5 Non-Linear Sigma Models in $d = 1 + 1$

Having explored the superpotential in the previous chapter, here I'd like to explore the role of the Kähler potential.

5.1 The Geometry of Kähler Manifolds

A **Kähler manifold** is a manifold M with three mutually compatible structures: it has a Riemannian metric g, a symplectic form ω and a complex structure J. Thus Kähler geometry is interesting because it sits at the intersection of three major branches of geometry; Riemannian, complex and symplectic. It can be viewed from all three perspectives.

You know what a Riemannian metric is. A 2-form $\omega \in \Omega^2(M)$ is **symplectic** if it is closed, $\mathrm{d}\omega = 0$, and non-degenerate in the sense that for a given vector field X and arbitrary vector field Y,

$$\omega(X, Y) = 0 \qquad \forall\, Y \text{ iff } X = 0\,. \tag{5.1}$$

Equivalently, picking a basis and writing $\omega = \omega_{ab}(x)\,\mathrm{d}x^a \wedge \mathrm{d}x^b$, the antisymmetric matrix $\omega_{ab}(x)$ is invertible at each $x \in M$. Symplectic geometry is at the heart of classical mechanics.

An **almost complex structure** J is a map $J : TM \otimes \mathbb{C} \to TM \otimes \mathbb{C}$ that obeys $J^2 = -1$, so its eigenvalues are $\pm\mathrm{i}$. At each point $p \in M$, we can use J to project tangent vectors at p into their holomorphic and antiholomorphic parts. In particular, we define the *holomorphic* and *antiholomorphic* tangent spaces at p by

$$\begin{aligned} T_p^{1,0}M &= \left\{ X \in T_pM \otimes \mathbb{C} \ : X = \frac{1}{2}(1 - \mathrm{i}J)X \right\} \\ T_p^{0,1}M &= \left\{ X \in T_pM \otimes \mathbb{C} \ : X = \frac{1}{2}(1 + \mathrm{i}J)X \right\}\,. \end{aligned} \tag{5.2}$$

For example, if $M = \mathbb{R}^2$, then $\{\partial/\partial x, \partial/\partial y\}$ form a basis of TM. We have

$$J\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y} \qquad \text{and} \qquad J\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x} \tag{5.3}$$

Thus J here rotates $\mathbb{R}^2$ anticlockwise through $\pi/2$ around the origin. Its eigenvectors are

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)$$

with eigenvalues $\pm\mathrm{i}$, respectively. Notice that the eigenvectors are *complex* linear combinations of our basis vectors.

An almost complex structure is said to be integrable, and hence be a **complex structure** iff the Lie bracket of any two holomorphic vector fields is again a holomorphic vector field. In other words, if

$$(1 + \mathrm{i}J)\,[(1 - \mathrm{i}J)X, (1 - \mathrm{i}J)Y] = 0 \tag{5.4}$$

for any $X, Y \in TM$. Taking real and imaginary parts shows that this is equivalent to the condition

$$N(X, Y) = -J^2([X, Y]) + J([JX, Y] + [X, JY]) - [JX, JY] = 0 \tag{5.5}$$

for all X, Y. Despite the presence of the Lie brackets, $N(X,Y)$ depends only on the values (not the derivatives) of X and Y, as well as J and its derivatives. It is thus a tensor on M, known as the **Nijenhuis tensor**. If our almost complex structure is integrable, then globally over M

$$TM \otimes \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M\,, \tag{5.6}$$

saying that we can always split a complex vector field into its holomorphic and antiholomorphic pieces. Similarly, we can split complex cotangent vectors $T^*M \otimes \mathbb{C}$ as

$$T^*M \otimes \mathbb{C} = T^{*(1,0)}M \oplus T^{*(0,1)}M \tag{5.7}$$

where $\bar{\alpha}(X) = 0$ for any $X \in T^{(1,0)}M$ and $\bar{\alpha} \in T^{*(0,1)}M$. It then follows that the space of complex k-forms $\Omega^k(M,\mathbb{C}) = \bigwedge^k T^*M \otimes \mathbb{C}$ also splits as

$$\Omega^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{(p,q)}(M)\,, \tag{5.8}$$

where

$$\Omega^{(p,q)}(M) = \bigwedge^p T^{*(1,0)}M \;\bigwedge^q T^{*(0,1)}M$$

is the space of (p,q)-forms on M. (If $\rho \in \Omega^{(p,q)}(M)$ then we can expand it in terms of components as

$$\rho = \rho_{a_1\ldots a_p\bar{b}_1\ldots\bar{b}_q}(z,\bar{z})\,\mathrm{d}z^{a_1} \wedge \cdots \mathrm{d}z^{a_p} \wedge \mathrm{d}\bar{z}^{\bar{b}_1} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\bar{b}_q}$$

where the functions $\rho_{a_1\ldots a_p\bar{b}_1\ldots\bar{b}_q}(z,\bar{z})$ have p holomorphic and q antiholomorphic indices.) This is known as the **Hodge decomposition**.

