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Calculus of Variations

Existence and regularity for generalised harmonic maps associated to a nonlocal polyconvex energy of Skyrme type

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Abstract We prove existence and regularity of critical points of arbitrary degree for a generalised harmonic map problem, in which there is an additional nonlocal polyconvex term in the energy, heuristically of the same order as the Dirichlet term. The proof of regularity hinges upon a special nonlinear structure in the Euler–Lagrange equation similar to that possessed by the harmonic map equation. The functional is of a type appearing in certain models of the quantum Hall effect describing nonlocal Skyrmions.

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1 Introduction and statement of results

1.1 Introductory discussion

We study weak solutions of the Euler–Lagrange equations for the functional

$$\mathcal{V}(\phi) = \frac{1}{2}\int\limits_{\Omega}\left(|\nabla\phi|^2 + 2\phi\cdot\mathbf{B}(x) + \kappa\int\limits_{\Omega}(j_\phi(x)-\sigma(x))K(x,y)(j_\phi(y)-\sigma(y))\,dy\right)dx \tag{1}$$

where $\Omega = (\mathbb{R}/2\pi\mathbb{Z})^2$ is the two dimensional torus. Smooth functions on Ω coincide with smooth, 2π-periodic functions on $\mathbb{R}^2$. The unknown ϕ is a map from Ω to the unit sphere S^2, embedded in $\mathbb{R}^3$. The constant $\kappa > 0$ is given, and $\mathbf{B} : \Omega \to \mathbb{R}^3$ and $\sigma : \Omega \to \mathbb{R}$ are given smooth periodic functions. In the third term

$$K(x,y) = (-\Delta)^{-1}(x,y)$$

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is the integral kernel of the inverse negative Laplacian while j_ϕ is the topological charge density

$$j_\phi(x) = \frac{1}{2}\epsilon_{ab}\phi \cdot \partial_a\phi \times \partial_b\phi.$$

Here we write $\times$ for the cross product in $\mathbb{R}^3$ and ϵ_{ab} for the antisymmetric tensor, with $\epsilon_{12} = +1$, which defines the standard complex structure on Ω. Thus the nonlocal term can be regarded as a kind of Coulomb interaction energy for the charge density $j_\phi - \sigma$, which is made up from a topological density j_ϕ and a given background density σ. It is an immediate consequence of degree theory ([1, Sect. 7.5A]) that for $\phi \in C^\infty(\Omega; S^2)$

$$\int j_\phi dx \in 4\pi\mathbb{Z}$$

(since it is the pull-back of the area of the sphere). Further by density of $C^\infty(\Omega; S^2)$ in $H^1(\Omega; S^2)$ ([22, Sect. 4]) the degree is a well defined integer valued function on $H^1(\Omega; S^2)$, which is constant on connected components. We study existence and regularity of critical points of (1) subject to the additional topological constraint

$$\int_\Omega j_\phi(x) = \int_\Omega \sigma(x)dx = 4\pi d \;\in\; 4\pi\mathbb{Z} \tag{2}$$

for fixed d. Action functionals of this type have been introduced in [15] for the continuum description of magnetisation in the fractional quantum Hall effect. The quantum Hall effect refers to the phenomenon first observed by Hall in which an effectively two-dimensional conducting sample is subjected to an electromagnetic field and the induced current (or measured conductivity) is perpendicular to the electric field. More recently phycisists observed that the conductivity in fact takes only quantised values (which are rational multiples of a quantity depending only on the electric charge and a universal constant). The explanation and modelling of this phenomenon has been linked to the observation that the spins also do not align with the induced magnetic field but form magnetic microstructure. In a continuum model the magnetisation can be described as an S^2 valued function and it is in this context that functionals of the type of (1) have been discussed. In particular in [15] a rather general energy functional is written down which contains, amongst other terms, the nonolocal Skyrme (i.e. the third) term in (1). Many variants have appeared in the literature, for example in [24] the functional (1) but with $K(x, y) = |x - y|^{-1}$ appears. Also there are other nonlocal variants of the harmonic map problem describing magnetic microstructure in different physical situations, for example [5] in which regularity results are proved.

If $\mathbf{B} = 0$ and $\kappa = 0$ the functional (1) is just the Dirichlet energy, whose corresponding critical points are called harmonic maps. Thus in mathematical terms the functional (1) involves a modification of the harmonic map energy by a nonlocal term of Skyrme or polyconvex type [2,23]: indeed, if $K(x, y) = \delta(x - y)$, the additional energy is proportional to $|d\phi \wedge d\phi|^2$. Harmonic map problems involving additional terms in the energy like this were introduced by Skyrme [23] and are studied as a phenomenological theory of nucleons. Existence theorems for minimisers of the Skyrme functional have been proved in [10,11] in the original three space dimensional case, and in [17] in the two dimensional case, via the direct variational method. There are many open problems regarding Skyrmions, particularly symmetry and regularity [12,16,18,19]. We show in this article the existence and C^∞ regularity of critical points of the nonlocal Skyrme functional (1) of arbitrary degree. This existence statement is in contrast to the case of harmonic maps where critical points do not exist for degree ± 1 (see the discussion following Theorem 1 and in Sect. 4). We therefore include a discussion of the limit $\kappa \to 0^+$, showing weak, but generally not strong, convergence of critical points.

For harmonic maps on two dimensional domains regularity was proved by Helein [14]. The proof hinged upon a special jacobian structure occuring in the Euler–Lagrange equation. Of the additional terms in (1) the middle term, which is linear in ϕ, can easily be handled as a mild perturbation. However, the additional nonlocal term is (heuristically speaking) of the same order as the Dirichlet term $|\nabla\phi|^2$ (since, as discussed in Sect. 1.3, the fundamental solution $\mathbb{K} = (-\Delta)^{-1}$ defines an operator

$$\mathbb{K} : f \mapsto \int K(\cdot, y) f(y) dy$$

of order -2 and each j_ϕ involves two derivatives). In spite of this we show that the same jacobian structure is still present in the Euler–Lagrange equation for (1) and deduce consequences for regularity. Thus the functional (1) is to be thought of heuristically as another scale invariant energy whose structure implies regularity through the special jacobian type nature of the Euler–Lagrange equation. (Of course the lower order terms involving $\mathbf{B}$, σ break the scale invariance but they are less important from the regularity perspective.)

Regarding existence, direct minimisation of (1) seems to fail in 2 dimensions. Instead we prove existence of critical points following a scheme used by Sacks and Uhlenbeck in [21]: we approach minimisers of $\mathcal{V}$ by minimisers of weakly lower semi-continuous functionals $\mathcal{V}_p$ obtained by replacing the Dirichlet energy term by (essentially) the p-energy, see (19). This gives the proof of existence of critical points with arbitrary degree $d \in \mathbb{Z}$.

1.2 Statement of results

We now write down the equations explicitly and state our main result. There are two useful formulations of the Euler–Lagrange equation. Firstly, the standard form of the Euler–Lagrange equation is

$$-\Delta\phi - |\nabla\phi|^2\phi - \kappa\partial_j\phi \times \left(\epsilon_{ij}\phi\partial_j\mathbb{K}(j_\phi - \sigma)\right) = -(\mathbf{B} - \phi\cdot\mathbf{B}\phi). \tag{3}$$

The alternative formulation of the Euler–Lagrange equation is as a conservation law ([14, theorem 1.3.1]):

$$\nabla \cdot \mathcal{J} = \phi \times \mathbf{B} \tag{4}$$

where $\mathcal{J}$ is the $\mathbb{R}^3$ valued vector field given by

$$\mathcal{J}_i = \phi \times \partial_i\phi + \kappa\epsilon_{ij}\phi\partial_j\mathbb{K}(j_\phi - \sigma). \tag{5}$$

Remark Equation (4) can be obtained by using variations of the form

$$\phi_\epsilon = \phi + \epsilon\zeta \wedge \phi$$

with $\zeta \cdot \phi = 0$; note $|\phi_\epsilon|^2 = 1 + \epsilon^2|\zeta \wedge \phi|^2$ and so $|\phi_\epsilon| = 1 + O(\epsilon^2)$ so that further normalisation is not necessary for computing the Euler–Lagrange equation. Equivalently, (4) arises when the Eq. (3) is projected on the tangent space of S^2 (c.f. [14, Chap. 1.3.1]):

$$-\phi \times \Delta\phi - \phi \times (\partial_i\phi \times \epsilon_{ij}\phi\partial_j\mathbb{K}(j_\phi)) = -\phi \times \mathbf{B} \tag{6}$$

which is equivalent to (4). Noting also that (3), projected in the direction in $\mathbb{R}^3$ parallel to ϕ, is automatically zero we deduce that $\phi \in C^\infty(\Omega; S^2)$ is a solution of (3) if and only if it is a solution of (4). The vector field $\mathcal{J}$ also emerges from the Noether theorem as a consequence of the symmetries of S^2 (with the standard metric induced from the Euclidean metric of $\mathbb{R}^3$) and of the functional density in (1).

This is our main theorem:

Theorem 1 *Given $d \in \mathbb{Z}$, $\kappa > 0$ and smooth periodic functions $\mathbf{B} \in C^\infty(\Omega; \mathbb{R}^3)$ and $\sigma \in C^\infty(\Omega)$ such that $(4\pi)^{-1} \int_\Omega \sigma = d$ there exists a smooth critical point of $\mathcal{V}$ of degree d, i.e. there exists $\phi \in C^\infty(\Omega; S^2)$ satisfying (2) and (3).*

As discussed in Sect. 4, the corresponding result is not true for $\kappa = 0, \mathbf{B} = 0, d = \pm 1$ as can be deduced by comparison with a classical non-existence result of Eells and Wood [9]. To explore this further we study the limit as $\kappa \to 0^+$. We show in Sect. 4 that critical points of $\mathcal{V}$ in $H^1(\Omega, S^2)$ tend weakly in H^1, as $\kappa \to 0^+$, to a critical point of

$$\mathcal{E}(\phi) = \frac{1}{2} \int_\Omega \left(|\nabla \phi|^2 + 2\phi \cdot \mathbf{B}(x) \right) \, dx. \tag{7}$$

However, the degree can, and in certain cases must, change in this limit and we show this occurs by bubbling off of harmonic spheres.

