Ambitwistors and the Scattering Equations

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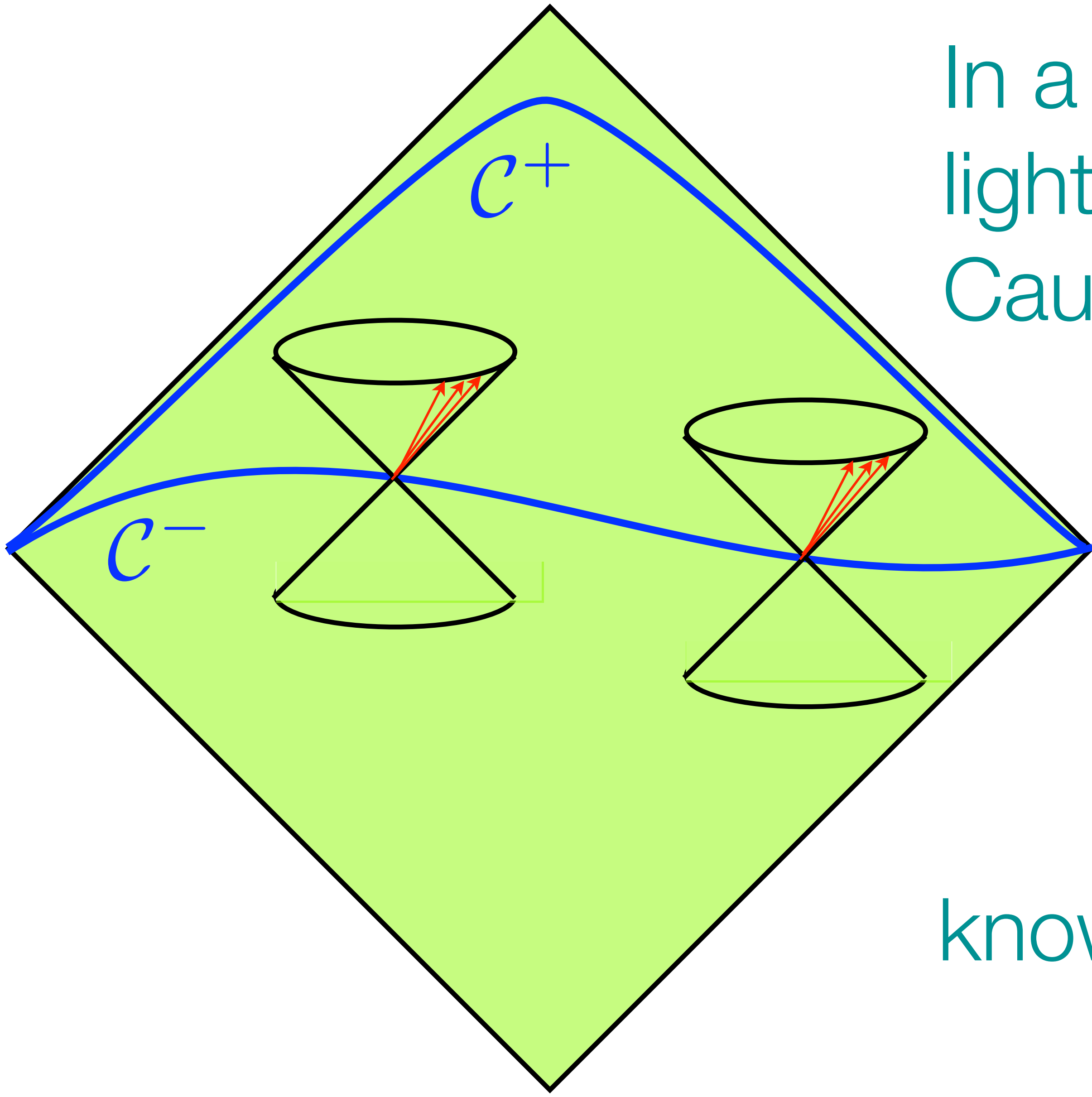
based on work with L. Mason
and with T. Adamo & E. Casali

- Spaces of complex null geodesics
- Scattering equations and CHY formulae

- Chiral ambitwistor strings
- Curved backgrounds
- Scattering and null infinity

- Conclusions

Spaces of complex null geodesics



In a Lorentzian manifold, the space of light rays is a sphere bundle over any Cauchy surface $\mathcal{C}$:

$$\begin{array}{ccc} S^{d-2} & \longrightarrow & \mathbb{PA}_{\mathbb{R}} \\ & & \downarrow \\ & & \mathcal{C}^{d-1} \end{array}$$

known as ***ambitwistor space***

- exists for arbitrary (geodesically convex) space-time $(M_{\mathbb{R}}, g_{\mathbb{R}})$
- space-time point x corresponds to a quadric $Q_x \subset \mathbb{PA}_{\mathbb{R}}$, with $\dim(Q_x) = d - 2$ (generators of the null cone)

We'll mostly work with complex (M, g) where metric is hol^phc^ in the sense that Levi-Civita $\nabla : \Gamma(T_M^{1,0}) \to \Gamma(T_M^{1,0} \otimes T_M^{*1,0})$

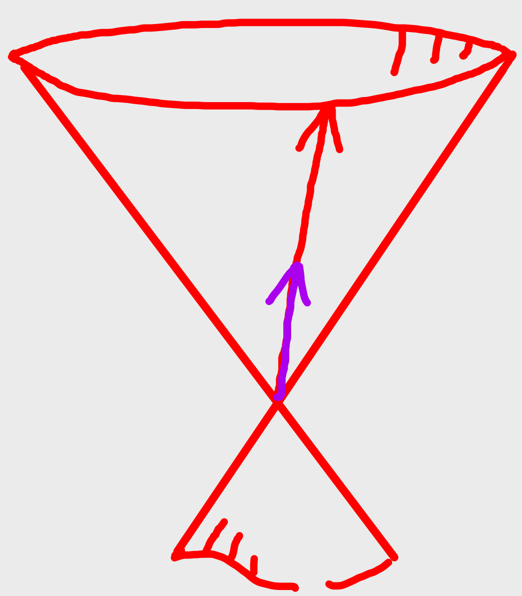
- $\mathbb{C}$-null geodesic: holomorphic integral curve of v where $g(v,v) = 0$ and $\nabla_v(v) = 0$

The space of complex null geodesics also has a useful symplectic description:

$$\mathbb{A} = T^*M//\{D\}$$

$D = P \cdot \nabla$ is geodesic spray

Hamiltonian $H = g^{\mu\nu}(X) P_\mu P_\nu$



- Quotienting by scale of P yields $\mathbb{PA}$ as a (non-degenerate) contact manifold, dim $2d - 3$
- $\theta = p_\mu dx^\mu$ descends to $\mathbb{A}$ and represents contact 1-form $\theta \in \Omega^1(\mathbb{PA}, L)$ on $\mathbb{PA}$

[LeBrun, 1983] proved that one can recover $(M,[g])$ from knowledge of the $\mathbb{C}$-structure on $\mathbb{PA}$. Correspondence is stable under deformations preserving θ

- Complex structure determined by contact structure as $T^{0,1}_{\mathbb{PA}} := \ker\theta\wedge(d\theta)^{d-2}$

Deformations of contact structure	$\Leftrightarrow$	Deformations of metric
$\delta\theta\in H^{0,1}(\mathbb{PA},L)$	$\Leftrightarrow$	$\delta g\in \mathrm{Sym}^2\, T^{*1,0}_M$

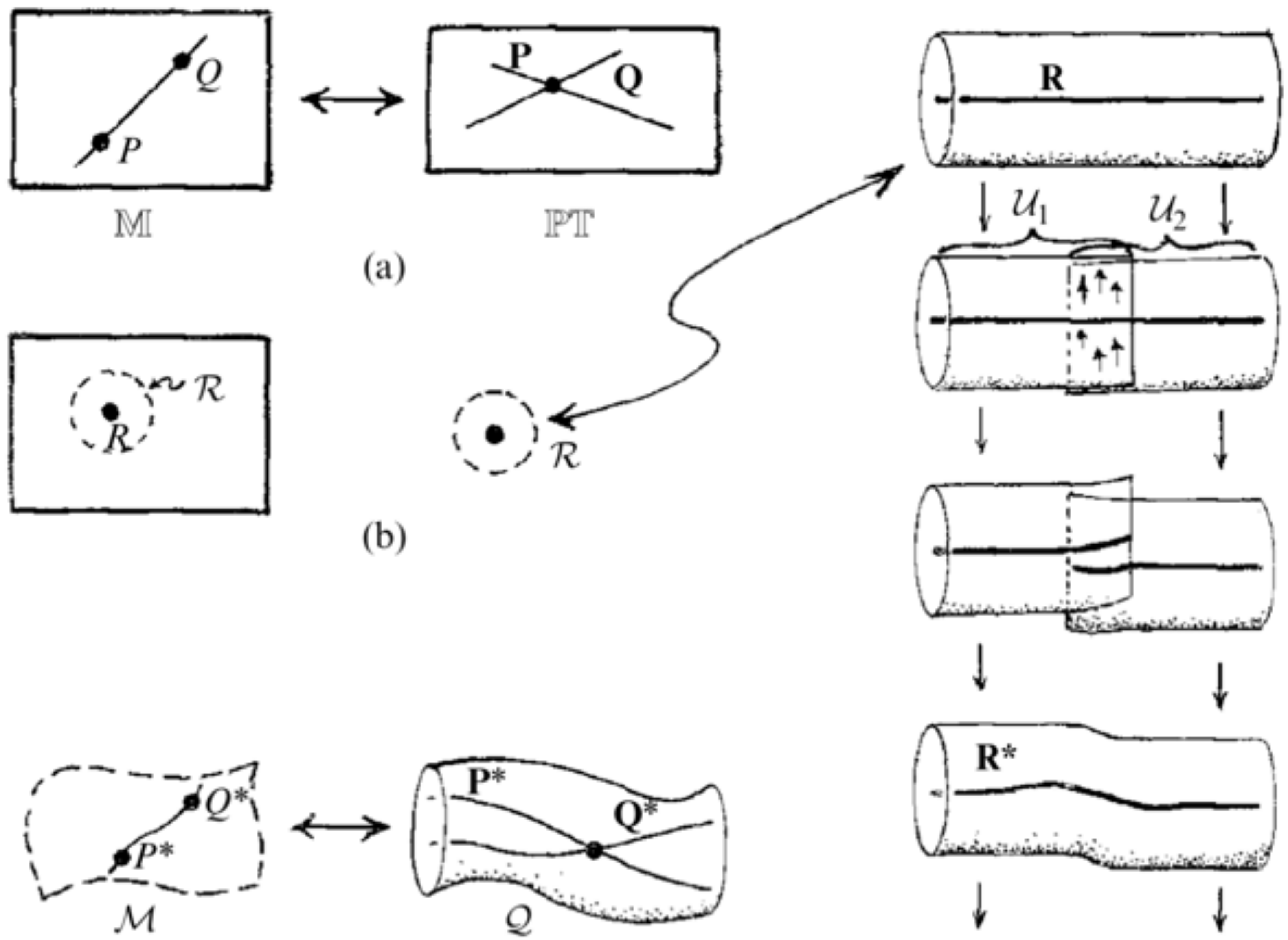
$$\begin{tikzcd} & \mathbb{P}(T^*_N M) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{PA} & & M \end{tikzcd}$$