In fact, if a manifold M possesses any *two* of the above structures, in a way that are compatible, the third is automatic. For example, from the symplectic viewpoint, a symplectic manifold (M,ω) is *compatible* with a complex structure J if $\omega(JX,JY) = \omega(X,Y)$. This says that under the decomposition

$$\Omega^2(M) = \Omega^{2,0} \oplus \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$$

of 2-forms into their holomorphic and antiholomorphic parts, the symplectic form ω actually lies in $\Omega^{1,1}(M)$. Any such symplectic manifold also has a natural Hermitian metric, defined by

$$g(X,Y) = \omega(X,JY) \tag{5.9}$$

for all $X, Y \in TM \otimes \mathbb{C}$. To see that this is a metric, note first that since ω is non-degenerate and $J^2 = -1$, so too g is non-degenerate. We also have

$$g(Y,X) = \omega(Y,JX) = -\omega(JX,Y) = \omega(JX,J^2Y) = \omega(X,JY) = g(X,Y)\,,$$

so g is symmetric. The metric is also compatible with the complex structure, since

$$g(JY,JX) = \omega(JY,J^2X) = -\omega(JY,X) = \omega(X,JY) = g(X,Y) = g(Y,X)\,,$$

and the metric is also of type (1,1). We say such metrics are **Hermitian**. The metric is Riemannian if and only if the symplectic form is **positive**, meaning $\omega(X, JX) > 0$ for all $X \in TM$.

The Poincaré lemma in a real manifold states that locally, any closed form α is exact, so if $\mathrm{d}\alpha = 0$ then $\alpha = \mathrm{d}\beta$ at least in some open region $U \subset M$. On a complex manifold, we can split $\mathrm{d} = \partial + \bar{\partial}$ into the exterior derivative in the holomorphic and antiholomorphic directions (*e.g.*, on $\mathbb{R}^2 = \mathbb{C}$ we have

$$\mathrm{d} = \mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y} = \mathrm{d}z\frac{\partial}{\partial z} + \partial\bar{z}\frac{\partial}{\partial \bar{z}} = \partial + \bar{\partial}\,,$$

where $\mathrm{d}z = \mathrm{d}x + \mathrm{i}\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - \mathrm{i}\mathrm{d}y$.) We then have

$$0 = \mathrm{d}^2 = \partial^2 + (\partial\bar{\partial} + \bar{\partial}\partial) + \bar{\partial}^2\,.$$

But since $\partial^2 : \Omega^{p,q} \to \Omega^{p+2,q}$, while $\partial\bar{\partial} + \bar{\partial}\partial : \Omega^{p,q} \to \Omega^{p+1,q+1}$ and $\bar{\partial}^2 : \Omega^{p,q} \to \Omega^{p,q+2}$, acting on any form of fixed type (p, q), the image of these three operators lies in different vector spaces. Thus the only way for them to sum to zero is to be zero separately.

In particular, since $0 = \mathrm{d}\omega = \partial\omega + \bar{\partial}\omega$, we must have $\partial\omega = 0$ and $\bar{\partial}\omega = 0$ separately, as these are $(3, 0)$- and $(2, 1)$-forms. Combined with the Poincaré lemma, this implies that

$$\omega = \mathrm{i}\,\partial\bar{\partial}K \tag{5.10}$$

for some function K, at least locally on the Kähler manifold. (The factor of i ensures that ω is real if K is a real function. Notice that K is defined upto the transformations

$$K(z, \bar{z}) \mapsto K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}) \tag{5.11}$$

where f is holomorphic. The function K is known as the Kähler potential.

