4 Supersymmetric Quantum Field Theory

We now seek to generalize the supersymmetry algebra $\{Q, Q^\dagger\} = 2H$ to field theories living on $d > 1$ space-times.

We will be particularly interested in the case $d = 2$, and we take $\mathbb{R}^2$ to have coordinates (t, s) with a Minkowski metric $\eta_{\mu\nu}$ of signature $(+, -)$. As we learned in section **??**, a Dirac spinor in $d = 2$ has 2 complex components, and the two Dirac γ-matrices can be represented by

$$\gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma^s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.1}$$

acting on the spinor $\psi = (\psi_-, \psi_+)^{\mathrm{T}}$. The basic action for a free Dirac spinor in $d = 2$ is thus

$$S[\psi] = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{i}\bar{\psi}\partial\!\!\!/\psi \, \mathrm{d}t\, ds = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\mathrm{i}\bar{\psi}_-(\partial_t + \partial_s)\psi_- + \mathrm{i}\bar{\psi}_+(\partial_t - \partial_s)\psi_+ \right) \mathrm{d}t\, \mathrm{d}s\,.$$

The equations of motion $(\partial_t + \partial_s)\psi_- = 0$ and $(\partial_t - \partial_s)\psi_+ = 0$ imply that, on-shell, $\psi_-(t, s) = f(t - s)$ is right moving, whilst $\psi_+(t, s) = g(t + s)$ is left moving.

4.1 Superspace and Superfields in $d = 2$

Consider a theory on $\mathbb{R}^{2|4}$ with bosonic coordinates $(t, s) = (x^0, x^1)$ and fermionic coordinates $(\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)$, where $\bar{\theta}^\pm = (\theta^\pm)^*$. The θs are spinors in two dimensions, and the $\pm$ index tells us their chirality: under a Lorentz transform, the coordinates transform as

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \mapsto \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}, \qquad \theta^\pm \mapsto \mathrm{e}^{\pm\gamma/2}\theta^\pm, \qquad \bar{\theta}^\pm \mapsto \mathrm{e}^{\pm\gamma/2}\bar{\theta}^\pm. \tag{4.2}$$

We sometimes call $\mathbb{R}^{2|4}$ $\mathcal{N} = (2, 2)$ **superspace** in $d = 2$.

We introduce the fermionic differential operators

$$\begin{aligned} \mathcal{Q}_\pm &= \frac{\partial}{\partial\theta^\pm} + \mathrm{i}\bar{\theta}^\pm \frac{\partial}{\partial x^\pm} \\ \bar{\mathcal{Q}}_\pm &= -\frac{\partial}{\partial\bar{\theta}^\pm} - \mathrm{i}\theta^\pm \frac{\partial}{\partial x^\pm} \end{aligned} \tag{4.3}$$

where

$$\frac{\partial}{\partial x^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right).$$

The $\mathcal{Q}$s obey the anti-commutation relations

$$\left\{ \mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm \right\} = -2\mathrm{i}\partial_\pm \tag{4.4}$$

(with correlated subscripts) and all other anticommutators vanish. Since $-\mathrm{i}\partial_\pm$ are generators of translations on $\mathbb{R}^2$, and will be represented on the space of fields by $P_\pm = H \pm P$, we recognise this as our supersymmetry algebra in $1 + 1$ dimensions. The idea of these derivatives is that supersymmetry transformations act on $\mathbb{R}^{2|4}$ geometrically, with infinitesimal transformations generated by

$$\delta = \epsilon_+ \mathcal{Q}_- - \epsilon_- \mathcal{Q}_+ - \bar{\epsilon}_+ \bar{\mathcal{Q}}_- - \bar{\epsilon}_- \bar{\mathcal{Q}}_+ \,. \tag{4.5}$$