The proof of the main theorem involves the construction of weak solutions which are then shown to be smooth. Here is the definition of weak solution:

Definition 2 A weak solution of (3) is a function $\phi \in H^1(\Omega; \mathbf{R}^3)$ such that $|\phi| = 1$ almost everywhere, $j_\phi \in H^{-1}$ and with the property that if $\eta \in H^1(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ then

$$\int_\Omega \nabla \eta \cdot \nabla \phi - \eta \cdot \Big[|\nabla \phi|^2 \phi + \kappa \partial_i \phi \times \left(\epsilon_{ij} \phi \partial_j \mathbb{K}(j_\phi - \sigma) \right) \Big] + \eta \cdot (\mathbf{B} - \phi \cdot \mathbf{B}\phi) dx = 0, \tag{8}$$

or, equivalently,

$$\int_\Omega \left(\nabla \eta \cdot \nabla \phi - \eta \cdot \partial_j \phi \times \mathcal{J}_j + \eta \cdot (\mathbf{B} - \phi \cdot \mathbf{B}\phi) \right) dx = 0 \tag{9}$$

where $\mathcal{J}$ is defined in (5), and is automatically square integrable under the above assumptions.

Remark (3) implies (4) weakly, i.e. a weak solution of (3) satisfies (4) in the sense that

$$\int_\Omega \nabla \zeta \cdot \mathcal{J} + \phi \times \mathbf{B} \cdot \zeta \, dx = 0$$

$\forall \zeta \in H^1(\Omega; \mathbb{R}^3)$.

Proof Take $\eta = \phi \times \zeta$ as test function in (8) with $\zeta \in C^1(\Omega; \mathbb{R}^3)$ and use $\phi \cdot \nabla \phi = 0$ to deduce the result for such ζ. Then the statement follows by the density of $C^1(\Omega; \mathbb{R}^3)$ in $H^1(\Omega; \mathbb{R}^3)$ and the observation that $\mathcal{J}$ is automatically square integrable. □

An important feature of the harmonic map equation, identified by Helein, and also shared by (3), is the jacobian structure. For (3) this amounts to the fact that it is possible to rewrite it as

$$\begin{aligned} -\Delta \phi &= \partial_i \phi \times (\phi \times \partial_i \phi) + \partial_i \phi \times \kappa \epsilon_{ij} \phi \partial_j \mathbb{K}(j_\phi - \sigma) - (\mathbf{B} - \phi \cdot \mathbf{B}\phi) \\ &= \partial_i \phi \times (\phi \times \partial_i \phi + \kappa \epsilon_{ij} \phi \partial_j \mathbb{K}(j_\phi - \sigma)) - (\mathbf{B} - \phi \cdot \mathbf{B}\phi) \\ &= \partial_i \phi \times \mathcal{J}_i - (\mathbf{B} - \phi \cdot \mathbf{B}\phi) \end{aligned} \tag{10}$$

where $\mathcal{J}$ is as in (4) and (5). This has consequences for regularity used in the Sect. 3.

Finally it is also useful to consider (ϕ, u), where $u = \underline{u}(\phi)$ and

$$\underline{u}(\phi) \equiv -\Delta^{-1}(j_\phi - \sigma) = \mathbb{K}(j_\phi - \sigma), \tag{11}$$

as solutions of the system

$$-\Delta\phi = \nabla\phi \times \mathcal{J} - (\mathbf{B} - \phi \cdot \mathbf{B}\phi) \tag{12}$$

$$\mathcal{J}_i = \phi \times \partial_i\phi + \kappa\phi\epsilon_{ij}\partial_j u \tag{13}$$

$$-\Delta u = (j_\phi - \sigma). \tag{14}$$

Definition 3 A pair $(\phi, u) \in H^1(\Omega; S^2) \times H^1(\Omega)$ is a weak solution of (12)–(14) if $j_\phi \in H^{-1}$, if (14) holds as an equality in H^{-1} and if (12) holds in the sense that

$$\int_\Omega \nabla\eta \cdot \nabla\phi - \eta \cdot \left[|\nabla\phi|^2\phi + \kappa\partial_i\phi \times \left(\epsilon_{ij}\phi\partial_j u\right)\right] + \eta \cdot (\mathbf{B} - \phi \cdot \mathbf{B}\phi)dx = 0 \tag{15}$$

for all $\eta \in H^1(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$.

1.3 Notation

We work with the spaces $H^s(\Omega)$ of periodic distributions $T : C^\infty(\Omega) \to \mathbb{R}$ whose fourier coefficients $\hat{T}(\mathbf{n}) = (2\pi)^{-2}T(e^{i\mathbf{n}\cdot\mathbf{x}})$, $\mathbf{n} \in \mathbb{Z}^2$, satisfy

$$|T|_s^2 = \sum(1 + |\mathbf{n}|^2)^s|\hat{T}(\mathbf{n})|^2 < \infty.$$

We also denote by $\dot{H}^s$ the corresponding spaces of distributions with zero mean, i.e.

$$\dot{H}^s = \{T \in H^s : \hat{T}(\mathbf{0}) = 0\}.$$

These spaces have obvious generalisations to spaces of vector valued distributions. For $s \geq 0$ $H^s(\Omega; S^2)$ is made up on $\mathbb{R}^3$ valued functions : $\Omega \to \mathbb{R}^3$ such that $|u(x)| = 1$ almost everywhere.

We use the following basic facts:

- For any $T \in \dot{H}^s$ there is a unique distribution $u \in \dot{H}^{s+2}$ which satisfies $-\Delta u = f$ with fourier coefficients $\hat{u}(\mathbf{n}) = \hat{T}(\mathbf{n})/\mathbf{n}\cdot\mathbf{n}$. In particular for $s = -1$ we have a bounded linear map,

$$\mathbb{K} = (-\Delta)^{-1} : \dot{H}^{-1} \to \dot{H}^1,$$

  which is used to define $\underline{u}(\phi)$ in the text.
- For integrable f we write

$$\mathbb{K}f(x) = \int_\Omega K(x, y)f(y)dy$$

  and $\langle f, \mathbb{K}f\rangle = \int_\Omega\int_\Omega K(x, y)f(x)f(y)dxdy$ is equivalent to the H^{-1} norm on periodic functions $f \in \dot{H}^{-1} \cap L^1$ of zero mean.

We also use spaces $W^{s,q}(\Omega)$, $s \in \mathbb{Z}^+$, $1 \le q \le \infty$ of measurable periodic functions whose weak derivatives up to order s can be represented by L^p functions. The corresponding spaces of vector and S^2 valued functions are defined in the usual way as above. The subscript $_0$ is used to indicate zero boundary values in the trace sense as usual.

We make use of the Sobolev inequality

$$|w|_{L^{\frac{2q}{2-q}}(B(x_0,r))} \le C_1(q)|\nabla w|_{L^q(B(x_0,r))} \qquad q < 2 \tag{16}$$

valid for $w \in W_0^{1,q}(B(x_0,r))$, and the Calderon–Zygmund estimate

$$|w|_{W^{2,q}(B(x_0,r))} \le C_2(q)|\Delta w|_{L^q(B(x_0,r))}, \qquad 1 < q < \infty \tag{17}$$

valid for $w \in W^{2,q}(B(x_0,r)) \cap W_0^{1,q}(B(x_0,r))$ where $C_1(q), C_2(q)$ depend on q but are independent of r for $r < 1$.

Lemma 4 *Assume $u \in W_0^{1,q}(B(r))$, is the weak solution of*

$$-\Delta u = \nabla \cdot F + f \qquad F \in L^q(B(r)),\ f \in L^\infty(B(r))$$

in a ball $B(r)$ of radius $r < 1$ with zero boundary data. For $1 < q < \infty$ there exists a number $c = c(q)$, independent of $r < 1$, such that

$$|\nabla u|_{L^q}^q \le c\Big(|F|_{L^q}^q + r^{2+q}|f|_{L^\infty}^q\Big).$$

all norms being taken on $B(r)$).

Proof This can be proved by scaling from the case $r = 1$. For $r = 1$ the inequality involving F is just the Calderon–Zygmund estimate, while that involving f can be proved, for example, by writing down the integral kernel explicitly and applying the generalised Young inequality. □

2 Existence

We use an approximation scheme which is a natural extension of that used in [21]. We aim to construct solutions of (3) with specified degree

$$deg(\phi) = \frac{1}{4\pi}\int_\Omega j_\phi(x)dx = \frac{1}{4\pi}\int_\Omega \sigma(x)dx = d \in \mathbb{Z} \tag{18}$$

by means of a study of the convergence properties of minimisers ϕ_p of the functional

$$\mathcal{V}_p(\phi) = \frac{1}{2}\int_\Omega \big((1+|\nabla\phi|^2)^{\frac{p}{2}} - 1 + 2\mathbf{B}\cdot\phi + \kappa|\nabla \underline{u}(\phi)|^2\big)dx \tag{19}$$

subject to the constraint

$$\phi \in \mathcal{S}_p \equiv \Big\{\phi \in W^{1,p}(\Omega; S^2) : \frac{1}{4\pi}\int_\Omega j_{\phi_p}(x)dx = d.\Big\} \tag{20}$$

Observe that $\mathcal{V}_2 = \mathcal{V}$ since

$$\underline{u}(\phi) = \mathbb{K}(j_\phi - \sigma).$$

We will show that as $p \to 2^+$ the minimisers converge to $\phi \in H^1(\Omega, S^2)$ with the same degree d, which solves (3) in the sense of Definition 2. The first step is the construction of minimisers for $p > 2$.