$\pi_1^*(\delta\theta)=\bar\partial j$ by a vanishing argument

$P\cdot\nabla(j)$ a globally holomorphic section of $\pi_1^*(L^2)$ hence $P\cdot\nabla(j)=(\pi_2^*\delta g)(P,P)$

[Baston,Mason;LeBrun]

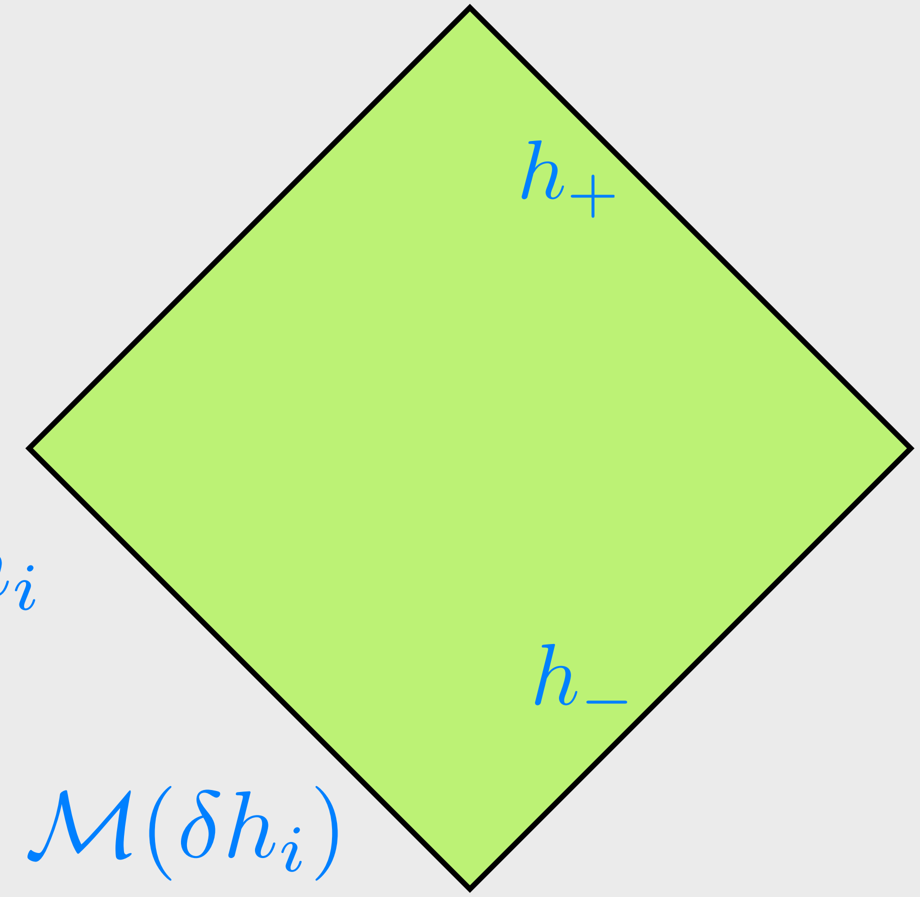
Situation is similar to Penrose twistor correspondence, where the $\mathbb{C}$-structure of twistor space $\mathcal{Z}$ also determines $(M, [g])$



- in twistor case, contact structure $\theta_{\mathcal{Z}}$ gives $g \in [g]$ with $\mathrm{Riem}(g) \in \Omega^{2+}(\mathrm{End}\, T_M)$, so automatically Einstein
- in ambitwistor case, no field equations implied / understood

Scattering amplitudes & CHY formulae

Seek an Einstein metric g with $g \sim h_\pm$
for prescribed asymptotic data $h_\pm$



- Perturbatively, take $h = g_0 + \sum_{i=1}^{n} \varepsilon_i \, \delta h_i$

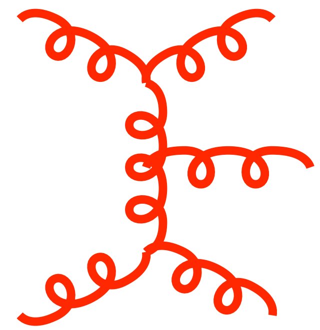
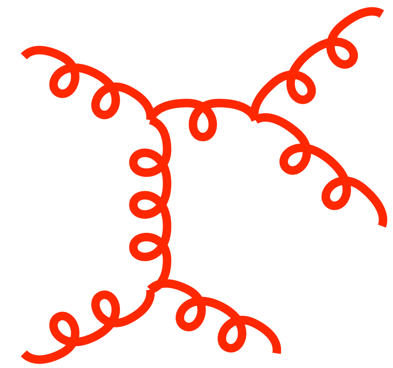
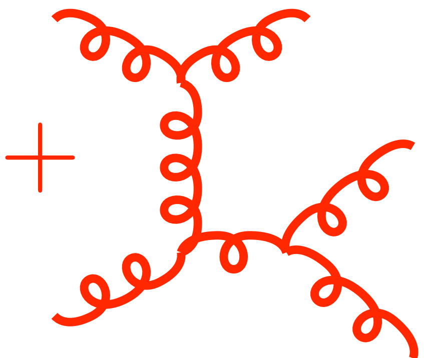
The n-particle, tree-level $\mathcal{S}$-matrix $\mathcal{M}(\delta h_i)$
is the coefficient of $\prod_i \varepsilon_i$ in $S_{\mathrm{EH}}[g]$

- We usually choose $g_0 = \eta$, the flat Minkowski metric,
and $\delta h_i = \epsilon_i \, \mathrm{e}^{i k_i \cdot x}$, with

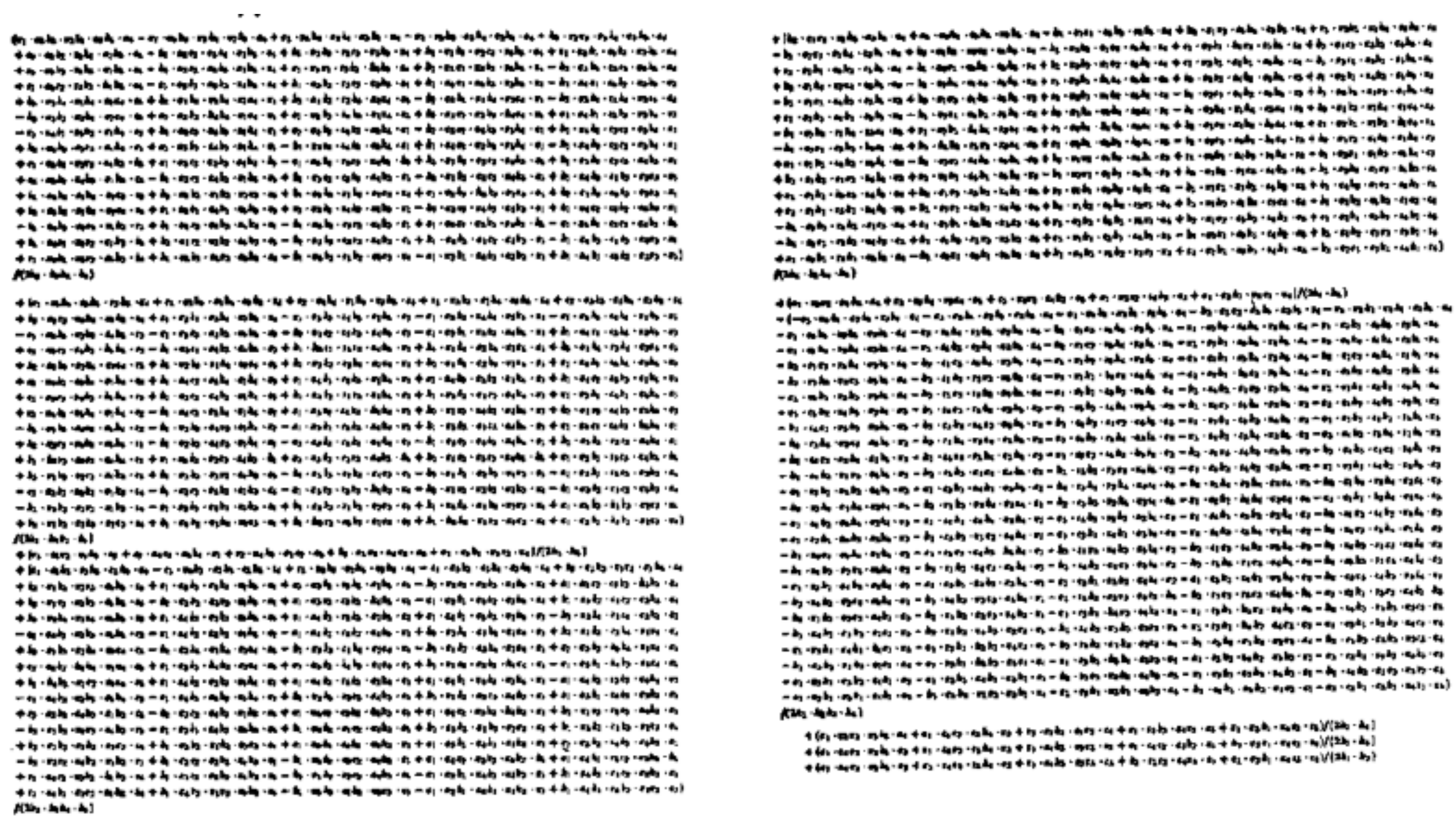
$$\eta^{-1}(k_i, k_i) = 0 \qquad \epsilon_i(k_i, \cdot\,) = 0 \qquad \epsilon_i \sim \epsilon_i + v \odot k_i$$