The relation (5.9) implies that, in terms of local holomorphic coordinates $(z^a, \bar{z}^{\bar{a}})$ on M, the metric has components $g_{a\bar{b}} = \partial_a\partial_{\bar{b}}K$. For example $\mathbb{C}^n$ can be treated as a Kähler manifold with $K(z, \bar{z}) = \sum_a |z^a|^2$. The resulting metric is just the flat metric $g = \sum_a \delta_{a\bar{a}}\,\mathrm{d}\bar{z}^{\bar{a}}\mathrm{d}z^a = \sum_a (\mathrm{d}x^a)^2 + (\mathrm{d}y^a)^2$ on $\mathbb{R}^{2n}$, and $\omega = \mathrm{i}\sum_a \delta_{a\bar{a}}\mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{a}} = \sum_i \mathrm{d}x^i \wedge \mathrm{d}y^i$ the usual symplectic form. As a second example, $\mathbb{CP}^n$ is a Kähler manifold with $K = \ln\left(1 + \sum_{a=1}^n |z^a|^2\right)$ on the coordinate patch $\mathbb{C}^n \subset \mathbb{CP}^n$ (this patch covers the complement of a hyperplane in $\mathbb{CP}^n$). The resulting metric is known as the **Fubini-Study metric** on $\mathbb{CP}^n$.

On any Kähler manifold, since $g_{a\bar{b}} = \partial_a\partial_{\bar{b}}K$, it's straightforward to check that the only non-vanishing components of the Levi–Civita connection are

$$\Gamma^a_{bc} = g^{a\bar{d}}\partial_b g_{c\bar{d}} \qquad \text{and} \qquad \Gamma^{\bar{a}}_{\bar{b}\bar{c}} = g^{\bar{a}d}\partial_{\bar{b}} g_{d\bar{c}}\,, \tag{5.12}$$

with all Christoffel symbols that involve both holomorphic & antiholomorphic indices vanishing.

5.2 Supersymmetric NLSM on Kähler Manifolds

We can now understand why Kähler geometry is the natural arena for (2,2) supersymmetry in $d = 2$. (It is also the natural arena for $\mathcal{N} = 1$ supersymmetry in $d = 4$.) If Φ^a are chiral superfields, then the kinetic terms

$$\int_{\mathbb{R}^{2|4}} K(\Phi, \bar{\Phi})\, \mathrm{d}^4\theta\, \mathrm{d}^2x$$

depend on a real function K that is defined up to the transformations

$$K(\Phi, \bar{\Phi}) \mapsto K(\Phi, \bar{\Phi}) + f(\Phi) + \bar{f}(\bar{\Phi}),$$

since these functions of chiral superfields will not survive the integration $\mathrm{d}^4\theta\, \mathrm{d}^2x$ (provided the fields decay sufficiently rapidly at $|x| \to \infty$ on the worldsheet). Performing the fermionic integrals, one finds

$$\begin{aligned} S_{\text{kin}}[\Phi^a, \bar{\Phi}^{\bar{a}}] = \int_{\mathbb{R}^2} \Big[&- g_{a\bar{b}}\, \partial^\mu \phi^a \partial_\mu \bar{\phi}^{\bar{b}} + \mathrm{i} g_{a\bar{b}}\, \bar{\psi}_+^{\bar{b}} \nabla_- \psi_+^a + \mathrm{i} g_{a\bar{b}}\, \bar{\psi}_-^{\bar{b}} \nabla_+ \psi_-^a \\ &+ R_{a\bar{b}c\bar{d}} \psi_+^a \psi_-^c \bar{\psi}_-^{\bar{b}} \bar{\psi}_+^{\bar{d}} + g_{a\bar{b}} \left(F^a - \Gamma^a_{cd} \psi_+^c \psi_-^d \right) \left(\bar{F}^{\bar{b}} - \Gamma^{\bar{b}}_{\bar{c}\bar{d}} \bar{\psi}_-^{\bar{c}} \bar{\psi}_+^{\bar{d}} \right) \Big]\, \mathrm{d}^2 x\,, \end{aligned} \tag{5.13}$$

where, as usual,

$$\nabla_\mu \psi_\pm^a = \partial_\mu \psi_\pm^a + \Gamma^a_{bc}\, \partial_\mu \phi^b\, \psi_\pm^c$$

in terms of the connection coefficients Γ^a_{bc}, and $R_{a\bar{b}c\bar{d}}$ is the (Riemann) curvature of Γ. The kinetic terms are non–singular provided the metric is positive definite. Combining this with a superpotential term

$$\frac{1}{2} \int_{\mathbb{R}^{2|2}} W(\Phi)\, \mathrm{d}^2\theta\, \mathrm{d}^2x \ + \ \text{c.c.} = \frac{1}{2} \int_{\mathbb{R}^2} \left[F^a \frac{\partial W}{\partial \phi^a} - \psi_+^a \psi_-^b \frac{\partial^2 W}{\partial \phi^a \partial \phi^b} \right] \mathrm{d}^2 x \ + \ \text{c.c.} \tag{5.14}$$

