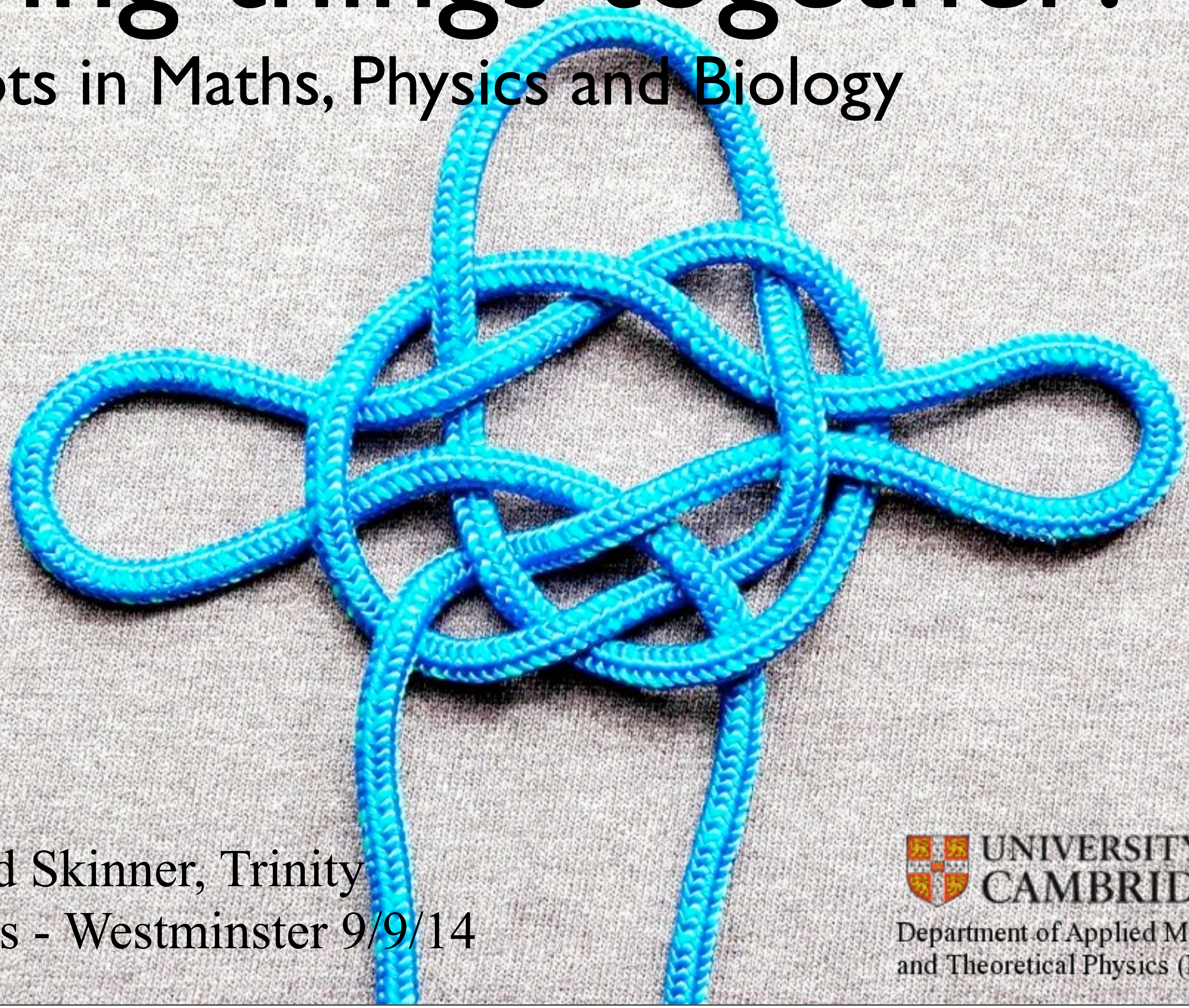
Tying things together:

Knots in Maths, Physics and Biology



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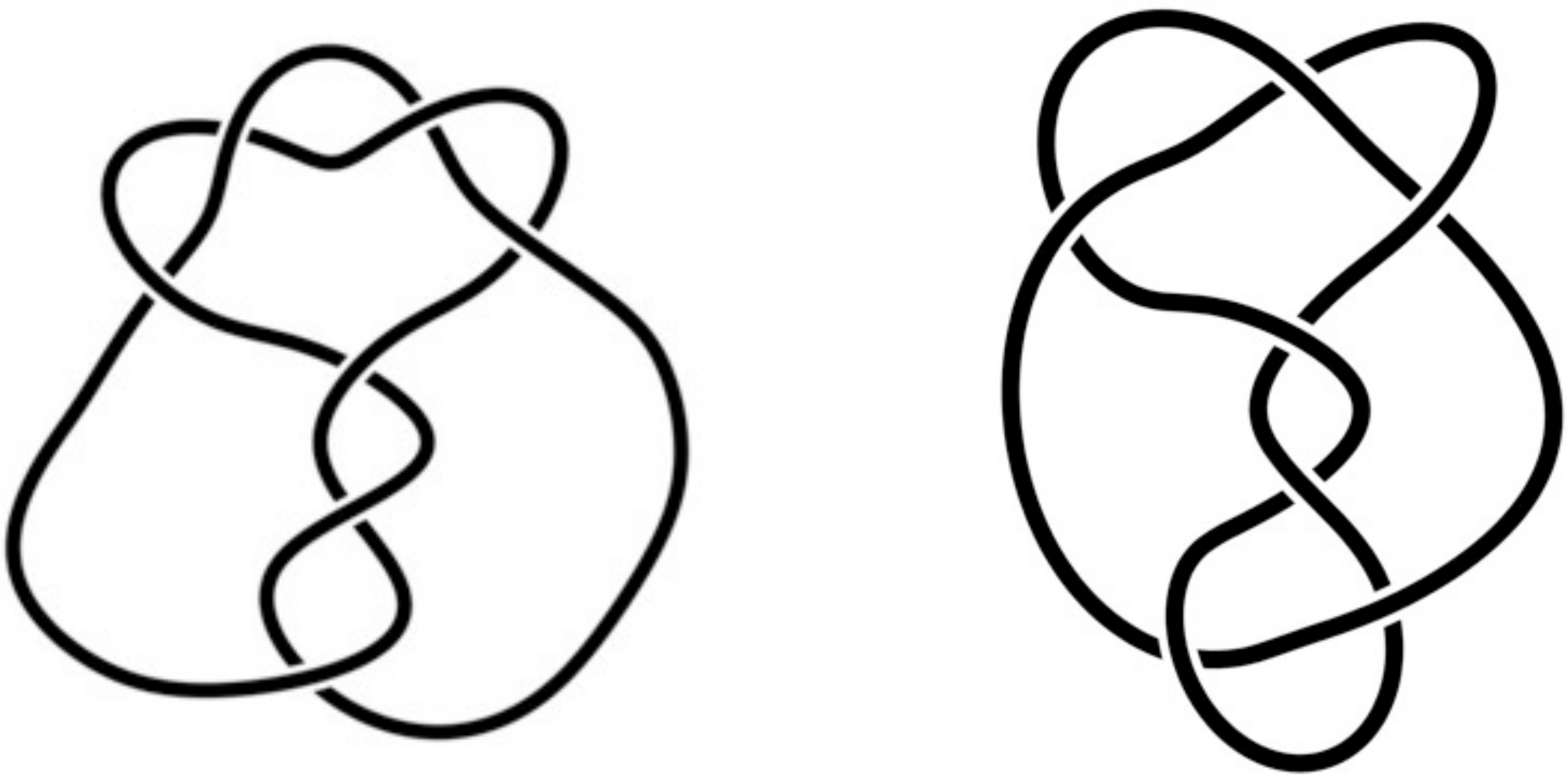
People around the world have been fascinated by knots for well over a thousand years

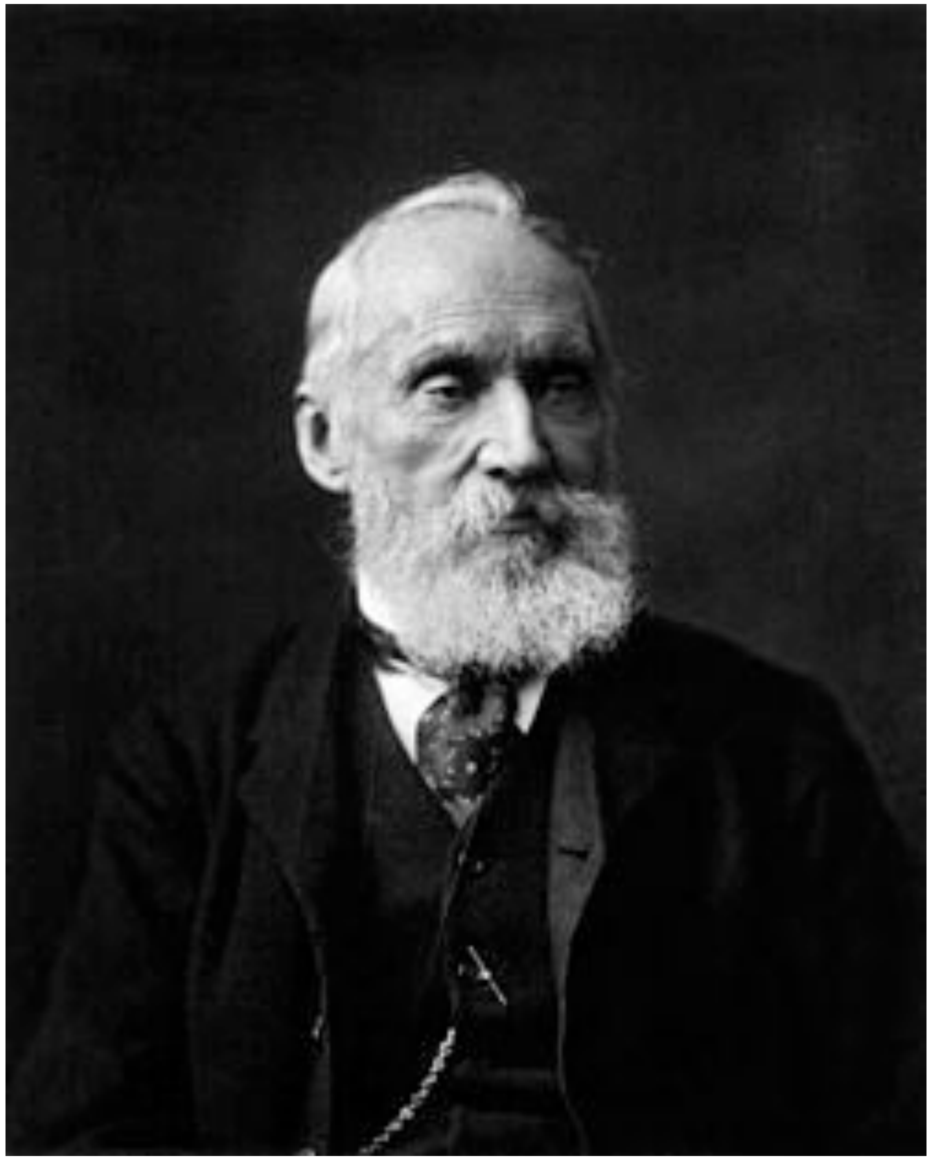
They’re interesting for the same reason they’re useful: knots are difficult to untangle.

For a mathematician, a knot is a continuous loop that sits inside 3d space in a perhaps complicated way

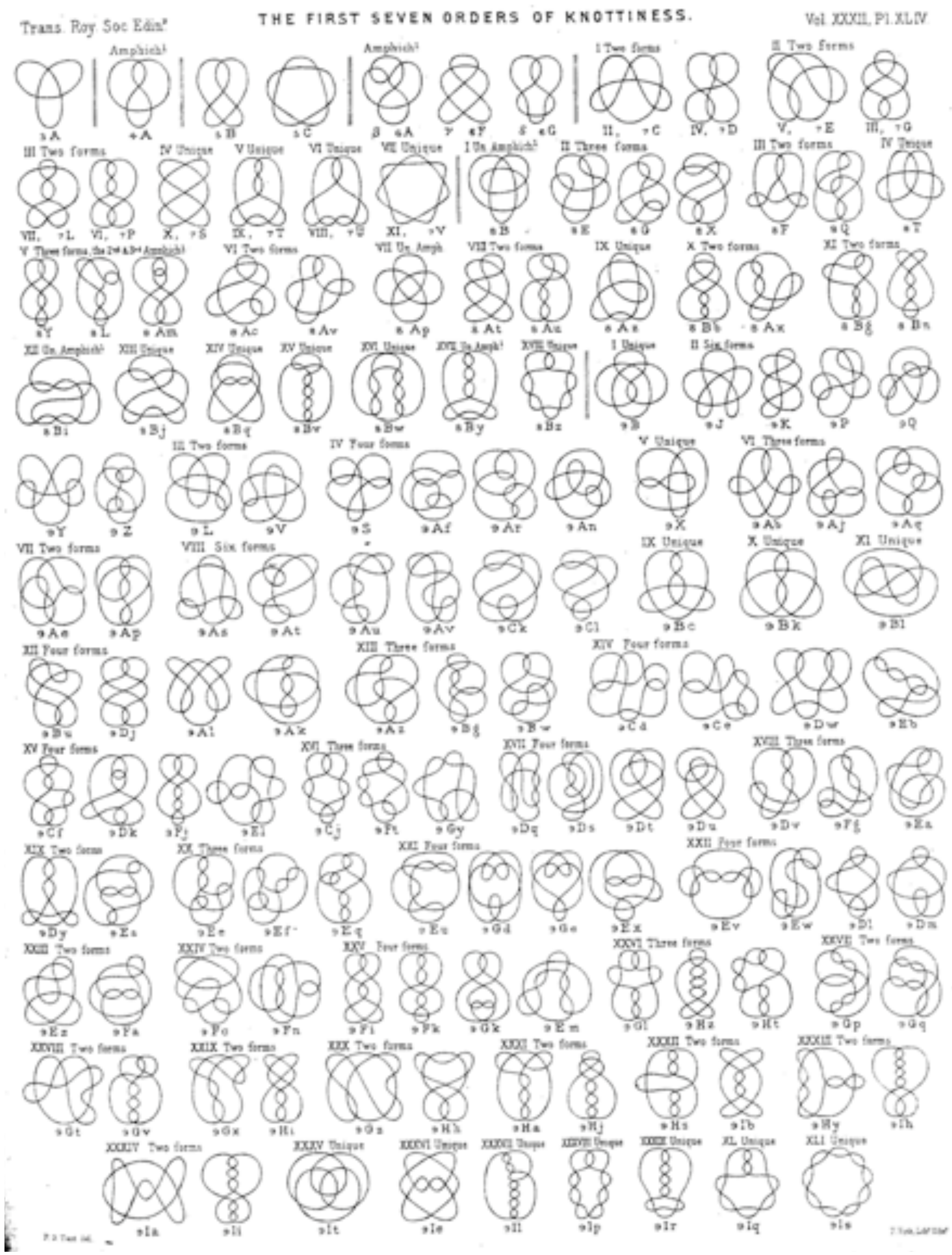


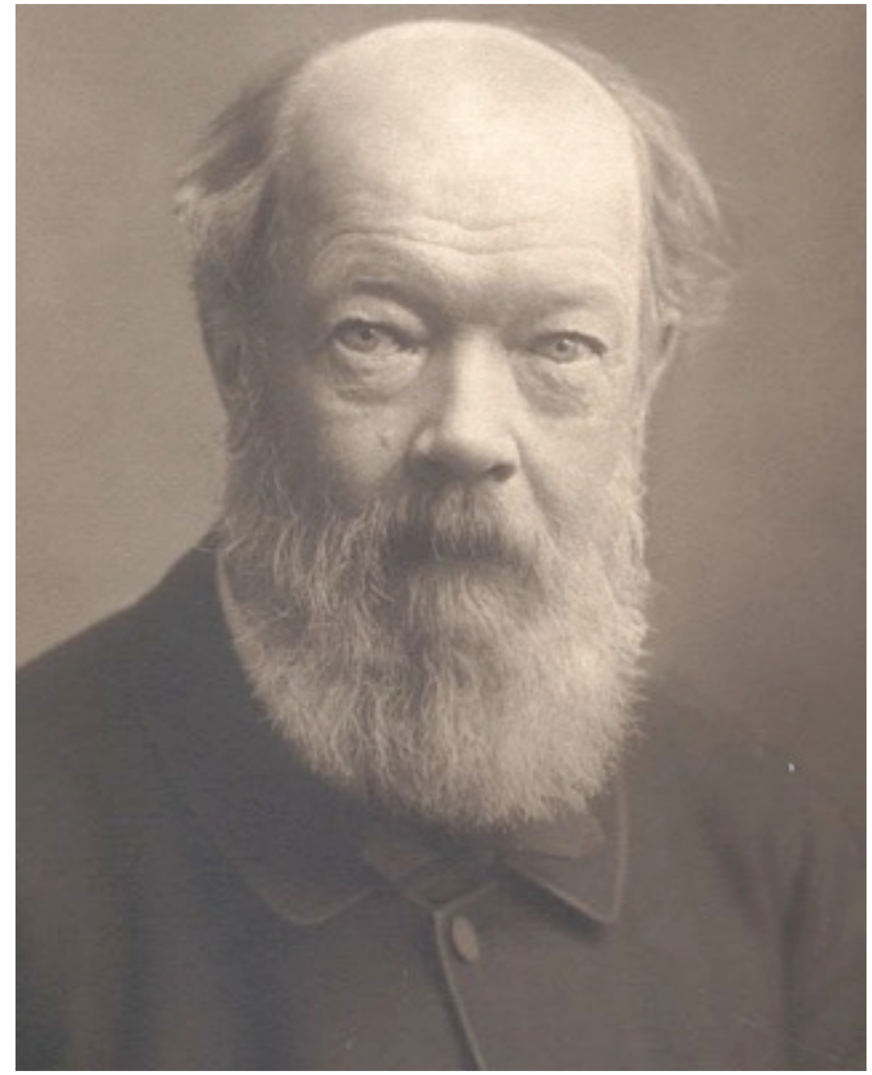
An obvious question is to decide whether two different looking pictures in fact represent the same knot