Lemma 5 (Minimisation of $\mathcal{V}_p$) *For* $2 < p$ *there exists* $\phi_p \in C^\infty(\Omega; S^2)$ *satisfying* (20) *such that*

$$\mathcal{V}_p(\phi_p) = min\{\mathcal{V}_p(\phi) : \phi \in W^{1,p}(\Omega; S^2) \ and \ deg\, \phi = \frac{1}{4\pi}\int j_\phi = d\}$$

and

$$-\partial_j\big(\frac{p}{2}(1+|\nabla\phi_p|^2)^{\frac{p-2}{2}}\partial_j\phi_p\big) - \partial_j\phi_p \times \mathcal{J}_j^p(\phi_p) + (\mathbf{B} - \phi_p\cdot\mathbf{B}\phi_p) = 0 \qquad (21)$$

where

$$\mathcal{J}_i^p(\phi) = \frac{p}{2}(1+|\nabla\phi|^2)^{\frac{p-2}{2}}\phi\times\partial_i\phi + \kappa\epsilon_{ij}\phi\partial_j\mathbb{K}(j_\phi - \sigma).$$

Furthermore, the minimiser satisfies

$$\partial_j\mathcal{J}_j^p(\phi_p) = \phi_p \times \mathbf{B}. \qquad (22)$$

Proof Observe that, for each fixed $p > 2$,

$$\mathcal{V}_p(\phi) \geq -2\Big(|\Omega| + |\mathbf{B}|^2_{L^2(\Omega)}\Big) + \frac{1}{2}|\nabla\phi|^p_{L^p(\Omega)} + \frac{1}{2}\kappa|\nabla\underline{u}(\phi)|^2_{L^2(\Omega)} \qquad (23)$$

since $|\phi| = 1$. Therefore, we may assume the existence of a minimising sequence of smooth S^2 valued functions $\phi_n \in \mathcal{S}_p$ such that

$$\mathcal{V}_p(\phi_n) \to \inf_{\phi\in\mathcal{S}_p}\mathcal{V}_p(\phi),$$

ϕ_n is weakly convergent in $W^{1,p}(\Omega)$, and with j_{ϕ_n} weakly convergent in $H^{-1}(\Omega)$ (or equivalently $\underline{u}(\phi_n) = -\Delta^{-1}(j_{\phi_n} - \sigma)$ is weakly convergent in $H^1(\Omega)$). By Morrey's lemma and the assumption $p > 2$ we have a uniform bound on the Holder semi-norm

$$|\phi_n(x) - \phi_n(y)| \leq c|x-y|^{1-\frac{2}{p}}$$

with c independent of n. Consequently the sequence is equi-continuous and by the Arzela-Ascoli theorem there exists a subsequence (also called $(\phi_n)_n$) and $\phi_p \in W^{1,p}\cap C(\Omega)$ such that, as $n \to \infty$,

$$\begin{aligned} \phi_n &\longrightarrow \phi_p \quad \text{uniformly on } \Omega \\ \phi_n &\rightharpoonup \phi_p \quad \text{weakly in } W^{1,p}(\Omega). \end{aligned}$$

As $\phi_n \rightharpoonup \phi_p$ in $W^{1,p}$, it follows that the sequence $j_{\phi_n} = \phi_n\cdot\partial_1\phi_n\times\partial_2\phi_n$ is bounded in $L^{\frac{p}{2}}$ and so for a subsequence $j_{\phi_n} \rightharpoonup g$ in $L^{\frac{p}{2}}$ for some g, which then also implies the L^1 weak convergence of j_{ϕ_n} since $\frac{p}{2} > 1$. In addition (j_{ϕ_n}) is bounded in $H^{-1}(\Omega)$ (as follows by the functional in (19)). We now identify g as $\phi_p\cdot\partial_1\phi_p\times\partial_2\phi_p$. For this, the antisymmetrising property of the jacobian is useful:

$$\epsilon_{ij}\partial_i\phi_n\times\partial_j\phi_n = \epsilon_{ij}\partial_i(\phi_n\times\partial_j\phi_n).$$

(This is evident as an identity for C^2 functions. For H^1 functions it holds as an equality of distributions, by density of smooth S^2 valued functions [22, Sect. 4]). Since the left hand side is bounded in $L^{\frac{p}{2}}$, while the right hand side is bounded in $W^{-1,p}$, we have for a subsequence

$$\partial_i(\phi_n\times\partial_j\phi_n) \rightharpoonup \tau_{ij} \quad \text{weakly in} \quad L^{\frac{p}{2}}\cap W^{-1,p}$$

for some $\tau_{ij} \in L^{\frac{p}{2}} \cap W^{-1,p}$. We first show that the weak limit of $\phi_n \times \partial_j \phi_n$ is $\phi_p \times \partial_j \phi_p$, from which the limit of j_{ϕ_n} will follow. Since $\phi_n \longrightarrow \phi_p$ uniformly it follows that $\phi_n \times \partial \phi_n \rightharpoonup \phi_p \times \partial \phi_p$ weakly in L^p (if $\chi \in L^{p'}(\Omega)$ then $\int \chi \phi_n \times \partial_j \phi_n \longrightarrow \int \chi \phi_p \times \partial_j \phi_p$ since $\chi \phi_n \longrightarrow \chi \phi_p$ strongly in $L^{p'}$ because $\phi_n \longrightarrow \phi_p$ uniformly and $\partial_j \phi_n \rightharpoonup \partial_j \phi_p$ weakly in L^p). Thus $\partial_i(\phi_n \times \partial_j \phi_n) \rightharpoonup \partial_i(\phi_p \times \partial_j \phi_p) \equiv \tau_{ij}$ weakly in $W^{-1,p}$ and also weakly in $L^{\frac{p}{2}}$ by the jacobian property above. Therefore $\epsilon_{ij} \partial_i \phi_n \times \partial_j \phi_n \rightharpoonup \epsilon_{ij} \partial_i \phi_p \times \partial_j \phi_p$ weakly in $W^{-1,p} \cap L^{\frac{p}{2}}$. From this follows that

$$j_{\phi_n} \rightharpoonup j_{\phi_p} \quad \text{weakly in} \quad L^{\frac{p}{2}}$$

(since if $\chi \in L^{(\frac{p}{2})'}$ then

$$\int \chi \epsilon_{ij} \phi_n \cdot \partial_i \phi_n \times \partial_j \phi_n \longrightarrow \int \chi \epsilon_{ij} \phi_p \cdot \partial_i \phi_p \times \partial_j \phi_p$$

using again the uniform convergence of ϕ_n). Thus we conclude that

$$\int_\Omega (1 + |\nabla \phi_p|^2)^{\frac{p}{2}} \, dx \le \liminf_n \int_\Omega (1 + |\nabla \phi_n|^2)^{\frac{p}{2}} \, dx \tag{24}$$

and

$$\frac{1}{4\pi} \int_\Omega j_{\phi_p}(x) dx = \lim_n \frac{1}{4\pi} \int_\Omega j_{\phi_n}(x) dx = d. \tag{25}$$

From this we now deduce information about the convergence of $\underline{u}(\phi_n) = -\Delta^{-1}(j_{\phi_n} - \sigma)$. Since the j_{ϕ_n} are bounded and weakly convergent in $H^{-1} \cap L^{p/2}$ the $\underline{u}(\phi_n)$ are bounded in $H^1 \cap W^{2,\frac{p}{2}}$ by the Calderon–Zygmund estimate. Furthermore,

$$\underline{u}(\phi_n) \rightharpoonup (-\Delta)^{-1}(j_{\phi_p} - \sigma) \equiv \underline{u}(\phi_p) \quad \text{weakly in } H^1 \cap W^{2,\frac{p}{2}} \tag{26}$$

(since if u_* is the weak limit of $\underline{u}(\phi_n)$ then since $j_{\phi_n} \rightharpoonup j_{\phi_p}$ weakly in $H^{-1} \cap L^{p/2}$ we have

$$\begin{aligned} \lim_n \int \big(\Delta \underline{u}(\phi_n) + (j_{\phi_n} - \sigma)\big)\chi \, dx &= \lim_n \int \big(\underline{u}(\phi_n)\Delta\chi + (j_{\phi_n} - \sigma)\chi\big) dx \\ &= \int \big(u_* \Delta \chi + (j_{\phi_p} - \sigma)\chi\big) dx \end{aligned}$$

for all $\chi \in C^\infty$. But also $\int_\Omega u_* = \lim \int_\Omega \underline{u}(\phi_n) = 0$ and hence $u_* = \underline{u}(\phi_p)$ as claimed). Thus we have proved that

$$\int |\nabla \underline{u}(\phi_p)|^2 dx \le \liminf_n \int |\nabla \underline{u}(\phi_n)|^2 \tag{27}$$

which together with (24),(25),(26) and the linearity of the third term in $\mathcal{V}_p$ proves that $\phi_p \in \mathcal{S}_p$ and

$$\mathcal{V}_p(\phi_p) \le \liminf_n \mathcal{V}_p(\phi_n) = \min_{\phi \in \mathcal{S}_p} \mathcal{V}_p(\phi)$$

completing the proof that the minimum of $\mathcal{V}_p$ in the constraint set $\mathcal{S}_p$ is attained at the point ϕ_p.

The minimiser $\phi_p \in W^{1,p}(\Omega; S^2)$ is a weak solution of the Euler–Lagrange equation (44) by the usual arguments. (The constraint (20) does not affect the derivation since the degree is unchanged by a small C^1 change in ϕ because it is integer valued). We now discuss regularity.