and eliminating the auxiliary field F via its equation of motion

$$F^a = \Gamma^a_{cd} \psi_+^c \psi_-^d - \frac{1}{2} g^{a\bar{b}} \partial_{\bar{b}} \overline{W}$$

allows us to write the full non-linear sigma model action as

$$\begin{aligned} S[\phi, \psi] = \int_{\mathbb{R}^2} &-g_{a\bar{b}}\, \partial^\mu \phi^a \partial_\mu \bar{\phi}^{\bar{b}} + \mathrm{i} g_{a\bar{b}}\, \bar{\psi}_+^{\bar{b}} \nabla_- \psi_+^a + \mathrm{i} g_{a\bar{b}}\, \bar{\psi}_-^{\bar{b}} \nabla_+ \psi_-^a + R_{a\bar{b}c\bar{d}} \psi_+^a \psi_-^c \bar{\psi}_-^{\bar{b}} \bar{\psi}_+^{\bar{d}} \\ &- \frac{1}{4} g^{a\bar{b}} \partial_a W \partial_{\bar{b}} \overline{W} - \frac{1}{2} \nabla_a \partial_b W \psi_+^a \psi_-^b - \frac{1}{2} \nabla_{\bar{a}} \partial_{\bar{b}} \overline{W} \bar{\psi}_-^{\bar{a}} \bar{\psi}_+^{\bar{b}}\ \mathrm{d}^2 x\,. \end{aligned} \tag{5.15}$$

Note that this action is invariant under holomorphic changes of coordinates on the target space (at least on the coordinate patch where the metric $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$). It's also invariant under the supersymmetry transformations

$$\begin{aligned} \delta\phi^a &= \epsilon_+ \psi_-^a - \epsilon_- \psi_+^a\,, & \delta\bar{\phi}^{\bar{a}} &= -\bar{\epsilon}_+ \bar{\psi}_-^{\bar{a}} + \bar{\epsilon}_- \bar{\psi}_+^{\bar{a}} \\ \delta\psi_\pm^a &= \pm 2\mathrm{i} \bar{\epsilon}_\mp \partial_\pm \phi^a + \epsilon_\pm F^a\,, & \delta\bar{\psi}_\pm^{\bar{a}} &= \mp 2\mathrm{i} \epsilon_\mp \partial_\pm \bar{\phi}^{\bar{a}} + \bar{\epsilon}_\pm \bar{F}^{\bar{a}} \end{aligned} \tag{5.16}$$

by construction.

Now let's set $W = 0$ to consider the model in the absence of a superpotential. Then, provided we make the obvious replacements $\mathrm{d}^2x \to \sqrt{h}\,\mathrm{d}^2x$,

$$g_{a\bar{b}}\partial^\mu\phi^a\partial_\mu\bar{\phi}^{\bar{b}} \to g_{a\bar{b}}h^{\mu\nu}\partial_\mu\phi^a\partial_\nu\bar{\phi}^{\bar{b}}$$

and include a worldsheet spin connection[26] in the fermion kinetic term, it makes sense to consider this theory not just on $\mathbb{R}^2$, but on a general Riemann surface Σ[27] Thus, given complex coordinates $(\phi^a, \bar{\phi}^{\bar{a}})$ on a patch U of a Kähler manifold N, we can construct a manifestly supersymmetric sigma model describing maps $\phi : \Sigma \to U$ and glue the models together at the level of the path integral to construct a model to describe maps $\Sigma \to N$. From this perspective, the worldsheet fermions should be thought of as sections

$$\psi_\pm \in \Gamma(\Sigma, \phi^*T^{(1,0)}N \otimes S_\pm) \qquad\qquad \bar{\psi}_\pm \in \Gamma(\Sigma, \phi^*T^{(0,1)}N \otimes S_\pm)\,,$$

where $S_\pm$ are the left- and right- spin bundles on Σ.

5.2.1 Anomalies in R-symmetries

The global R-symmetries acting as

$$U(1)_V : \Phi(x^\pm, \theta^\pm, \bar{\theta}^\pm) \mapsto \mathrm{e}^{2\mathrm{i}q\beta}\,\Phi(x^\pm, \mathrm{e}^{-\mathrm{i}\beta}\theta^\pm, \mathrm{e}^{\mathrm{i}\beta}\bar{\theta}^\pm)$$
$$U(1)_A : \Phi(^\pm, \theta^\pm, \bar{\theta}^\pm) \mapsto \Phi(x^\pm, \mathrm{e}^{\mp\mathrm{i}\alpha}\theta^\pm, \mathrm{e}^{\pm\mathrm{i}\alpha}\bar{\theta}^\pm)$$

of the classical models (with quasi-homogeneous superpotentials) may or may not be preserved at the quantum level. This is because although these transformations do preserve the classical action, $\delta S = 0$, they may not also preserve the path integral measure, $\delta([D\Phi]) \neq 0$. In general, when a symmetry that is valid in the classical theory is broken by the quantum theory, we say the symmetry is **anomalous**.