A **superfield** is simply a function on superspace. Thus, a generic superfield has an expansion

$$\begin{aligned}\mathcal{F}(x^\pm;\theta^\pm,\bar{\theta}^\pm) = f_0(x^\pm) &+ \theta^+ f_+(x^\pm) + \theta^- f_-(x^\pm) + \bar{\theta}^+ g_+(x^\pm) + \bar{\theta}^- g_-(x^\pm) \\ &+ \theta^+\theta^- h_{+-}(x^\pm) + \theta^+\bar{\theta}^+ h_{++}(x^\pm) + \cdots + \theta^+\bar{\theta}^+\theta^-\bar{\theta}^- D(x^\pm)\end{aligned} \tag{4.6}$$

containing $2^4 = 16$ components in all. Under an infinitesmal supersymmetry transformation, a generic (scalar) superfield changes as $\mathcal{F} \mapsto \mathcal{F} + \delta\mathcal{F}$ where δ is the vector field (4.5). It is messy (and usually unnecessary) to see what this does to each of the components. However, it is important to notice that the highest component $D(x^\pm)$ necessarily changes by total $x^\pm$ derivatives: any part of $\mathcal{Q}_\pm$ or $\bar{\mathcal{Q}}_\pm$ that differentiates wrt the θs or $\bar{\theta}$s cannot contribute to this term, and all other pieces in δ involve bosonic derivatives.

We'll often be interested in somewhat smaller superfields, which are constrained in some way. For this purpose, we introduce the further derivatives

$$\begin{aligned} D_\pm &= \frac{\partial}{\partial\theta^\pm} - \mathrm{i}\bar{\theta}^\pm \frac{\partial}{\partial x^\pm} \\ \bar{D}_\pm &= -\frac{\partial}{\partial\bar{\theta}^\pm} + \mathrm{i}\theta^\pm \frac{\partial}{\partial x^\pm} \end{aligned} \tag{4.7}$$

which obey $\{D_\pm, \bar{D}_\pm\} = +2\mathrm{i}\partial_\pm$ and all anticommutators between the Ds and $\mathcal{Q}$s vanish. A **chiral superfield** Φ is a superfield which is constrained to obey

$$\bar{D}_\pm \Phi = 0 \tag{4.8}$$

A chiral superfield can depend on the $\bar{\theta}$s only through the combinations $y^\pm = x^\pm - \mathrm{i}\theta^\pm\bar{\theta}^\pm$, each of which is annihilated by both $\bar{D}_+$ and $\bar{D}_-$. We can therefore expand a chiral superfield as

$$\begin{aligned}\Phi &= \phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm) \\ &= \phi(x^\pm) - \mathrm{i}\theta^+\bar{\theta}^+\partial_+\phi(x^\pm) - \mathrm{i}\theta^-\bar{\theta}^-\partial_-\phi(x^\pm) - \theta^+\bar{\theta}^+\theta^-\bar{\theta}^-\partial_+\partial_-\phi(x^\pm) \\ &\quad + \theta^+\psi_+(x^\pm) - \mathrm{i}\theta^+\theta^-\bar{\theta}^-\partial_-\psi_+(x^\pm) + \theta^-\psi_-(x^\pm) - \mathrm{i}\theta^-\theta^+\bar{\theta}^+\partial_+\psi_-(x^\pm) \\ &\qquad + \theta^+\theta^- F(x^\pm)\end{aligned} \tag{4.9}$$

where $\alpha = (+,-)$. Notice that the product $\Phi_1\Phi_2$ of two chiral superfields is again chiral, while the complex conjugate $\bar{\Phi}$ of a chiral superfield obeys $D_\pm\bar{\Phi} = 0$ and is called **antichiral**.

Under a supersymmetry transformation, $\Phi \mapsto \Phi + \delta\Phi$ with δ given by (4.5) as before. The significance of the constraint (4.8) is that, since all $\mathcal{Q}$s anticommute with $\bar{D}_\pm$,

$\bar{D}_\pm(\delta\Phi) = \delta(\bar{D}_\pm\Phi) = 0$, so chiral superfields remain chiral under supersymmetry transformations. To work out the supersymmetry transformations of the component fields in Φ, it is useful to note that

$$\begin{aligned}\mathcal{Q}_\pm = \left.\frac{\partial}{\partial\theta^\pm}\right|_{x,\bar\theta} + \mathrm{i}\bar\theta^\pm \left.\frac{\partial}{\partial x^\pm}\right|_{\theta,\bar\theta} &= \left.\frac{\partial}{\partial\theta^\pm}\right|_{y,\bar\theta} + \left.\frac{\partial y^\pm}{\partial\theta^\pm}\right|_{x,\bar\theta} \left.\frac{\partial}{\partial y^\pm}\right|_{\theta,\bar\theta} + \mathrm{i}\bar\theta^\pm \frac{\partial}{\partial y^\pm}|_{\theta,\bar\theta} \\ &= \left.\frac{\partial}{\partial\theta^\pm}\right|_{y,\bar\theta}\end{aligned}$$