- Leading term in asymptotic series in $\hbar$ coming from
Feynman path integral (divergent for gravity)

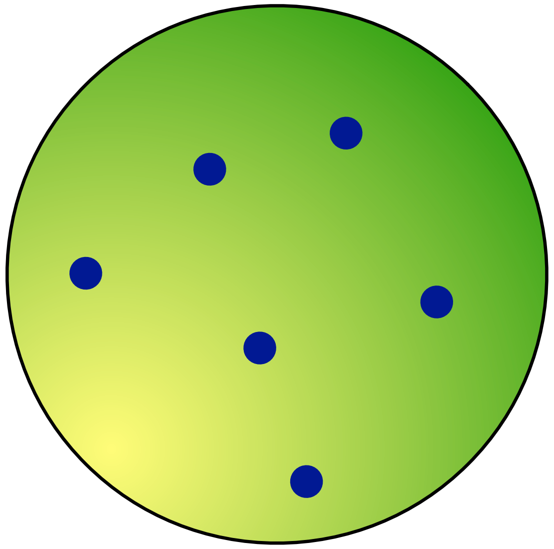
We may expand $S_{\mathrm{EH}}[g]$ in terms of tree-level Feynman diagrams, but in gravity, these are unpleasant:

$$\sum_{\text{graphs}} \left(\quad \cdots \quad \right) =$$  $+$  $+$  $+ \cdots$

$=$



Cachazo, He & Yuan have found that, for tree amplitudes in a wide range of massless field theories, Feynman diagrams may be replaced by the *scattering equations*:



$$S_i(z_j) := \sum_{i \neq j} \frac{k_i \cdot k_j}{z_i - z_j} = 0$$

- determine $\{z_i\} \in \mathbb{CP}^1$ in terms of the external momenta
- same equations dominate high energy, fixed angle string scattering

[Gross,Mende; *cf* Fairlie,Roberts]

In the CHY picture, tree amplitudes are given by

$$\mathcal{M} = \sum_{\{z_i \,|\, S_i(z_j)=0\}} \frac{\mathrm{Obj}(\epsilon_i, k_i, z_i, \cdots)}{\mathrm{Jac}(k_i, z_i)}$$

where $\mathrm{Jac} := \det\left(\partial S_i / \partial z_j\right)$ is the Jacobian obtained upon solving the scattering equations

Different theories correspond to different choices for “Obj”:

$$\Psi = \left[\begin{array}{cccc|cccc}
0 & \frac{k_1\cdot k_2}{z_{12}} & \cdots & \frac{k_1\cdot k_n}{z_{1n}} & C_{11} & \frac{k_1\cdot \epsilon_2}{z_{12}} & \cdots & \frac{k_1\cdot \epsilon_n}{z_{1n}} \\
\frac{k_2\cdot k_1}{z_{21}} & 0 & & & \frac{k_2\cdot \epsilon_1}{z_{21}} & \ddots & & \\
\vdots & & \ddots & & \vdots & & \ddots & \\
\frac{k_n\cdot k_1}{z_{n1}} & \cdots & & 0 & \frac{k_n\cdot \epsilon_1}{z_{n1}} & \cdots & & C_{nn} \\
\hline
-C_{11} & \frac{\epsilon_1\cdot k_2}{z_{12}} & \cdots & \frac{\epsilon_1\cdot k_n}{z_{1n}} & 0 & \frac{\epsilon_1\cdot \epsilon_2}{z_{12}} & \cdots & \frac{\epsilon_1\cdot \epsilon_n}{\epsilon_{1n}} \\
\frac{\epsilon_2\cdot k_1}{z_{21}} & \ddots & & & \frac{\epsilon_2\cdot \epsilon_1}{z_{21}} & 0 & & \\
\vdots & & \ddots & & \vdots & & \ddots & \\
\frac{\epsilon_n\cdot k_1}{z_{n1}} & \cdots & & -C_{nn} & \frac{\epsilon_n\cdot \epsilon_1}{z_{n1}} & \cdots & & 0
\end{array}\right]$$

$$C_{ii} = \sum_{j\neq i} \frac{\epsilon_i \cdot k_j}{z_{ij}}$$

$$\mathcal{M}_{\text{grav}} = \sum_{\text{sols}} \frac{\text{Pfaff}'\Psi \, \text{Pfaff}'\tilde{\Psi}}{\text{Jac}} \qquad \mathcal{M}_{\phi^3} = \sum_{\text{sols}} \frac{1}{\text{Jac}} \left[\frac{\text{tr}(T_1\cdots T_n)}{z_{12}z_{23}\cdots z_{n1}} + \cdots\right]^2$$

$$\mathcal{M}_{\text{YM}} = \sum_{\text{sols}} \frac{\text{Pfaff}'\Psi}{\text{Jac}} \left[\frac{\text{tr}(T_1\cdots T_n)}{z_{12}z_{23}\cdots z_{n1}} + \cdots\right]$$

A particularly sharp statement of “gravity x scalar = YM^2”

The CHY expressions have weird & wonderful properties:

- diffeo/gauge invariant sol^n-by-sol^n
- manifest permutation/cyclic invariance for gravity/YM
- transparent behaviour in soft limits sol^n-by-sol^n
- each contribution algebraic, but non-rational

They are also very much in Penrose's spirit: look like the output of a localisation calculation, but describe *dynamics*

- similar formulae also known for scattering amplitudes in Einstein-YM, DBI, NLSM, scalar ϕ^3 theory, …

Where do these 'magic' expressions come from? What does their very existence teach us about these theories?

Chiral strings in ambitwistor space

Ambitwistor strings are a chiral theory based (in flat space-time) on the worldsheet action

$$S = \int_\Sigma P_\mu \bar{\partial} X^\mu - \frac{1}{2} e P_\mu P^\mu \qquad \begin{array}{l} P_\mu \in \Gamma(\Sigma, K) \\ e \in \Omega^{0,1}(\Sigma, T_\Sigma) \end{array}$$

- complexification of worldline theory for massless particle

The constraint $P^2 = 0$ and associated gauge transformations

$$\delta X^\mu = \alpha P^\mu \qquad \delta P_\mu = 0 \qquad \delta e = \bar{\partial}\alpha \qquad \alpha \in \Gamma(\Sigma, T_\Sigma)$$

implement Marsden-Weinstein quotient. Identifying K with the pullback of L, the theory describes holomorphic maps $Y : \Sigma \to \mathbb{PA}$

- the action is just $\int_\Sigma Y^* \theta$

However simple it may appear, the chiral model $S = \int_\Sigma P\bar{\partial}X$ is actually inconsistent as it stands.

- problem lies with target space diffeomorphisms

$$X^\mu \to f^\mu(X) \qquad\qquad P_\mu \to \frac{\partial f^\nu}{\partial X^\mu} P_\nu$$

and the second transformation requires regularization

- from the path integral perspective, perturbing around a constant map $X : \Sigma \to x_0 \in M$, we get a chiral determinant

$$\frac{1}{\det(\bar{\partial}_{X^*TM})} = \int DP\, DX\, \mathrm{e}^{-\int P\bar{\partial}X}$$

which again is not $\mathrm{Diff}(M)$ invariant

Familiar in curved $\beta\gamma$-systems [Vaintrob,Malikov,Schechtman;Witten;Nekrasov]

The simplest way to cure this is to add in 2d real fermions

$$S = \int_\Sigma \left(P_\mu \bar{\partial} X^\mu - \frac{1}{2} e P^2 + \sum_{r=1}^{2} \psi_{r\mu} \bar{\partial} \psi_r^\mu + \chi_r \psi_r^\mu P_\mu \right)$$

- note that *both* sets of fermions are left-moving
- the corresponding fermionic currents $\psi_r^\mu P_\mu$ are also gauged, and the BRST operator is taken to be

$$Q = \oint \left(cT + \tilde{c} P^2 + \sum_r \gamma_r \psi_r \cdot P \right)$$

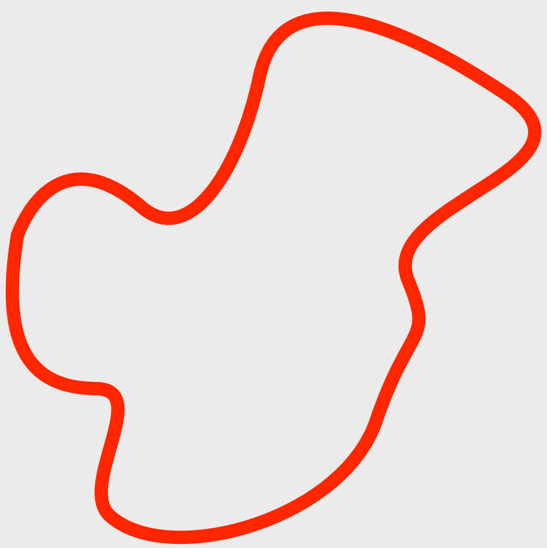
where c is the usual (holomorphic) reparametrization ghost, and $(\tilde{c}, \gamma_r)$ are ghosts associated to $(P^2, \psi_r \cdot P)$

The theory is anomaly free and $Q^2 = 0$ iff $d = 10$

The simplest BRST-closed (NS-NS) vertex operators are

$$U(z) := c\tilde{c}\,\delta^2(\gamma)\psi_1^\mu\psi_2^\nu\,\epsilon_{\mu\nu}\mathrm{e}^{ik\cdot X}$$

which look very similar to the RNS string, but again recall that everything in sight here is left-moving



Usually in string theory, worldsheet oscillators create an infinite (Regge) tower of extra states, entering at a scale set by the string tension $1/\alpha'$.

$$X^\mu(z)\,X^\nu(w) \sim 0 \qquad P_\mu(z)\,X^\nu(w) \sim \frac{\delta_\mu^{\ \nu}}{z-w} \qquad P_\mu(z)\,P_\nu(w) \sim 0$$

In particular, since $XX \sim 0$, $\mathrm{e}^{ik\cdot X}$ has vanishing conformal weight, no matter the value of k_μ. There is no Regge tower.

