PARTIAL DIFFERENTIAL EQUATIONS

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In addition to the sets of lecture notes written by previous lecturers ([1, 2]) the books [4, 7] are very good for the PDE topics in the course. For distributions [5] is most relevant. Also [3, 6] are also useful; the books [8] are more advanced, but the first volume may be helpful.

References

[1] T.W. Körner, Cambridge Lecture notes on PDE, available at *http://www.dpmms.cam.ac.uk.*

[2] M. Joshi and A. Wassermann, Cambridge Lecture notes on PDE,available at http://www.damtp.cam.ac.uk/user/dmas2/.

[3] G.B. Folland, Introduction to Partial Differential Equations, *Princeton 1995, QA 374 F6*

[4] L.C. Evans, Partial Differential Equations,*AMS Graduate Studies in Mathematics Vol 19, QA377.E93 1990*

[5] F.G. Friedlander, Introduction to the Theory of Distributions, *CUP 1982, QA324*

[6] F. John, Partial Differential Equations, *Springer-Verlag 1982, QA1.A647*

[7] Rafael Jos Iorio and Valria de Magalhes Iorio, Fourier analysis and partial differential equations *CUP 2001, QA403.5 .I57 2001*

[8] M.E. Taylor, Partial Differential Equations,Vols I-III*Springer 96, QA1.A647*

PARTIAL DIFFERENTIAL EQUATIONS
EXAMPLE SHEET I

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1. (a) Let A be the vector field $A(x_1, x_2) = (x_2, -x_1)$ on $\mathbb{R}^2$ in standard co-ordinates (x_1, x_2). Find the general solution (integral curve) $\mathbf{X}(\mathbf{t})$ with initial data $\mathbf{X}(0) \in \mathbb{R}^2$. Sketch the integral curves.

(b) Do the same for the vector field $B(x_1, x_2) = (x_2, x_1)$.

(c) Write down the flow map $\Phi(t, \mathbf{X}(0))$ for both cases (a) and (b) Discuss the invertibility properties of the restricted flow map: $\alpha(t, s) = \Phi(t, (s, 0))$ in (a) and (b) i.e. for given (x_1, x_2) try to find $(t, s) = (\tau(x_1, x_2), \sigma(x_1, x_2))$ such that $\alpha(t, s) = (x_1, x_2)$ and discuss the domains of definition etc of such (τ, σ).

2. Solve the linear PDE $x_1 u_{x_2} - x_2 u_{x_1} = u$ with boundary condition $u(x_1, 0) = f(x_1)$ for f a C^1 function. Where is your solution valid? Classify the f for which a global C^1 solution exists. (Global solution here means a solution which is C^1 on all of $\mathbb{R}^2$. Problems of this type, in which data are specified on a hypersurface, are referred to as initial value or Cauchy problems).

3. Solve Cauchy problem for the semi-linear PDE $u_{x_1} + u_{x_2} = u^4$, $u(x_1, 0) = f(x_1)$ for f a C^1 function. Where is your solution C^1?

4. For the quasi-linear Cauchy problem $u_{x_2} = x_1 u u_{x_1}$, $u(x_1, 0) = x_1$

(a) Verify that the non-characteristic condition holds at all points of the initial hypersurface $x_2 = 0$ in $\mathbb{R}^2$,

(b) Solve the characteristic ODE's, give the flow map Φ, and discuss invertibility of the restricted flow map $\alpha(t, s)$ (this may not be possible explicitly),

(c) give the solution to the Cauchy problem (implicitly).

5. For the quasi-linear Cauchy problem $u_{x_2} = A(u_{x_1} + 1)/(B - x_1 - u)$, $u(x_1, 0) = 0$:

(a) Find all points on the intial hypersurface where the non-characteristic condition holds.

(b) Solve the characteristic ODE and invert (where possible) the restricted flow map, relating your answer to (a).

(c) Give the solution to the Cauchy problem, paying attention to any sign ambiguities that arise.

(In this problem take A, B to be positive real numbers).