and similarly

$$\bar{\mathcal{Q}}_\pm = -\left.\frac{\partial}{\partial\bar\theta^\pm}\right|_{y,\theta} - 2\mathrm{i}\theta^\pm \left.\frac{\partial}{\partial y^\pm}\right|_{\theta,\bar\theta} .$$

with the first term vanishing on a chiral superfield $\Phi(y,\theta^\pm)$ that depends on $\bar\theta^\pm$ only through $y^\pm$. Using these, one finds that the component fields of a chiral superfield transform as

$$\begin{aligned}\delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+ \\ \delta\psi_\pm &= \pm 2\mathrm{i}\bar\epsilon_\mp\partial_\pm\phi + \epsilon_\pm F \\ \delta F &= -2\mathrm{i}\bar\epsilon_+\partial_-\psi_+ - 2\mathrm{i}\bar\epsilon_-\partial_+\psi_- .\end{aligned} \tag{4.10}$$

under a general supersymmetry transformation. In particular, notice that the supersymmetry transformation of F is a sum of total derivatives. This can also be understood by noting that $\bar{\mathcal{Q}}_\pm = \bar{D}_\pm - 2\mathrm{i}\theta^\pm\partial_\pm$. Thus, when acting on a chiral superfield

4.1.1 Supersymmetric Actions in $d = 2$

We can use the observation that the D-term of a general real superfield and the F-term of a chiral superfield vary under supersymmetr only by total x derivatives to build actions that are guaranteed to be supersymmetric.

Firstly, let $K(\mathcal{F}_i;\Phi,\bar\Phi)$ be a real, smooth function of the superfields $\mathcal{F}_i$, Φ^a and $\bar\Phi^a$. Then $K(\mathcal{F}_i)$ is itself a real superfield and has an expansion ending in $\theta^+\bar\theta^+\theta^-\bar\theta^-\mathrm{d}$ for some d, built from the components of the $\mathcal{F}^a$. The integral

$$\int_{\mathbb{R}^{2|4}} K(\mathcal{F}_i)\ \mathrm{d}^2x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta \tag{4.11}$$

is guaranteed to be invariant under supersymmetry, provided the component fields in the $\mathcal{F}_i(x^\pm,\theta^\pm,\bar\theta^\pm)$ behave appropriately as $x^\pm\to\infty$. This is because the four fermionic integrals extract the highest component d in $K(\mathcal{F}_i)$, which transforms by a bosonic total derivative. This form of action is known as a **D-term**.

Similarly, if $W(\Phi^a)$ is any *holomorphic* function of chiral superfields Φ^a, then $\bar{D}_\pm W = 0$ by the chain rule, so $W(\Phi^a)$ is itself a chiral superfield. Then the integral

$$\int_{\mathbb{R}^{2|2}} W(\Phi^a)\ \mathrm{d}^2y\,\mathrm{d}^2\theta \tag{4.12}$$

is again invariant under supersymmetry, since it extracts the F-term part (coefficient of $\theta^+\theta^-$) in $W(\Phi^a)$, which again transforms as a total derivative. Correspondingly, this type of action for chiral superfields is known as an **F-term**, while the function W is known as the **superpotential**. A generic action can involve both D-terms and F-terms, and we'll see that their character is very different, both classically and in the quantum theory.

4.1.2 The Wess–Zumino Model

As an example, let's conisder the simplest case of a theory of a single chiral superfield Φ and its conjugate $\bar{\Phi}$. We choose $K(\Phi, \bar{\Phi}) = \bar{\Phi}\Phi$, and from the component expansions

$$\begin{aligned}\Phi = \phi &- \mathrm{i}\theta^+\bar{\theta}^+\partial_+\phi - \mathrm{i}\theta^-\bar{\theta}^-\partial_-\phi - \theta^+\bar{\theta}^+\theta^-\bar{\theta}^-\partial_+\partial_-\phi \\ &+ \theta^+\psi_+ - \mathrm{i}\theta^+\theta^-\bar{\theta}^-\partial_-\psi_+ + \theta^-\psi_- - \mathrm{i}\theta^-\theta^+\bar{\theta}^+\partial_+\psi_- + \theta^+\theta^-F\end{aligned}$$