Chiral / holomorphic strings cannot oscillate

- requiring that these vertex operators are annihilated by the currents $(P^2, \psi_r \cdot P)$ inside BRST operator leads to the conditions $k^2 = 0$ and $\epsilon \cdot k = 0$

There are no massive states in the spectrum

- including the R-sectors, the spectrum of the theory is just Type II (A/B) supergravity

- integrated (NS-NS) vertex operators are slightly unusual:

$$U(z) := c\tilde{c}\,\delta^2(\gamma)\psi_1^\mu\psi_2^\nu\,\epsilon_{\mu\nu}\mathrm{e}^{ik\cdot X}$$

$$\int_\Sigma V := \int_\Sigma \bar{\delta}(k\cdot P)\,(P^\mu + \psi_1^\mu k\cdot\psi_1)\,(P^\nu + \psi_2^\nu k\cdot\psi_2)\,\epsilon_{\mu\nu}\mathrm{e}^{ik\cdot X}$$

and represent deformations of contact structure

Inserting n vertex operators, the X dependence becomes

$$\int_\Sigma \left(P_\mu \bar{\partial} X^\mu + \sum_{i=1}^{n} \delta(z - z_i) k_i \cdot X \right)$$

so performing the X path integral constrains

$$P_\mu(z) = \sum_{i=1}^{n} \frac{k_{i\mu}\, dz}{z - z_i}$$

This in turn implies $P^2(z) = (dz)^2 \sum_{i,j} \frac{k_i \cdot k_j}{(z - z_i)(z - z_j)}$

which is generically not zero, threatening our whole interpretation in terms of ambitwistors

- the scattering equations, coming from the factors of $\bar{\delta}(k_i \cdot P)$ in the integrated vertex operators, save the day

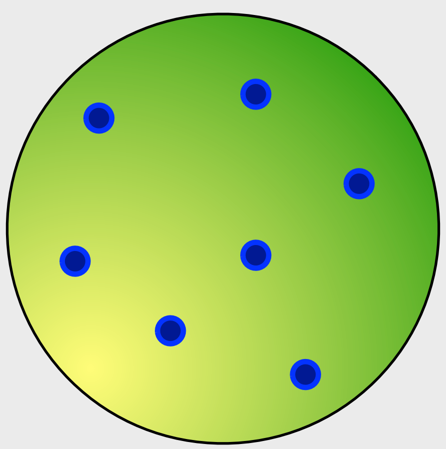
At $g=0$, any meromorphic quadratic differential vanishes identically if it has fewer than four (simple) poles

$$\mathrm{Res}_{z_i} P^2 = \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} \;\; (= k_i \cdot P(z_i))$$

the scattering equations!

These equations simply state that our theory indeed lives on ambitwistor space (even in the presence of vertex operators)

- Pfaffians of Ψ and $\tilde{\Psi}$ come from fermion correlators

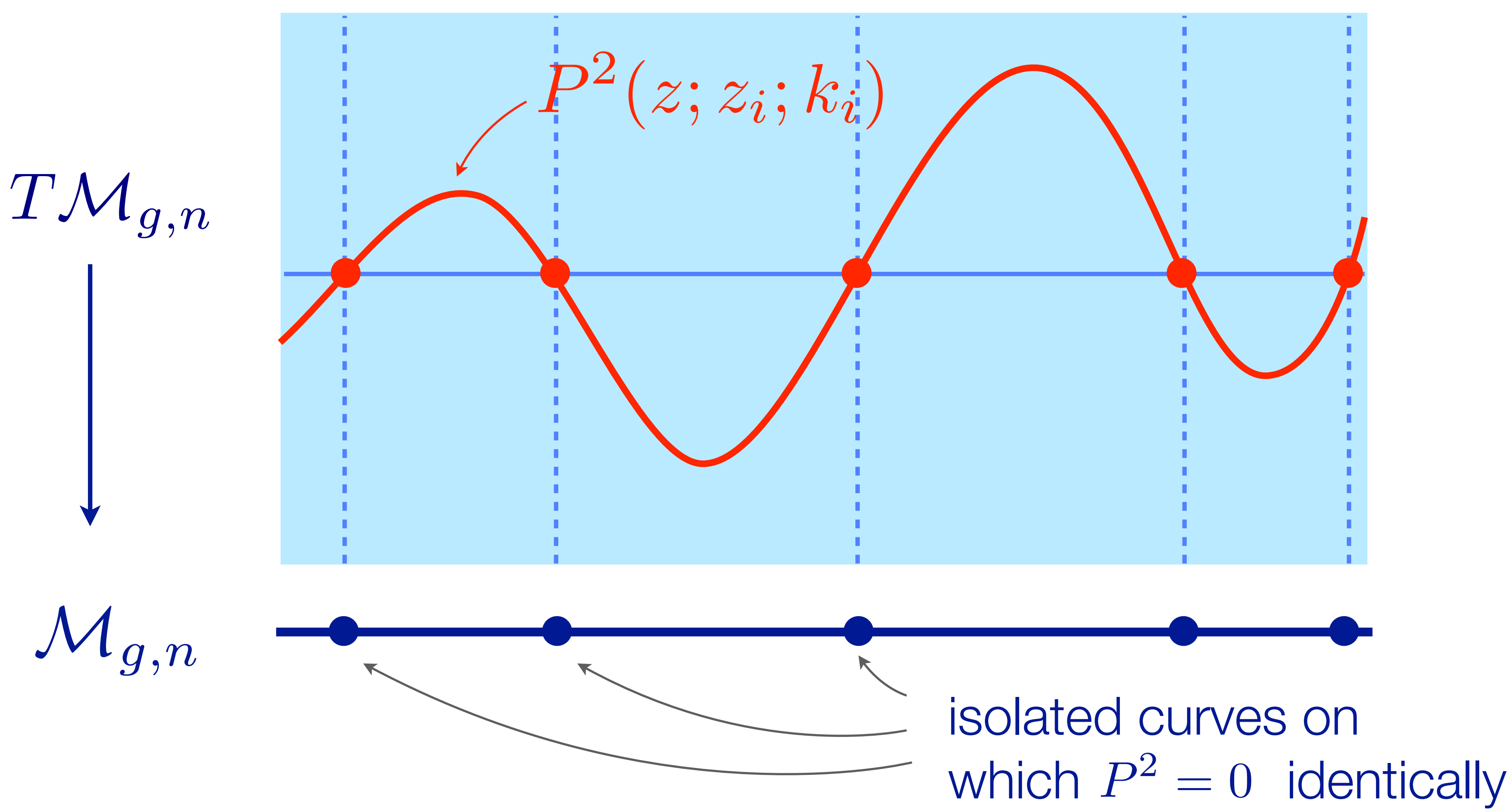


$$\mathcal{M}_{\mathrm{ambi}} = \sum_{\mathrm{sols}} \frac{\mathrm{Pfaff}'\Psi\,\mathrm{Pfaff}'\tilde{\Psi}}{\mathrm{Jac}} = \mathcal{M}_{\mathrm{grav}}$$

- integral over $\mathcal{M}_{0,n}$ is completely localized by the scattering equations

Since it plays such a key role, it's worth understanding how this localization arises. There are two main facts[Ohmori]:

- first, since we independently gauge both T and P^2, the (bosonic) moduli space of the ambitwistor string is really $T\mathcal{M}_{g,n}$, rather than just $\mathcal{M}_{g,n}$



Let t represent local coords in a neighbhd of $p \in \mathcal{M}_{g,n}$ with complex structure corresponding to some $\bar{\partial}$-operator on Σ

- nearby we have $\bar{\partial} \to \bar{\partial} + \mu(t)\partial$ and also

$$e = e(s,t) = \sum_{\alpha} s_\alpha e_\alpha(t)$$

where the e_α form a basis of $H^1(\Sigma, T_\Sigma(-\{z_i\})) \cong T\mathcal{M}_{g,n}|_{\Sigma(z_i)}$ (the moduli of the field e on the marked curve)

Near p, the worldsheet action becomes

$$S = \int_\Sigma P_\mu \bar{\partial} X^\mu + \mu(t) T + \delta\mu\, b - \frac{1}{2} e P^2 - \frac{1}{2} \delta e\, \tilde{b}$$

where $(\delta\mu, \delta e)$ are constant fermionic parameters added to cancel BRST variations of the moduli (μ, e)

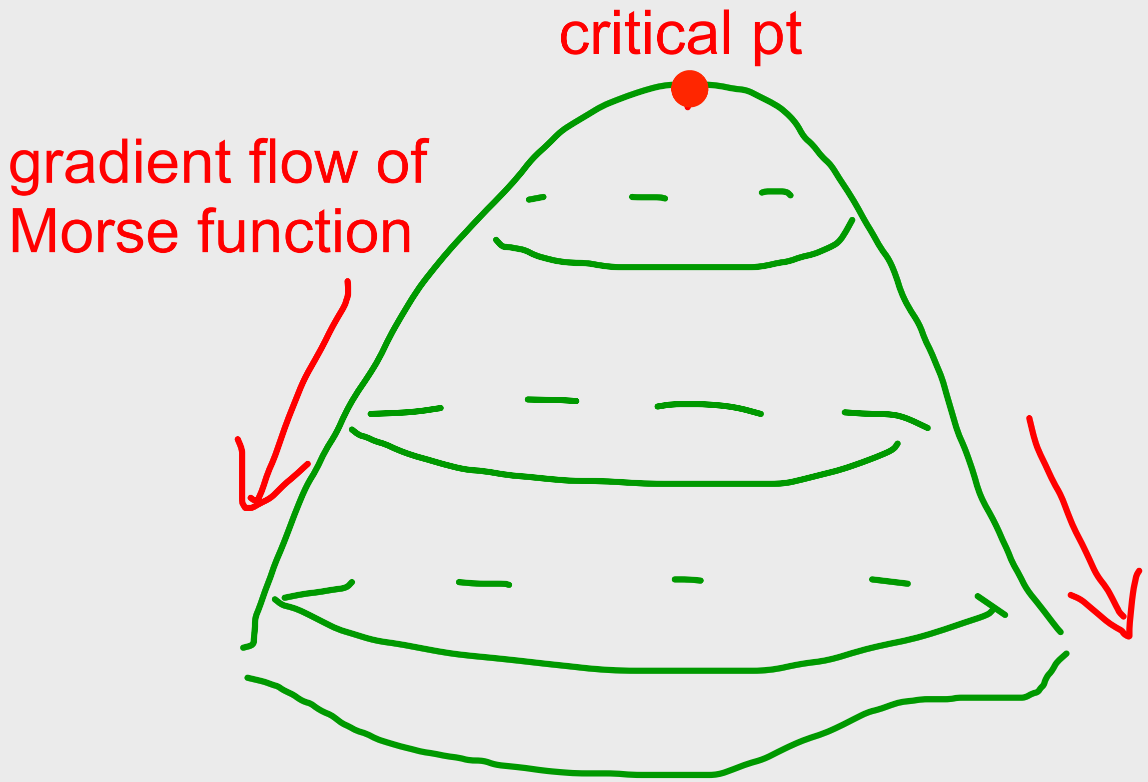
[Witten; Ohmori]