Modern knot theory really got going around 1867 when Lord Kelvin suggested that chemical elements could be interpreted as ‘vortices in the ether’

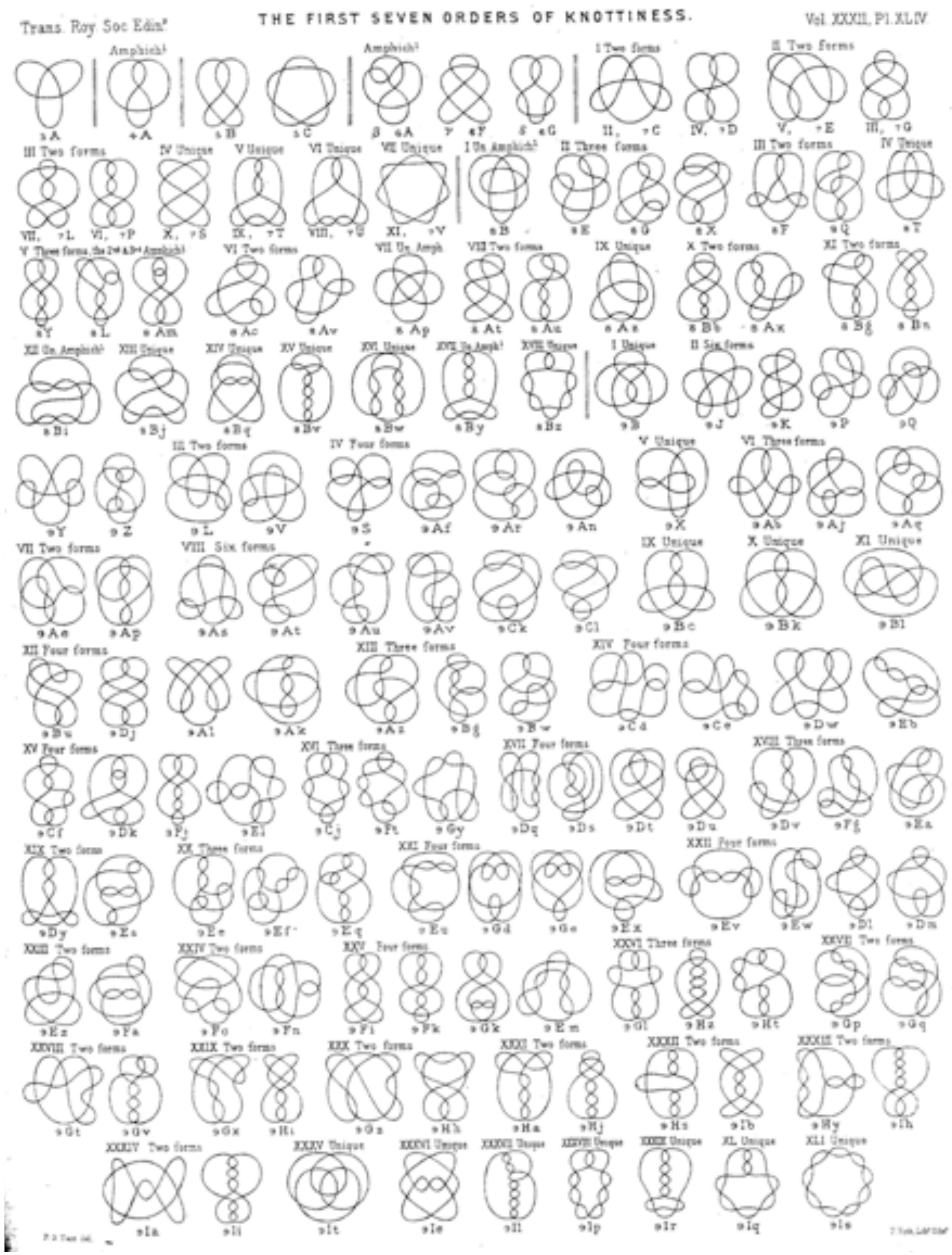


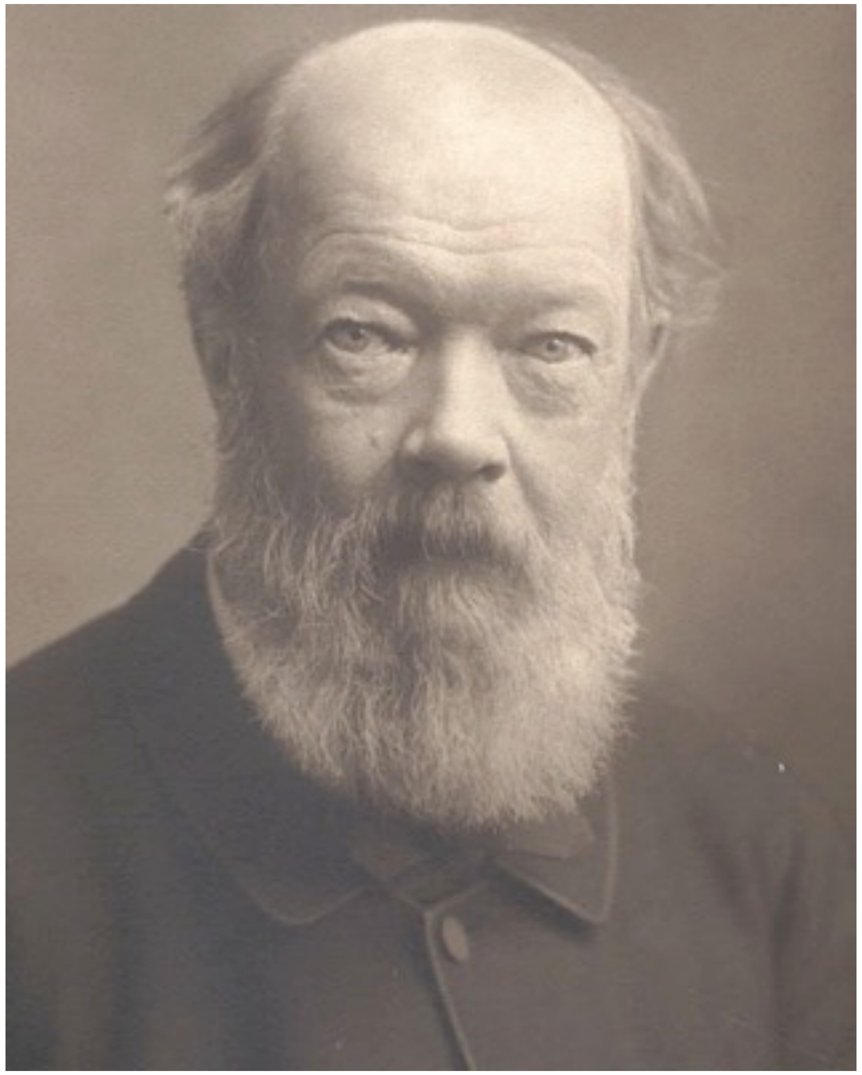
- Stability of matter
  knots are difficult to untangle
- Variety
  there are lots of different knots
- Emission Spectra
  vibrations of the knot





Peter Guthrie Tait decided he'd try to classify different knot types

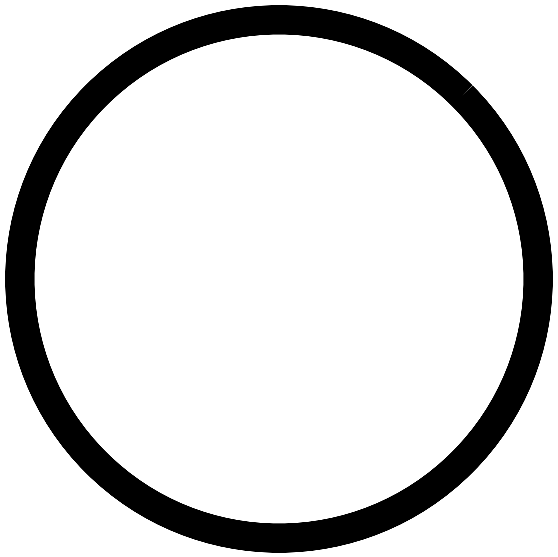




Peter Guthrie Tait decided he'd try to classify different knot types

How should the list be organised?

Tait organised knots in terms of their *crossing index* c[*K*]: the smallest number of crossings of any plane projection



c[*unknot*] = 0

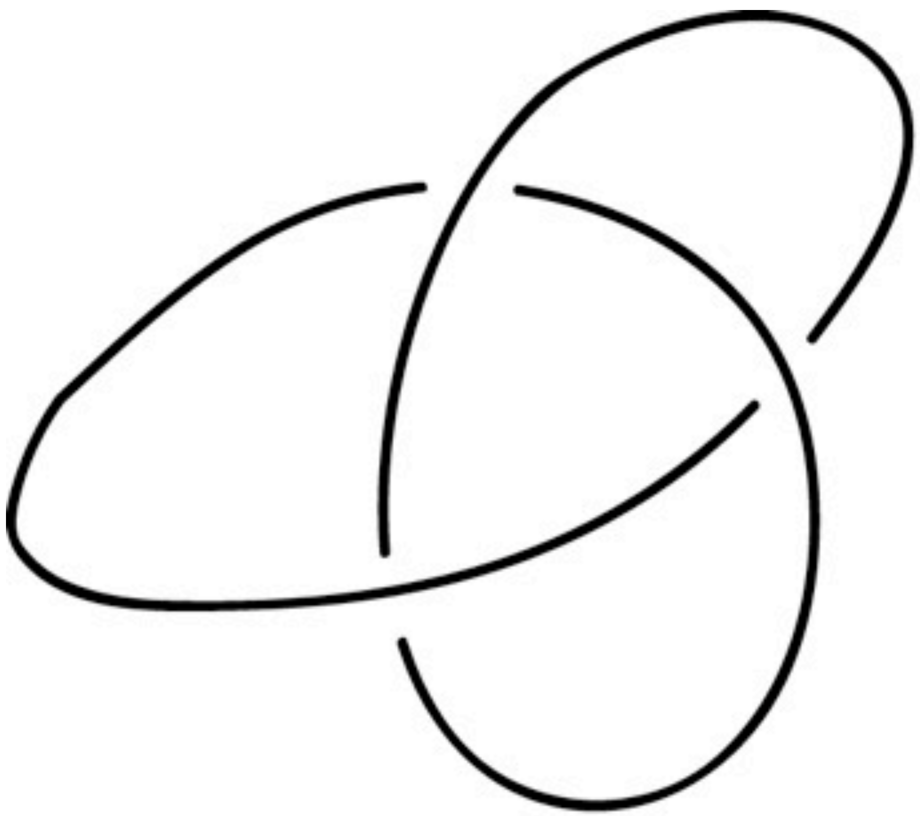
c[*trefoil*] = 3

The crossing index is an example of a *knot invariant*

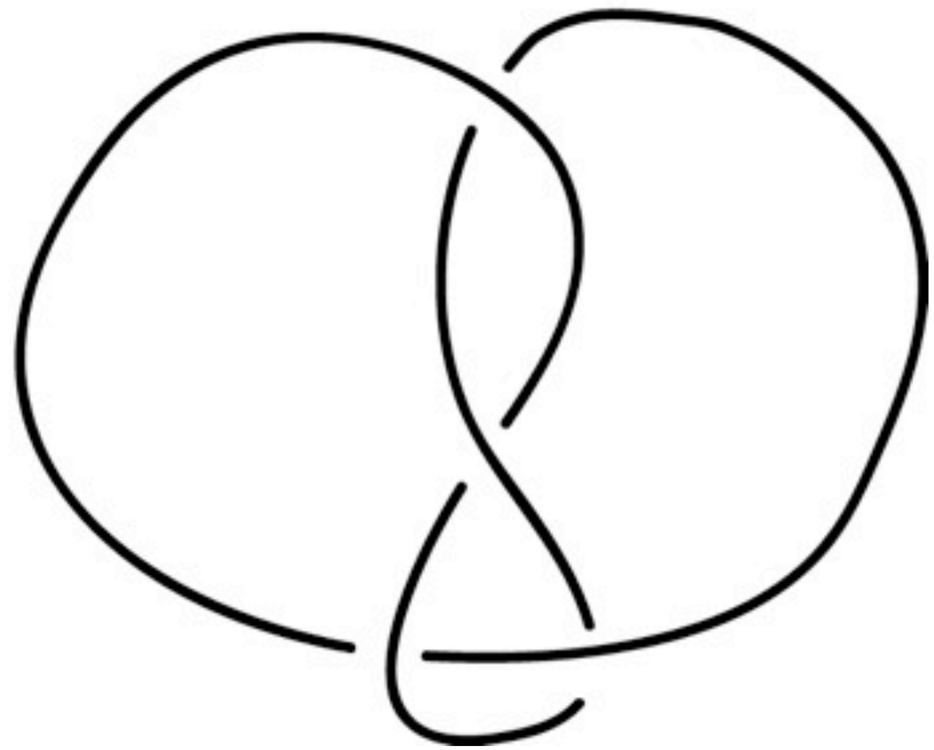
- If two knots have different crossing indices, they *can't* be smoothly deformed into eachother
- Different knots *can* have the same crossing index

Not only does the crossing index fail to tell all knots apart, it can also be very difficult to compute.

For example, we can combine two knots K_1 and K_2 to make a bigger one, called $K_1\#K_2$



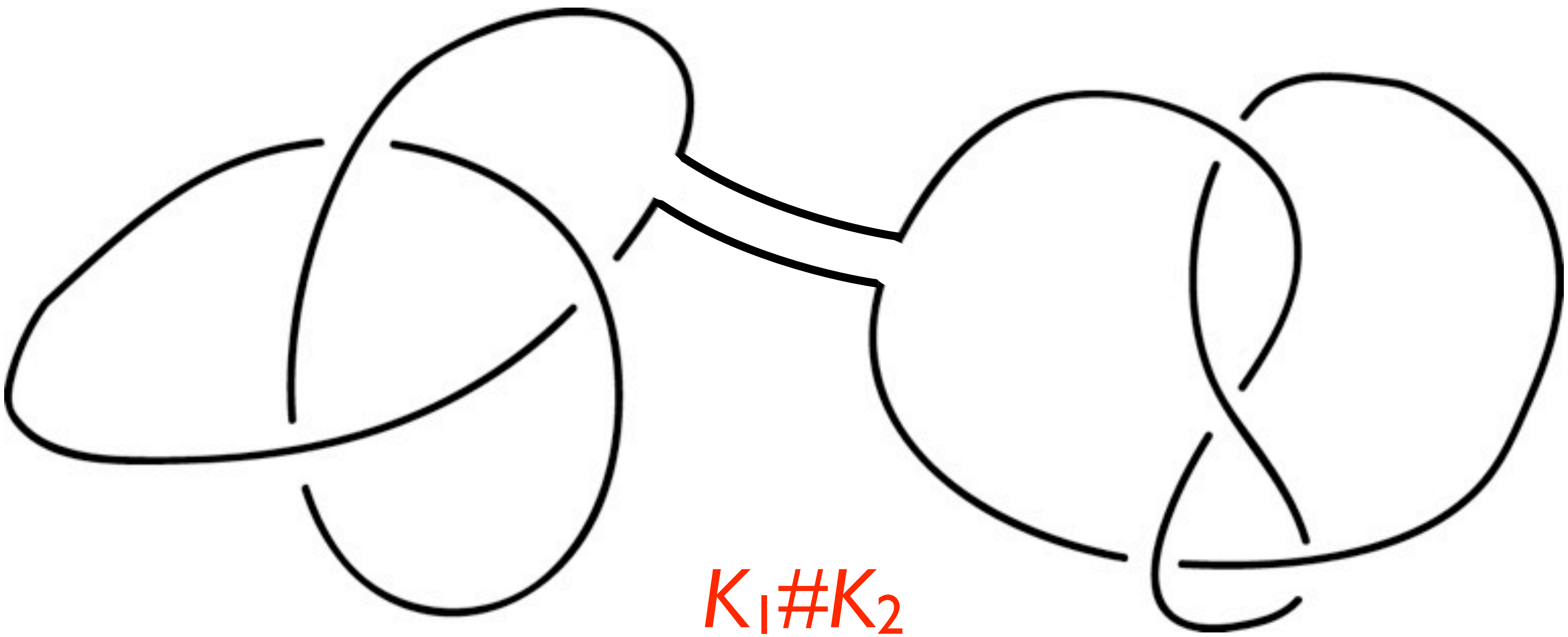
K_1



K_2

Not only does the crossing index fail to tell all knots apart, it can also be very difficult to compute.

For example, we can combine two knots K_1 and K_2 to make a bigger one, called $K_1 \# K_2$



$K_1 \# K_2$

No one knows whether $c[K_1 \# K_2] = c[K_1]+c[K_2]$ in general!

A more sophisticated 'list' is provided by labelling each knot by a *polynomial* instead of just one number

e.g.

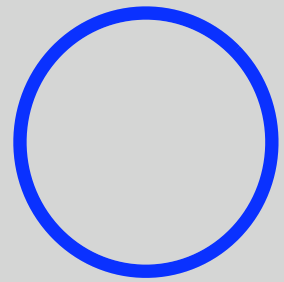
$$1 - 2x + 9x^4$$

$$\frac{7}{x^3} + \frac{2}{x} - 4x + x^2 + 8x^5$$

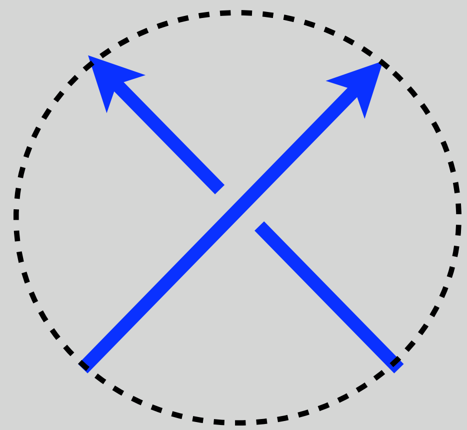
Different polynomials are supposed to correspond to different knots.