and

$$\begin{aligned}\bar{\Phi} = \bar{\phi} &+ \mathrm{i}\theta^+\bar{\theta}^+\partial_+\bar{\phi} + \mathrm{i}\theta^-\bar{\theta}^-\partial_-\bar{\phi} - \theta^+\bar{\theta}^+\theta^-\bar{\theta}^-\partial_+\partial_-\bar{\phi} \\ &- \bar{\theta}^+\bar{\psi}_+ - \bar{\theta}^+\theta^-\bar{\theta}^-\partial_-\bar{\psi}_+ - \bar{\theta}^-\bar{\psi}_- - \mathrm{i}\bar{\theta}^-\theta^+\bar{\theta}^+\partial_+\bar{\psi}_- + \bar{\theta}^-\bar{\theta}^+\bar{F}\,,\end{aligned}$$

one finds that

$$\begin{aligned}\bar{\Phi}\Phi\big|_{\theta^4} = &-\bar{\phi}\partial_+\partial_-\phi + \partial_+\bar{\phi}\partial_-\phi + \partial_-\bar{\phi}\partial_+\phi - \partial_+\partial_-\bar{\phi}\,\phi \\ &+ \mathrm{i}\bar{\psi}_+\partial_-\psi_+ - \mathrm{i}\partial_-\bar{\psi}_+\,\psi_+ + \mathrm{i}\bar{\psi}_-\partial_+\psi_- - \mathrm{i}\partial_+\bar{\psi}_-\,\psi_i + |F|^2\,.\end{aligned}$$

Hence, after bosonic integrations by parts (and discarding any boundary terms) we obtain

$$\begin{aligned}S_{\mathrm{kin}} &= \int_{\mathbb{R}^{2|4}} \bar{\Phi}\Phi\,\mathrm{d}^2x\,\mathrm{d}^4\theta \\ &= \int_{\mathbb{R}^2} \left(|\partial_0\phi|^2 - |\partial_1\phi|^2 + \mathrm{i}\bar{\psi}_-\partial_+\psi_- + \mathrm{i}\bar{\psi}_+\partial_-\psi_+ + |F|^2\right)\,\mathrm{d}^2x\end{aligned} \tag{4.13}$$

which are the familiar kinetic terms for a complex scalar and Dirac fermion in 1+1 dimensions. Note that the component field F has turned out to be auxiliary – it's derivatives do not appear in the action, and the equation of motion for F will be purely algebraic.

To get an inteacting theory, we can also include a superpotential term. The θ^2 component of $W(\Phi)$ is

$$W(\Phi)\big|_{\theta^2} = W'(\phi)F - W''(\phi)\psi_+\psi_-\,.$$

For the action to be real, we include both this F-term and its complex conjugate:

$$\begin{aligned}S_{\mathrm{pot}} &= \int_{\mathbb{R}^{2|2}} W(\Phi)\,\mathrm{d}^2y\,d^2\theta + \int_{\mathbb{R}^{2|2}} \overline{W}(\bar{\Phi})\,\mathrm{d}^2\bar{y}\,\mathrm{d}^2\bar{\theta} \\ &= \int \Big(W'(\phi)F - W''(\phi)\psi_+\psi_- + \overline{W}'(\bar{\phi})\bar{F} - \overline{W}''(\bar{\phi})\bar{\psi}_-\bar{\psi}_+\Big)\,\mathrm{d}^2x\,,\end{aligned} \tag{4.14}$$

where in going to the final line we noted that integrating over all values of the bosonic coordinates $y^\pm$ (or $\bar{y}^\pm$ is the same as integrating over all values of $x^\pm$. Combining both pieces, the action we have obtained is

