamma^c_{ab} e^i{}_c + \omega^i_{aj} e^j{}_b = 0.$$

Since $e^{j}_{\ b}$ is invertible, this fixes the spin connection to be

$$\begin{split} \omega_{aj}^{i} &= e^{b}{}_{j}\Gamma_{ab}^{c}e^{i}{}_{c} - e^{b}{}_{j}\partial_{a}e^{i}{}_{b} \\ &= \frac{1}{2}e^{b}{}_{j}\left(\partial_{a}e^{i}{}_{b} - \partial_{b}e^{i}{}_{a}\right) - \frac{1}{2}e^{bi}\left(\partial_{a}e_{bj} - \partial_{b}e_{aj}\right) - \frac{1}{2}e^{bi}e^{cj}\left(\partial_{b}e_{kc} - \partial_{c}e_{kb}\right)e^{k}{}_{a}\,. \end{split}$$

where tangent space indices (on ω and e) are raised and lowered using the metric δ_{ij} on the tangent space.

We can use the spin connection to construct a curved space Dirac operator

$$\nabla \psi = \gamma^a \left(\partial_a \psi + \omega_a^{jk} \Sigma_{jk} \psi \right)$$

which allows us to parallel transport spinors on N. Since ∇ anticommutes with γ^{n+1} , the Dirac operator maps positive chirality spinors to negative chirality spinors, and vice versa. We thus write

$$\nabla = \begin{pmatrix} 0 & D^{\dagger} \\ D & 0 \end{pmatrix}$$

with respect to the decomposition $S(N) = S^+(N) \oplus S^-(N)$. Notice that $D^2 = (D^{\dagger})^2 = 0$. We define the index of the Dirac operator to be

$$\operatorname{index}(D) = \dim \ker(D) - \dim \ker(D^{\dagger})$$

Again, we can get an alternative expression for this index by studying the path integral representation. As usual, with both x and ψ periodic, the path integral over fields on S^1 is independent of the circumference β . In the limit $\beta \to 0$ it is dominated by constant field configurations (x_0, ψ_0) . We expand around these as $x^a = x_0^a + \delta x^a$ and $\psi^a = \psi_0^a + \delta \psi^a$, where $\oint \delta x^a \, d\tau = 0 = \oint \delta \psi^a \, d\tau$. Using Riemann normal coordinates

$$g_{ab}(x) = \delta_{ab} - \frac{1}{3} R_{acbd}(x_0) \delta x^c \delta x^d + \mathcal{O}(\delta x^3)$$

$$\Gamma^a_{bc}(x) = \partial_d \Gamma^a_{bc}(x_0) \delta x^d + \mathcal{O}(\delta x^2) = -\frac{1}{3} \left(R^a_{\ bcd}(x_0) + R^a_{\ cbd}(x_0) \right) \delta x^d + \mathcal{O}(\delta x^2) \,.$$

for the metric and connection, the action becomes

$$S[x_0, \psi_0, \delta x, \delta \psi] = \oint -\frac{1}{2} \delta x_a \frac{d^2}{d\tau^2} \delta x^a + \frac{1}{2} \delta \psi_a \frac{d}{d\tau} \delta \psi^a - \frac{1}{4} R_{abcd} \psi_0^a \psi_0^b \delta x^c \frac{d\delta x^d}{d\tau} d\tau \qquad (3.93)$$

to quadratic order in the fluctuations, where indices are raised and lowered with the flat metric, and we have made use of algebraic Bianchi identities to bring the curvature in this form.

Integrating over the fluctuations gives³⁰

$$\frac{\sqrt{\det'(\delta^a_{\ b}\partial_{\tau})}}{\sqrt{\det'(-\delta^a_{\ b}\partial_{\tau}^2 + \mathcal{R}^a_{\ b}\partial_{\tau})}} = \frac{1}{\sqrt{\det'(-\delta^a_{\ b}\partial_{\tau} + \mathcal{R}^a_{\ b})}}$$

where $\mathcal{R}^a_{\ b} = R^a_{\ bcd}(x_0)\psi^c_0\psi^d_0$ is an antisymmetric matrix with entries depending on x_0 as well as ψ_0 . We can decompose the tangent space TN_{x_0} to N at x_0 into two-dimensional

³⁰The primes on the determinants are to remind us that these are just the non-zero modes. Note that the path integral over fermion non-zero modes gives a square root of the determinant of $-\partial_{\tau}$ (or the Pfaffian of this operator), rather than the determinant we would obtain from complex fermions).