- The second fact is that the chiral theory depends only on the holomorphic moduli, and provides a $(\mathrm{top}, 0)$-form on $T\mathcal{M}_{g,n}$

Ohmori picks an integration cycle defined by downward gradient flow of the Morse function

$$-\mathrm{Re}\left(\int_\Sigma \mu(t)T + e(t,s)P^2\right)$$

This Morse function is exactly the real part of the (bosonic) moduli dependence in the action



- The integration cycles start from a critical point, then flow away to infinity

The dependence on the moduli is BRST exact:

$$S = \int_\Sigma P\cdot\bar\partial X + \mu(t)T + \delta\mu\, b - \frac{1}{2}eP^2 - \frac{1}{2}\delta e\,\tilde b$$
$$= \int_\Sigma P\cdot\bar\partial X + \left\{Q, \int_\Sigma \mu(t)\left(b - \frac{1}{2}e(t,s)\,\tilde b\right)\right\}$$

so we localise on critical points of the Morse function. These are determined by

$$\frac{\partial}{\partial s^\alpha}\int_\Sigma e(s,t)\,P^2 = 0 \qquad \frac{\partial}{\partial t^\alpha}\left(\int_\Sigma \mu(t)\,T + e(s,t)\,P^2\right) = 0$$

- in a standard basis, $\frac{\partial}{\partial s^\alpha}\int e(s,t)P^2 = \int e_\alpha(t)P^2 = \mathrm{Res}_{z_\alpha}P^2$ whose vanishing is exactly the scattering equations

Curved backgrounds

Because there are no oscillators / no α' corrections, the Einstein eqns should be *exact* conditions for consistency on a curved background

- linearized eoms for vertex operators came from algebra

$$G(z)\,\tilde{G}(w) \sim \frac{H}{z-w} \qquad \begin{aligned} G &= (\psi_1 + i\psi_2)^\mu P_\mu \\ \tilde{G} &= (\psi_1 - i\psi_2)^\mu P_\mu \end{aligned} \qquad H = P^2$$

rather than OPE with T, so expect Einst. eqs. from anomaly here, not from β-function.

For the curved theory, we take the action to be

$$S = \int_\Sigma P_\mu \bar{\partial} X^\mu + \tilde{\psi}_\mu \bar{D}\psi^\mu = \int_\Sigma \pi_\mu \bar{\partial} X^\mu + \tilde{\psi}_\mu \bar{\partial}\psi^\mu$$

$$\bar{D}\psi^\mu = \bar{\partial}\psi^\mu + \Gamma^\mu_{\nu\lambda}\psi^\nu \bar{\partial} X^\lambda \qquad \Pi_\mu = P_\mu + \tilde{\psi}_\nu \gamma^\nu_{\mu\lambda}\psi^\lambda$$

This is a 'curved $\beta\gamma$-system'. Bosonic versions are subtle, due to anomalies in chiral determinants, but easier with SUSY

[Malikov,Schechtman,Vaintrob; Nekrasov; Witten; Frenkel,Nekrasov,Losev]

- action remains free, but *currents* are deformed. Classically:

$$\mathcal{G}^0 = \psi^\mu \Pi_\mu \qquad \tilde{\mathcal{G}}^0 = g^{\mu\nu}\tilde{\psi}_\mu \left(\Pi_\nu - \Gamma^\rho_{\nu\sigma}\tilde{\psi}_\rho\psi^\sigma\right)$$

$$\tilde{\mathcal{H}}^0 = g^{\mu\nu}\left(\Pi_\mu - \gamma^\kappa_{\mu\lambda}\tilde{\psi}_\kappa\psi^\lambda\right)\left(\Pi_\nu - \Gamma^\rho_{\nu\sigma}\tilde{\psi}_\rho\psi^\sigma\right) - \frac{1}{2}R^{\kappa\lambda}{}_{\mu\nu}\tilde{\psi}_\kappa\tilde{\psi}_\lambda\psi^\mu\psi^\nu$$

- operator $\mathcal{O}_V = V^\mu(X)\Pi_\mu + \partial_\nu V^\mu\,\tilde{\psi}_\mu\psi^\nu$ obeys OPE $\mathcal{O}_V(z)\,\mathcal{O}_W(w) \sim \dfrac{\mathcal{O}_{[V,W]}}{z-w}$ with no higher poles, so generates target space diffeomorphisms

While all the basic fields transform correctly, $\mathcal{O}_V$ shows that the composite currents $(\mathcal{H}^0, \mathcal{G}^0, \tilde{\mathcal{G}}^0)$ have anomalous behaviour under target space diffeomorphisms

To get something sensible, we must add quantum corrections to the currents

- modifications of $\mathcal{G}^0$ & $\tilde{\mathcal{G}}^0$ turn out to be

$$\mathcal{G} = \mathcal{G}^0 + \partial(\mathcal{L}_{\psi^\mu \partial_\mu} \log \Omega) \qquad \tilde{\mathcal{G}} = \tilde{\mathcal{G}}^0 + \partial(\mathcal{L}_{g^{\mu\nu} \tilde{\psi}_\mu \partial_\nu} \log \Omega)$$

with $\Omega = \mathrm{e}^{-2\Phi(X)} \sqrt{g}\, dX^1 \wedge \cdots \wedge dX^{10}$

These modified currents have the desired OPEs

$$\mathcal{O}_V(z)\,\mathcal{G}(w) \sim \cdots + \frac{\mathcal{L}_V \mathcal{G}}{z-w} \qquad \mathcal{O}_V(z)\,\tilde{\mathcal{G}}(w) \sim \cdots + \frac{\mathcal{L}_V \tilde{\mathcal{G}}}{z-w}$$

respecting target space diffeomorphisms

- there are also modifications of the worldsheet stress tensor that ensure the new operators are primaries

$$S \to S + \frac{1}{8\pi} \int_\Sigma R_\Sigma \log(\mathrm{e}^{-2\Phi} \sqrt{g})$$

Because the curved space action is trivial, we can compute OPEs *exactly*. One finds

$$\mathcal{G}(z)\,\mathcal{G}(w) \sim 0 \qquad\qquad \tilde{\mathcal{G}}(z)\,\tilde{\mathcal{G}}(w) \sim 0$$

just requires the usual Bianchi identities on ${R^\kappa}_{\lambda\mu\nu}$, while (also allowing for a B-field)

$$\mathcal{G}(z)\,\tilde{\mathcal{G}}(w) \sim \frac{2}{(z-w)^3}\left(R + 4\nabla_\mu\nabla^\mu\Phi - 4\nabla_\mu\nabla^\mu\Phi - \frac{1}{12}H^2\right)$$
$$+2g^{\nu\lambda}\frac{(\Gamma^\mu_{\kappa\nu}\partial X^\kappa + \psi^\mu\tilde{\psi}_\nu)}{(z-w)^2}\left(R_{\mu\lambda} + 2\nabla_\mu\nabla_\lambda\Phi - \frac{1}{4}H_{\mu\rho\sigma}{H_\lambda}^{\rho\sigma}\right)$$
$$+\frac{\psi^\mu\psi^\nu - \tilde{\psi}^\mu\tilde{\psi}^\nu)}{(z-w)^2}\left(\nabla_\kappa {H^\kappa}_{\mu\nu} - 2{H^\kappa}_{\mu\nu}\nabla_\kappa\Phi\right) + \frac{\mathcal{H}}{z-w}$$

where $\mathcal{H}(= \ \mathcal{H}^0$ + quantum corrections) generalizes P^2

The Einstein, B-field and dilaton eqns are the *exact* conditions for a consistent background

- requiring the flat space algebra

$$G(z)\,\tilde{G}(w) \sim \frac{H}{z-w} = \frac{P^2}{z-w}$$

to hold in the presence of vertex operators imposed the scattering equations

Vertex operators are infinitesimal deformations of these currents. Requiring the same algebra to hold nonlinearly amounts to the full nonlinear field equations

Quantum Scattering Equations $\longleftrightarrow$ Einstein Field Equations

Quantum corrections

It's natural to expect quantum corrections to scattering amplitudes to arise from higher genus curves

- at higher genus, still have $\bar{\partial} P_\mu(z) = \sum_{i=1}^{n} k_{i\mu}\, \delta(z - z_i)\, \mathrm{d}z$
  but this now implies

$$P_\mu(z) = \sum_{a=1}^{g} \ell_\mu \omega_a - \sum_{i=1}^{n} k_{i\mu}\, \partial \ln E(z - z_i;\tau)$$

prime form

basis of $H^0(\Sigma, K_\Sigma)$

- again $P^2(z)$ is a meromorphic quadratic differential, now with $3g - 3 + n$ moduli