6. For the Cauchy problem

$$u_{x_1} + 4u_{x_2} = \alpha u \qquad u(x_1, 0) = f(x_1), \tag{0.1}$$

with C^1 initial data f, obtain the solution $u(x_1, x_2) = e^{\alpha x_2/4} f(x_1 - x_2/4)$ by the method of characteristics. For fixed x_2 write $u(x_2)$ for the function $x_1 \mapsto u(x_1, x_2)$ i.e. the solution restricted to "time" x_2. Derive the following *well-posedness* properties for solutions $u(x_1, x_2)$ and $v(x_1, x_2)$ corresponding to data $u(x_1, 0)$ and $v(x_1, 0)$ respectively:

- for $\alpha = 0$ there is *global well-posedness* in the supremum (or L^∞) norm *uniformly in time* in the sense that if for fixed x_2 the distance between u and v is taken to be

  $$\|u(x_2) - v(x_2)\|_{L^\infty} \equiv \sup_{x_1} |u(x_1, x_2) - v(x_1, x_2)|$$

  then

  $$\|u(x_2) - v(x_2)\|_{L^\infty} \leq |u(0) - v(0)|_{L^\infty} \qquad \text{for all } x_2.$$

  Is the inequality ever strict?
- for all α there is *well-posedness* in supremum norm *on any finite time interval* in the sense that for any time interval $|x_2| \leq T$ there exists a number $c = c(T)$ such that

  $$\|u(x_2) - v(x_2)\|_{L^\infty} \leq c(T) \|u(0) - v(0)\|_{L^\infty}.$$

  and find $c(T)$. Also, for different α, when can c be assumed independent of time for positive (respectively negative) times x_2?
- Try to do the same for the L^2 *norm*, i.e. the norm defined by

  $$\|u(x_2) - v(x_2)\|^2_{L^2(dx_1)} = \int |u(x_1, x_2) - v(x_1, x_2)|^2 dx_1.$$

7(a). Consider the equation

$$u_{x_1} + n u_{x_2} = f$$

where n is an integer and f is a smooth function which is 2π- periodic in both variables:

$$f(x_1 + 2\pi, x_2) = f(x_1, x_2 + 2\pi) = f(x_1, x_2).$$

Apply the method of characteristics to find out for which f there is a solution which is also 2π- periodic in both variables:

$$u(x_1 + 2\pi, x_2) = u(x_1, x_2 + 2\pi) = u(x_1, x_2).$$

(b) Consider the problem in part (a) using fourier series representations of f and u (both 2π- periodic in both variables) and compare your results.

8. Verify that $f(x) = \|x\|^{2-n}$ $(n > 2)$ and $f(x) = \log \|x\|$ $(n = 2)$ are solutions of Laplace's equation $\sum \partial^2 f/\partial x_i^2 = 0$ in $\mathbb{R}^n \backslash 0$. Verify that $K(x,t) = \exp(-\|x\|^2/4t)/t^{n/2}$ is a solution of the heat equation $\partial K/\partial t = \sum \partial^2 K/\partial x_i^2$ in $\mathbb{R}^n$ for $t > 0$. [These solutions appear later in the course as "fundamental solutions".]

9. For which real numbers a can you solve the Cauchy problem

$$u_{x_1} + u_{x_2} = 0 \qquad u(x_1, ax_1) = f(x_1)$$

for any C^1 function f. Explain in terms of inverting the (restricted) flow map. Interpret your answer in relation to the line $x_2 = ax_1$ on which the initial data is given.

10. Use the method of characteristics to solve $u_{x_2} + uu_{x_1} = 1, \quad u(x_1, 0) = x_1$, showing that the solution is smooth for $x_2 > 0$. Also solve $u_{x_2} + uu_{x_1} = u^2$ with data $u(0, x_2) = x_2$.

PARTIAL DIFFERENTIAL EQUATIONS
EXAMPLE SHEET II

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Some questions involve the type of formulae and results which you may well have met in IB Methods: make sure that the answers you give now follow from the mathematical definitions given of distributions and their derivatives.

0. Which of the following functions of x lie in Schwartz space $\mathcal{S}(\mathbb{R})$: (a) $(1+x^2)^{-1}$,(b) e^{-x}, (c)$e^{-x^4}/(1+x^2)$? Show that if $f \in \mathcal{S}(\mathbb{R})$ then so is $f(x)/P(x)$ where P is any strictly positive polynomial (i.e. $P(x) \geq \theta > 0$ for some real θ.