spaces that are invariant under the action of $\mathcal{R}^a_{\ b}$, such that the restriction of $\mathcal{R}^a_{\ b}$ to the i^{th} such subspace takes the form

$$\mathcal{R}\big|_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}$$

for some ω_i . Let $-D_i$ denote the restriction of $-\delta^a_{\ b}\partial_\tau + \mathcal{R}^a_{\ b}$ to this subspace. Expanding $\delta x^a(\tau)$ as a Fourier series

$$\delta x^a(\tau) = \sum_{k \neq 0} \delta x^a_k \,\mathrm{e}^{2\pi \mathrm{i} k \tau} \,,$$

(and applying an \mathfrak{so}_n transformation to the δx_k^a s if necessary) we see that the eigenvalues of $-D_i$ on this subspace are $2\pi i k \pm \omega$. Therefore

$$\sqrt{\det'(-D_i)} = \prod_{k \neq 0} \sqrt{-(2\pi k)^2 - \omega_i^2} = \prod_{k=1}^\infty (2\pi k)^2 \prod_{k=1}^\infty \left(1 - \frac{\omega_i^2}{(2\pi k)^2}\right)$$

The first infinite product diverges, and can be understood via ζ -function regularization as

$$\prod_{n=1}^{\infty} (2\pi k)^2 = (4\pi^2)^{\zeta(0)} e^{-2\zeta'(0)} = 1.$$

The more interesting factor is the remaining one involving ω_i^2 . Using the infinite product expansion

$$\sinh(z) = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 n^2} \right) \,,$$

we recognize this factor as $\frac{\sinh(\omega_i/2)}{\omega_i/2}$. Combining the factors from all the n/2 orthogonal subspaces, we have finally

$$\sqrt{\det'(-\delta^a_{\ b}\partial_\tau + \mathcal{R}^a_{\ b})} = \prod_i \frac{\sinh(\omega_i/2)}{\omega_i/2} = \det\left(\frac{\sinh(\mathcal{R}/2)}{\mathcal{R}/2}\right), \quad (3.94)$$

and hence the path integral gives us the Atiyah-Singer index theorem

$$\operatorname{index}(\mathscr{D}^{+}) = \int \det\left(\frac{\sinh(\mathcal{R}_{ab}\psi_{0}^{a}\psi_{0}^{b}/2)}{\mathcal{R}_{ab}\psi_{0}^{a}\psi_{0}^{b}/2}\right) \,\mathrm{d}^{n}x_{0} \,\mathrm{d}^{n}\psi_{0} = \int_{N} \det\left(\frac{\sinh(\mathscr{R}/2)}{\mathscr{R}/2}\right) \tag{3.95}$$

for the Dirac operator on N. Here, $\mathscr{R}^{c}_{\ d} = R^{\ c}_{ab\ d} dx^{a} \wedge dx^{b}$ is the curvature 2-form, and the last integral is understood to mean we should extract the *n*-form part of the determinant as a Taylor series in \mathscr{R} .

Consider the action

$$S[x,\psi,\eta] = \oint \left[\frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b + \frac{1}{2}g_{ab}\psi^a(\nabla_\tau\psi)^b + \bar{\eta}_\alpha(D_\tau\eta)^\alpha - \frac{1}{2}\bar{\eta}_\alpha(F_{ab})^\alpha_{\ \beta}\psi^a\psi^b\eta^\beta\right]\mathrm{d}\tau \quad (3.96)$$

where the covariant derivatives are

$$(\nabla_{\tau}\psi)^{a} = \partial_{\tau}\psi^{a} + \Gamma^{a}_{bc}\dot{x}^{b}\psi^{c} (D_{\tau}\eta)^{\alpha} = \partial_{\tau}\eta^{\alpha} + A^{\alpha}_{a\beta}\dot{x}^{a}\eta^{\beta}$$

$$(3.97)$$

and the $(F_{ab})^{\alpha}_{\ \beta} = (\partial_a A_b - \partial_b A_a + [A_a, A_b])^{\alpha}_{\ \beta}$ is the curvature of D. This action is invariant under the supersymmetry transformations

$$\delta x^{a} = \epsilon \psi^{a}, \qquad \delta \psi^{a} = -\epsilon \dot{x}^{a}$$

$$\delta \eta^{\alpha} = -\epsilon \psi^{a} (A_{a})^{\alpha}{}_{\beta} \eta^{\beta}, \qquad \delta \bar{\eta}_{\beta} = -\epsilon \bar{\eta}_{\alpha} (A_{a})^{\alpha}{}_{\beta} \psi^{a} \qquad (3.98)$$

that generalize (3.69).

$$\psi^{a}(\tau) = \sum_{n \in \mathbb{Z}} \psi^{a}_{n} \mathrm{e}^{2\pi \mathrm{i} n\tau/\beta} = \psi^{a}_{0} + \sum_{n \neq 0} \psi^{a}_{n} \mathrm{e}^{2\pi \mathrm{i} n\tau/\beta}$$

we have that the path integral