Since $p > 2$ the function ϕ_p is Holder continuous by Morrey's lemma. Also $j_{\phi_p} \in L^{p/2}$ so that

$$u_p = (-\Delta)^{-1}(j_{\phi_p} - \sigma) = \mathbb{K}(j_{\phi_p} - \sigma) \in W^{2,p/2}$$

by the Calderon–Zygmund estimate since $p > 2$, so that $u_p \in W^{1,\frac{2p}{4-p}}$. But $4 > p > 2$ implies $\frac{2p}{4-p} > 2$ so that $u_p \in C^{\frac{2(p-2)}{p}}$. Thus (ϕ_p, u_p) is a Holder continuous solution of the second order elliptic system

$$\begin{aligned} &-\partial_j \cdot \big(\frac{p}{2}(1+|\nabla\phi_p|^2)^{\frac{p-2}{2}}\partial_j\phi_p\big) - \partial_j\phi_p \times \mathcal{J}_j^p(\phi_p) + (\mathbf{B} - \phi_p \cdot \mathbf{B}\phi_p) = 0,\\ &-\Delta u_p = j_{\phi_p} - \sigma. \end{aligned} \tag{28}$$

Since $p > 2$ it follows (as in [21]) by general theory [13,20] that (ϕ_p, u_p) is smooth. □

The next lemma implies strong convergence of the minimisers as $p \to 2^+$ in the absence of energy concentration. To state it we introduce

$$\eta = \eta(r, x_0; \phi, u, \kappa) \equiv |\nabla u|_{L^2(B(x_0,r))} + \kappa|\nabla\phi|_{L^2(B(x_0,r))}$$

as a measure of "energy" concentrated on the ball $B(x_0, r) \in \Omega$. We make use of the following inequalities, valid respectively for $w \in W_0^{1,\frac{4}{3}}(B(x_0, r))$ and for $w \in W^{2,\frac{4}{3}}(B(x_0, r)) \cap W_0^{1,\frac{4}{3}}(B(x_0, r))$:

$$|w|_{L^4(B(x_0,r))} \le C_1|\nabla w|_{L^{\frac{4}{3}}(B(x_0,r))} \qquad \text{(Sobolev)} \tag{29}$$

$$|w|_{W^{2,\frac{4}{3}}(B(x_0,r))} \le C_2|\Delta w|_{L^{\frac{4}{3}}(B(x_0,r))} \qquad \text{(Calderon–Zygmund)} \tag{30}$$

where C_1, C_2 independent of r for $r < 1$. Below let $b \in C_0^\infty(B(x_0, r))$ be a cut-off function satisfying $0 \le b \le 1$, $b = 1$ for $|x - x_0| \le \rho < r$, $b = 0$ for $|x - x_0| \ge \frac{\rho+r}{2}$ and $(r - \rho)|\nabla b(x)| + (r - \rho)^2|\nabla^2 b(x)| \le \beta$.

Lemma 6 (ϵ- regularity) *Let $\phi \in C^\infty(\Omega)$ be a solution of* (21). *Then for $0 \le \rho < r$, and the cut-off b (depending on ρ) as above,*

$$\begin{aligned} &(1 - (p-2)C_2 - C_1C_2\eta)|b\phi|_{W^{2,\frac{4}{3}}}\\ &\quad\le\ C_2|\mathbf{B}|_{L^{\frac{4}{3}}} + c(\beta)C_2\left(\frac{1}{r-\rho}|\nabla\phi|_{L^{\frac{4}{3}}} + \frac{1}{r-\rho^2}|\phi|_{L^{\frac{4}{3}}}\right) \end{aligned} \tag{31}$$

where all norms are taken on $B(x_0, r)$.

Proof Multiplying ϕ by the cut-off b gives a smooth function $b\phi$ which solves the equation

$$\begin{aligned} -\Delta(b\phi) &= (p-2)b\frac{\langle\partial_i\phi, \partial_j\partial_i\phi\rangle}{1+|\nabla\phi|^2}\partial_j\phi\\ &\quad - 2\nabla b\cdot\nabla\phi - (\Delta b)\phi + \frac{\partial_j\phi \times \mathcal{J}_j^p}{(1+|\nabla\phi|^2)^{\frac{p-2}{2}}}\, b - \frac{(\mathbf{B} - \phi\cdot\mathbf{B}\phi)b}{(1+|\nabla\phi|^2)^{\frac{p-2}{2}}} \end{aligned}$$

which, together with the pointwise inequalities:

$$\begin{aligned} |\mathcal{J}^p(\phi)| &\le (1+|\nabla\phi|^2)^{\frac{p-2}{2}}|\nabla\phi| + \kappa|\nabla u|,\\ |\phi\times\mathcal{J}^p(\phi)| &\le (1+|\nabla\phi|^2)^{\frac{p-2}{2}}|\nabla\phi| \end{aligned}$$

and (22), gives (pointwise)

$$\left|-\Delta(b\phi) - (p-2)\frac{\langle\partial_i\phi, \partial_j\partial_i\phi\rangle}{1+|\nabla\phi|^2}\partial_j\phi - \frac{\partial_j(b\phi)\times\mathcal{J}_j^p}{(1+|\nabla\phi|^2)^{\frac{p-2}{2}}} + \frac{b(\mathbf{B}-\phi\cdot\mathbf{B}\phi)}{(1+|\nabla\phi|^2)^{\frac{p-2}{2}}}\right|$$
$$\leq c(\beta)\left(\frac{|\nabla\phi|}{r-\rho} + \frac{|\phi|}{(r-\rho)^2}\right).$$

Holder's inequality, with $3/4 = 1/2 + 1/4$, and (29)–(30), imply

$$\left|\frac{\partial_j(b\phi)\times\mathcal{J}_j^p}{(1+|\nabla\,\phi|^2)^{\frac{p-2}{2}}}\right|_{L^{\frac{4}{3}}} \leq |\nabla(b\phi)|_{L^4}(|\nabla\phi|_{L^2} + \kappa|\nabla u|_{L^2})$$
$$\leq C_1\eta|(b\phi)|_{W^{2,\frac{4}{3}}} \tag{32}$$

where all the norms are on $B(x_0, r)$. Consequently,

$$|\Delta(b\phi)|_{L^{\frac{4}{3}}} \leq (p-2)|(b\phi)|_{W^{2,\frac{4}{3}}}$$
$$+C_1\eta|(b\phi)|_{W^{2,\frac{4}{3}}} + |\mathbf{B}|_{L^{\frac{4}{3}}} + c(\beta)\left(\frac{|\nabla\phi|_{L^{\frac{4}{3}}}}{r-\rho} + \frac{|\phi|_{L^{\frac{4}{3}}}}{(r-\rho)^2}\right)$$

which implies the inequality as claimed. □

We can now prove existence of weak solutions:

Theorem 7 *Given $d \in \mathbb{Z}$ and smooth periodic functions $\mathbf{B} \in C^\infty(\Omega; \mathbb{R}^3)$ and $\sigma \in C^\infty(\Omega)$ such that $(4\pi)^{-1}\int_\Omega \sigma = d$ there exists a weak solution of* (3) *in the sense of Definition 2 which satisfies* (18).

Proof of theorem 7 We consider for $2 < p \leq 5/2$ the minimisers ϕ_p just obtained. By (23) they satisfy the bounds,

$$\int_\Omega |\nabla\phi_p|^2 + \kappa|\nabla u_p|^2 \leq c(d, |\mathbf{B}|_{L^2}, |\Omega|) \tag{33}$$

uniformly in $p \in (2, 5/2]$ and consequently there exists a subsequence as $p \downarrow 2$ a function $\phi \in W^{1,2}$ and measures μ, ν such that

$$|\nabla\phi_p|^2 + \kappa|\nabla u_p|^2 \rightharpoonup \mu \quad \text{weakly in } \mathcal{M}_+(\Omega) \tag{34}$$
$$j_{\phi_p} \rightharpoonup \nu \quad \text{weakly in } \mathcal{M}(\Omega)\cap H^{-1}(\Omega) \tag{35}$$
$$\phi_p \rightharpoonup \phi \quad \text{weakly in } W^{1,2}(\Omega) \text{ and weak* in } L^\infty(\Omega) \tag{36}$$
$$\phi_p(x) \longrightarrow \phi(x) \quad \text{for a.e.}\, x \in \Omega \tag{37}$$

where $\mathcal{M}(\Omega)$ is the space of signed Radon measures on Ω and $\mathcal{M}_+(\Omega)$ the space of non-negative Radon measures on Ω ([4, p.75–76]). We now show that this convergence is strong on the complement of a finite set of points and identify the non-singular parts of μ and ν.

Lemma 8 *There exists a finite point set $\{s_1, \ldots, s_N\}$ such that*

$$\mu = \big(|\nabla\phi|^2 + \kappa|\nabla u|^2\big)dx + \sum_{i=1}^N \mu(\{s_i\})\delta_{s_i} \tag{38}$$
$$\nu = j_\phi dx \tag{39}$$

and if $x_0 \notin \{s_1, \dots, s_N\}$ *there exists* $\rho > 0$ *such that* $\phi_p \to \phi$ *strongly in* $W^{1,q}(B(x_0, \rho))$ *for* $q < 4$. *Furthermore* $\int_\Omega j_\phi dx = 4\pi d$.

Proof of lemma 8 We assume we have restricted to a subsequence in $p \to 2^+$ for which (34)–(37) hold. Using the energy bound (33) and the $\epsilon-$regularity lemma 6 we first obtain that the absolutely continuous part of μ is given by $|\nabla\phi|^2 + \kappa|\nabla u|^2$. Fix arbitrary $\epsilon > 0$ such that $2(1+\kappa)\epsilon < 1/(4C_1C_2)^2$. Let

$$f_p = |\nabla\phi_p|^2 + \kappa|\nabla u_p|^2.$$

Define the set

$$S_\epsilon = \cap_{r>0}\{x \in \Omega : \mu(B(x, r)) \geq \epsilon/2\}.$$

Clearly S_ϵ is a finite set of cardinality less than $\frac{2\mu(\Omega)}{\epsilon} \leq \frac{c(d,|\mathbf{B}|_{L^2},|\Omega|)}{\epsilon}$. Consider a point $x_0 \notin S_\epsilon$; there exists $r > 0$ such that $\mu(B(x_0, 2r)) < \epsilon/2$. Take a cut-off function $\chi \in C_0^\infty(B(x_0, 2r)$ with $0 \leq \chi \leq 1$ on $B(x_0, 2r)$ and $\chi = 1$ on $B(x_0, r)$. The sequence $(f_p)_{p>2}$ is bounded in L^1 and weak* convergent in $\mathcal{M}_+(\Omega)$ to μ by (34), so that if $p-2 > 0$ is sufficiently small $|\int_\Omega f_p\chi - \mu(\chi)| < \epsilon/2$, which implies

$$\int\limits_{B(x_0,r)} f_p dx \leq \int\limits_\Omega f_p\chi \leq \mu(\chi) + \epsilon/2 < \mu(B(x_0, 2r) + \epsilon/2 < \epsilon.$$

It follows from the fact that $\eta_p^2 \leq 2(1+\kappa)\int_{B(x_0,r)} f_p$ that

$$\eta_p \equiv |\nabla\phi_p|_{L^2(B(x_0,r)} + \kappa|\nabla u_p|_{L^2(B(x_0,r)} \leq \sqrt{2(1+\kappa)\epsilon}.$$