$$\begin{aligned}S[\Phi,\bar{\Phi}] &= \int_{\mathbb{R}^{2|4}} \bar{\Phi}\Phi\,\mathrm{d}^2x\,\mathrm{d}^4\theta + \left[\int_{\mathbb{R}^{2|2}} W(\Phi)\,\mathrm{d}^2y\,\mathrm{d}^2\theta \;+\; \mathrm{c.c}\right] \\ &= \int_{\mathbb{R}^2} \Big(|\partial_0\phi|^2 - |\partial_1\phi|^2 + \mathrm{i}\bar{\psi}_-\partial_+\psi_- + \mathrm{i}\bar{\psi}_+\partial_-\psi_+ \\ &\qquad - |W'(\phi)|^2 - W''(\phi)\psi_+\psi_- - \overline{W}''(\bar{\phi})\bar{\psi}_-\bar{\psi}_+ + |F + \overline{W}'(\bar{\phi})|^2\Big)\,\mathrm{d}^2x\end{aligned} \tag{4.15}$$

Eliminating F via its algebraic equation of motion $F = -\overline{W}'(\bar{\phi})$, we see that we have a theory of a complex scalar ϕ with potential $|W'(\phi)|^2$, interacting with a Dirac fermion coupled via the Yukawa interaction $W''(\phi)\psi_+\psi_- +$c.c.. This model is known as the **Wess–Zumino** model in 1+1 dimensions.

By construction, the action is invariant under supersymmetry transformations. One can check that the Noether currents – known as **supercurrents** – associated to these transformations are

$$\begin{aligned} G^0_\pm &= 2\partial_\pm\bar{\phi}\,\psi_\pm \mp \mathrm{i}\bar{\psi}_\mp\overline{W}'(\bar{\phi}) \\ G^1_\pm &= \mp 2\partial_\pm\bar{\phi}\,\psi_\pm - \mathrm{i}\bar{\psi}_\mp\overline{W}'(\bar{\phi}) \\ \bar{G}^0_\pm &= 2\bar{\psi}_\pm\partial_\pm\phi \pm \mathrm{i}\psi_\mp W'(\phi) \\ \bar{G}^1_\pm &= \mp 2\bar{\psi}_\pm\partial_\pm\phi \pm \mathrm{i}\psi_\mp W'(\phi) \end{aligned} \tag{4.16}$$

and the corresponding **supercharges** are

$$Q_\pm = \int G^0_\pm\,\mathrm{d}x^1 \qquad\qquad \bar{Q}_\pm = \int \bar{G}^0_\pm\,\mathrm{d}x^1\,. \tag{4.17}$$

Under SO(1,1) transformations of the worldsheet

$$Q_\pm \mapsto \mathrm{e}^{\mp\gamma/2}Q_\pm \qquad\qquad \bar{Q}_\pm \mapsto \mathrm{e}^{\mp\gamma/2}\bar{Q}_\pm\,,$$

so these charges transform as spinors

It's important that the Wess–Zumino model in fact has further global symmetries. Firstly, consider the U(1) transformation

$$\Phi(x^\pm,\theta^\pm,\bar{\theta}^\pm) \mapsto \Phi(x^\pm,\mathrm{e}^{\mp\mathrm{i}\alpha}\theta^\pm,\mathrm{e}^{\pm\mathrm{i}\alpha}\bar{\theta}^\pm)\,. \tag{4.18}$$

Since this acts oppositely on $\theta^\pm$ and on $\bar{\theta}^\pm$, it leaves both $\theta^2\bar{\theta}^2$ and θ^2 invariant. Hence $\int W(\Phi)\,\mathrm{d}^2\theta$ and $\int|\Phi|^2\,\mathrm{d}^4\theta$ are each invariant under this transformation, so the transformation is a symmetry of the action, called **axial R-symmetry**. Instead of thinking of this as acting geometrically on the coordinates of $\mathbb{R}^{2|4}$, we can equivalently take this transformation to act on the component fields as

$$\phi(x^\pm) \mapsto \phi(x^\pm)\,, \qquad\qquad \psi_\pm(x^\pm) \mapsto \mathrm{e}^{\mp\mathrm{i}\alpha}\psi_\pm(x^\pm)\,. \tag{4.19}$$

This is readily checked to be a global symmetry of the component action in (4.15), and is generated by the Noether current

$$\begin{aligned} J^0_A &= \bar{\psi}_+\psi_+ - \bar{\psi}_-\psi_- \\ J^1_A &= -\bar{\psi}_+\psi_+ - \bar{\psi}_-\psi_- \end{aligned} \tag{4.20}$$

with corresponding charge

$$F_A = \int J^0_A\,\mathrm{d}x^1\,. \tag{4.21}$$

(The subscript 'A' is for 'axial').