Higher genus analogue of the scattering equations is

$$\mathrm{Res}_{z_i} P^2 = 0 \qquad P^2(z_j) = 0$$

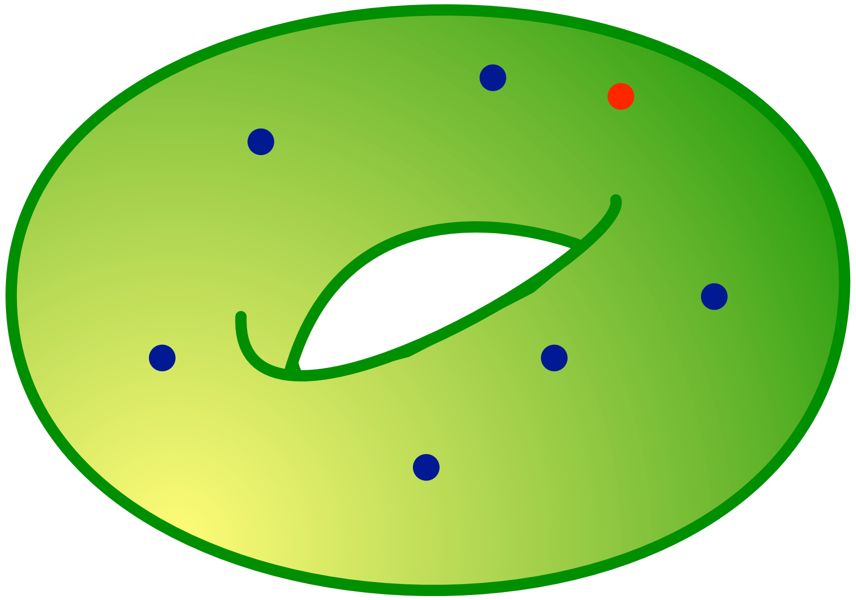
at exactly enough points to ensure $P^2(z) = 0$ identically

[Adamo,Casali,DS]

For example, at genus 1

$$P_\mu(z) = \left(\ell_\mu + \sum_{i=1}^{n} k_{i\mu} \frac{\theta'_{11}(z - z_i, \tau)}{\theta_{11}(z - z_i, \tau)} \right) dz$$

and the scattering equations enforce



$\mathrm{Res}_{z_i} P^2 = 0$ for $i \in \{1, 2, \ldots, n-1\}$

$P^2(z_j) = 0$ at any one point z_j

- again, these eq^{n}s arise from the moduli of e
- agrees with Gross-Mende saddle point when $n = 4$

[Casali, Tourkine]

The (even spin structure part of the) genus 1 amplitude is

$$\mathcal{M}^1 = \int d^{10}\ell \, d\tau \, \bar{\delta}(P^2(z_n;\tau)) \prod_{i=1}^{n} \bar{\delta}(k_i \cdot P(z_i;\tau))$$
$$\times \sum_{a,b} (-1)^{a+b} Z_{a,b}(\tau) \, \mathrm{Pfaff}'(\Psi) \mathrm{Pfaff}'(\tilde{\Psi})$$

the Pfaffians coming from free fermion correlators on the torus

- modular invariant when $d = 10$
- correct behaviour under factorizations & IR limits

[Adamo,Casali,DS]

However, these elliptic objects seem far removed from the ***rational*** function we'd expect for the Feynman loop integrand of a field theory

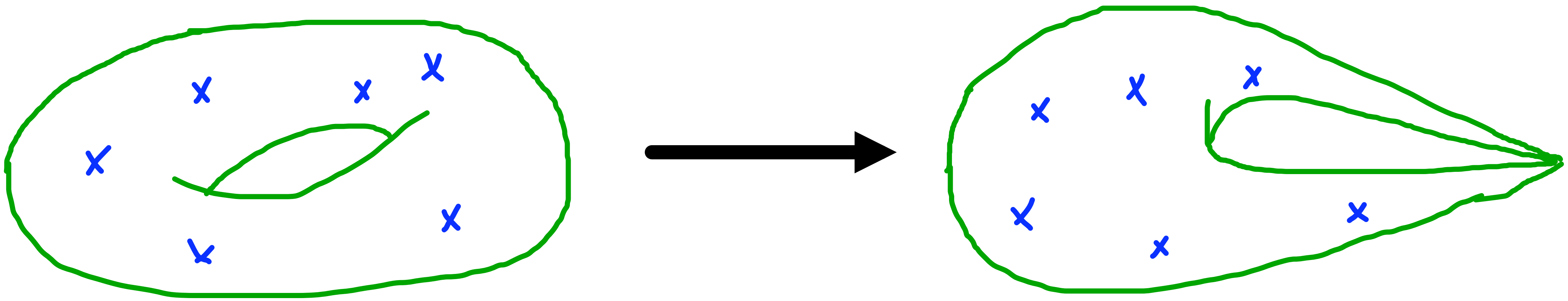
- explicitly shown to be rational when $n = 4$
- correct R^4 tensor structure

[Casali,Tourkine]

The proof that this is indeed 1-loop supergravity came in a beautiful paper[Geyer,Mason,Monteiro,Tourkine]

- main idea was to integrate by parts in moduli space

$$d\tau\,\bar{\delta}(P^2(z_n)) = \frac{dq}{q}\,\bar{\delta}(P^2(z_n)) = -dq\,\bar{\delta}(q)\,\frac{1}{P^2(z_n)}$$



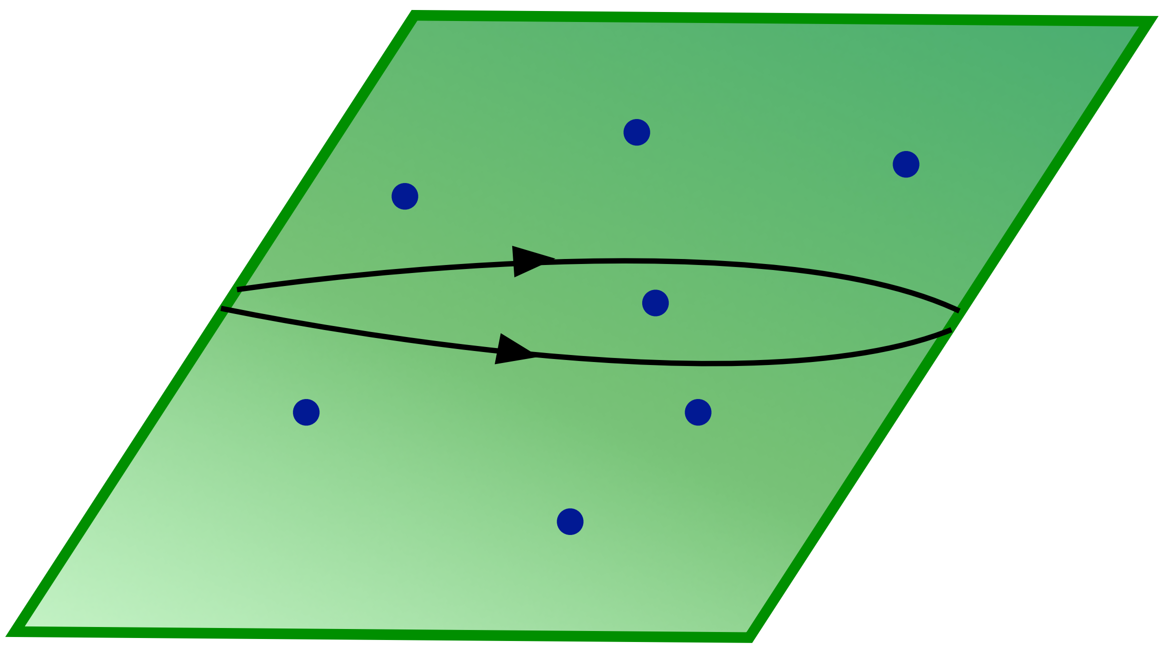
- remaining scattering equations now become

$$0 = \frac{k_i\cdot\ell}{z-z_0} - \frac{k_i\cdot\ell}{z-z_\infty} - \sum_{j\neq i}\frac{k_i\cdot k_j}{z_i-z_j}$$

and kill the polar part of $P(z)$, leaving us with $P^2(z)=\ell^2$

$$\mathcal{M}^{4,1} = t^8\tilde{t}^8 R^4 \int \frac{d^{10}\ell}{\ell^2} \sum_{\sigma\in S_4} \frac{1}{\ell\cdot k_{\sigma(1)}\,(\ell\cdot(k_{\sigma(1)}+k_{\sigma(2)})+k_{\sigma(1)}\cdot k_{\sigma(2)})\,\ell\cdot k_{\sigma(4)}}$$

There's some (expected) ambiguity in the definition of ℓ_μ :



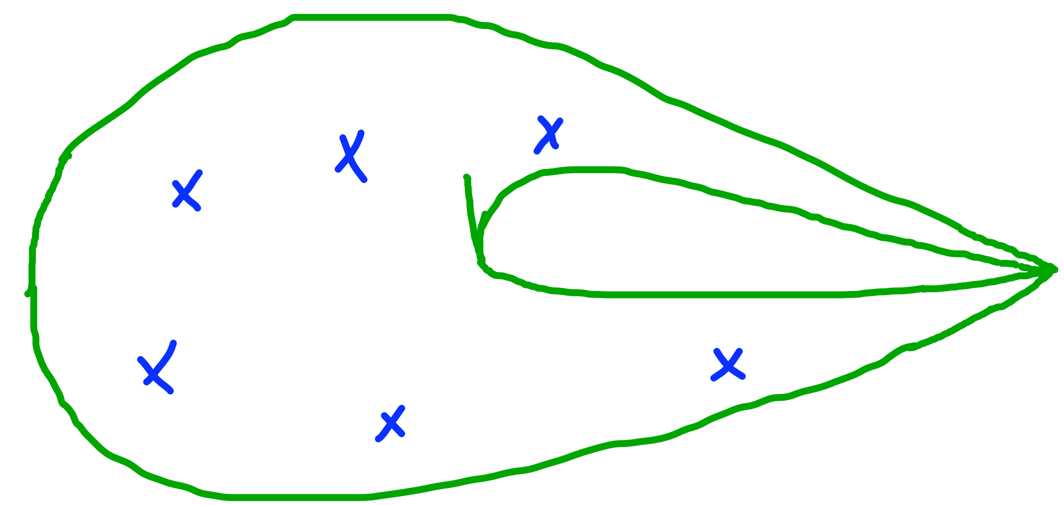
which propagator does it represent?