1. Solve the following initial value problem

$$\partial_t u = \partial_x^3 u \qquad u(0,x) = f(x)$$

for $x \in [-\pi, \pi]$ with periodic boundary conditions $u(t,-\pi) = u(t,\pi)$ and f smooth and 2π-periodic. Discuss well-posedness properties of your solutions for the L^2 norm, i.e. $\|u(t)\|_{L^2} = (\int_{-\pi}^{+\pi} |u(t,x)|^2 dx)^{\frac{1}{2}}$, using the Parseval-Plancherel theorem.

2. Show that the heat equation $\partial_t u = \partial_x^2 u$, with boundary conditions as in 1 is well-posed forwards in time in L^2 norm, but not backwards in time (even locally). (Hint compute the L^2 norm of solutions u_n for negative t corresponding to initial values $u_n(0,x) = n^{-1}e^{inx}$.)

3. Define, for positive integral s, the norm $\|\cdot\|_s$ on the space of smooth 2π-periodic function of x by

$$\|f\|_s^2 \equiv \sum (1+n^2)^s |\hat{f}(n)|^2$$

where $\hat{f}(n)$ are the fourier coefficients of f. (This is called the Sobolev H^s norm).

(i) What are these norms if $s = 0$. Write down a formula for these norms in terms of $f(x)$ and its derivatives directly. (Hint Parseval).

(ii) If $u(t,x)$ is the solution you obtained for the heat equation in 2 then for $t > 1$ and $s = 0, 1, 2, , ,$ find a number $C_s > 0$ such that

$$\|u(t,\cdot)\|_s \leq C_s \|u(0,\cdot)\|_0$$

(iii) Show that there exists a number $C > 0$ which does not depend on f so that $\max|f(x)| \leq C\|f\|_1$ for all smooth 2π-periodic f.

4. (i) Use fourier series to solve the Schrodinger equation

$$\partial_t u = i\partial_x^2 u \qquad u(0,x) = f(x)$$

for initial value f smooth and periodic. Prove there is only one smooth solution.

(ii) Use the fourier transform to solve the Schrodinger equation for $x \in \mathbb{R}$ and initial value $f \in \mathcal{S}(\mathbb{R})$. Find the solution for the case $f = e^{-x^2}$.

5. Verify that the distribution u on the real line defined by the function $(2m)^{-1}e^{-m|x|}$, (for positive m), solves

$$\Big(\frac{-d^2}{dx^2} + m^2\Big)u = \delta_0$$

in $\mathcal{S}'(\mathbb{R})$.

6.Verify that the function on the real line $g(x) = 1$ for $x \leq 0$ and $g(x) = e^{-x}$ for $x > 0$ defines a tempered distribution T and that it solves in $\mathcal{S}'(\mathbb{R})$

$$T'' + T' = -\delta_0.$$

7. Write down the precise distributional meaning of the equation

$$-\Delta(|x|^{-1}) = 4\pi\delta_0 \qquad \text{in } \mathcal{S}'(\mathbb{R}^3)$$

in terms of test functions, and then use the divergence theorem to verify that it holds. (Hint: apply the divergence theorem on the region $\{0 < |x| < R\} - \{0 < |x| < \epsilon\}$ for R sufficiently large and take the limit $\epsilon \to 0$ carefully). (Optional extra: try to find the corresponding solution of $-\Delta T = \delta_0$ in the $n-$dimensional case.)

8. Compute by contour integration the inverse Fourier transform of the function $g(\xi) = (m^2 + \xi^2)^{-1}$ for $\xi \in \mathbb{R}$ and m a positive constant. Comment on the relation with question 5.

9. Following the strategy in class give a definition of $fT \in \mathcal{S}'(\mathbb{R})$ given $T \in \mathcal{S}'(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$. Now obtain a formula for $(fT)'$ (using the definition of distributional derivative.) This is the product rule for multiplication of distributions and test functions.