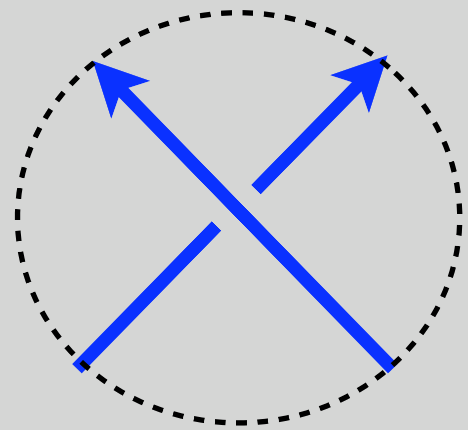
We need a rule to decide which polynomial to attach to which knot

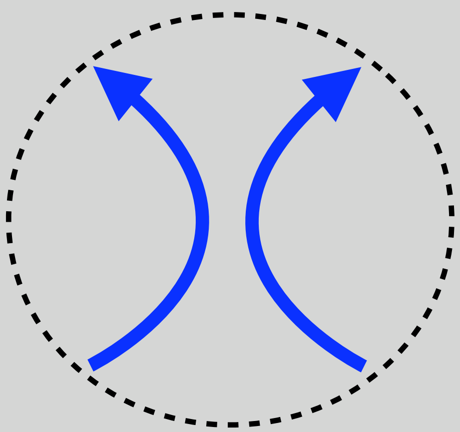
1) Pick a starting point

 $\Leftrightarrow \quad 1$

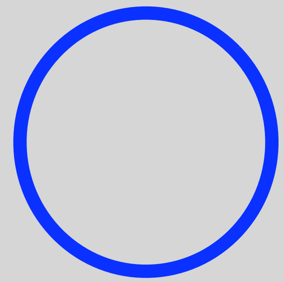
2) Choose a *skein relation*


L_+

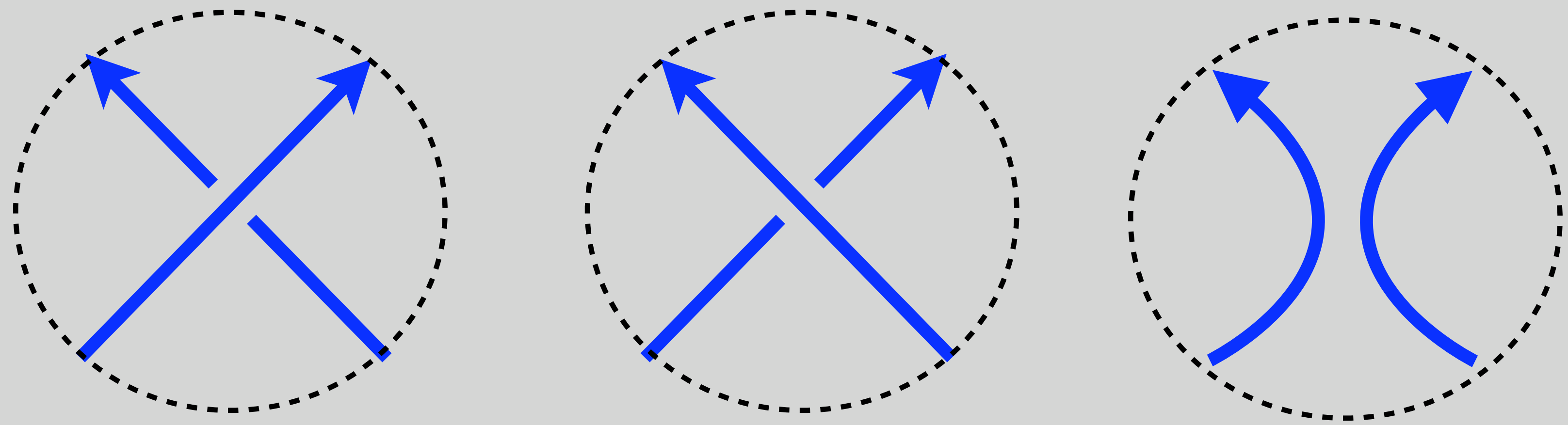

L_-


L_0

1) Pick a starting point

 $\Leftrightarrow \quad 1$

2) Choose a *skein relation*



L_+ $\qquad$ L_- $\qquad$ L_0

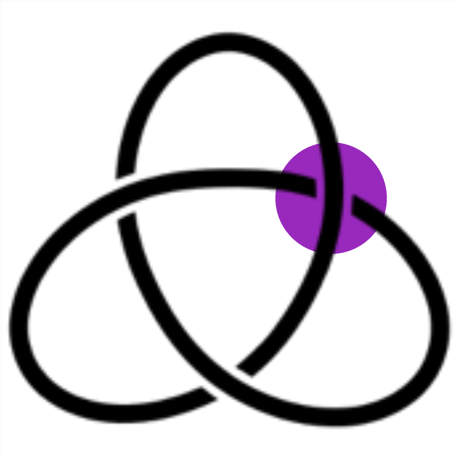
John Conway chose $C_{L_+} - C_{L_-} = xC_{L_0}$
to define the *Alexander-Conway Polynomial*

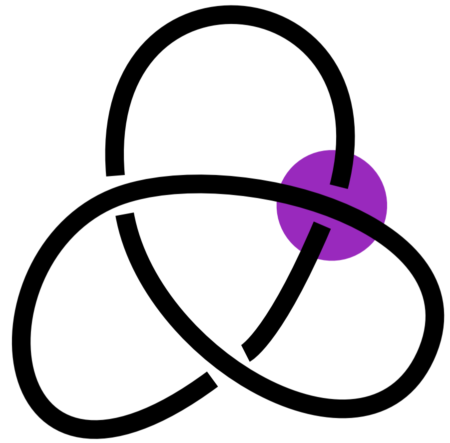
Vaughan Jones chose

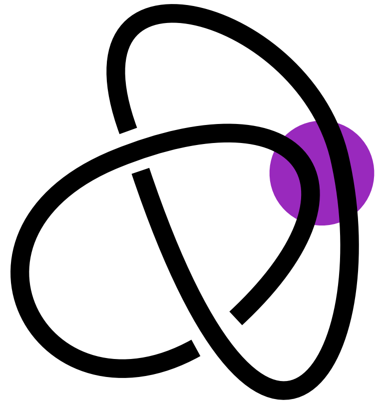
$$\frac{1}{x}J_{L_+} - xJ_{L_-} = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) J_{L_0}$$

and defined the *Jones Polynomial*

To see how this works, let's look at an example:







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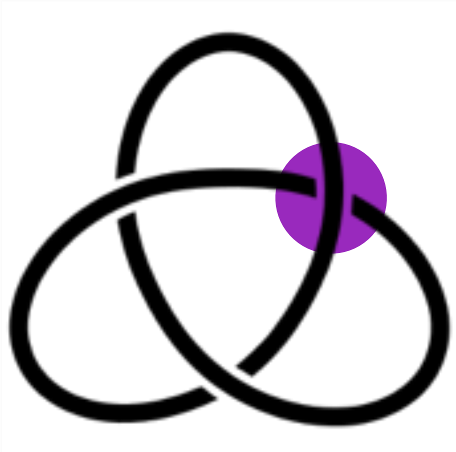
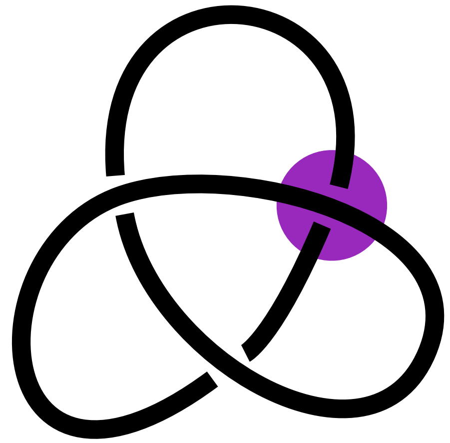
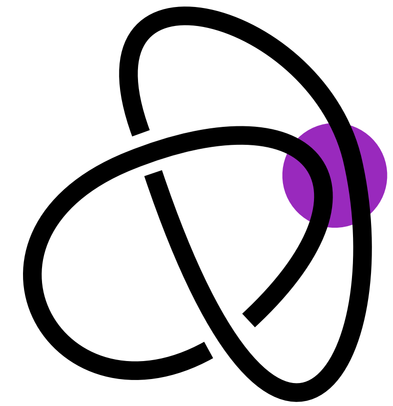
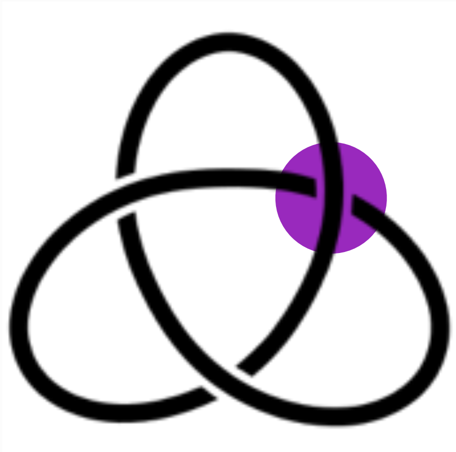


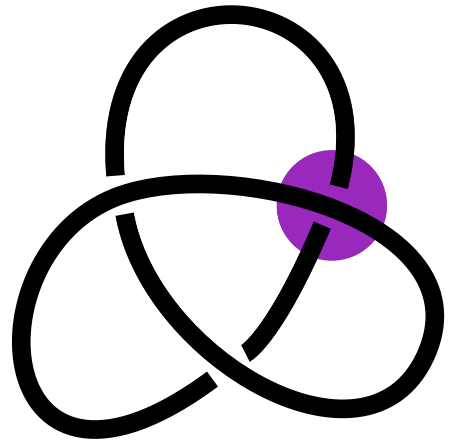


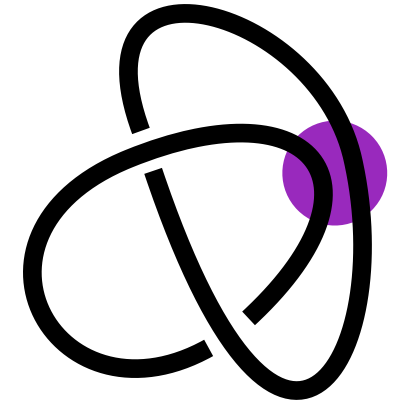


$$C_{\text{trefoil}} = C_A + x \times C_B$$

To see how this works, let's look at an example:

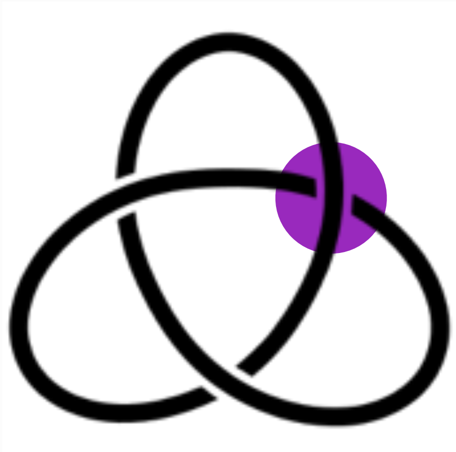
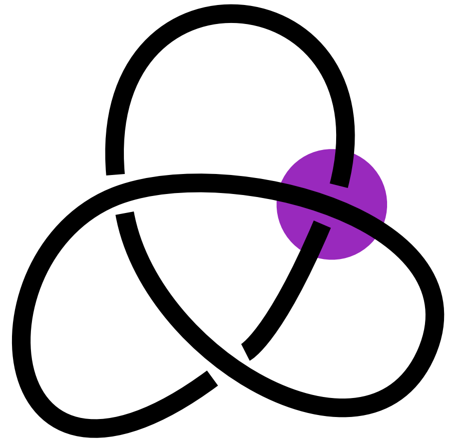
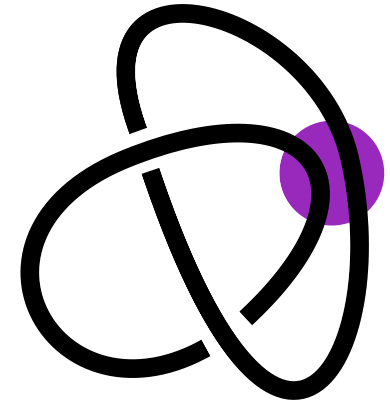




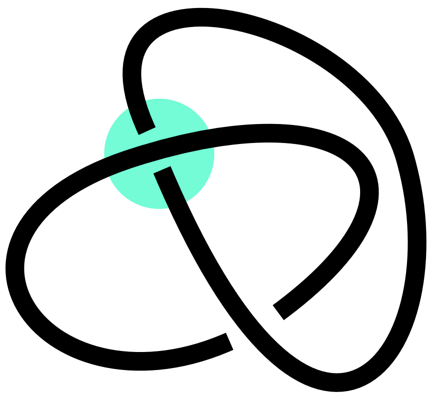


$$C_{\text{trefoil}} \quad = \quad 1 \quad + \quad x \times C_B$$

To see how this works, let's look at an example:

$$C_{\mathrm{trefoil}} \quad = \quad 1 \quad + \quad x \times C_B$$



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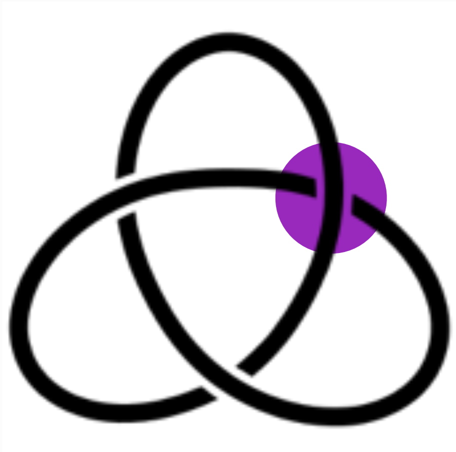
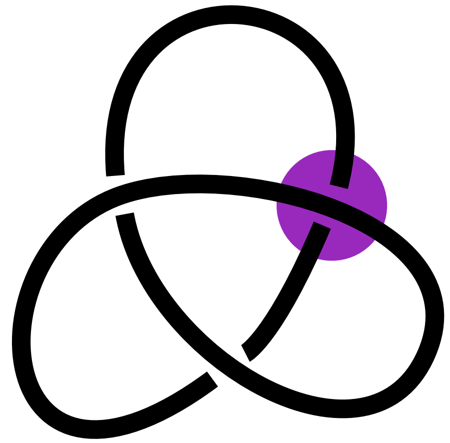
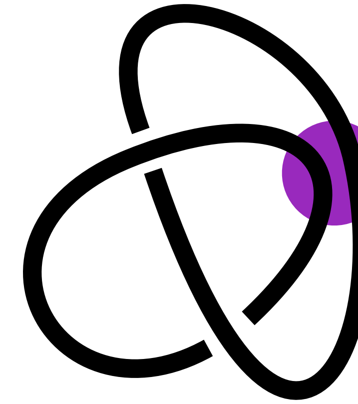
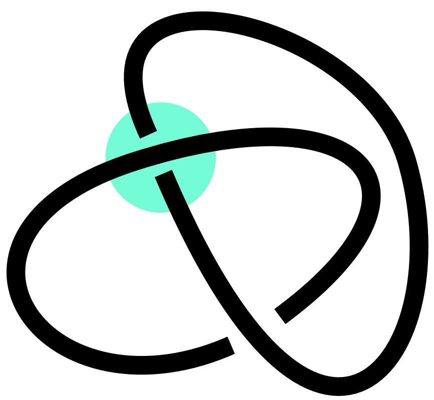
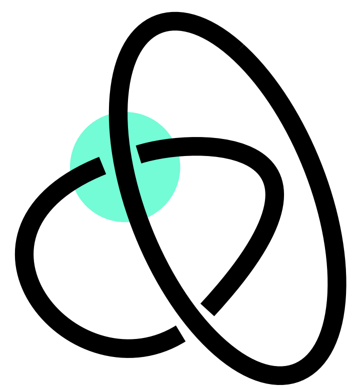
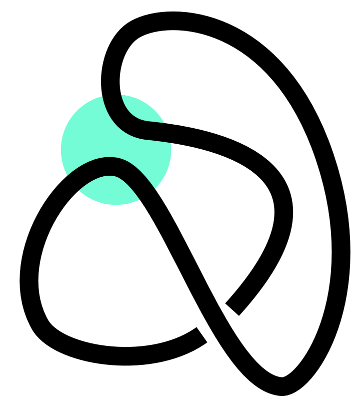
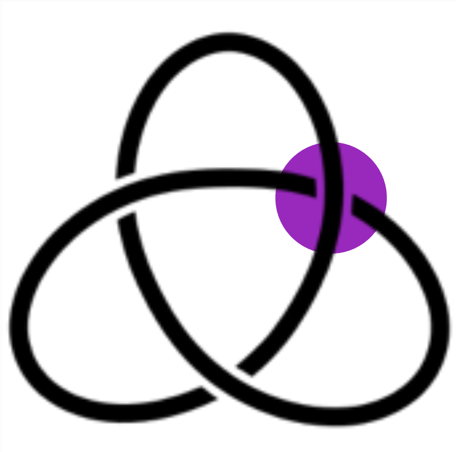
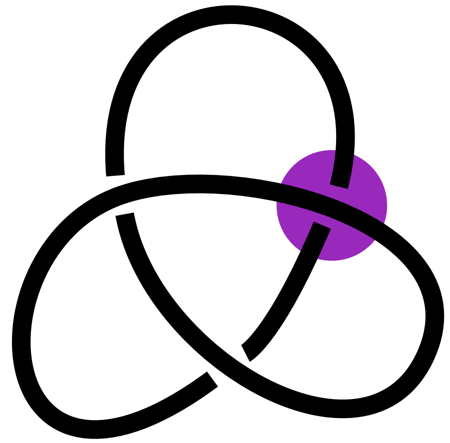
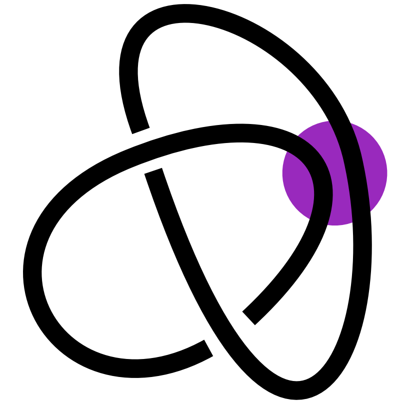
  