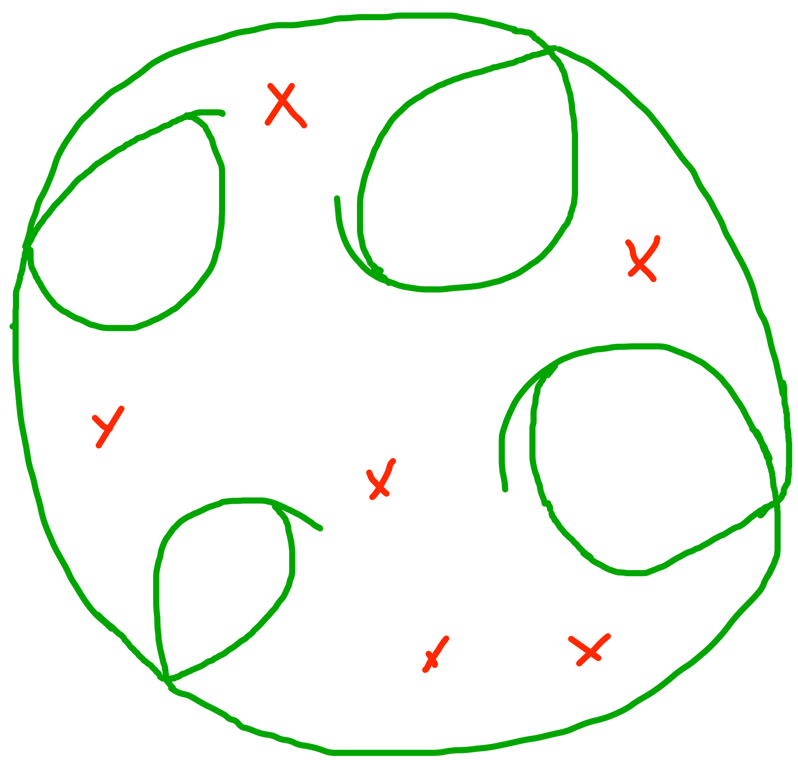
$$\ell \to \ell + k_i$$

Exploiting this in a smart way, GMMT showed that this indeed agrees with the known 1-loop integrand for Type II supergravity

[Brink,Green,Schwarz]

What's so striking about GMMT's derivation is that by localizing to $q = 0$, the story ends up being no more complicated than for trees!





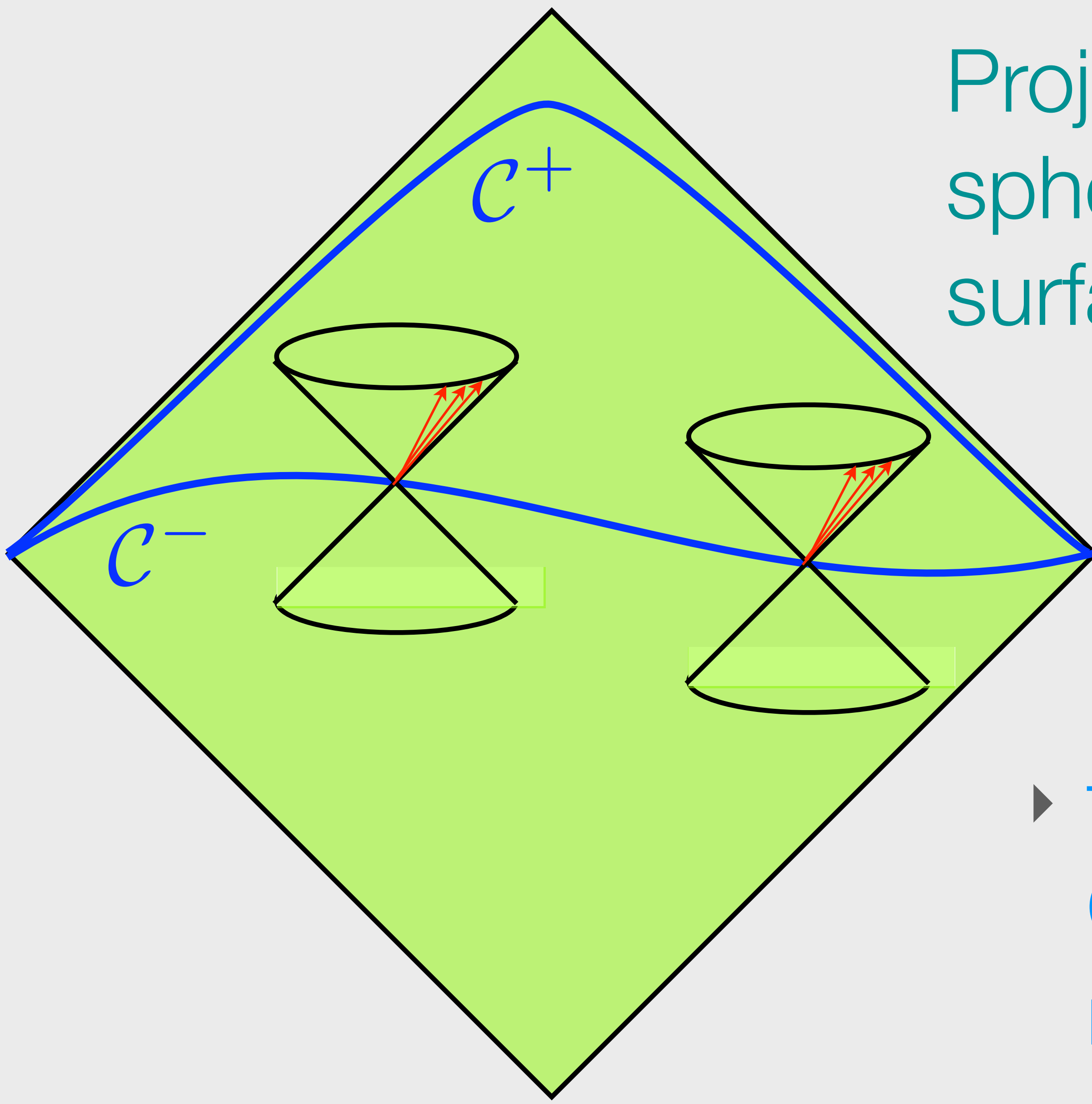
Generalising, they give natural looking conjectures for multi-loop expressions, shown to be correct now also for $(g, n) = (2, 4)$ & $(1, n)$

We're left with an integral over the zero modes ℓ_μ. This is (of course!) UV divergent - it's $d = 10$ supergravity.

Scattering and null infinity

Amplitudes are inherently holographic; they are meaningful, diffeo invariant observables in asymptotically flat space-times

- we usually compute them non-holographically, by evolving metric fluctuations through the bulk



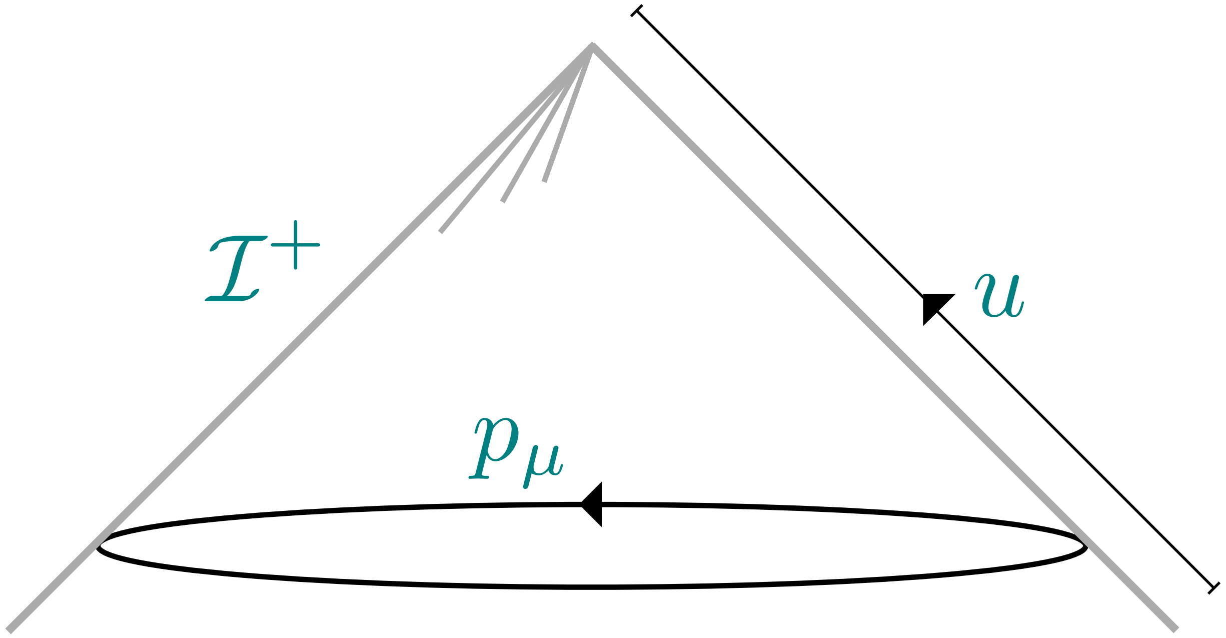
Projective ambitwistor space is a sphere bundle over any Cauchy surface $\mathcal{C}$:

$$\begin{tikzcd} S^{d-2} \arrow[r] & \mathbb{PA} \arrow[d] \\ & \mathcal{C}^{d-1} \end{tikzcd}$$

- the ambitwistor spaces of two Cauchy surfaces $\mathcal{C}^{\pm}$ are related by a diffeomorphism

For scattering theory, it's natural to take limiting case $\mathcal{C}^{\pm} = \mathcal{I}^{\pm}$

- choose coordinates adapted to $\mathcal{I}^+$ as follows:



$$(u, p_\mu) \in \mathbb{R}^{1+d}$$

with constraint $p^2 = 0$

and equivalence relation

$$(u, p_\mu) \sim (\alpha u, \alpha p_\mu)$$

- coordinates for $\mathbb{A} \cong T^*\mathcal{I}^+$ are then given by

$$(u, p_\mu; w, q^\mu) \sim (\alpha u, \alpha p_\mu; w/\alpha, q^\mu/\alpha + \beta p^\mu)$$

$$p^2 = 0 \qquad\qquad uw - p \cdot q = 0$$

and the symplectic potential on ambitwistor space is

$$\Theta = w du - q^\mu dp_\mu$$

Any $V \in \mathit{Diff}(\mathcal{I}^+)$ (such as a BMS transformation) can be lifted to a $\tilde{V} \in \mathit{Diff}(T^*\mathcal{I}^+) \cong \mathit{Diff}(\mathbb{A})$, generated by $H_{\tilde{V}} := \tilde{V} \lrcorner \Theta$