10. (i) For each of the following equations find the most general tempered distribution T which satisfies it (see Friedlander §2.7)

$$xT = 0, \;\; x dT/dx = 0, \;\; x^2 T = \delta_0, \;\; x dT/dx = \delta_0$$

$$dT/dx = \delta_0, \;\; dT/dx + T = \delta_0 \;\; T - (d/dx)^2 T = \delta_0.$$

(ii) Solve the equation $x^m T = 0$ in $\mathcal{S}'(\mathbb{R})$.

PARTIAL DIFFERENTIAL EQUATIONS
EXAMPLE SHEET III

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1. Using the Green identities show that if f_1, f_2 both lie in $\mathcal{S}(\mathbb{R}^n)$ then the corresponding Schwartzian solutions u_1, u_2 of the equation $-\Delta u + u = f$, i.e.

$$(-\Delta + 1)u_1 = f_1 \qquad (-\Delta + 1)u_2 = f_2$$

satisfy

$$(*) \qquad \int |\nabla(u_1 - u_2)|^2 + |u_1 - u_2|^2 \leq c \int |f_1 - f_2|^2$$

where the integrals are over $\mathbb{R}^n$. (This is interpreted as implying the equation $-\Delta u + u = f$ is well-posed in the H^1 norm (or "energy" norm) defined by the right hand side of (*).) Now try to improve the result so that the H^2 norm:

$$\|u\|_{H^2}^2 \equiv \sum_{|\alpha| \leq 2} \int |\partial^\alpha u|^2 dx,$$

appears on the left. (The sum is over all multi-indices of order less than or equal to 2).

2. Prove a maximum principle for solutions of $-\Delta u + V(x)u = 0$ (on a bounded domain Ω with smooth boundary $\partial\Omega$) with $V > 0$: if $u|\partial\Omega = 0$ then $u \leq 0$ in Ω. (Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Hint: exclude the possibility of u having a strictly positive interior maximum).

What does the maximum principle reduce to for one dimensional harmonic functions i.e. C^2 functions such that $u_{xx} = 0$?

3. Let $\phi \in C(\mathbb{R}^n)$ be absolutely integrable with $\int \phi(x)dx = 1$. Assume $f \in C(\mathbb{R}^n)$ is bounded with $\sup |f(x)| \leq M < \infty$. Define $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ and show

$$\phi_\epsilon * f(x) - f(x) = \int \Big(f(x - \epsilon w) - f(x)\Big)\phi(w)dw$$

(where the integrals are over $\mathbb{R}^n$). Now deduce the approximation lemma that $\phi_\epsilon * f(x) \to f(x)$ as $\epsilon \to 0$, and uniformly if f is uniformly continuous. (Hint: split up the w integral into an integral over the ball $B_R = \{|w| < R\}$ and its complement B_R^c for large R).

4. (i) Derive, from the definition of Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$, a formula for the Fourier transform of δ_0 and its derivatives. For $n = 1$ find the Fourier transform of the Heaviside distribution.
(ii) If $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, prove that $\widehat{\phi T} = (2\pi)^{-n}\widehat{\phi} \star \widehat{T}$.
5. (i) Use the Fourier transform to obtain a representation for the solution to the initial value problem

$$u_t = u_{xxx} \qquad u(0,x) = u_0(x)$$

for u_0 a Schwartz function. Deduce that the solution obtained satisfies the well-posedness estimate for all times t:

$$\int_{-\infty}^{+\infty} |u(t,x) - v(t,x)|^2 dx \le \int_{-\infty}^{+\infty} |u(0,x) - v(0,x)|^2 dx$$

(Global well-posedness in L^2 uniformly in time).