Let's investigate whether the $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ symmetries above can be anomalous. To begin, we'll consider a simpler example of a single, massless, charged Dirac fermion ψ living on a worldsheet torus $T^2 = \mathbb{C}/\Lambda$. We let the gauge field be A and take the usual action

$$S = \mathrm{i}\int_{T^2} \bar{\psi}_+ D_z \psi_+ + \bar{\psi}_- \bar{D}_{\bar{z}}\psi_-\; d^2z\,, \tag{5.17}$$

where $z = x + \mathrm{i}y$ is a local coordinate on T^2, and $D_z = \partial_z + A_z$ while $\bar{D}_{\bar{z}} = \partial_{\bar{z}} + A_{\bar{z}}$.

This action is invariant under global transformations

$$\psi_\pm \to \mathrm{e}^{-\mathrm{i}(\alpha\pm\beta)}\psi_\pm \qquad\qquad \bar{\psi}_\pm \to \mathrm{e}^{+\mathrm{i}(\alpha\pm\beta)}\bar{\psi}_\pm \qquad\qquad A \to A \tag{5.18}$$

[26]In two dimensions, with Euclidean signature, the Lorentz group is $\mathrm{SO}(2) \cong \mathrm{U}(1)$, so this spin connection is just an Abelian connection, with spinors changing just by a phase under worldsheet Lorentz transformations.

[27]Recall that a **Riemann surface** is a 1-dimensional complex manifold. Thought of as a real 2-manifold, it's metric can always be put in the form

$$h = h_{\mu\nu}(x)\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{e}^{2\Omega}\big((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\big)$$

in any coordinate patch $U \subset \Sigma$. Thus, locally, Riemann surfaces are always conformally flat.

of the fermions. However, suppose

$$k = \int_{T^2} c_1(E) = \frac{\mathrm{i}}{2\pi}\int_{T^2} F > 0$$

so that the gauge field A has non-zero instanton number. Then, by the index theorem

$$\dim(\ker \bar{D}_{\bar{z}}) - \dim(\ker D_z) = \int_{T^2} c_1(E) > 0$$

so that the number of ψ_+ zero modes is k larger than the number of ψ_- zero modes. Now, complex conjugation both swaps the sign of the $U(1)$ charge of a particle, and exchanges $z \leftrightarrow \bar{z}$. Therefore $(\psi_+)^* = \bar{\psi}_-$, and likewise $(\psi_-)^* = \bar{\psi}_+$. Thus complex conjugation implies that the number of ψ_- zero modes is the same as the number of $\bar{\psi}_+$ zero modes, and likewise for ψ_+ and $\bar{\psi}_i$. Because the path integral measure $[D\psi]$ is an instruction to integrate over all modes of the fields $\psi_\pm$ and $\bar{\psi}_\pm$, it transforms as

$$[D\psi] \to \mathrm{e}^{2\mathrm{i}k\beta}\,[D\psi]$$

under the U(1) transformations (5.18), with the transformations of all the (infinitely many) non-zero modes cancelling out. Hence the vector U(1) transformation $\psi_\pm \to \mathrm{e}^{-\mathrm{i}\alpha}\psi_\pm$ is not anomalous and remains a symmetry of the quantum theory, whilst the axial U(1) $\psi_\pm \to \mathrm{e}^{\mp\mathrm{i}\beta}\psi_\pm$ is broken quantum mechanically in the presence of an instanton with $k \neq 0$.

If $k \neq 0$ then at least some of the fermion fields must have zero modes. By definition, these zero modes do not contribute to the action, and so the partition function (on a worldsheet T^2) vanishes by Grassmann integration over these modes in the path integral. To get something non-vanishing, we must compute a correlation function with enough fermions inserted to saturate the zero-mode integral. Although our index computation above only told us the *difference* between the number of ψ_+ and ψ_- zero modes, It turns out that, for a generic U(1) gauge field of instanton number k on T^2, there will be *exactly* k zero modes of ψ_+ (and $\bar{\psi}_-$) and no ψ_- (or $\bar{\psi}_+$) zero-modes. In this generic case, the correlator

$$\langle \psi_+(z_1)\psi_+(z_2)\cdots\psi_+(z_k)\bar{\psi}_-(w_1)\bar{\psi}_-(w_2)\cdots\bar{\psi}_-(w_k)\rangle$$

is non-vanishing, with the fermion insertions at points $\{z_1,\ldots,z_k,w_1,\ldots,w_k\} \in T^2$ exactly saturating the zero-mode integrals. This is again a common theme in QFT: anomalies in global symmetries impose **selection rules** on which correlation functions can be non-zero. These selection rules are really just a modification of the usual selection rules attendant to any symmetry via Ward identities, but take into account the non-trivial transformation of the path integral measure (perhaps in the presence of a gauge field instanton).