$$C_{\text{trefoil}} = 1 + x \times C_B$$

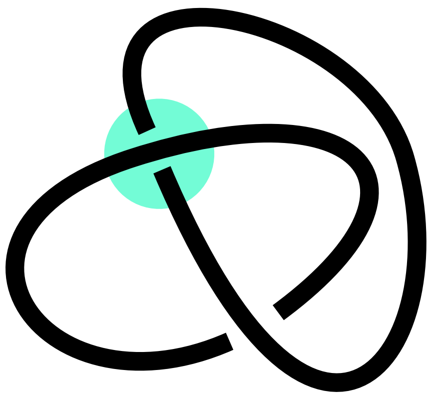
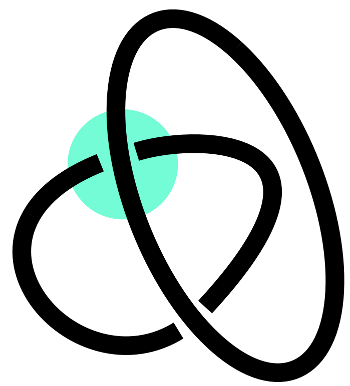
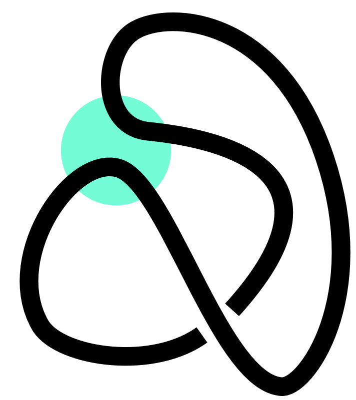
  

$$C_B = C_D + x \times C_E$$

To see how this works, let's look at an example:

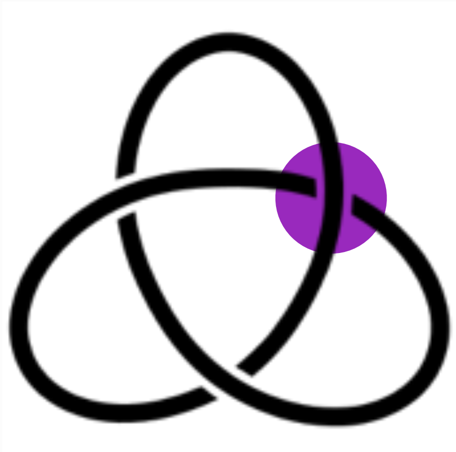
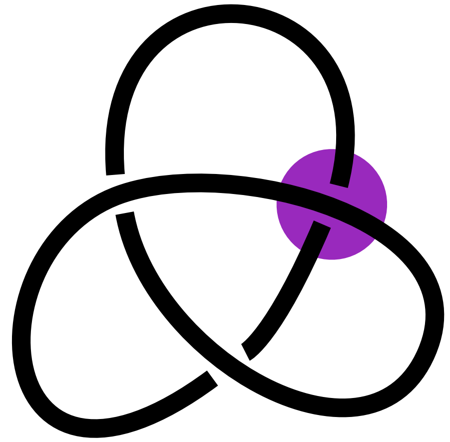
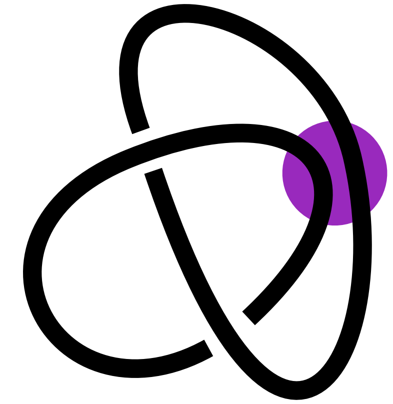




$$C_{\text{trefoil}} \quad = \quad 1 \quad + \quad x \times C_B$$

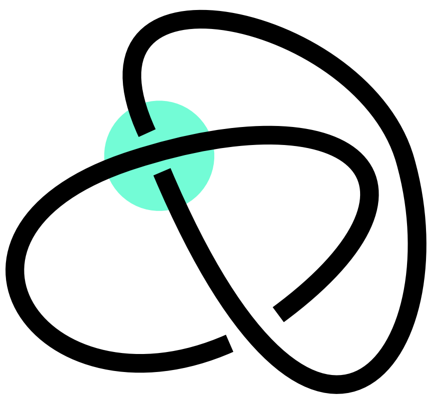
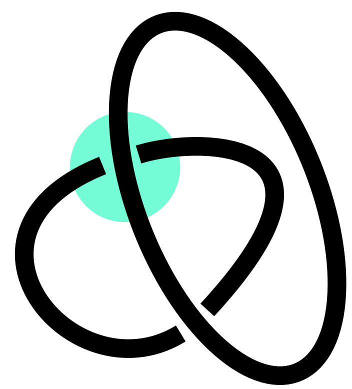
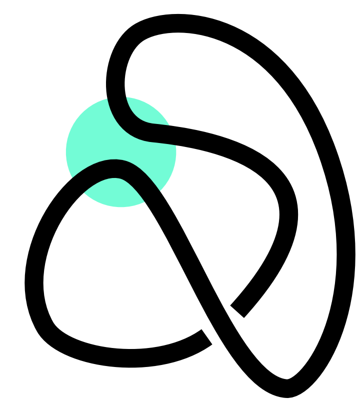




$$C_B \quad = \quad C_D \quad + \quad x \times 1$$

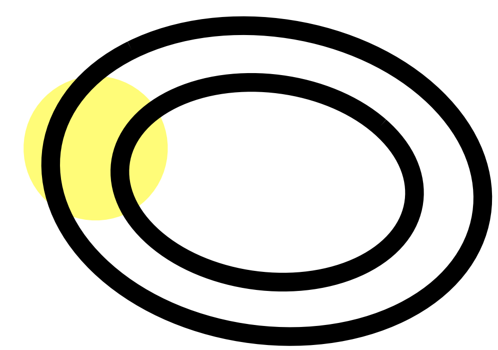
To see how this works, let's look at an example:

$$C_{\text{trefoil}} = 1 + x \times C_B$$

$$C_B = C_D + x \times 1$$



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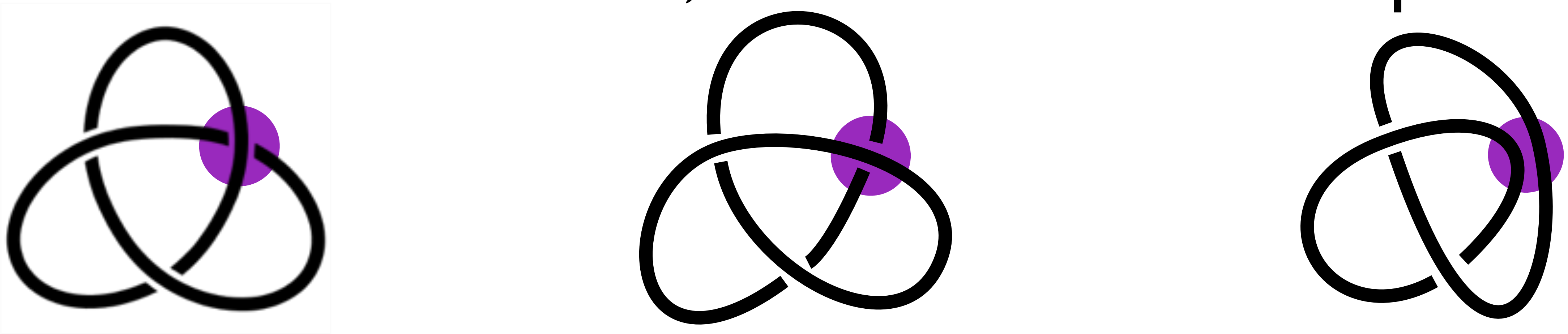
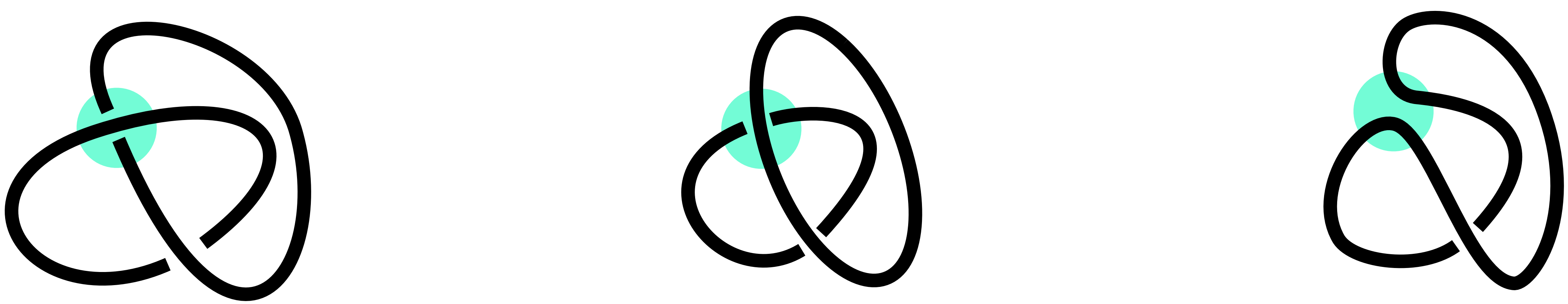
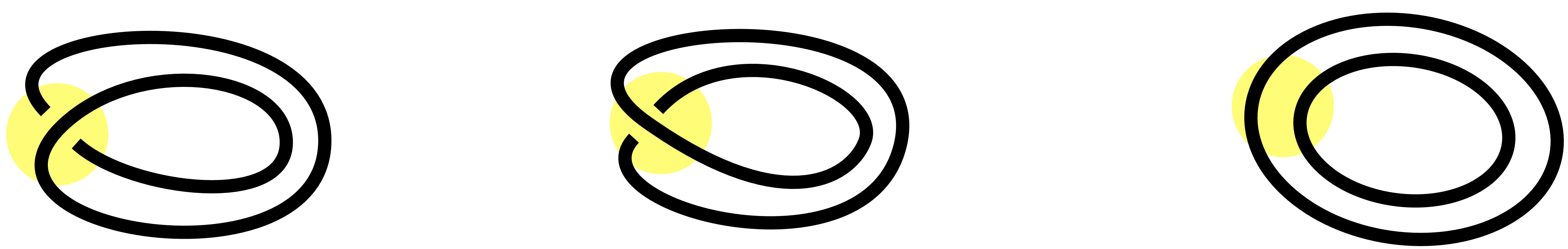


$$C_{\text{trefoil}} = 1 + x \times C_B$$

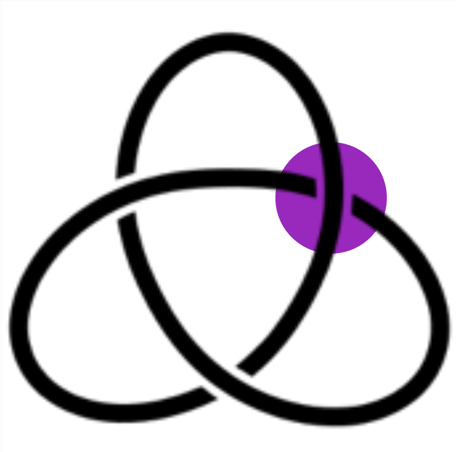
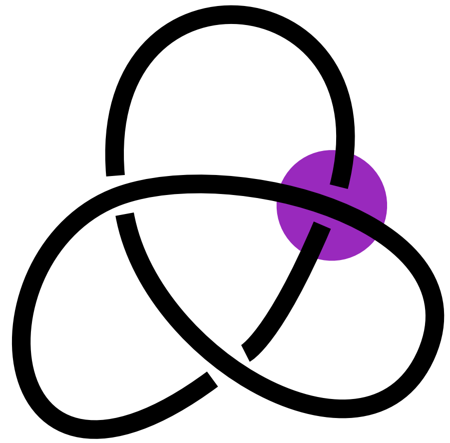
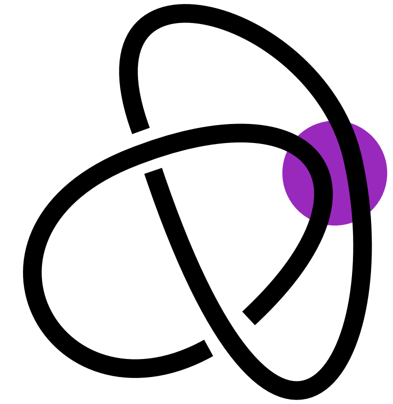


$$C_B = C_D + x \times 1$$

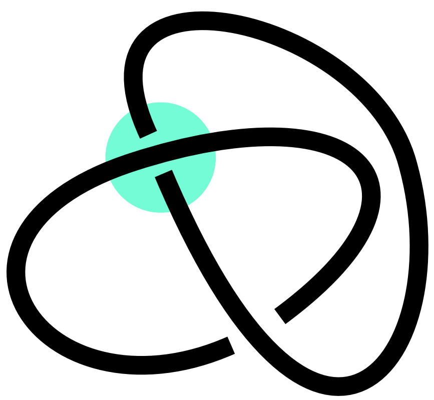
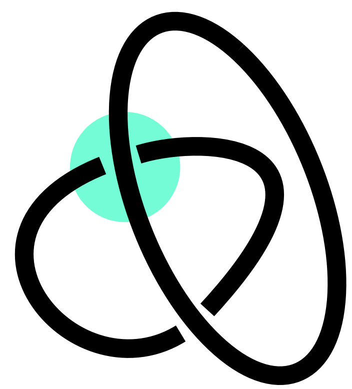
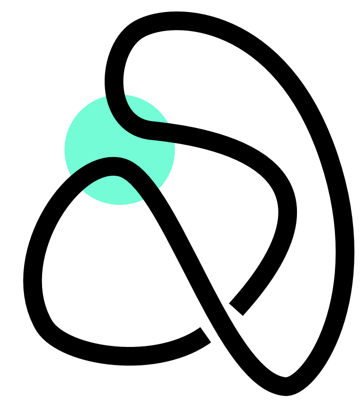


$$1 = 1 + x \times C_D$$

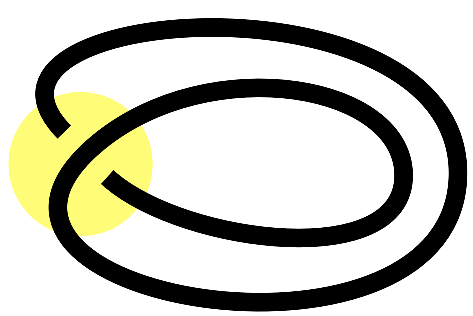
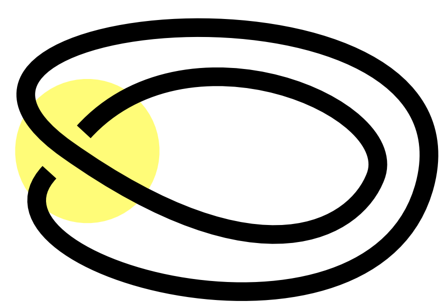
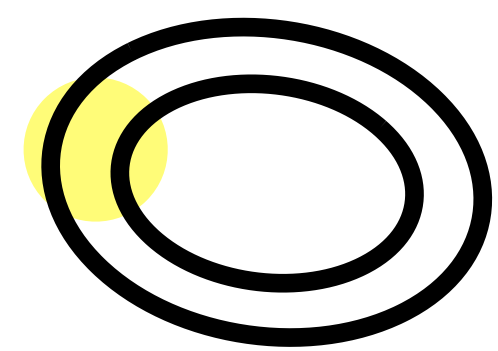
To see how this works, let's look at an example:

$$C_{\text{trefoil}} = 1 + x \times C_B$$

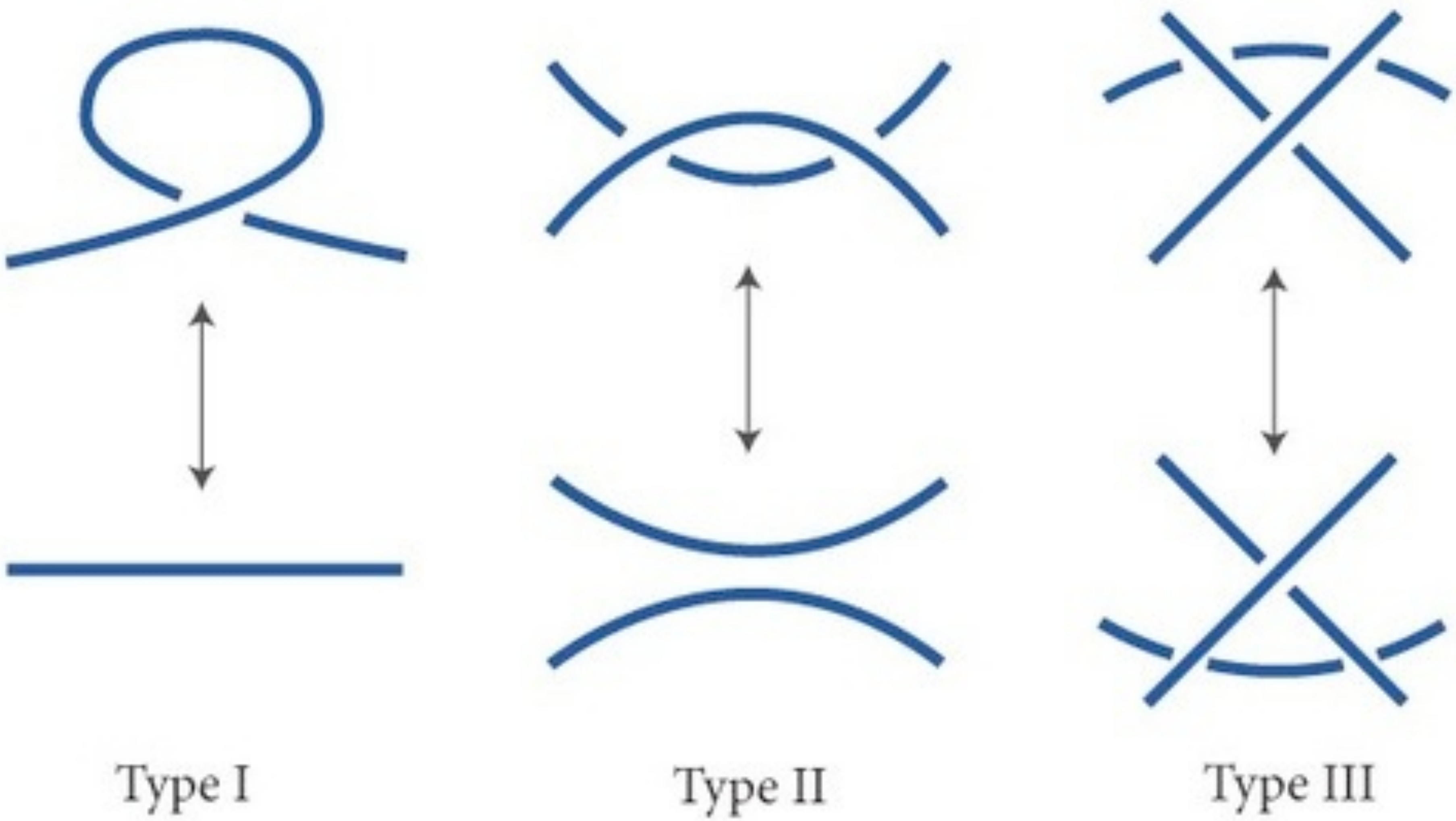
$$C_B = C_D + x \times 1$$

$$1 = 1 + x \times C_D$$

So the Conway polynomial for the trefoil $C_{\text{trefoil}} = 1 + x^2$

If you want to have a go at defining your own knot polynomial, there's just one thing you need to check:

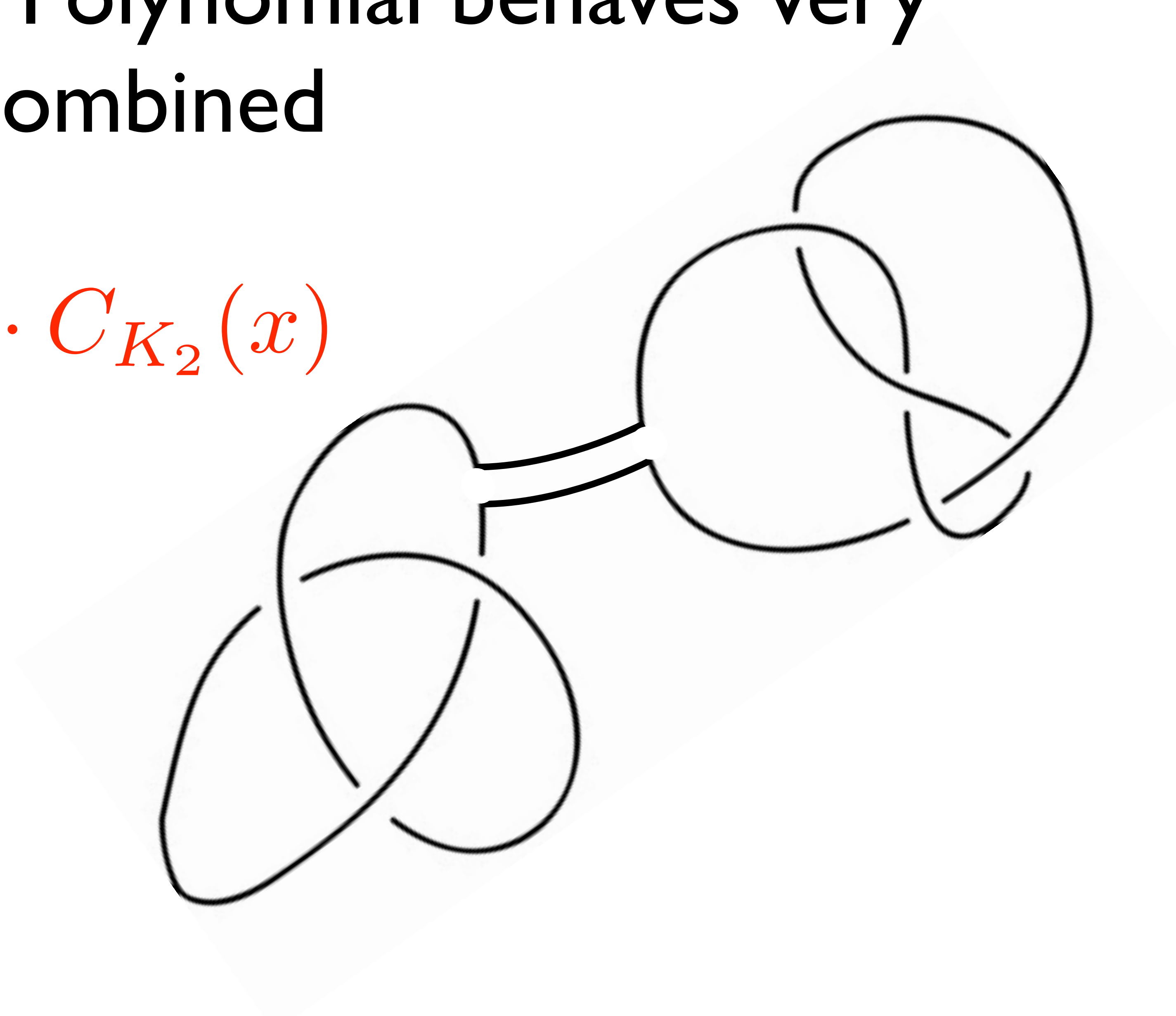


The *Reidemeister moves* are ways to change your knot picture - you have to check your polynomial *doesn't* change when you do a Reidemeister move (it's not easy!)

The Alexander-Conway Polynomial behaves very nicely when knots are combined

$$C_{K_1 \# K_2}(x) = C_{K_1}(x) \cdot C_{K_2}(x)$$

(so too does the Jones Polynomial)



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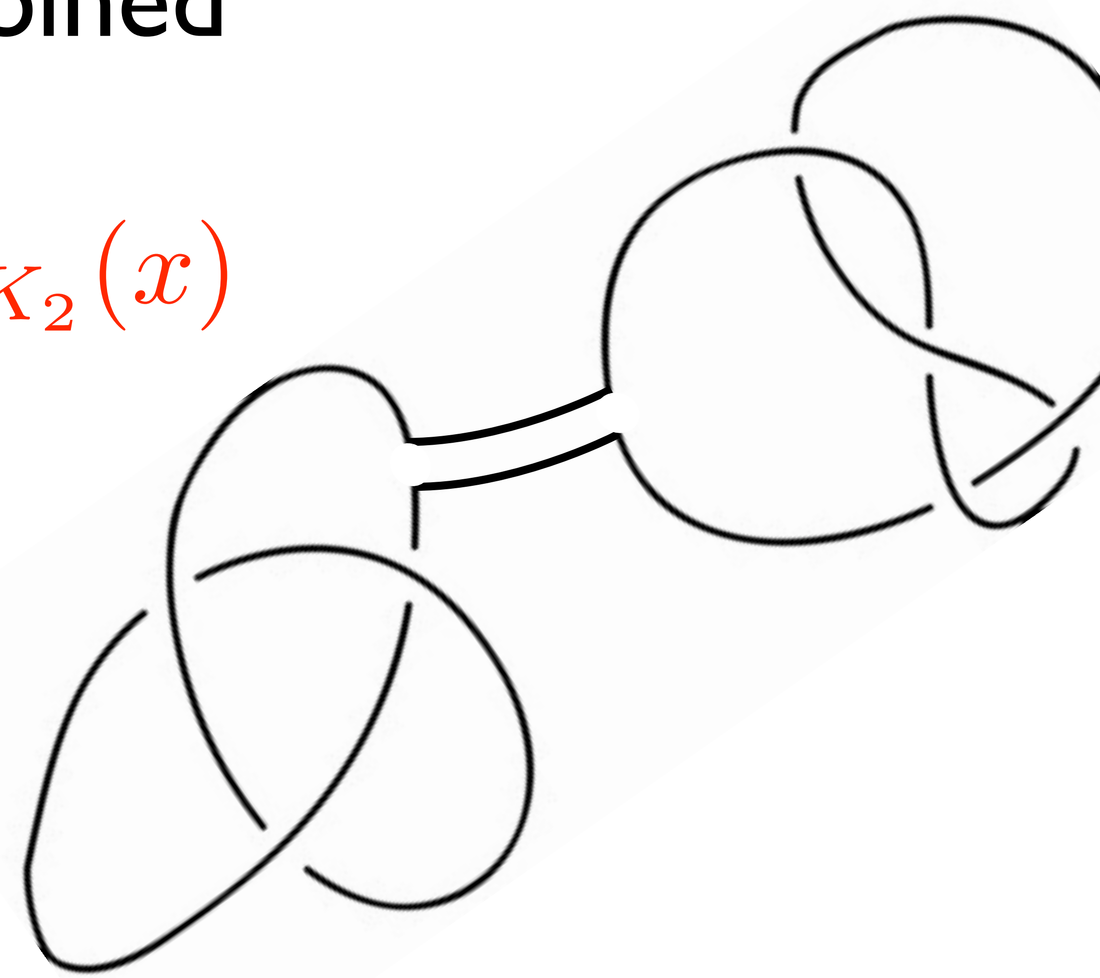

(so too does the Jones Polynomial)

In fact, just like the positive integers, knots obey a beautiful 'prime factorization theorem':

Any knot can be written as a combination (under #) of prime knots, in a way that's unique up to ordering.

However, even these knot polynomials still can't distinguish all knots

The Alexander-Conway polynomial of each of these knots is 1, so it can't even tell they're knotted at all!

The most powerful knot invariants come from a surprising direction - *Quantum Field Theory*

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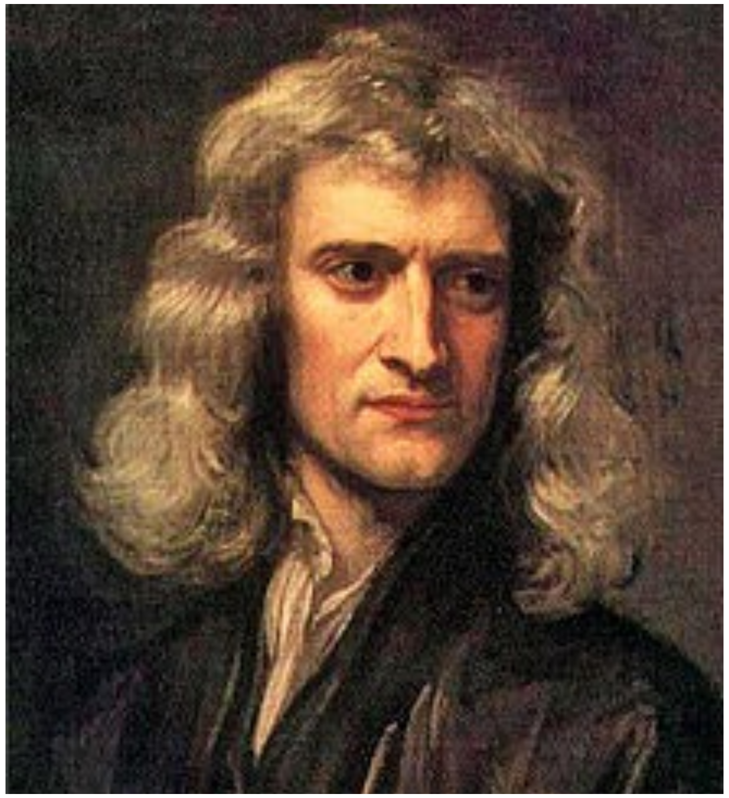

In classical physics, Newton taught us to use *F = ma* to work out how a particle travels from A to B

In quantum physics, we instead use the *Feynman path integral*

$$\langle A|B\rangle = \int_{\mathrm{paths}} \mathcal{D}x \ \mathrm{e}^{\mathrm{iS}}$$

Classically, once we know the forces acting, a particle's path is uniquely determined

B

A

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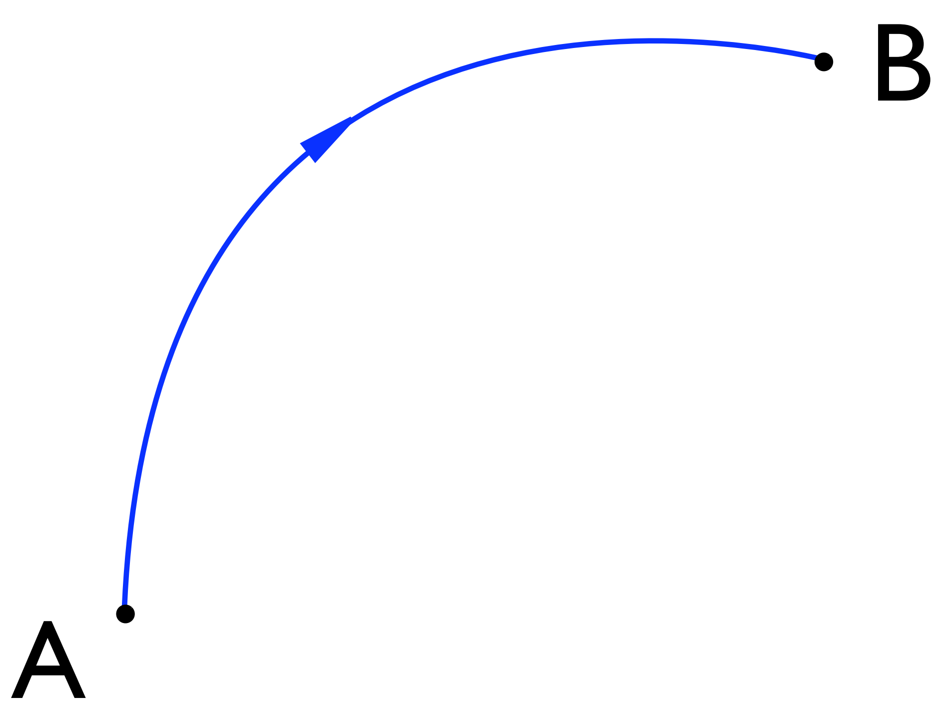