6. The Dirichlet problem in half-space:
Let $H = \{(x,y) : x \in \mathbb{R}^n, y > 0\}$ be the half-space in $\mathbb{R}^{n+1}$. Consider the problem $\Delta_x u + \partial_y^2 u = 0$, where Δ_x is the Laplacian in the x variables only) and $u(x,0) = f(x)$ with f a bounded and uniformly continuous function on $\mathbb{R}^n$. Define

$$u(x,y) = P_y * f(x) = \int_{\mathbb{R}^n} P_y(x-z) f(z) dz$$

where $P_y(z) = \frac{2y}{\omega_n (|x|^2+y^2)^{\frac{n+1}{2}}}$ for $x \in \mathbb{R}^n$ and $y > 0$. (This is the Poisson kernel for half-space.) Show that for an appropriate choice of ω_n u is harmonic on the half-space H and is equal to f for $y = 0$.
(Hint: first differentiate carefully under the integral sign; then note that $P_y(x) = y^{-n} P_1(\frac{x}{y})$ where $P_1(x) = \frac{2}{\omega_n (1+|x|^2)^{\frac{n+1}{2}}}$, i.e. an approximation to the identity) and use the approximation lemma to obtain the boundary data).

(ii) Assume instead that $f \in \mathcal{S}(\mathbb{R}^n)$. Take the Fourier transform in the x variables to prove the same result.

7. Using the representation of the solution of the initial value problem

$$(**) \qquad \partial_t u - \Delta u = 0 \qquad u(0,x) = f(x), \quad f \in \mathcal{S}(\mathbb{R}^n)$$

given by the fundamental solution show that if $\sup_{x \in \mathbb{R}^n} f(x) \le M$ then $\sup_{x \in \mathbb{R}^n} u(t,x) \le M$ for all $t > 0$.

8. Let u be a smooth solution of the initial value problem (**) of qu. 7 which lies in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ for each fixed $t > 0$. Compute, assuming you can differentiate under the integral sign, $\frac{d}{dt}\int_{\mathbb{R}^n} |u(t,x)|^2 dx$

and hence prove that there is only one such solution of (**). Compute also

$$\frac{d}{dt}\int_{\mathbb{R}^n} \phi\Big(u(t,x)\Big)\,dx$$

where $\phi \in C^2(\mathbb{R})$ is a positive function. For which ϕ is your answer ≤ 0?

9. If $u \in C^2$ and $\Delta u \geq 0$ then u is called subharmonic (e.g. referred to in the weak maximum principle). State and prove a mean value property for subharmonic functions. (Similarly if $\Delta u \leq 0$.)

10. Show that if $u \in C([0,\infty) \times \mathbb{R}^n) \cap C^2((0,\infty) \times \mathbb{R}^n)$ satisfies (i) the heat equation, (ii) $u(0,x) = 0$ and (iii) $|u(t,x)| \leq M$ and $|\nabla u(t,x)| \leq N$ for some M, N then $u \equiv 0$. (Hint: multiply heat equation by $K_{t_0-t}(x-x_0)$ and integrate over $|x| < R, a < t < b$. Apply the divergence theorem, carefully let $R \to \infty$ and then $b \to t_0$ and $a \to 0$ to deduce $u(t_0, x_0) = 0$.)

PARTIAL DIFFERENTIAL EQUATIONS EXAMPLE SHEET IV

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1. Show that a bounded C^2 solution with u and $|\nabla u|$ square integrable of the equation

$$-\Delta u + u = f \qquad f \in \mathcal{S}(\mathbb{R}^n)$$

minimizes the functional

$$I_f[u] = \frac{1}{2}\int |\nabla u|^2 + u^2 - 2fu$$

amongst such functions. Formulate the notion of a weak or distributional solution of this equation. Finally, show conversely that any minimizer of this type is a weak solution.

2. (i) Let K_t be the heat kernel on $\mathbb{R}^n$ at time t and show that

$$K_t * K_s = K_{t+s}$$

for $t, s > 0$ (semi-group property). (Is the proof easier in the x-space or in Fourier space?).
(ii) Show that the solution operator $S : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ for the heat initial value problem satisfies for positive t $\|S\|_{\mathcal{L}(L^1,L^\infty)} \leq ct^{-\frac{n}{2}}$. This means that the solution satisfies $\|u(t)\|_{L^\infty} \leq ct^{-\frac{n}{2}}\|u(0)\|_{L^1}$, or more explicitly:

$$\sup_x |u(t,x)| \leq ct^{-n/2}\int |u(0,x)|dx$$

for some positive number c, which should be found.
(iii) Now let $n = 4$. Deduce, by considering $v = u_t$, that if $f \in \mathcal{S}(\mathbb{R}^4)$ is a function of x only, the solution of $u_t - \Delta u = f$ with zero initial data converges to some limit as $t \to \infty$. Try to identify the limit.