Restrict further p to be such that $(p-2)C_2 \leq 1/4$, then, together with the choice of ϵ above, this implies that $1-(p-2)C_2 - C_1C_2\eta \geq 1 - 1/4 - C_1C_2\sqrt{2(1+\kappa)\epsilon} > 1/2$. We apply (31) on $B(x_0, r)$, with cut-off function b as defined prior to Lemma 6, to deduce that for sufficiently small $p-2$

$$|\phi_p|_{W^{2,\frac{4}{3}}(B(x_0,\rho))} < M = M(r, \rho, |\nabla\phi_p|_{L^{\frac{4}{3}}}) < c \tag{40}$$

where the constant c is independent of p because the $L^{\frac{4}{3}}$ norms of ϕ_p are bounded independently of p by the bound in (33). Since $W^{1,\frac{4}{3}} \subset L^4$ is compactly embedded into $\subset L^q$ for $q < 4$ we deduce that (possibly after redefinition on a set of measure zero)

$$\phi \in W^{2,\frac{4}{3}}(B(x_0, \rho)) \cap C\left(\overline{B(x_0, \rho)}\right)$$

and that, for $1 \leq q < 4$,

$$\nabla\phi_p \longrightarrow \nabla\phi \text{ strongly in } L^q(B(x_0, \rho)) \tag{41}$$

$$\phi_p \longrightarrow \phi \text{ uniformly in } \overline{B(x_0, \rho)}. \tag{42}$$

The bound (40) allows us to deduce that j_{ϕ_p}, which a priori converges to ν as a signed Radon measure, in fact converges in L^1 strongly near any point $x_0 \notin S_\epsilon$, and hence that near such points ν can be represented by an L^1 function. Write

$$\begin{aligned} j_{\phi_p} - j_\phi &= (\phi_p - \phi)(\partial_1\phi_p \times \partial_2\phi_p) \\ &\quad + \phi\left((\partial_1\phi_p - \partial_1\phi) \times \partial_2\phi_p + \partial_1\phi \times (\partial_2\phi_p - \partial_2\phi)\right) \end{aligned}$$

and then estimate

$$|j_{\phi_p} - j_\phi|_{L^1} \leq |\phi_p - \phi|_{L^\infty} |\partial_1 \phi_p|_{L^2} |\partial_2 \phi_p|_{L^2}$$
$$+ |\phi|_{L^\infty} \Big(|\partial_2 \phi_p|_{L^2} |\partial_1 \phi_p - \partial_1 \phi|_{L^2} + |\partial_1 \phi|_{L^2} |\partial_2 \phi_p - \partial_2 \phi|_{L^2} \Big)$$

with all the norms taken on $B(x_0, \rho)$. By (41) and (42) the limit as $p \to 2$ is zero. Thus for every $x_0 \notin S_\epsilon$ there exists $\rho > 0$ such that

$$j_{\phi_p} \longrightarrow j_\phi \text{ strongly in } L^1(B(x_0, \rho))$$

and hence $\nu - j_\phi dx$ is a measure supported on the finite set $\{s_1, \ldots, s_N\}$, i.e. a finite combination of Dirac measures δ_{s_i}. We now show that in fact $\nu = j_\phi dx$ is absolutely continuous with respect to Lebesgue measure, and further that

$$4\pi \; deg(\phi_p) = \int j_{\phi_p} \overset{p \to 2^+}{\longrightarrow} \int j_\phi = 4\pi \; deg(\phi). \tag{43}$$

To see this recall that, by the minimisation of $\mathcal{V}_p$ in Lemma 5, $(j_{\phi_p})_p$ is bounded in $H^{-1}(\Omega)$. But by the above $\nu = j_\phi dx + \sum \nu(\{s_i\}) \delta_{s_i}$ and the singular set of $\{s_i : \nu(\{s_i\}) \neq 0\}$ is a subset of S_ϵ which is finite. In fact,
Claim $\nu(\{s_i\}) = 0$ for all $s_i \in S_\epsilon$.
To prove the claim recall that H^1 contains unbounded functions in two dimensions (such as $f : r \to \ln \ln |x - s|^{-2} \in H^1$). Smoothing these gives smooth functions χ_ϵ, supported in a neighbourhood of s_i, which are bounded independent of ϵ in H^1 but with $\chi_\epsilon(s_i)$ arbitrarily large for small ϵ. Integration then gives a contradiction since $|\int j_{\phi_p} \chi_\epsilon| \leq |j_{\phi_p}|_{H^{-1}} |\chi_\epsilon|_{H^1}$ but $\int j_{\phi_p} \chi_\epsilon \to \nu(\{s_i\}) \chi_\epsilon(s_i) + O(1)$.

From this claim we deduce that $\nu = j_\phi dx$ and hence

$$\int_\Omega j_\phi dx = \nu(\Omega) = \lim \int_\Omega j_{\phi_p} dx$$

by definition of weak convergence of measures, and (43) follows.

Completion of proof of theorem 7 It remains to prove that the weak limit ϕ just constructed is in fact a weak solution, i.e. satisfies (8). We start with the weak form of (21):

$$\int_\Omega \Big(\frac{p}{2} (1 + |\nabla \phi_p|^2)^{\frac{p-2}{2}} \partial_j \phi_p \cdot \partial_j \eta - \eta \cdot \partial_j \phi_p \times \mathcal{J}_j^p + \eta \cdot (\mathbf{B} - \phi_p \cdot \mathbf{B} \phi_p) \Big) dx = 0 \tag{44}$$

for all $\eta \in C^\infty(\Omega; \mathbb{R}^3)$, which is automatically satisfied since ϕ_p is a smooth classical solution of (21). It is sufficient to show, by taking the limit $p \to 2^+$ of this equation, that

$$\int_\Omega \big(\partial_j \phi \cdot \partial_j \eta - \eta \cdot \partial_j \phi \times \mathcal{J}_j + \eta \cdot (\mathbf{B} - \phi \cdot \mathbf{B} \phi) \big) dx = 0 \tag{45}$$

for all $\eta \in C^\infty(\Omega; \mathbb{R}^3)$; it will then automatically hold for $\eta \in L^\infty \cap H^1$. We will prove (45) by showing that the first and second terms in (44) converge to the corresponding terms in (45), using Lemma 6. The convergence of the third term in (44) to the corresponding term in (45) is clear from the bounded convergence theorem.

Consider the second term: using the final equation in Lemma 5 and the L^q-strong convergence of $\phi_p \to \phi\,\forall q < \infty$, reduces the problem to showing that

$$\int_\Omega \partial_j \eta \phi_p \times \mathcal{J}_j^p dx \to \int_\Omega \partial_j \eta \phi \times \mathcal{J}_j dx \tag{46}$$

as $p \to 2^+$. Decompose $\mathcal{J}^p$ as follows:

$$\begin{aligned}\mathcal{J}_i^p(\phi) &= \kappa \epsilon_{ij} \phi \partial_j \mathbb{K}(j_\phi - \sigma) + \frac{p}{2}(1 + |\nabla \phi|^2)^{\frac{p-2}{2}} \phi \times \partial_i \phi \\ &= \mathcal{K}(\phi) + \mathcal{L}^p(\phi).\end{aligned}$$

Again the L^q-strong convergence of $\phi_p \to \phi\,\forall q < \infty$, implies that (46) holds once the following assertions are proved:

$$\begin{aligned}\mathcal{K}(\phi_p) &\rightharpoonup \mathcal{K}(\phi) \quad \text{weakly in } L^{\frac{4}{3}}, \quad \text{and} \\ \mathcal{L}^p(\phi_p) &\longrightarrow \mathcal{L}(\phi) \quad \text{strongly in } L^{\frac{5}{4}}.\end{aligned}$$

To prove the first observe that $j_{\phi_p} \rightharpoonup j_\phi$ weakly in H^{-1} and hence $\partial_j \mathbb{K}(j_{\phi_p} - \sigma) \rightharpoonup \partial_j \mathbb{K}(j_\phi - \sigma)$ weakly in L^2. But again using the strong L^4 convergence of ϕ_p this implies that $\mathcal{K}(\phi_p)$ converges to $\mathcal{K}(\phi)$ weakly in $L^{\frac{4}{3}}$.

To prove the second it is necessary to take a covering to allow different treatment near and away from the singular points s_i. Around each singular point s_i take an open ball $B(s_i, r_i)$ and by choosing the radii small enough they may be assumed disjoint; the complement in Ω of all these balls $\cap B(s_i, r_i)^c$ is compact. Around any point x of this complement there is a ball $B(x, \rho_x)$ on which (40) holds for p sufficiently close to 2. By compactness there exists a finite sub-cover of the complement

$$\cup_{\alpha=1}^M B(x_\alpha, \rho_\alpha) \supset \cap B(s_i, r_i)^c \qquad (\text{where } \rho_\alpha = \rho_{x_\alpha})$$

and a number L such that for $p < 2 + 1/L$

$$|\phi_p|_{W^{2,\frac{4}{3}}(B(x_\alpha, \rho_\alpha))} \le c_\alpha < \infty \tag{47}$$

for all $\alpha \in \{1, \ldots, M\}$. Then by the Sobolev and Rellich theorems we may assume

$$\max_\alpha \sup_{2<p<1/L} |\nabla \phi_p|_{L^4(B(x_\alpha, \rho_\alpha))} = N < \infty, \tag{48}$$

$$\lim_{p \to 2^+} \max_\alpha |\nabla \phi_p - \nabla \phi|_{L^{10/3}(B(x_\alpha, \rho_\alpha))} = 0. \tag{49}$$

(The 10/3 exponent is chosen for convenience of use in the next paragraph).