Now let's return to the sigma model on a Kähler manifold M. The fermion terms in the action are the kinetic term

$$\mathrm{i}\int_{T^2} g_{a\bar{b}}\bar{\psi}^{\bar{b}}_-\nabla_z\psi^a_- + g_{a\bar{b}}\bar{\psi}^{\bar{b}}_+\bar{\nabla}_{\bar{z}}\psi^a_-\ \mathrm{d}^2z\,,$$

together with the 4-fermion interaction $R_{a\bar{b}c\bar{d}}\psi^a_+\psi^c_-\bar{\psi}^{\bar{b}}_-\bar{\psi}^{\bar{d}}_+$. For the purposes of computing anomalies, we can ignore this 4-fermi interaction, because we can always choose to write the path integral measure in terms of integrals over the eigenmodes of the Dirac operator $\gamma^\mu \nabla_\mu$ coupled to the pullback of the holomorphic[28] tangent bundle $\phi^* T^{(1,0)} M$. In this case, the relevant instanton number is given by

$$k = \int_{T^2} c_1(\phi^* T^{(1,0)} M) = \frac{\mathrm{i}}{2\pi} \int_{T^2} \mathrm{tr}(R)$$

where R is the curvature 2-form[29] of the (pullback to T^2 of) the Levi-Civita connection on M.

Just as in the Abelian case above, whenever $k \neq 0$ there is a mismatch between the numbers of ψ^a_+ and ψ^a_- zero-modes, leading to an anomaly in the $\mathrm{U}(1)_A$ symmetry of the sigma model. An important class of target spaces for which this anomaly vanishes are **Calabi-Yau manifolds**, which are Kähler manifolds with $\mathrm{tr}(R) = 0$.

5.2.2 The β-function of a NLSM

On a two-dimensional worldsheet, scalar fields ϕ^a have mass dimension zero, whilst all fermions ψ have mass dimension $\frac{1}{2}$. Thus (when $W = 0$) the NLSM

$$S[\phi, \psi] = \int_{T^2} -g_{a\bar{b}}\, h^{\mu\nu} \partial_\mu \phi^a \partial_\nu \bar{\phi}^{\bar{b}} + \mathrm{i} g_{a\bar{b}}\, \bar{\psi}^{\bar{b}} \gamma^\mu \nabla_\mu \psi^a_+ + R_{a\bar{b}c\bar{d}}\, \psi^a_+ \psi^c_- \bar{\psi}^{\bar{b}}_- \bar{\psi}^{\bar{d}}_+ \; \sqrt{h}\, d^2 x$$

is invariant under scale transformations

$$h_{\mu\nu} \to \lambda^2 h_\mu\,, \qquad \gamma^\mu \to \lambda^{-1} \gamma^\mu\,, \qquad \phi^a \to \phi^a\,, \qquad \psi^a_\pm \to \lambda^{-\frac{1}{2}} \psi^a_\pm \tag{5.19}$$

for $\lambda \in \mathbb{R}_{>0}$. (Note that the scaling behaviour of the Dirac matrices γ^μ is fixed by $\{\gamma^\mu, \gamma^\nu\} = 2h^{\mu\nu}$.)

Whether or not this scale invariance is preserved in the quantum theory will again turn out to depend on $c_1(T^{(1,0)} M)$, just as for the $\mathrm{U}(1)_A$ symmetry above. We saw above that if $k = \frac{\mathrm{i}}{2\pi} \int_{T^2} \mathrm{tr}(R) \neq 0$, then the correlators

$$f(g, h) = \langle (\psi_-)^k (\bar{\psi}_+)^k \rangle_h$$

(with the target space indices chosen in some way, and with the fields inserted at various points on the worldsheet T^2) would be able to saturate the $\mathrm{U}(1)_A$ anomaly. We call this correlator $f(g, h)$ to emphasize that it is computed with target space metric $g_{a\bar{b}}$ and worldsheet metric $h_{\mu\nu}$.