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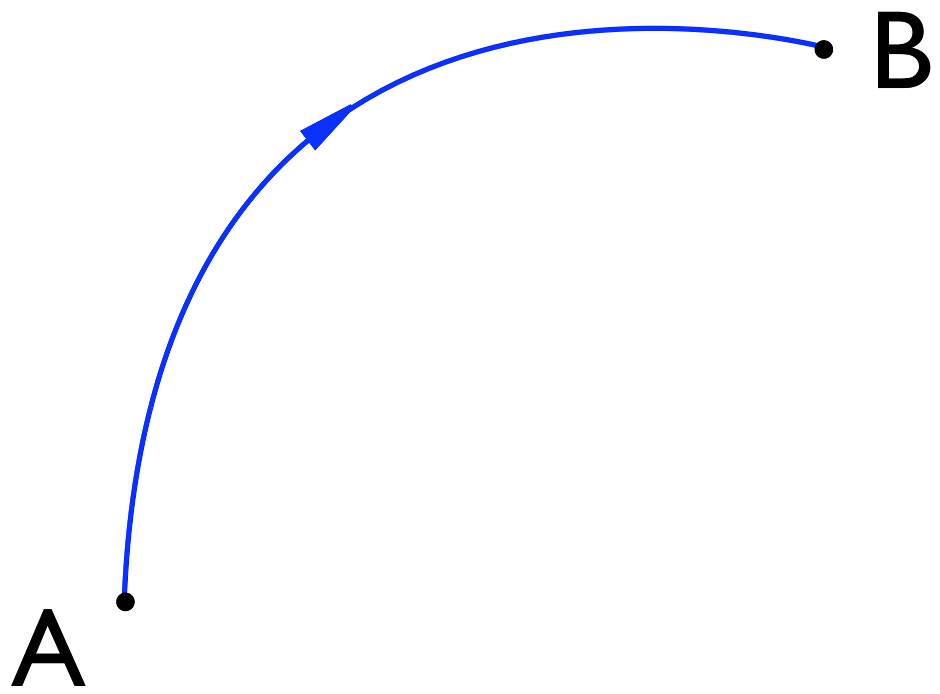
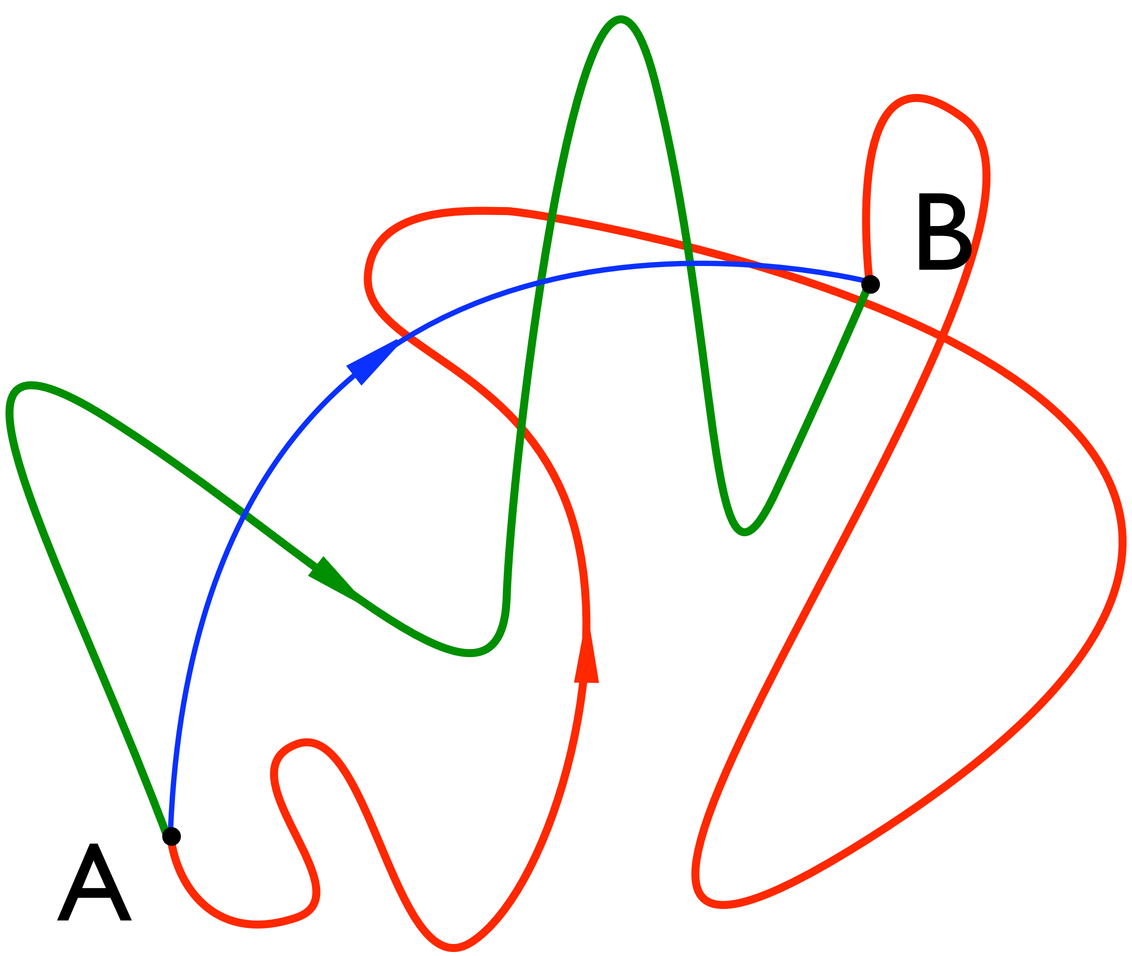
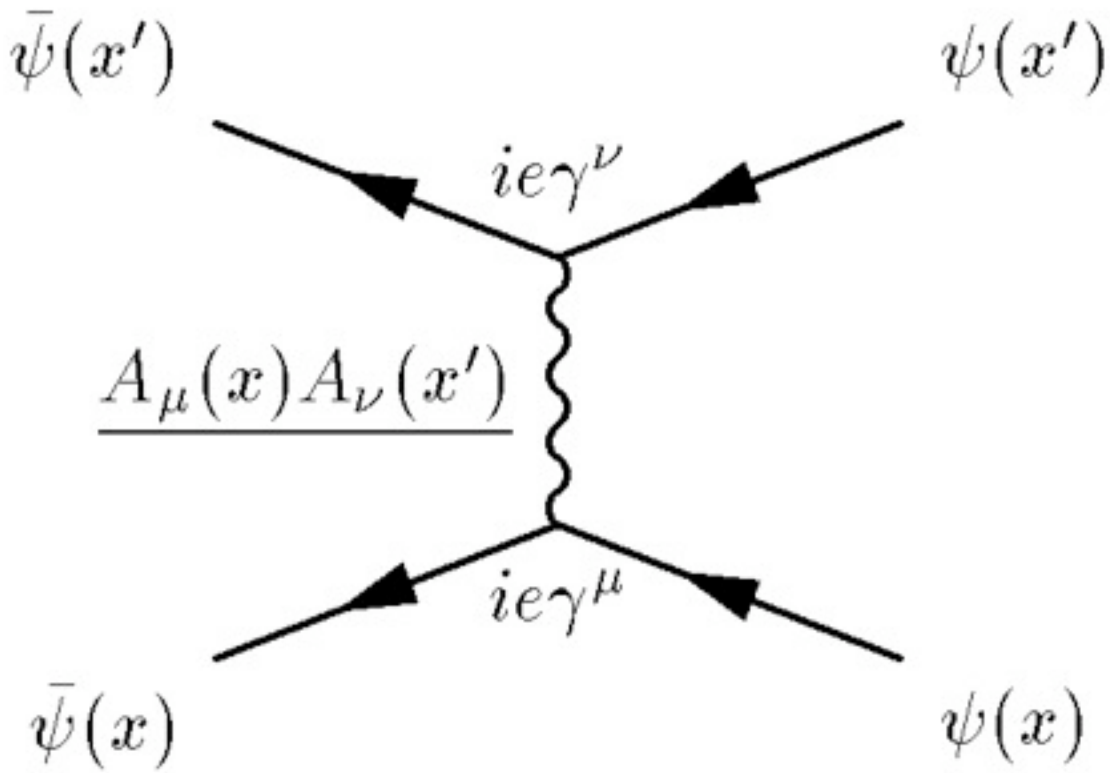


In quantum theory, there's always a chance for it to take *any* path, no matter how crazy

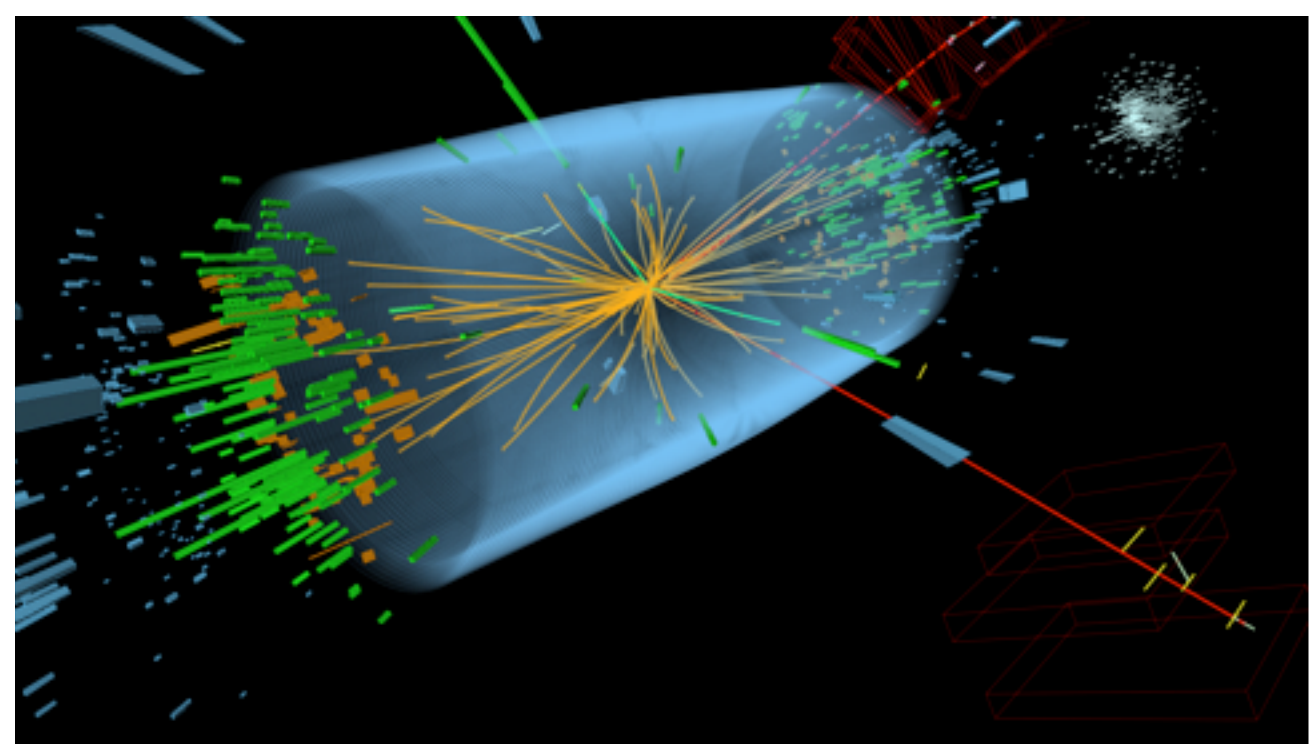


Exactly what the probability is for any given process depends on the forces acting



By changing what you think the force may be, you change the predicted outcome of experiments

Physicists at CERN use this to determine the structure of subnuclear forces



Usually, the force between two charges decreases as the charges move apart

e.g. Coulomb's law $\mathbf{F} = \dfrac{q_1 q_2}{r^2}\hat{\mathbf{r}}$ states that the force between two electrical charges falls as the inverse square of the distance between them

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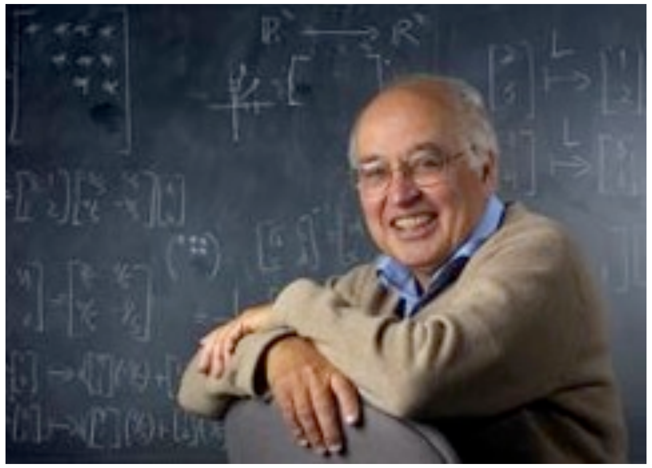
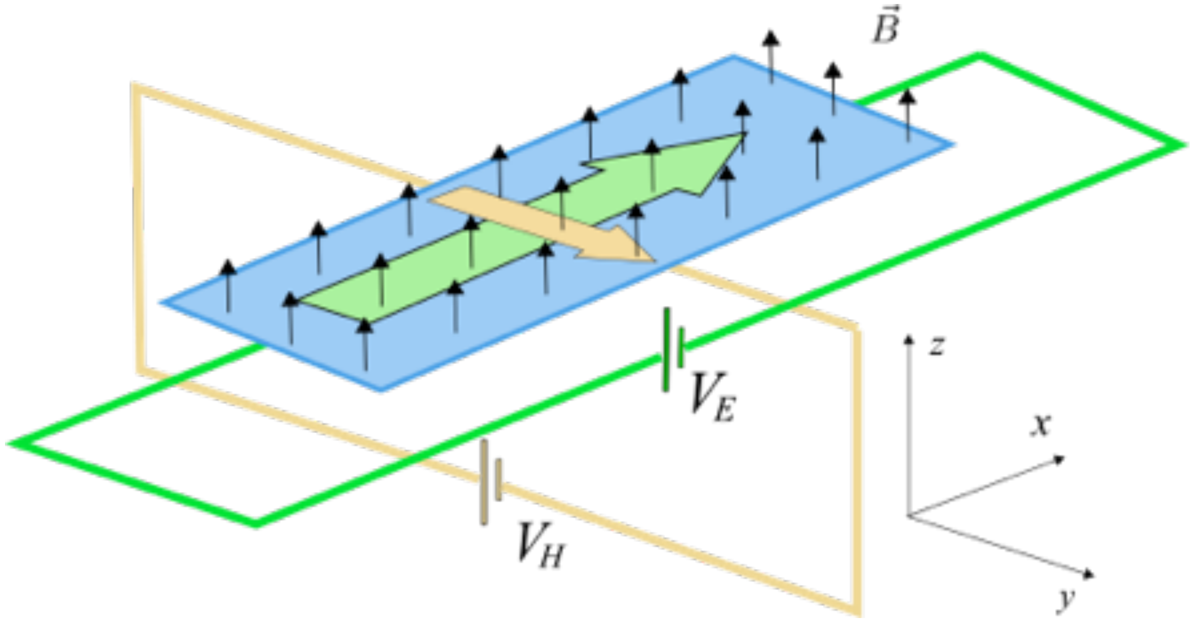
But there are exotic theories in which distance doesn't matter - *only whether the particle travels in a knot*!



The simplest version of this theory is actually a cousin of electomagnetism, and is important in the *Quantum Hall Effect*



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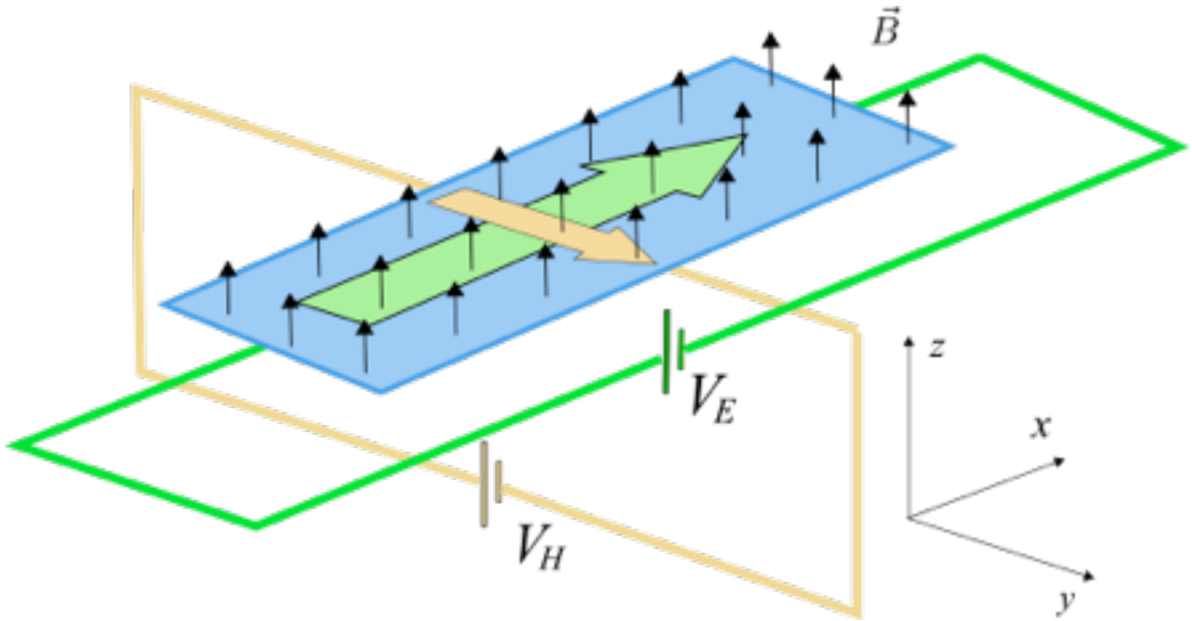
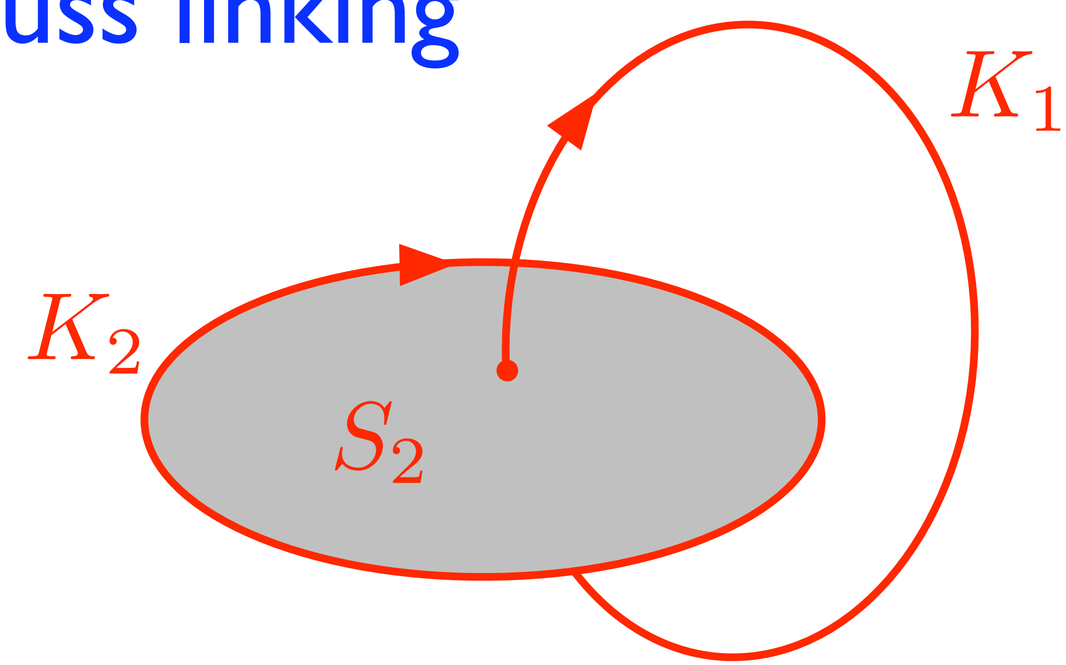


It also computes classical link invariants such as Gauss linking



$$\mathrm{lk} = \sum_{K_1 \cup S_2} \pm 1$$

The next simplest version is a relative of the *weak nuclear force* responsible for much of radioactivity

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$$J_K = \int \mathrm{D}A\, \mathrm{e}^{-S_{\mathrm{CS}}[A]}\, \mathrm{tr}_R\, \mathrm{Hol}_K[A]$$

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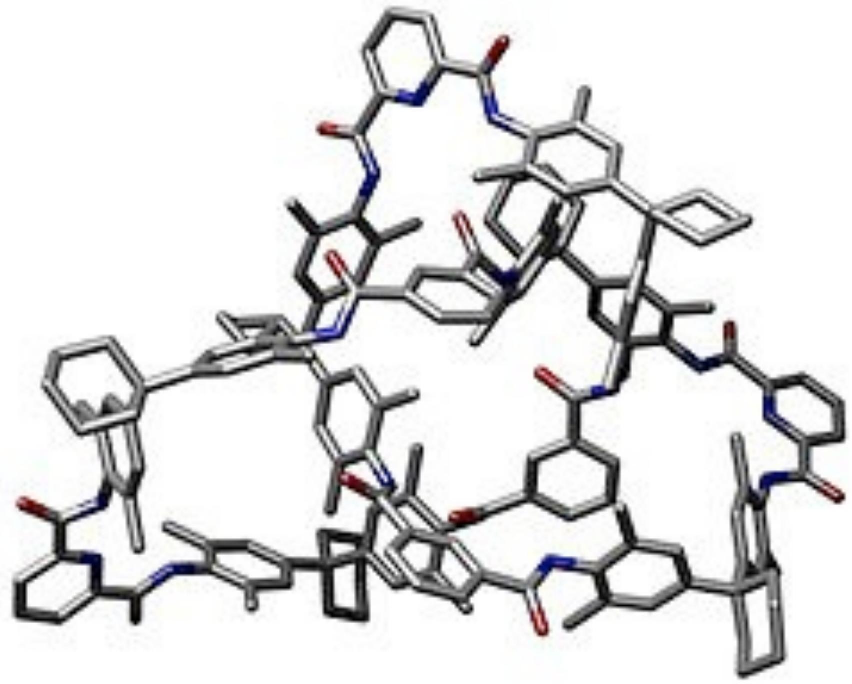
There are more complicated versions still, which compute knot invariants such as the HOMFLYPT Polynomials, Quantum Groups, Khovanov Homology...