On the singular balls we can estimate e.g.

$$|\mathcal{L}^p(\phi_p)|_{L^{\frac{5}{4}}(B(s_i, r))} \le c |\nabla \phi_p|_{L^2}^{p-1} r^{\frac{8}{5} - (p-1)}$$

by Holder's inequality. Restricting to $p < 12/5$ and $P \le 5/4$ the exponent $\frac{2}{P} - (p-1) > 1/5$. Consequently for any $\delta > 0$ it is possible to choose $\max_{1 \le i \le N}\{r_i\}$ sufficiently small that

$$\sum_{i=1}^N \int_{B(s_i, r_i)} \Big(|\mathcal{L}^p(\phi_p)|^{5/4} + |\mathcal{L}(\phi)|^{5/4} \Big) dx < \delta/4,$$

uniformly in $p < 12/5$ by (33).

Now, pointwise a.e.,

$$|\mathcal{L}^p(\phi_p) - \mathcal{L}(\phi)| \le c_1\Big(1 + |\nabla\phi_p|^{p-2}\Big)|\nabla\phi_p - \nabla\phi| + c_2(p-2)|\nabla\phi_p|^2|\nabla\phi|$$

which can be estimated in $L^{5/4}$ using Holder's inequality with $4/5 = 1/2 + 3/10$ and (48)–(49). It follows that for arbitrary $\delta > 0$ it is possible to choose $p-2$ sufficiently small (and positive) that

$$\int_{B(x_\alpha,\rho_\alpha)} |\mathcal{L}^p(\phi_p) - \mathcal{L}(\phi)|^{5/4} dx \le \delta/(2M).$$

Therefore, using $(a+b)^{5/4} \le 2^{1/4}(a^{5/4} + b^{5/4})$,

$$\begin{aligned}\int_\Omega |\mathcal{L}^p(\phi_p) - \mathcal{L}(\phi)|^{5/4} &\le 2^{1/4}\sum_{i=1}^{N}\int_{B(s_i,r_i)} \Big(|\mathcal{L}^p(\phi_p)|^{5/4} + |\mathcal{L}(\phi)|^{5/4}\Big)dx\\ &\quad + \sum_{\alpha=1}^{M}\int_{B(x_\alpha,\rho_\alpha)} |\mathcal{L}^p(\phi_p) - \mathcal{L}(\phi)|^{5/4}dx\\ &< \delta\end{aligned}$$

which proves the strong $L^{5/4}$ convergence of $\mathcal{L}^p(\phi_p)$ since δ was arbitrary. To conclude the proof of theorem 7 we apply an identical argument to show that $(1 + |\nabla\phi_p|^2)^{\frac{p-2}{2}}\nabla\phi_p$ converges to $\nabla\phi$ strongly in $L^{5/4}$ and hence deduce that the first term in (44) converges to the first term in (45). □

3 Regularity

We consider $(\phi, u) \in H^1(\Omega; S^2) \times H^1(\Omega)$ which are weak solutions of the system (12)-(14), as in definition 3, and prove that $(\phi, u) \in C^\infty(\Omega)$. We first show continuity and then improve it to Holder continuity and thence smoothness.

3.1 Continuity

We show continuity of ϕ as a consequence of Wente's lemma and an observation analogous to that made for harmonic maps by Helein:

Theorem 9 *Given a weak solution (ϕ, u) of (12)–(14), as in definition 3, ϕ is continuous and in fact $\phi \in C(\Omega) \cap W^{2,1}(\Omega) \cap H^1(\Omega)$.*

Proof The crucial point is that (3) can be rewritten in the jacobian form (10). This will allow us to apply an immediate extension of Wente's lemma to deduce the continuity of ϕ. This works because (4) implies that $\nabla \cdot \mathcal{J} \in L^\infty$, so that the gradient part of $\mathcal{J}$ appearing in the Hodge decomposition is more regular than L^2 (which is all that is known a priori). Here are the details.

By the Hodge decomposition

$$\mathcal{J} = \nabla a + \nabla^\perp b$$

where $\Delta a = \nabla \cdot \mathcal{J}$ and $\Delta b = \nabla^{\perp} \cdot \mathcal{J}$. (Here $a, b \in \mathbb{R}^3$ and $\nabla^{\perp} \cdot \mathcal{J} \equiv \partial_2 \mathcal{J}_1 - \partial_1 \mathcal{J}_2$.) Thus (10) can be written equivalently as

$$-\Delta\phi = \nabla\phi \times \nabla^{\perp} b + f \tag{50}$$

where $f = \nabla\phi \times \nabla a - (\mathbf{B} - \phi \cdot \mathbf{B}\phi)$. By (4) it follows that $\nabla \cdot \mathcal{J} \in L^{\infty}$, and so by (17) $a \in W^{2,q}$ for all $1 < q < \infty$ which implies $a \in C^1(\Omega)$. Therefore $f \in L^2$ whereas initially one only has $f \in L^1$. It follows that a weak solution ϕ of (3) and (50) can be decomposed as $\phi = \psi + \eta$ where

$$-\Delta\psi = \nabla\phi \times \nabla^{\perp} b$$
$$-\Delta\eta = f.$$

By Wente's lemma [14,27] $\psi \in C(\Omega)$ and by elliptic theory $\eta \in H^2 \subset C(\Omega)$. Therefore

$$\phi \in C(\Omega) \cap H^1(\Omega).$$

Thus the jacobian structure together with Wente's lemma yields continuity as for harmonic maps. In fact, for harmonic maps it is also true that $\phi \in W^{2,1}$, again by virtue of the jacobian determinant in (50), using [7] or [26, Chap. 13, Proposition 12.5]: as $\nabla^{\perp} b \in L^2$ is divergence-free, we have that $\nabla\phi \times \nabla^{\perp} b \in \mathcal{H}^1_{loc}$, the local Hardy space. Therefore, if, as above, ψ solves

$$-\Delta\psi = \nabla\phi \times \nabla^{\perp} b \in \mathcal{H}^1_{loc}$$

then $\psi \in W^{2,1}(\Omega)$ by the definition of the Hardy space as the subset of L^1 stable under action of singular integrals ([25, Chap. 3] or [14, Theorem 3.2.9]). Recalling that $W^{2,1}(\Omega)$ is continuously embedded in $C(\Omega)$ this also gives an alternative proof of continuity. □

3.2 Holder continuity and smoothness

In the case of harmonic maps ($\kappa = 0$ and $\mathbf{B} = 0$) it can be deduced from general elliptic theory, once continuity is known, that the harmonic map is smooth (see [3] for an argument specific to S^2 valued harmonic maps, or [26, Chap. 13] for a more general framework). In the general $\kappa > 0$ case it seems to be necessary to prove Holder continuity in order to deduce smoothness from general theory. This is due to the structure of the term j_ϕ, which is not evidently in the Hardy space as a consequence of Theorem 9, and so continuity of $u = \underline{u}(\phi)$ is not assured without further work. A technique to exploit the jacobian structure present in the harmonic map equation to prove Holder continuity directly was given by Chang et al. in [6]. Here we show that this technique can be modified to prove regularity for weak solutions of (12)–(14) as in Definition 3. The main step towards establishing Holder continuity is achieved by the following Morrey growth type estimate:

Lemma 10 *Fix*[1] $p \in (2, \infty)$. *For ϕ as in Theorem 9 there exist positive numbers* $\theta_0,\ \beta,\ \gamma,\ s < \frac{1}{4}$ *and a sequence* $\{A_k\}_{k=0}^{\infty}$ *of vectors in* $\mathbb{R}^3$ *such that if*

$$\int_{B_{R_0}} |\nabla\phi|^2 + |\nabla u|^2 \ < \theta_0$$

[1] In this section we study only critical points of (1) and the exponent p has nothing to do with that appearing in the modified energy (19).

then for any $R \in (0, R_0)$

$$\int_{B_{s^{k+1}R}} |\phi - A_{k+1}|^p \leq s^\gamma \int_{B_{s^k R}} |\phi - A_k|^p \, dx + \beta(s^k R)^{2+2p} \tag{51}$$

and

$$|A_{k+1} - A_k| \leq c \left(\frac{1}{|B(s^k R)|} \int_{B_{s^k R}} |\phi - A_{k+1}|^p \right)^{\frac{1}{p}}. \tag{52}$$

Proof Given $A_0 \in \mathbb{R}^3$ there exists $r \in [\frac{R}{2}, R]$ such that

$$\int_{\partial B_r} |\phi - A_0|^p \leq \frac{4}{R} \int_{B_R} |\phi - A_0|^p. \tag{53}$$

Let h satisfy

$$-\Delta h = 0$$
$$h - \phi \in W_0^{1,2}(B_r)$$

which implies

$$sup_{z \in B(\frac{r}{2})} |\nabla h(z)|^p \leq \frac{c_1}{r^{1+p}} \int_{\partial B_r} |h - A_0|^p \leq \frac{c_2}{R r^{1+p}} \int_{B(R)} |\phi - A_0|^p \tag{54}$$

by the Cauchy representation and Holder's inequality. Therefore,

$$-\Delta(\phi - h) = \nabla \cdot ((\phi - A_0) \times \mathcal{J}) - ((\phi - A_0) \times \phi \times \mathbf{B}) - (\mathbf{B} - \phi \cdot \mathbf{B}\phi).$$

Now apply Lemma 4 with $q = \frac{2p}{2+p} < 2$, estimating the first term using Holder's inequality with $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$ and

$$|\mathcal{J}|^2_{L^2(B_r)} \leq c \int_{B_r} (|\nabla \phi|^2 + |\nabla u|^2) \leq c\theta_0, \qquad \text{for } r < R_0,$$

to deduce,

$$\int_{B_r} |\nabla(\phi - h)|^q \leq c \left(\theta_0^{\frac{q}{2}} \left(\int_{B_r} |\phi - A_0|^p \right)^{\frac{q}{p}} + r^{q+2} \right). \tag{55}$$

Now for $s < \frac{1}{4}$ we have $sR < \frac{R}{4} < r \leq R \leq R_0$ and the Poincare and Sobolev inequalities give

$$\frac{1}{(sR)^2} \int_{B_{sR}} |\phi - h(0)|^p \leq c \left(\frac{1}{(sR)^2} \int_{B_r} |\phi - h|^p + \frac{1}{(sR)^2} \int_{B_{sR}} |h - h(0)|^p \right)$$