- *e.g.* bulk Poincare transform $\delta X^\mu = \omega^\mu{}_\nu X^\nu + a^\mu$ gives

$$\delta u = a \cdot p \qquad \delta w = 0$$

$$\delta p_\mu = -\omega^\nu{}_\mu p_\nu \qquad \delta q^\mu = \omega^\mu{}_\nu q^\nu + w a^\mu$$

with associated vector field & Hamiltonian

$$\tilde{V} = a^\mu \left(p_\mu \frac{\partial}{\partial u} + w \frac{\partial}{\partial q^\mu} \right) + \omega^\mu{}_\nu \left(q^\nu \frac{\partial}{\partial q^\mu} - p_\mu \frac{\partial}{\partial p_\nu} \right)$$

$$H_{\tilde{V}} = w(a \cdot p) - \omega^\mu{}_\nu q^\nu p_\mu \quad = H_T \ + \ H_R$$

BMS supertranslations and superrot$^{\mathrm{n}}$s instead correspond to

$$H_{\mathrm{ST}} = w f(p) \qquad H_{\mathrm{SR}} = -\omega^\mu{}_\nu(p)\, q^\nu p_\mu$$

so the translation/rotation varies around cuts of $\mathcal{I}^+$

The Hamiltonians are worldsheet charges for the ambitwistor string, generating transformations of the target:

$$e.g. \qquad Q[a] = \oint H_a = \oint w\, a \cdot p = \oint a^\mu P_\mu$$

The vertex operators correspond to Hamiltonian deformations of the contact structure, so can also be thought of as charges

$$\int_\Sigma V = \oint \frac{e^{ik\cdot X}}{k \cdot P} \epsilon^{\mu\nu} P_\mu P_\nu + \text{fermions} \; = \oint w \frac{e^{ik\cdot q/w}}{k\cdot p} \epsilon^{\mu\nu} p_\mu p_\nu + \cdots$$

$$= \oint w \left(\frac{\epsilon^{\mu\nu} p_\mu p_\nu}{k \cdot p} \right) + i \left(\frac{p_\mu \epsilon^{\mu[\nu} k^{\lambda]}}{k\cdot p} \right) p_\nu q_\lambda + \cdots$$

generates supertranslations

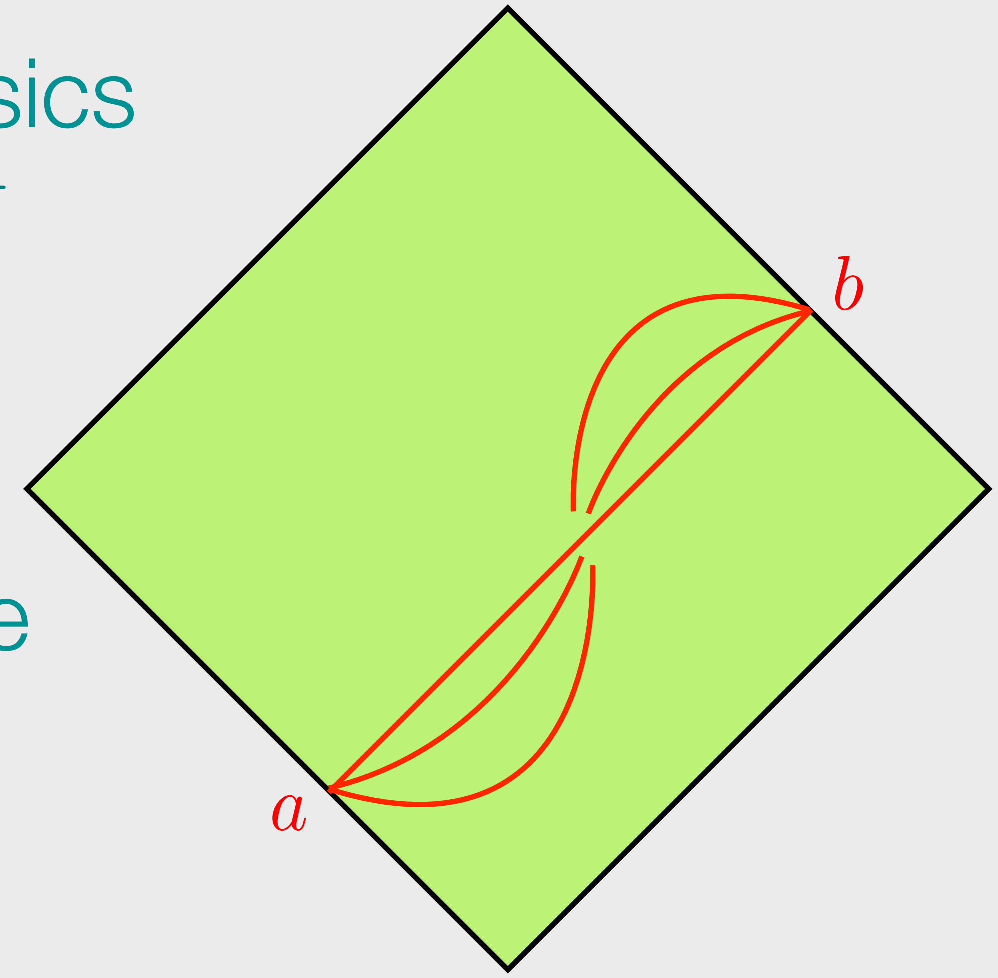
generates superrotations

A particularly direct illustration of the relation between BMS transformations and soft gravitons [Strominger *et al.*]

- Sub-subleading terms generate diffeomorphisms of $\mathbb{A} \cong T^*\mathscr{I}^+$, but not of $\mathscr{I}^+$ itself

In flat space, all the null geodesics emanating from a point $a \in \mathscr{I}^-$ reconverge at a point $b \in \mathscr{I}^+$

This is no longer the case in the presence of gravitational radiation / black holes / *etc…*



- The ambitwistor string realises a picture of scattering as a diffeomorphism $\mathbb{A}_{\mathscr{I}^-} \xrightarrow{\cong} \mathbb{A}_{\mathscr{I}^+}$ telling us where light rays emanating from $\mathscr{I}^-$ end up on $\mathscr{I}^+$

[Adamo,Casali,DS; Geyer,Lipstein,Mason]

Conclusions

The CHY formulation of massless amplitudes really means there's an underlying theory in the space of light rays

- spectrum contains only massless states - Type II sugra
- worldsheet correlators give CHY formula for gravity amplitudes
- generalizations to loop level now known & understood
- scattering equations are the avatars of nonlinear field equations

Of course there are many open questions

- is there an ambitwistor string for Einstein-Yang-Mills?
- what is ambitwistor string field theory?
- not a super Riemann surface, but a super bundle over a bosonic surface...
- how is all this related to standard string theory?

Thank you

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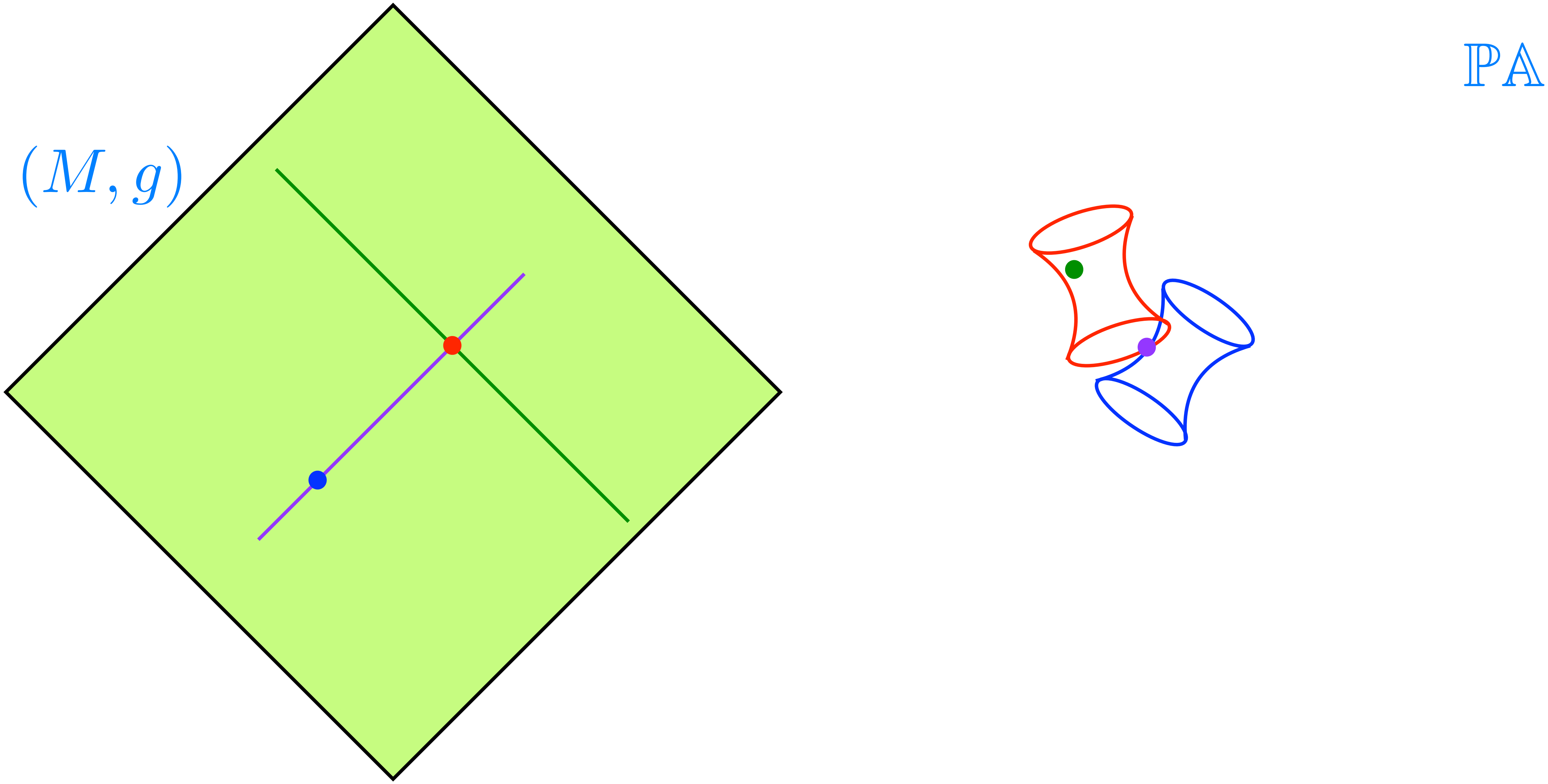
(M, g)
$\mathbb{P}\mathbb{A}$