The most powerful of these - Khovanov homology - is finally believed to distinguish all knots

Recently, knots have also begun to play a role in chemistry



Chemists can now design new complex molecules that are knotted.

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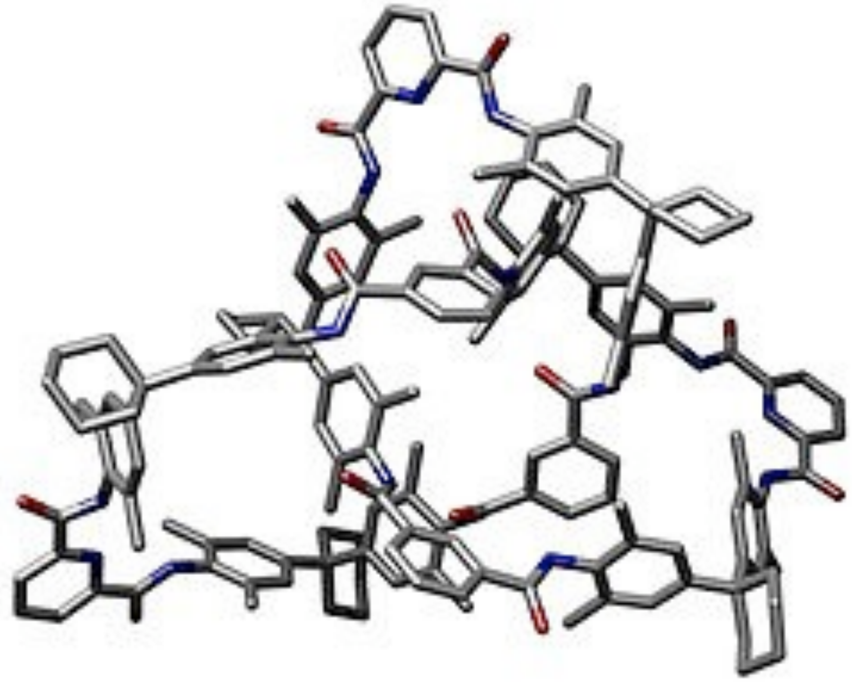
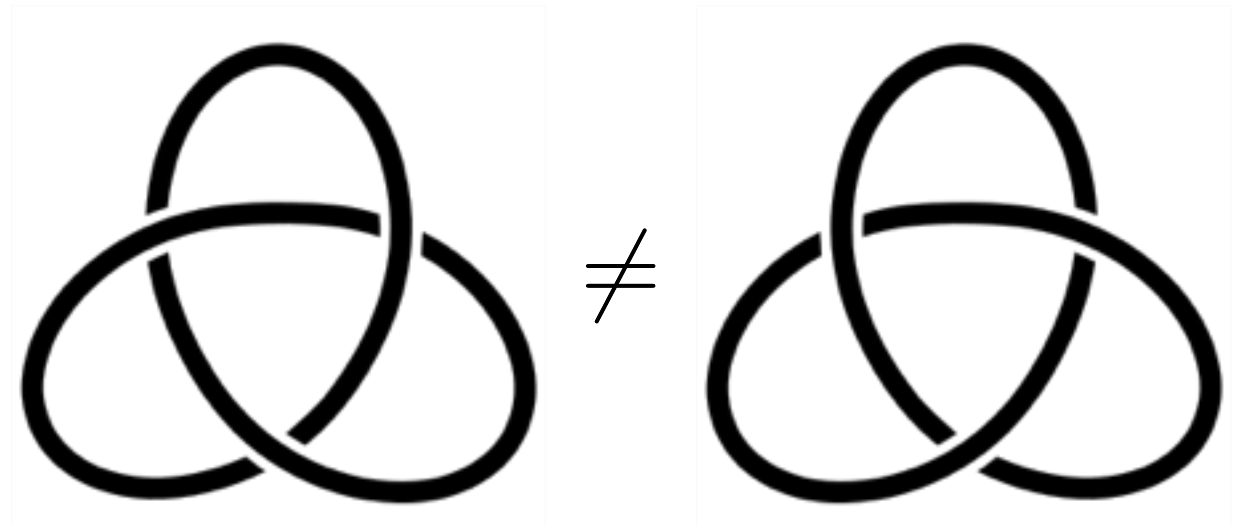


Chemists can now design new complex molecules that are knotted.

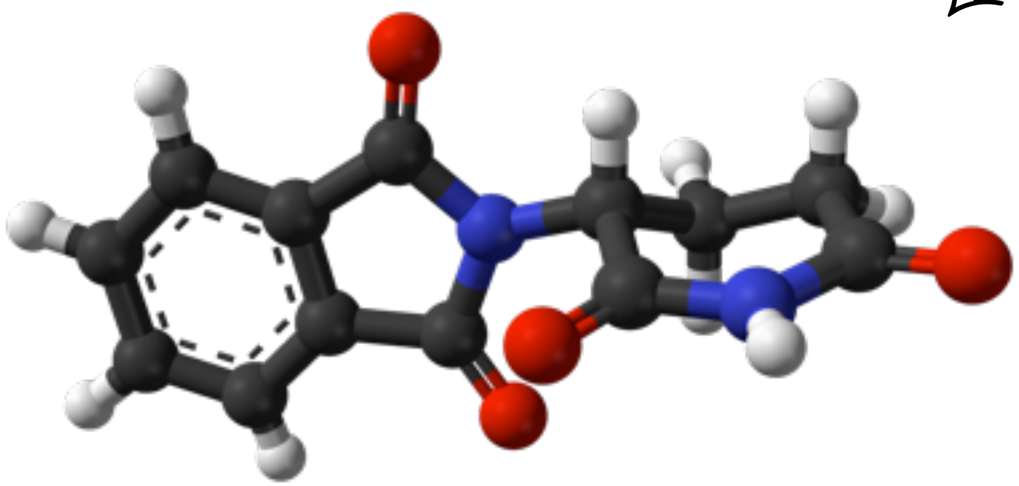
Could the topological properties of the knot affect the chemical properties of the molecule?

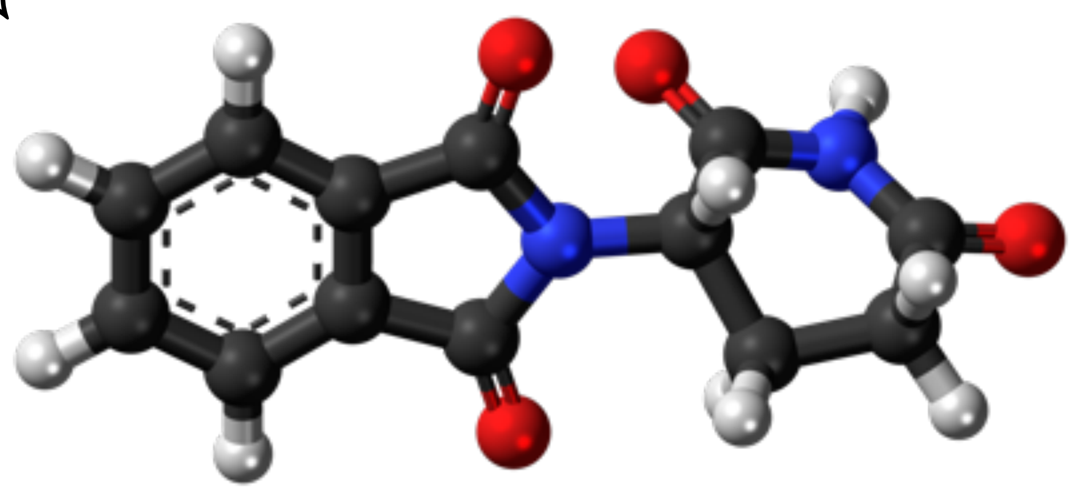
The trefoil knot is *chiral*. The Jones polynomials of the two chiralities are different - they are different knots



The chirality of chemical isomers is extremely important

$C_{13}H_{10}N_2O_4$





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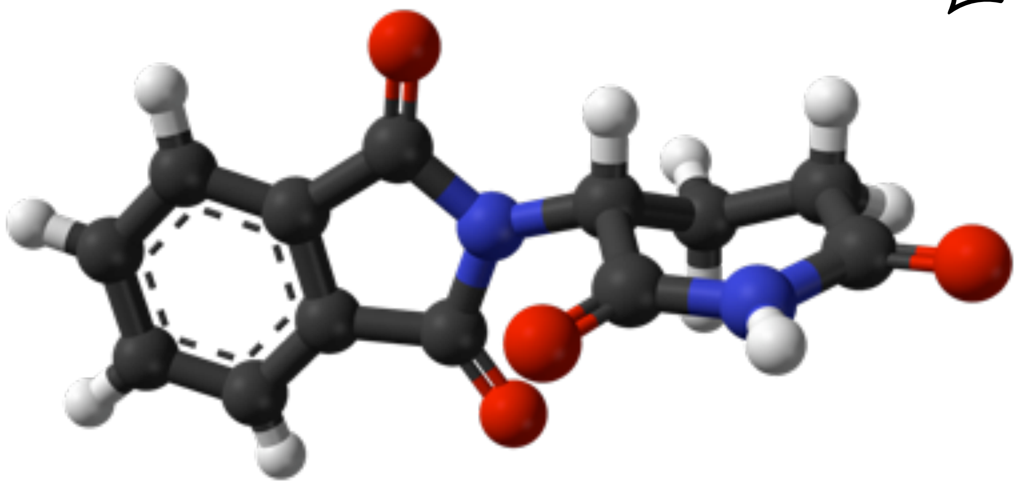
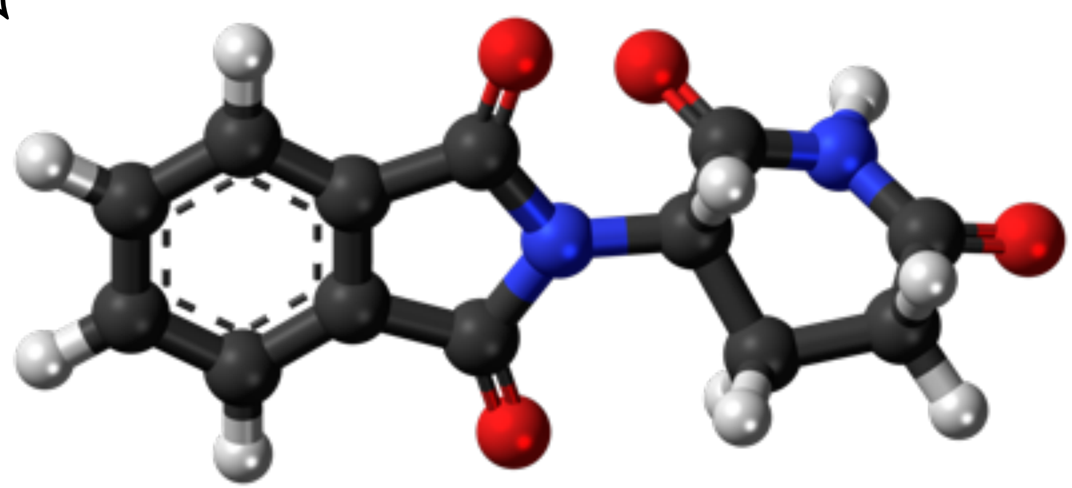
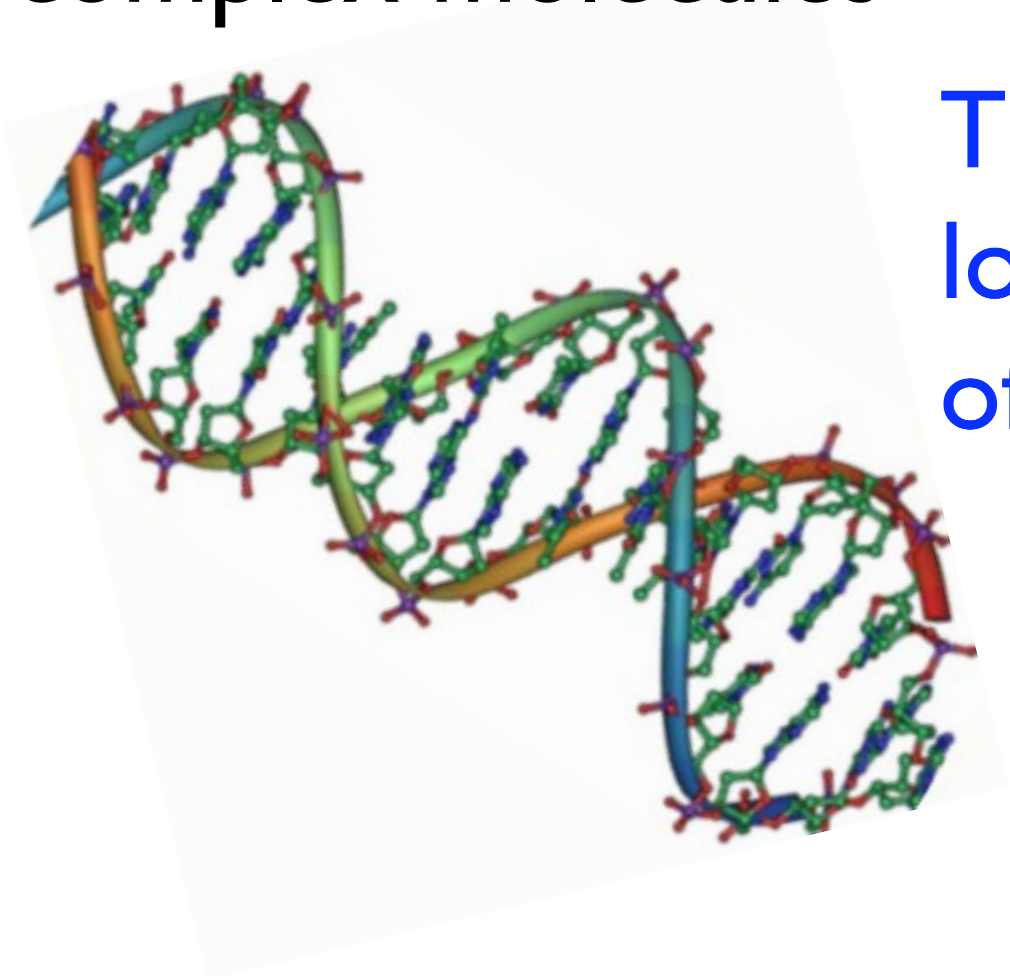




Nature herself provides us with a rich source of knotted complex molecules



The human genome is about 1m long, yet is curled up inside a nucleus of diameter about a micrometer