Next, consider the U(1) **vector** R transformations

$$\Phi(x^\pm,\theta^\pm,\bar{\theta}^\pm) \mapsto \mathrm{e}^{\mathrm{i}q\beta}\,\Phi(x^\pm,\mathrm{e}^{-\mathrm{i}\beta}\theta^\pm,\mathrm{e}^{+\mathrm{i}\beta}\bar{\theta}^\pm)$$

where we have allowed the whole superfield Φ to have some possibly non-zero $\mathrm{U}(1)_V$ charge q. Since the $\bar\theta$s transform in the opposite sense to the θs, $\theta^2\bar\theta^2$ is invariant under this transformation. Similarly, the overall phase of Φ is not seen by $|\Phi|^2$, so the Kähler potential term $\int |\Phi|^2\,\mathrm{d}^4\theta$ is invariant under this transformation, for any q. However, because both $\theta^\pm$ transform the same way, $\theta^2 \mapsto \mathrm{e}^{-2\mathrm{i}\beta}\theta^2$ under the $\mathrm{U}(1)_V$ transformation. Therefore, the superpotential term $\int W(\Phi)\,\mathrm{d}^2\theta$ will only be invariant if the superpotential itself has charge -2. This will be the case if $W(\Phi)$ is a monomial in Φ, say $W(\Phi) = c\Phi^k$, and we assign charge $q = 2/k$ to Φ. At the level of the component fields, these transformations are represented by

$$\phi(x^\pm) \mapsto \mathrm{e}^{2\mathrm{i}\beta/k}\phi(x^\pm)\,, \qquad\qquad \psi_\pm(x^\pm) \mapsto \mathrm{e}^{(2/k-1)\mathrm{i}\beta}\psi_\pm(x^\pm)\,, \tag{4.22}$$

with corresponding Noether currents

$$\begin{aligned} J_V^0 &= \frac{2\mathrm{i}}{k}(\partial_0\bar\phi\,\phi - \bar\phi,\partial_0\phi) - \left(\frac{2}{k}-1\right)(\bar\psi_+\psi_+ + \bar\psi_-\psi_-) \\ J_V^1 &= \frac{2\mathrm{i}}{k}(-\partial_1\bar\phi\,\phi + \bar\phi\,\partial_1\phi) + \left(\frac{2}{k}-1\right)(\bar\psi_+\psi_+ - \bar\psi_-\psi-) \end{aligned} \tag{4.23}$$

and charge

$$F_V = \int J_V^0\, dx^1\,. \tag{4.24}$$

4.1.3 Vacuum Moduli Spaces

We've seen that in theories of chiral superfields, the potential for the scalar fields takes the form $V(\phi,\bar\phi) = |W'(\phi)|^2$ for a single complex scalar, and more generally one has

$$V(\phi^a,\bar\phi^{\bar a}) = \sum_a \left|\frac{\partial W}{\partial\phi^a}\right|^2 \tag{4.25}$$

for several scalars. Thus the potential is a sum of non-negative terms. In any supersymmetric theory, we know the Hamiltonian is non-negative, and that a ground state $|\Omega\rangle$ is supersymmetric iff $H|\Omega\rangle = 0$. The Hamiltonian is a sum of kinetic and potential contributions, each of which is non-negative, so the lowest energy field configuration will come from a configuration ϕ_0^a that sits in the minimum of the potential throughout space. If this configuration is to have zero energy, the minimum must obey $V(\phi_0) = 0$. We conclude that a model can have a supersymmetric vacuum if and only if

$$\frac{\partial W}{\partial\phi^a}(\phi_0^a) = 0 \tag{4.26}$$

for all fields ϕ^a.