Since ψ^a_- and $\bar{\psi}^{\bar{b}}_+$ are each invariant under both the $\overline{Q}_+$ and Q_- supersymmetry transformations[30], we expect that this correlator is amenable to localization. In particular, the

[28] Note that the connection ∇ here always acts on a fermion with a holomorphic target space index.

[29] In general, the curvature of a connection should be thought of as a 2-form (the antisymmetric field-strength tensor) whose entries take values in endomorphisms of the gauge bundle (so can themselves be thought of as matrices). This perspective should be familiar in non-Abelian gauge theories, and is equally true of the curvature of a tangent bundle.

[30] Recall that these transformations are associated with the parameters $\bar{\epsilon}_-$ and ϵ_+, respectively, in (5.16). One reason for choosing the worldsheet to be a torus is that, unlike on a generic genus g Riemann surface, these parameters can be treated as constants globally. In particular, we choose fermions to be periodic around each of the two non-contractible 1-cycles of T^2.

correlator $f(g,h)$ turns out to be independent of the scalings (5.19), so that

$$f(g,h) = f(g,\lambda^2 h)\lambda^k \tag{5.20a}$$

Localisation also implies that the correlator receives contributions only from holomorphic maps $\phi : T^2 \to M$, and then

$$f(g,h) = n_h \,\mathrm{e}^{-A_g}\,, \tag{5.20b}$$

where $A_g = \int_{T^2} g_{a\bar{b}}\, h^{\mu\nu}\partial_\mu\bar{\phi}^{\bar{b}}\partial_\nu\phi^a\ \sqrt{h}\,d^2x$ is the area of the image of the worldsheet in M, and n_h is some number that depends on the worldsheet metric h, but not on the details of the map ϕ or target space metric g. Combining these two properties shows that

$$f(g,h) = f(g,\lambda^2 h)\lambda^k = n_{\lambda^2 h}\mathrm{e}^{-(A_g - k\ln\lambda} = f(\lambda^2 h, g')\,, \tag{5.21}$$

where g' is a new metric on the target space such that $A_{g'} = A_g - k\ln\lambda$. In other words, if we rescale the worldsheet metric as $h \to \lambda^2 h$ (as in the renormalization group) then we must also change the target space metric from $g \to g'$ in order for the correlation function to remain the same.

Furthermore, for the flat metric $h = \delta$, we have

$$\begin{aligned} A_g &= \int_{T^2} g_{a\bar{b}}\, h^{\mu\nu}\partial_\mu\bar{\phi}^{\bar{b}}\partial_\nu\phi^a\ \sqrt{h}\,d^2x \\ &= \mathrm{i}\int_{T^2} g_{a\bar{b}}\left(\partial_z\bar{\phi}^{\bar{b}}\partial_{\bar{z}}\phi^a + \partial_{\bar{z}}\bar{\phi}^{\bar{b}}\partial_z\phi^a\right)\,\mathrm{d}z\wedge\mathrm{d}\bar{z} \\ &= 2\mathrm{i}\int_{T^2} g_{a\bar{b}}\partial_z\bar{\phi}^{\bar{b}}\partial_{\bar{z}}\phi^a\mathrm{d}z\wedge\mathrm{d}\bar{z} + \int_{T^2}\phi^*\omega \end{aligned}$$

in terms of the pullback of the Kähler form on M. For holomorphic maps, the first term in the last line vanishes, so we have

$$A_g = \int_{T^2}\phi^*\omega \tag{5.22}$$

and thus the effect of the scaling transformation $h \to \lambda^2 h$ is to change the Kähler class of the target as

$$[\omega] \to [\omega'] = [\omega] - \frac{\mathrm{i}}{2\pi}\ln\lambda\,[\mathrm{tr}(R)] \tag{5.23}$$

using our earlier expression for k. (The square brackets here indicate that it is the *cohomology class* of ω must change: adding an exact form on to ω will have no effect in the integral (5.22).) Again we see that if M is Calabi-Yau, so that $\mathrm{tr}(R) = 0$, the model is invariant under worldsheet scaling transformations even at the quantum level.