(because $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ and $sR < r$)

$$\leq c \left(\left(\frac{1}{(sR)^2} \int_{B_r} |\nabla(\phi - h)|^q \right)^{\frac{p}{q}} + (sR)^p \sup_{B_{\frac{R}{4}}} |\nabla h|^p \right)$$

(because $h - \phi \in W_0^{1,2}(B_r)$)

$$\leq c\left(\frac{\theta_0^{\frac{p}{2}}}{(sR)^2}\int_{B_R}|\phi - A_0|^p + \frac{r^{\frac{(q+2)p}{q}}}{(sR)^2} + \frac{s^p R^{p-1}}{r^{1+p}}\int_{B_R}|\phi - A_0|^p\right),$$

by (54) and (55). Therefore, recalling that $R < 4r$,

$$\begin{aligned}\int_{B_{sR}}|\phi - h(0)|^p &\leq c\left(\theta_0^{\frac{p}{2}}\int_{B_R}|\phi - A_0|^p + \frac{s^{p+2}R^{p+1}}{r^{1+p}}\int_{B_R}|\phi - A_0|^p + r^{\frac{(q+2)p}{q}}\right)\\ &\leq C\left(\theta_0^{\frac{p}{2}}\int_{B_R}|\phi - A_0|^p + s^{p+2}\int_{B_R}|\phi - A_0|^p + r^{\frac{(q+2)p}{q}}\right),\end{aligned}$$

for some constant C, which may be assumed to satisfy $C > 1$ without loss of generality. Given $s < \frac{1}{4}$ and define $\gamma = \gamma(s)$ by

$$s^\gamma = 2Cs^{p+2}.$$

Observe that by choosing s small we may (and will) ensure that

$$\gamma = p + 2 + \frac{\ln 2C}{\ln s} \in (2, p+2). \tag{56}$$

Choose θ_0 sufficiently small so that $\theta_0^{\frac{p}{2}} < s^{p+2}$, so that $C\theta_0^{p/2} + Cs^{p+2} < 2Cs^{p+2} = s^\gamma$ and then we have

$$\int_{B_{sR}}|\phi - A_1|^p \leq s^\gamma\int_{B_R}|\phi - A_0|^p + \beta R^{\frac{(q+2)p}{q}}$$

with $A_1 = h(0)$ and $\beta = C$. Since this applies to any $R \leq R_0$ and $(q+2)p = 2q(1+p)$ we obtain (51) with $k = 0$. Finally,

$$\begin{aligned}|A_1 - A_0| &= |h(0) - A_0|\\ &= \frac{1}{|\partial B_r|}\left|\int_{\partial B_r}|\phi - A_0|\right|\\ &\leq c\left(\frac{1}{|\partial B_r|}\int_{\partial B_r}|\phi - A_0|^p\right)^{\frac{1}{p}}.\end{aligned}$$

From this we deduce

$$R^2|A_1 - A_0|^p = \int_0^R 2r|A_1 - A_0|^p dr \leq c\int_{B_R}|\phi - A_0|^p dx$$

and hence

$$|A_1 - A_0| \le \left(\frac{c}{|B_R|} \int_{B_R} |\phi - A_0|^p \right)^{\frac{1}{p}} \tag{57}$$

and this completes the proof of the lemma, for some sequence A_k of vectors in $\mathbb{R}^3$, since for $k \ge 1$ (14) is obtained by repeating the proof of (57) with R replaced by $s^k R$. □

As a corollary we obtain Holder continuity for ϕ:

Corollary 11 *The solution $\phi \in C^{0,\alpha}$ for some $\alpha > 0$ determined in (59) below.*

Proof Fix $x \in \Omega$ and let $X_k = X_k(x)$ be given by

$$X_k = \int_{B(x,s^k R)} |\phi - A_k(x)|^p$$

where $(A_k(x))_k$ is the sequence of vectors of the lemma above. Then

$$X_{k+1} \le s^\gamma X_k + \beta (s^k R_0)^{2+2p}.$$

By (56) above there exists $\gamma_1 \in (2, \gamma)$ so that we can write $\tilde{\beta} = \beta R_0^{2+2p-\gamma_1}$ and then

$$X_{k+1} \le s^\gamma X_k + \tilde{\beta} R_0^{\gamma_1} s^{k\gamma_1}$$

so that for the first three terms we have

$$\begin{aligned} X_1 &\le s^\gamma X_0 + \tilde{\beta} R_0^{\gamma_1} \\ X_2 &\le s^\gamma (s^\gamma X_0 + \tilde{\beta} R_0^{\gamma_1}) + \tilde{\beta} s^{\gamma_1} R_0^{\gamma_1} \\ X_3 &\le s^\gamma (s^{2\gamma} X_0 + s^\gamma \tilde{\beta} R_0^{\gamma_1} + \tilde{\beta} s^{\gamma_1} R_0^{\gamma_1}) + s^{2\gamma_1} \tilde{\beta} R_0^{\gamma_1}. \end{aligned}$$

By induction in general

$$\begin{aligned} X_{k+1} &\le s^{(k+1)\gamma} X_0 + \tilde{\beta} (s^k R_0)^{\gamma_1} \sum_{j=0}^{k} s^{(\gamma-\gamma_1) j} \\ &\le s^{(k+1)\gamma} X_0 + \tilde{\beta} (s^k R_0)^{\gamma_1} \frac{1}{1 - s^{\gamma-\gamma_1}} \end{aligned}$$

so that

$$\begin{aligned} X_{k+1} &\le s^{(k+1)\gamma} X_0 + \frac{\tilde{\beta} s^{\gamma_1 k} R_0^{\gamma_1}}{1 - s^{\gamma-\gamma_1}} \\ &\le C s^{k\gamma_1} \end{aligned} \tag{58}$$

where $C = C(S, \gamma, \gamma_1, X_0, R_0)$. Since $\gamma_1 > 2$ this also implies by (52)

$$|A_{k+1}(x) - A_k(x)| \le C s^{(\gamma_1 - 2)k} \longrightarrow 0$$

and so the sequence $\{A_k(x)\}_{k=1}^\infty$ has a limit which will be denoted $A(x)$.

Claim 1: $A(x) = \phi(x)$ for a.e. x

Proof By Lebesgue differentiation, we have for a.e. $x \in \Omega$:

$$
\begin{aligned}
|\phi(x) - A(x)| &= \left| \lim_{r\to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \Big(\phi(z) - A(x)\Big) dz \right| \\
&= \left| \lim_{k\to\infty} \frac{1}{|B(x,s^k R)|} \int\limits_{B(x,s^k R)} \Big(\phi(z) - A_k(x)\Big) dz \right| \\
&\le \lim_{k\to\infty} \left(\frac{1}{|B(x,s^k R)|} \int\limits_{B(x,s^k R)} |\phi(z) - A_k(x)|^p dz \right)^{\frac{1}{p}} \\
&= 0,
\end{aligned}
$$

and the claim is proved. □

We now show that, redefining ϕ on a set of zero measure, $A_k(x)$ converges uniformly to a Holder continuous function $\phi(x)$.

Claim 2: $|A_k(x) - A_k(y)| \le C|x-y|^{\frac{(\gamma_1 - 2)}{p}}$.

Proof Let $R = |x - y|$ then

$$
\begin{aligned}
&|B(x,2R) \cap B(y,2R)|\ |A_k(x) - A_k(y)| \\
&\quad= \int\limits_{\{|z-x|\le 2R\}\cap\{|z-y|\le 2R\}} |A_k(x) - \phi(z) + \phi(z) - A_k(y)| dz \\
&\quad= \int\limits_{B(x,2R)} |A_k(x) - \phi(z)| dz + \int\limits_{B(y,2R)} |A_k(y) - \phi(z)| dz \\
&\quad\le |A_k(x) - \phi|_{L^p(B(x,2R))} |B(x,2R)|^{1-\frac{1}{p}} \\
&\qquad+ |A_k(y) - \phi|_{L^p(B(y,2R))} |B(y,2R)|^{1-\frac{1}{p}} \\
&\quad\le 2cR^{\frac{\gamma_1}{p}} |B_{2R}|^{1-\frac{1}{p}},
\end{aligned}
$$

where the last line follows by (58). Dividing by $|B(x,2R) \cap B(y,2R)| \ge \pi R^2$ leads to $|A_k(x) - A_k(y)| \le 2cR^{\frac{(\gamma_1-2)}{p}}$ and we conclude that the continuous representative of ϕ is $C^{0,\alpha}$ with

$$
\alpha = \frac{(\gamma_1 - 2)}{p}, \tag{59}
$$

concluding the proof of the claim and of Corollary 11. □

Given Holder continuity smoothness now follows by general theory:

Lemma 12 *If the pair is a weak solution of* (12)–(14) *as in definition* 3 *and* ϕ *is Holder continuous then* $(\phi, u) \in C^\infty$.

Proof First of all observe that the topological density

$$
j_\phi(x) = \frac{1}{2}\epsilon_{ab}\phi \cdot \partial_a\phi \times \partial_b\phi
$$

is the product of a Holder continuous function and a function in the local Hardy space $\mathcal{H}^1_{loc}$, and as such is itself in $\mathcal{H}^1_{loc}$ by [26, Chap. 13, Proposition 12.5]. It follows that u is also continuous by [26, Chap. 13, Corollary 12.12] since an $\mathcal{H}^1_{loc}$ function differs from an element of $\mathcal{H}^1$ by a C_0^∞ function ([25, Sect. 5.17]). Therefore, (u, ϕ) form a continuous weak solution of the elliptic system (12)–(14) in which the first derivatives appear at most quadratically: this system satisfies the conditions for [26, Chap. 14, Corollary 12B.5] from which smoothness follows. □

4 The harmonic map limit $\kappa \to 0^+$

In this section we discuss the behaviour of the solutions in the limit $\kappa \to 0^+$ in which the functional becomes (at least for $\mathbf{B} = 0$) the harmonic map functional. This is motivated by a comparison of theorem 1 with the following classical result of Eells and Wood:

Theorem 13 ([9]) *There is no harmonic map of degree ± 1 from the two dimensional torus Ω to the sphere S^2.*

This is proved by showing that any such map would have to be holomorphic (or anti-holomorphic), and recalling from complex analysis that there is no degree ± 1 holomorphic map $\Omega \to S^2$ (since this would give a doubly periodic meromorphic function with exactly one pole and one zero, which is impossible by [8, Lemma 4.2]). In fact it is proved more generally in [9] that a degree d harmonic map $\phi : X \to Y$ between two closed Riemann surfaces is automatically holomorphic (or anti-holomorphic) if $e(X) + |d| > 0$, where $e(X)$ is the Euler characteristic of X.