In the quantum theory, possible solutions to these equations determine the possible *vacuum expectation values* (VEVs) $\phi_0^a = \langle\phi^a\rangle$ of the scalars. If $W(\phi^a)$ is a polynomial, then since it is holomorphic, the vacuum conditions (4.26) are a system of polynomial equations over $\mathbb{C}$. This is exactly the realm of **algebraic geometry** (over $\mathbb{C}$) : given a set $\{p_1, p_2, \ldots, p_n\}$, of polynomials, we wish to understand its **zero set**

$$p_1(\phi^a) = p_2(\phi^a) = \cdots = p_n(\phi^a) = 0$$

for $\phi^a \in \mathbb{C}$. In physics, the space $\mathcal{M}$ of solutions to these equations is called the **vacuum moduli space**. When $\mathcal{M}$ is not just a set of isolated points, we sometimes say the potential has **flat directions** – it is possible to continuously change the vevs of the scalar fields whilst preserving $V(\phi_0) = 0$.

For example, if $W(\phi) = m\phi^2/2 + \lambda\phi^3/3$ then the vacuum equations are

$$W'(\phi) = m\phi + \lambda\phi^2 = 0\,,$$

which has two isolated solutions

$$\phi = 0 \qquad \text{and} \qquad \phi = -\frac{m}{\lambda}\,.$$

These are the (degenerate) local minima of the potential.

On the other hand, consider the case of three complex scalars ϕ^a with $W(\phi^a) = \phi^1\phi^2\phi^3$. Then the vacuum equations give

$$0 = \partial_1 W = \phi^2\phi^3\,, \qquad 0 = \partial_2 W = \phi^1\phi^2\,, \qquad 0 = \partial_3 W = \phi^1\phi^2\,.$$

We get a solution by taking any pair of the fields to vanish, with the other left arbitrary. Thus the vacuum moduli space is

$$\mathcal{M} = \{\phi^1 \neq 0,\ \phi^2 = \phi^3 = 0\} \cup \{\phi^2 \neq 0,\ \phi^1 = \phi^3 = 0\} \cup \{\phi^3 \neq 0,\ \phi^1 = \phi^2 = 0\}$$

with the three branches intersecting at the origin. More generally, the vacuum moduli space is the affine variety

$$\mathcal{M} = \mathbb{C}[\phi^1, \ldots, \phi^n]/(\partial_i W)$$

i.e., the ring of all polynomials in the fields, with the derivatives $\partial_a W$ as ideals.

In non-supersymmetric theories, renormalization of the potential typically changes the form of the scalar potential, with RG flow both running the values of the couplings in the original potential and (in a Wilsonian effective theory) generating an infinite series of new, higher dimension interactions. Generically, flat directions are lifted by these quantum corrections, so the form of the classical vacuum moduli space is of limited interest. However, in supersymmetric theories, the form of the superpotential is not altered by quantum corrections, so the vacuum moduli space of the classical and quantum theories agree.

4.2 Seiberg Non-Renormalization Theorems

In a generic quantum field theory, we know that the classical action receives quantum corrections, with the dynamics at energy scale $\mu \ll \Lambda_{\rm UV}$ described instead by a Wilsonian effective action that takes account of the behaviour of quantum fields with energies $\mu \leq |k| \leq \Lambda_{\rm UV}$. This effective action often has very different interactions to those in the microscopic theory we start with, with couplings that were present in the microscopic theory 'running' according to their β-functions, and new couplings to higher dimension operators emerging. This story is just as true in 1+1 dimensions as in higher dimensions.

However, supersymmetry provides protection: the F-term does not receive any quantum corrections. To understand how this works, let's consider a simple example where the superpotential

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3\,,$$

and we note that m will indeed correspond to a mass term for ϕ in the potential $V(\phi) = |W'(\phi)|^2$. This term is not homogeneous, so it appears that the $\mathrm{U}(1)_V$ symmetry is explicitly broken, leaving is just with $\mathrm{U}(1)_A$ (as well as supersymmetry). Since the $\mathrm{U}(1)_A$ transformations act trivially on the scalar field ϕ, it does not appear that we can use this to constrain the form of any quantum corrections.

However, Seiberg found a beautiful argument that shows in fact there can be no quantum corrections to the superpotential, so $W_{\text{eff}}(\Phi) = W(\Phi)$. The first step is to promote the couplings m and λ to *chiral superfields* M and Λ. (Note that since they appear in $W(\Phi, M, \Lambda)$, these fields must indeed be chiral superfields, with the complex conjugate couplings $\bar{m} \to \bar{M}$ and $\bar{\lambda} \to \bar{\Lambda}$ replaced by antichiral superfields in $\overline{W}(\bar{\Phi}, \bar{M}, \bar{\Lambda})$.) In promoting these couplings to fields, we also give them a kinetic term

$$K_M + K_\Lambda = \frac{1}{\epsilon}\left(\bar{M}M + \bar{\Lambda}\Lambda\right)$$

for some parameter ϵ. As $\epsilon \to 0$, all fluctuations (derivative terms) in M and Λ will be strongly suppressed, so the fields will take their vacuum expectation values, which we take to be m and λ.