3. Write down the Kirchoff formula for the solution of the wave equation in three space dimensions with initial data $u(0,x) = 0, u_t(0,x) = u_1(x)$ for $u_1 \in \mathcal{S}(\mathbb{R}^n)$. What can you say about the behaviour of $u(t,x)$ for large t?

4. (a) Use the change of variables $v(t,x) = e^t u(t,x)$ to obtain a formula for the solution to the initial value problem:

$$u_t + u = \Delta u \qquad u(0,\,\cdot\,) = u_0(\cdot) \in \mathcal{S}(\mathbb{R}^n).$$

Hence show that $|u(t,x)| \leq \sup_x |u_0(x)|$ and use this to deduce well-posedness in the supremum norm (for $t > 0$ and all x).

If $a \leq u_0(x) \leq b$ for all x what can you say about the possible values of $u(t,x)$ for $t > 0$.

(b) Solve the equation

$$u_{tt} - 2u_t + u = \Delta u \qquad u(0,\,\cdot\,) = u_0(\cdot) \in \mathcal{S}(\mathbb{R}^n),\ u_t(0,\cdot) = u_1(\cdot) \in \mathcal{S}(\mathbb{R}^n).$$

5. Take the Fourier transform of the wave equation in the space variables to show that the solution is given by

$$u(t,x) = (2\pi)^{-n} \int \exp^{i\xi\cdot x}(\cos(t|\xi|)\widehat{u}(0,\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u_t}(0,\xi))d\xi$$

for initial values $u(0,x) = u_0(x), u_t(0,x) = u_1(x)$ in $\mathcal{S}(\mathbb{R}^n)$.

6. (i) Let $u(t,x)$ be a twice continuously differentiable solution of the wave equation for $n = 3$ which is radial, i.e. a function of $r = \|x\|$ and t. By letting $w = ru$ deduce that u is of the form

$$u(t,x) = \frac{f(r-t)}{r} + \frac{g(r+t)}{r}.$$

(ii) Show that the solution with initial data $u(0,\cdot) = 0$ and $u_t(0,\cdot) = G$, where G is radial and even function, is given by

$$u(t,r) = \frac{1}{2r}\int_{r-t}^{r+t} \rho G(\rho)d\rho.$$

(iii) Hence show that for initial data $u(0,\cdot) \in C^3(\mathbb{R}^n)$ and $u_t(0,\cdot) \in C^2(\mathbb{R}^n)$ the solution $u = u(t,x)$ need only be in $C^2(\mathbb{R}\times\mathbb{R}^n)$. Contrast this with the case of one space dimension.

7. If u is a C^2 solution of the wave equation show that

$$\partial_t\Big(\frac{u_t^2 + |\nabla u|^2}{2}\Big) + \partial_i\Big(-u_t\partial_i u\Big) = 0$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

8. Deduce from 7. that if $u \in C^2$ solves the wave equation and $u(0,x)$ and $u_t(0,x)$ both vanish for $|x| < R$ then $u(t,x)$ vanishes for $|x| < R - |t|$ if $|t| < R$. (Hint: divergence theorem).

Deduce that for the wave equation the initial value problem, $\Box u = 0$, $u(0,t) = f$, $u_t(0,x) = g$, with $f, g \in \mathcal{D}(\mathbb{R}^n)$ given, has a unique smooth solution.

9. The Kirchhoff formula for solutions of the wave equation $n = 3$ was derived for initial data $u(0,\cdot) = 0, u_t(0,\cdot) = g \in \mathcal{S}(\mathbb{R}^n)$ i.e. in the Schwartz class. Show that the validity of the formula can be extended to any smooth function $g \in C^\infty(\mathbb{R}^n)$.(Hint: finite speed of propagation).

10. For the solution of the inhomogeneous Cauchy problem $\Box u = h$ with h a Schwartz function as obtained by the Duhamel principle determine the domain of dependence for a point (t_0, x_0) on the values of h. Comment on the difference between the cases of two and three space dimensions.