These scaling transformations are, of course, intimately connected to the behaviour of the theory under renormalization group flow. Let's first consider a purely bosonic NLSM describing maps $\phi : \Sigma \to M$ where M is any Riemannian manifold. The action is

$$S[\phi] = \frac{1}{2}\int g_{ij}\, h^{\mu\nu}\,\partial_\mu\phi^i\,\partial_\nu\phi^j\,\sqrt{h}\,\mathrm{d}^2x$$

as always, and is classically invariant under $h_{\mu\nu} \to \lambda^2 h_{\mu\nu}$ with ϕ unchanged. We choose Riemann normal coordinates centred on a point $\phi_0 \in M$ and expand the target metric as

$$g_{ij}(\phi_0 + \xi) = \delta_{ij} - \frac{1}{3} R_{ikjl}(\phi_0)\, \xi^k \xi^l + \mathcal{O}(\xi^3)$$

in terms of flucuations ξ around this point. Then, to lowest order in perturbation theory around constant maps $\Sigma \to \phi_0$, the action becomes

$$S[\xi] = \frac{1}{2} \int_\Sigma \left(\partial^\mu \xi^i \partial_\mu \xi_i - \frac{1}{3} R_{ikjl}(\phi_0) \xi^k \xi^l \partial^\mu \xi^i \partial_\mu \xi^j + \mathcal{O}(\xi^5) \right) , \mathrm{d}^2 x \, . \tag{5.24}$$

for a flat worldsheet metric. The first term gives the usual $1/k^2$ propagator in $2d$ momentum space, while the second term is a valency-4 vertex. We have the 2-point function

$$\langle \xi^i(x) \xi^j(y) \rangle^{1-\mathrm{loop}} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{\mathrm{e}^{\mathrm{i}k\cdot(x-y)}}{k^2} \left[\delta^{ij} + \frac{1}{3} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{1}{p^2} R^{ij}(\phi_0) \right]$$

to 1-loop accuracy, where the first term (proportional to δ^{ij}) is just the classical contribution from a single propagator joining the two ξ insertions, while the second term is a 1-loop correction arising from a further insertion of the 4-ξ vertex. As usual, the loop integral is divergent. We can regularize by introducing UV and IR cutoffs in momentum space. Then

$$\int_{leq|p|\leq\Lambda} \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{1}{p^2} = \frac{1}{2\pi} \int_\mu^\Lambda \frac{\mathrm{d}p}{p} = \frac{1}{2\pi} \ln \frac{\Lambda}{\mu}$$

Comparing this to the classical propagator, we see we can absorb the UV divergence $\sim \ln \Lambda$ if we include counterterms so that the metric is

$$g_{ij} + \delta g_{ij}(\Lambda) = \delta_{ij} - \frac{1}{6\pi} R_{ij}(\phi_0) \ln \frac{\Lambda}{\lambda}$$

in terms of some arbitrary renormalization scale λ. The renormalized metric is then

$$g_{ij}(\lambda, \mu) = \delta_{ij} + \frac{1}{6\pi} R_{ij}(\phi_0) \ln \frac{\lambda}{\mu} \tag{5.25}$$

and remains finite (at 1-loop) as the UV cutoff $\Lambda \to \infty$. This renormalized metric runs with scale λ, with β-function given by

$$\beta_{ij} = \lambda \frac{dg_{ij}}{d\lambda} = \frac{1}{6\pi} R_{ij}$$

and so is proportional to the Ricci curvature. We thus have three possible cases:

- If $R_{ij} > 0$ then the model is asymptotically free. The curvature of the target space is less and less important as we focus in on smaller and smaller distances on the worldsheet, and in the limit we have a free theory. Thus this asymptotically free theory makes sense.

- If $R_{ij} < 0$ then the theory is valid at most as an effective theory, and becomes increasingly weakly coupled in the IR.

- Finally, if $R_{ij} = 0$ the theory is scale (and in fact conformally) invariant, at least to 1-loop order. Of course, the β-functions may well receive corrections at higher loops. Having $R_{ij} = 0$ means the target space metric is Ricci-flat, and hence solves the vacuum Einstein equations.

The same calculations can be performed for a supersymmetric NLSM, and again one finds that at 1-loop the metric is renormalized as $g_{a\bar{b}} \to g_{a\bar{b}} + cR_{a\bar{b}}$ for some constant c. For a Kähler manifold, $\mathrm{tr}(R) \propto R_{a\bar{b}} \mathrm{d}\phi^a \wedge \mathrm{d}\bar{\phi}^{\bar{b}}$ and in particular vanishing Ricci tensor $R_{a\bar{b}} = 0$ implies vanishing first Chern class, so that NLSMs with Calabi-Yau targets are fixed under RG flow, at least at 1-loop. While there are indeed higher-loop corrections to this (in fact, for the (2,2) supersymmetric model above, they start only at 4-loops in worldsheet perturbation theory!), in the supersymmetric theory our exact localization calculation above shows that the *Kähler class* $[\omega]$ of the metric lways remains unchanged on a Calabi-Yau target.