The virtue of promoting the parameters to fields is that the new superpotential

$$W(\Phi, M, \Lambda) = \frac{M}{2}\Phi^2 + \frac{\Lambda}{3}\Phi^3$$

does have $\mathrm{U}(1)_V$ charge 2 if we assign (say) the charges $(1, 0, -1)$ to (Φ, M, Λ), respectively. In addition, there is a further global symmetry under which the superfields (Φ, M, Λ) have charges $(1, -2, -3)$ (and the θs are uncharged). Provided these symmetries are respected by the quantum theory[25], the effective superpotential $W_{\text{eff}}(\Phi, M, \Lambda)$ must also

- be a holomorphic function of the fields Φ, M and Λ,
- have $\mathrm{U}(1)_V$ charge 2 and be invariant under the global symmetry above, and
- reduce to the classical form $M\Phi^2/2 + \Lambda\Phi^3/3$ in the limit $\Lambda \to 0$ (*i.e.* the weak coupling limit of the quantum theory), and remain regular as $M \to 0$, because this is the massless limit.

The first two conditions constrain

$$W_{\text{eff}}(\Phi, M, \Lambda) = M\Phi^2\, f\left(\frac{\Lambda\Phi}{M}\right)$$

where $f(t)$ is a holomorphic function of its argument. In particular, $f(t)$ must be regular as $t \to 0$, while $f(t)/t$ must be regular as $t \to \infty$. Hence $f(t) = a + bt$ and the condition

[25]We shall consider possible anomalies later

that it reproduces the classical superpotential at small t fixes $a = 1/2$ and $b = 1/3$. Thus we see that

$$W_{\text{eff}}(\Phi, M, \Lambda) = \frac{M}{2}\Phi^2 + \frac{\Lambda}{3}\Phi^3 = W(\Phi, M, \Lambda)\,. \tag{4.27}$$

Finally, we take the limit $\epsilon \to 0$ to decouple the fields M and Λ. To see that this does not change the superpotential, we could again promote the parameters ϵ to superfields. As they only appear in the Kähler potential, we can choose them to be *real* superfields , but real superfields cannot appear in the holomorphic superpotential, so the superpotential is unchanged by decoupling the fields M and Λ. Thus we conclude that the superpotential $W(\Phi) = m\Phi^2 + \lambda\Phi^3$ receives no quantum corrections.

In fact, the superpotential receives no quantum corrections holds in general, for an arbitrary form of $W(\Phi^a)$ depending on an arbitrary (finite) number of chiral superfields Φ^a. It also holds for supersymmetric theories in higher dimensions, such as $d = 4$, for the same reasons. Unfortunately, this is not true of the Kähler potential – $K(\Phi, \bar{\Phi})$ *does* generically receive quantum corrections. In particular, since the Kähler potential governs the kinetic terms of the theory, there can be non-trivial wavefunction renormalization. Usually in QFT, one chooses to work with renormalized fields

$$\Phi_{\text{r}} = Z_{\Phi}^{\frac{1}{2}}\Phi$$

which have canonical kinetic terms. In terms of the renormalized fields, one would write

$$W(\Phi) = m\Phi^2 + \lambda\Phi^3 = m_{\text{r}}\Phi_{\text{r}}^2 + \lambda_{\text{r}}\Phi_{\text{r}}^3 = W_{\text{r}}(\Phi_{\text{r}})$$

where the couplings $m_{\text{r}} = Z_{\Phi}^{-1}m$ and $\lambda_{\text{r}} = Z_{\Phi}^{-\frac{3}{2}}\lambda$ receive corrections only from the wavefunction renormalization factors (*i.e.*, the anomalous dimensions of the fields). For this reason, working with canonically normalized fields somewhat obscures the non-renormalization theorems of supersymmetry, and it can often be worthwhile to use non canonical fields so as to keep the superpotential pristine.