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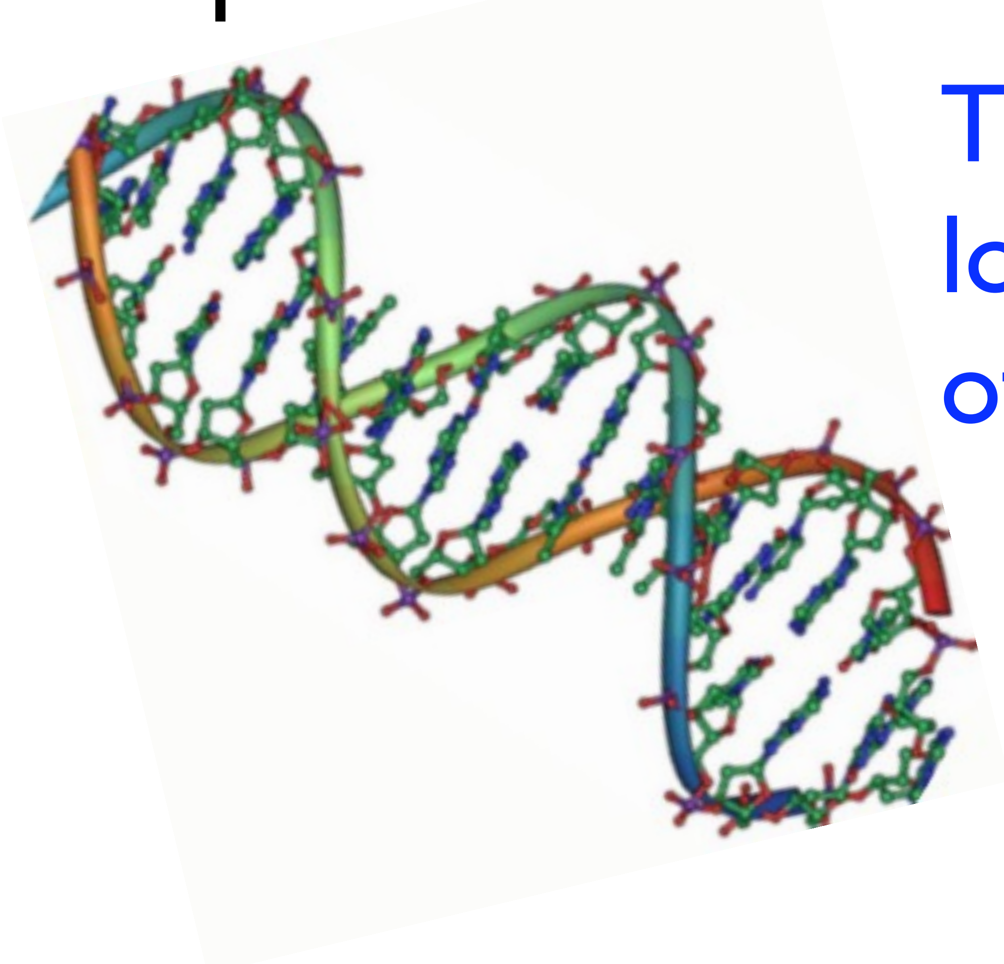

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Knots and tangles often form, particularly during replication

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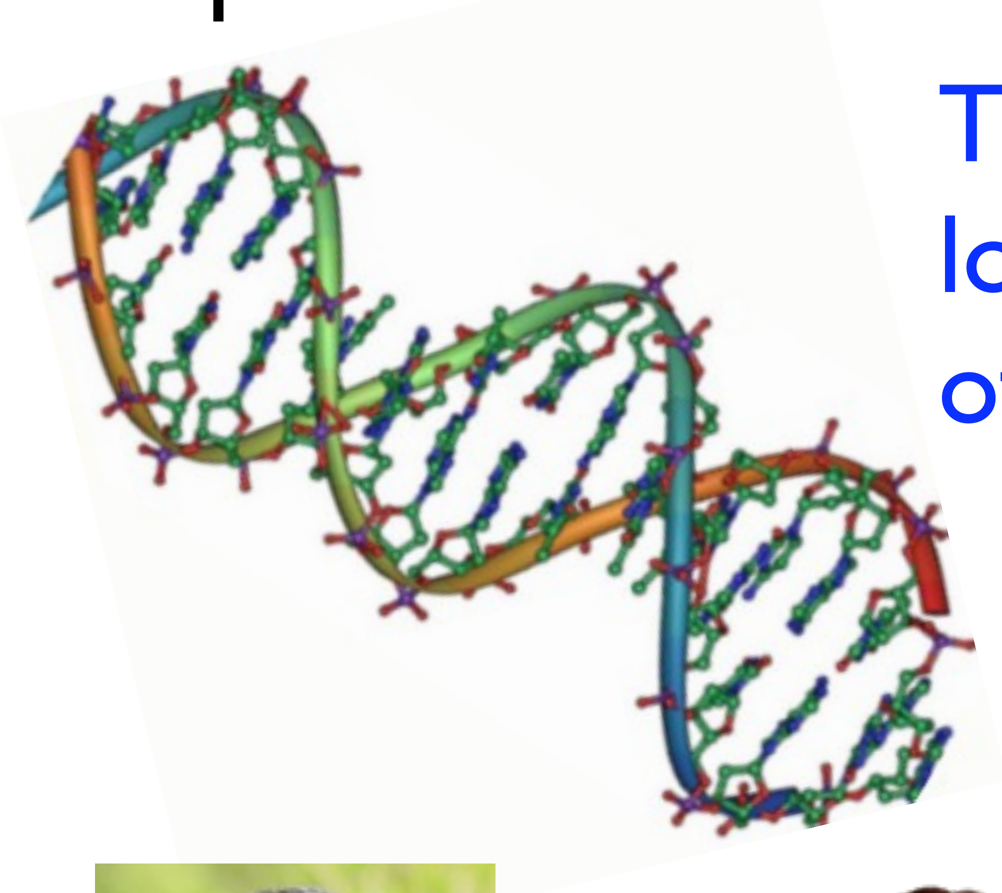

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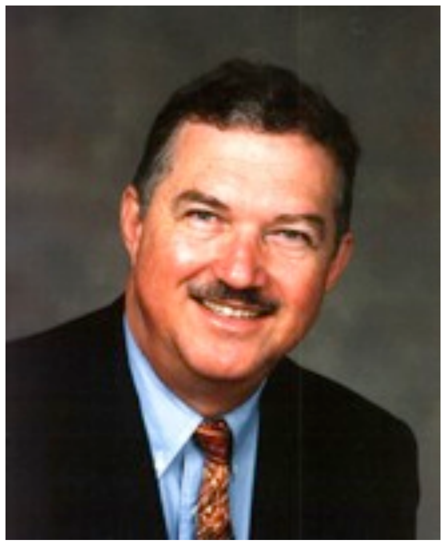
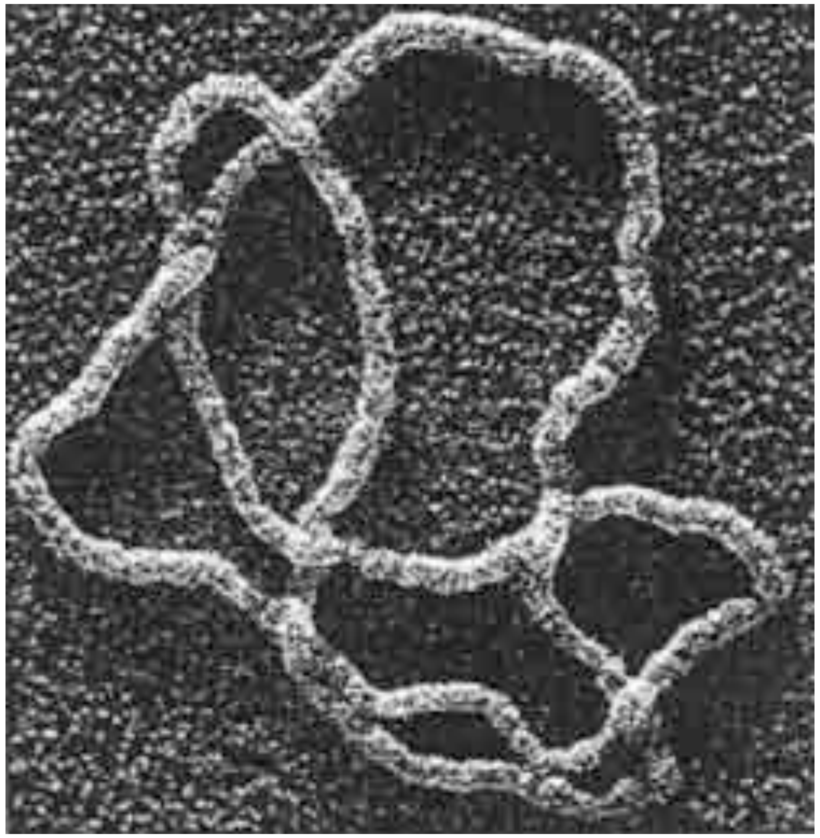
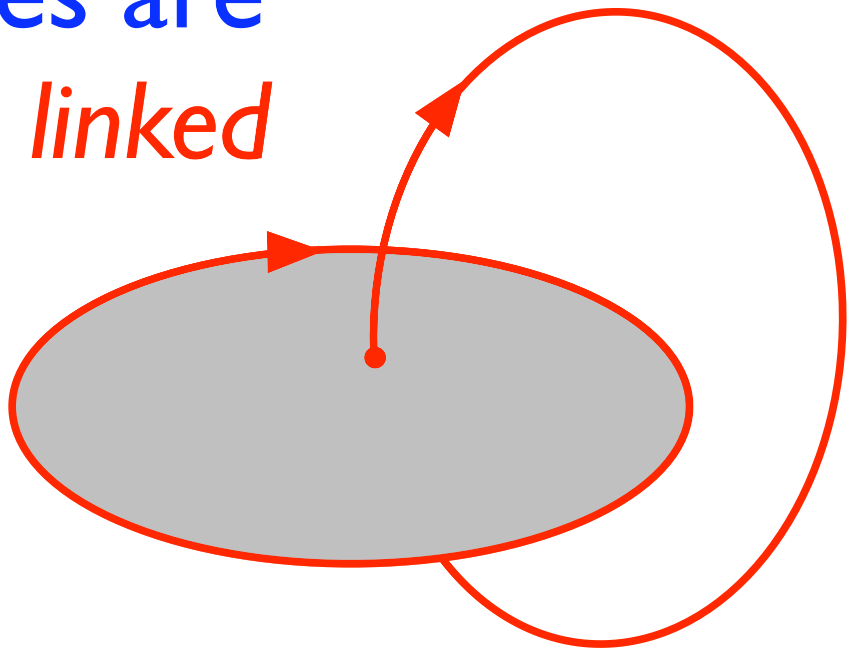


The interplay between knot theory & molecular biology is becoming an increasingly active field of research

Most bacterial genomic DNA is topologically a circle



After replication, the daughter DNA molecules are typically *linked*



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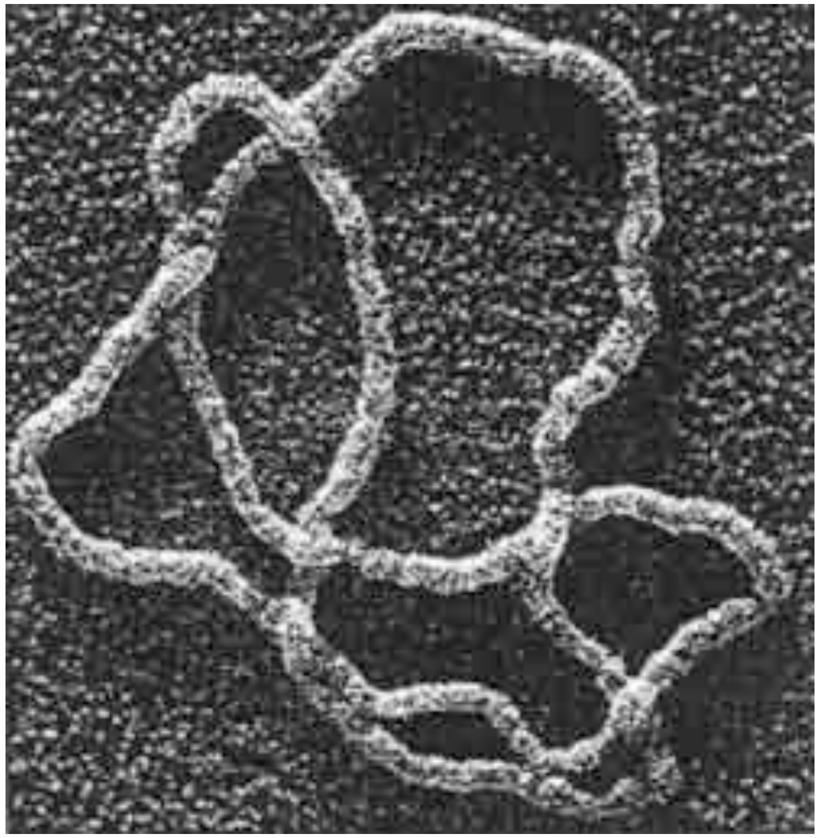

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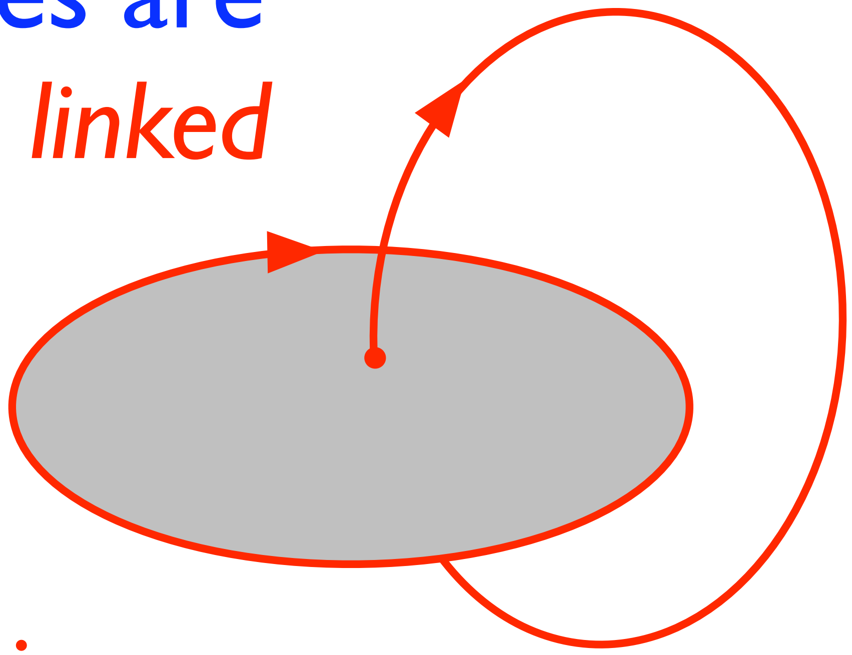

A certain type of enzyme - a *topoisomerase* - knows about the DNA topology! Its job is to set the daughters free

Many modern antibiotics work by inhibiting the action of these topoisomerases. *Without them, the bacterium aborts*

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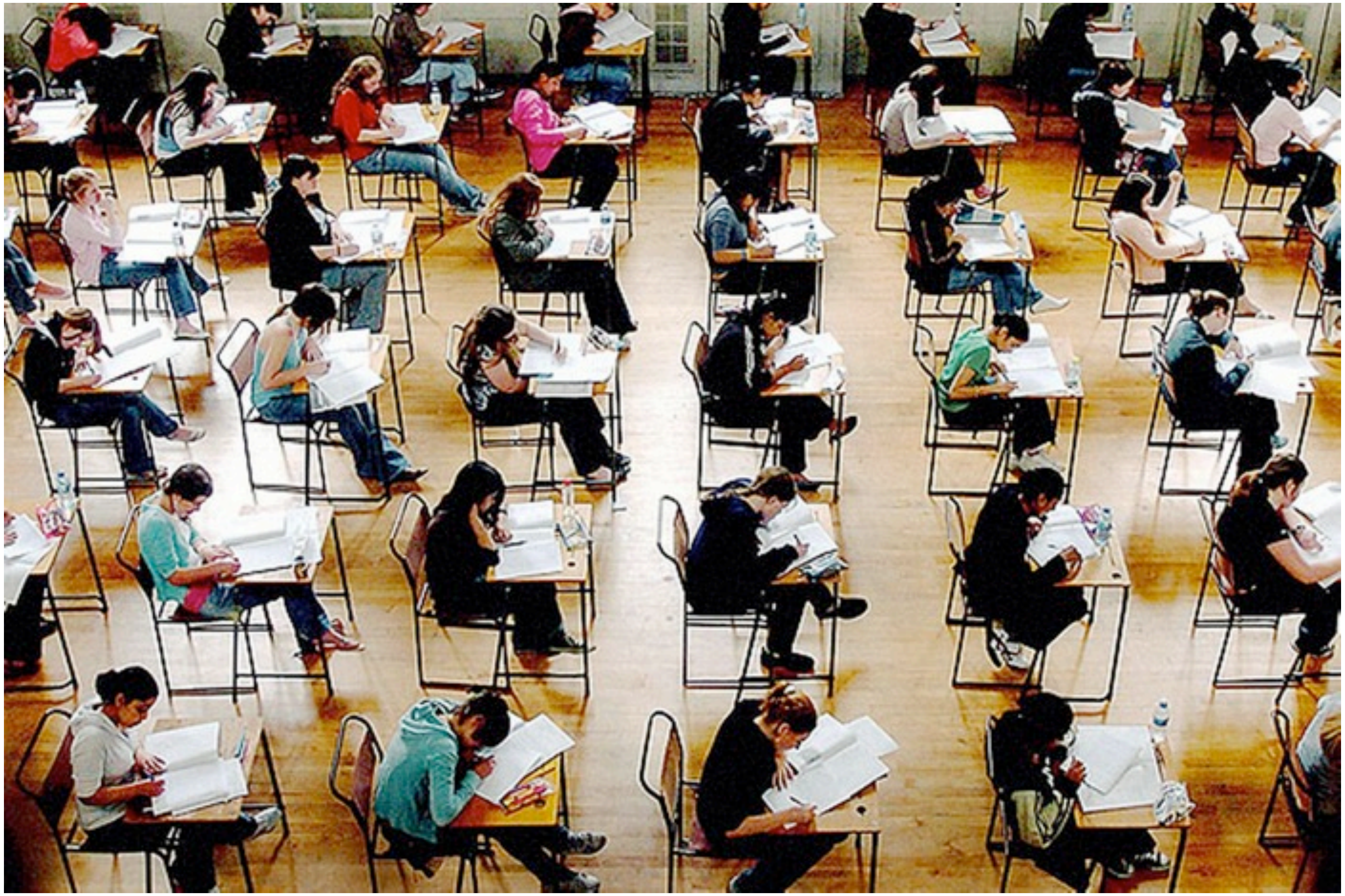

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You have the rest of your lives to do it... *Good luck!*