It follows from a comparison of Theorem 13 with Theorem 1 that the limit $\kappa \to 0^+$ is singular and the convergence of the corresponding solutions must fail to be strong (thus allowing for a change of the topological degree in the limit). In this section we investigate this process, showing that the change of topology occurs via bubbling off of harmonic maps $S^2 \to S^2$. As a setting for the discussion assume given a sequence $\kappa_\nu \to 0^+$ and corresponding smooth solutions $\phi_\nu, u_\nu = \underline{u}(\phi_\nu)$ to the system (12)–(14) satisfying $|\phi_\nu(x)| = 1$ and the uniform bounds

$$|\nabla\phi_\nu|^2_{L^2} + \kappa_\nu |\nabla u_\nu|^2_{L^2} \le M^2$$

with M independent of ν, and such that

$$\phi_\nu \rightharpoonup \phi \quad \text{weakly in } H^1(\Omega) \text{ and weak* in } L^\infty(\Omega).$$

Lemma 14 *The weak limit ϕ is a critical point of $\mathcal{E}$, the functional defined in* (7).

Proof Each ϕ_ν is a smooth solution of

$$-\Delta\phi_\nu = \nabla \cdot \big(\phi_\nu \times \mathcal{J}^\nu\big) - \big(\phi_\nu \times (\phi_\nu \times \mathbf{B})\big) - (\mathbf{B} - \phi_\nu \cdot \mathbf{B}\phi_\nu) \tag{60}$$

where

$$\mathcal{J}_i^\nu = \phi_\nu \times \partial_i\phi_\nu + \kappa_\nu \phi_\nu \epsilon_{ij}\partial_j u_\nu \tag{61}$$
$$-\Delta u_\nu = (j_{\phi_\nu} - \sigma). \tag{62}$$

Also recall that

$$\nabla \cdot \mathcal{J}^\nu = \phi_\nu \times \mathbf{B} \tag{63}$$

Noting that $\sqrt{\kappa_\nu}|\nabla u_\nu|_{L^2} \leq M$ we deduce that $\mathcal{J}^\nu$ converges to $\phi \times \nabla\phi$ weakly in L^q for $q < 2$ (since by Rellich's theorem ϕ_ν converges strongly in L^r, $r < \infty$). Then writing the weak formulation of (60):

$$\int_\Omega \nabla\eta \cdot \left(\nabla\phi_\nu - \phi_\nu \times \mathcal{J}^\nu\right) - \eta\Big(\left(\phi_\nu \times (\phi_\nu \times \mathbf{B})\right) + \left(\mathbf{B} - \phi_\nu \cdot \mathbf{B}\phi_\nu\right)\Big) = 0 \tag{64}$$

and letting $\nu \to \infty$ it follows again from weak L^2 convergence of $\nabla\phi_\nu$ and strong L^r convergence of ϕ_ν that the limit satisfies

$$\int_\Omega \nabla\eta \cdot \left(\nabla\phi - \phi \times (\phi \times \nabla\phi)\right) - \eta\Big(\left(\phi \times (\phi \times \mathbf{B})\right) + \left(\mathbf{B} - \phi \cdot \mathbf{B}\phi\right)\Big) = 0. \tag{65}$$

Taking the weak limit of (63) implies that

$$\int_\Omega \nabla\zeta \cdot (\phi \times \nabla\phi) + \zeta\phi \times \mathbf{B} = 0$$

for all $\zeta \in H^1$. Together these imply that ϕ is a critical point of $\mathcal{E}$. □

A crucial fact is the following $\epsilon-$regularity lemma in which we consider a fixed ball $B_r = B(x_0, r)$ and concentric sub-balls $B_{\rho'} \subset B_\rho \subset B_r$ for $\rho' < \rho < r$.

Lemma 15 **(ϵ- regularity)** *Let (ϕ_ν, u_ν) be as just dscribed. Then if $\sqrt{\kappa_\nu}C_1C_2M < \frac{1}{4}$ and $|\nabla\phi_\nu|_{L^2(B_r)} < \frac{1}{4C_1C_2}$ and $\rho' < \rho < r$:*

$$|b\phi_\nu|_{W^{2,4/3}(B_\rho)} \leq c = c(M, r, \rho, \beta, |\mathbf{B}|_{L^{\frac{4}{3}}(B_r)}) \tag{66}$$

$$|b\phi_\nu|_{W^{2,P}(B_{\rho'})} \leq c = c(M, r, \rho, \rho', \beta, P, |\mathbf{B}|_{L^{\frac{4}{3}}(B_r)}) \quad \textit{for all } P < \infty. \tag{67}$$

Proof The first bound is proved by considering the $p = 2$ version of (31):

$$\begin{aligned}(1 - C_1C_2\eta_\nu(r))&|b\phi_\nu|_{W^{2,\frac{4}{3}}} \\ &\leq |\mathbf{B}|_{L^{\frac{4}{3}}} + c(\beta)\left(\frac{1}{r-\rho}|\nabla\phi_\nu|_{L^{\frac{4}{3}}} + \frac{1}{r-\rho^2}|\phi_\nu|_{L^{\frac{4}{3}}}\right),\end{aligned} \tag{68}$$

where the cut-off b was defined just prior to Lemma 6, and

$$\eta_\nu(r) = |\nabla\phi_\nu|_{L^2(B(x_0,r))} + \kappa_\nu|\nabla u_\nu|_{L^2(B(x_0,r))}.$$

For ν large we may assume that $\sqrt{\kappa_\nu}C_1C_2M < 1/4$. Then if $|\nabla\phi_\nu|_{L^2(B(x_0,r))} < 1/(4C_1C_2)$ the first term in brackets is greater than $1/2$, leading to the inequality (66).

To derive (68) multiply ϕ_ν by the cut-off b giving a smooth function $b\phi_\nu$ which solves the equation

$$-\Delta(b\phi_\nu) = -2\nabla b\cdot\nabla\phi_\nu - (\Delta b)\phi_\nu + \partial_j(b\phi_\nu)\times\mathcal{J}_j^\nu(\phi_\nu) - \partial_j b\phi_\nu\times\mathcal{J}_j^\nu(\phi_\nu) - (\mathbf{B} - \phi\cdot\mathbf{B}\phi)b \tag{69}$$

which together with the pointwise inequalities

$$\begin{aligned}|\mathcal{J}^\nu(\phi_\nu)| &\leq |\nabla\phi_\nu| + \kappa_\nu|\nabla u_\nu|, \\ |\phi \times \mathcal{J}^\nu(\phi)| &\leq |\nabla\phi|\end{aligned}$$

gives (pointwise)

$$\left|-\Delta(b\phi_\nu) - \partial_j(b\phi_\nu) \times \mathcal{J}_j^\nu(\phi_\nu) + b(\mathbf{B} - \phi \cdot \mathbf{B}\phi)\right| \le c(\beta)\left(\frac{|\nabla\phi_\nu|}{r-\rho} + \frac{|\phi_\nu|}{(r-\rho)^2}\right). \quad (70)$$

Holder's inequality implies

$$\begin{aligned}\left|\partial_j(b\phi_\nu) \times \mathcal{J}_j^\nu(\phi_\nu)\right|_{L^{\frac{4}{3}}} &\le c|\nabla(b\phi_\nu)|_{L^4}(|\nabla\phi_\nu|_{L^2} + \kappa_\nu|\nabla u_\nu|_{L^2}) \\ &\le c_1\eta_\nu|(b\phi_\nu)|_{W^{2,\frac{4}{3}}} \end{aligned} \quad (71)$$

where all the norms are on $B(x_0, r)$. Consequently,

$$|\Delta(b\phi_\nu)|_{L^{\frac{4}{3}}} \le C_1\eta_\nu|(b\phi_\nu)|_{W^{2,\frac{4}{3}}} + |\mathbf{B}|_{L^{\frac{4}{3}}} + c(\beta)\left(\frac{|\nabla\phi_\nu|_{L^{\frac{4}{3}}}}{r-\rho} + \frac{|\phi_\nu|_{L^{\frac{4}{3}}}}{(r-\rho)^2}\right)$$

which implies (68).

To derive the second bound first apply the Sobolev inequality to deduce a bound for $b\phi_\nu$ in $W^{1,4}(B_r)$, and hence of ϕ_ν in $W^{1,4}(B_\rho)$. Now write down (69) but with cut-off supported in B_ρ instead of B_r and equal to 1 on $B_{\rho''}$ for $\rho'' \in (\rho', \rho)$. The Calderon–Zygmund estimate then gives a $W^{2,2}(B_{\rho''})$ bound, which by the Sobolev inequality implies a $W^{1,P}(B_{\rho''})$ bound for all $P < \infty$. Again using (69) (but with cut-off supported in $B_{\rho''}$ and equal to 1 on $B_{\rho'}$) the Calderon–Zygmund estimate gives (67). □

Lemma 15 indicates that strong convergence can only fail due to concentration of $|\nabla\phi_\nu|^2$, so we introduce a concentration measure $\lambda \in \mathcal{M}_+(\Omega)$ such that (restricting to a subsequence)

$$|\nabla\phi_\nu|^2 \rightharpoonup \lambda \quad \text{weakly in } \mathcal{M}_+(\Omega). \quad (72)$$

The idea is to show first that convergence is strong on the complement of a finite set of points, and then to analyse the behaviour at those points by a blow up argument.

Lemma 16 *There exists a finite point set* $\{s_1, \ldots, s_N\}$ *such that*

$$\lambda = |\nabla\phi|^2 + \sum_{i=1}^{N} \lambda(\{s_i\})\delta_{s_i} \quad (73)$$

and if $x_0 \notin \{s_1, \ldots, s_N\}$ *there exists* $\rho_0 > 0$ *such that* $\phi_\nu \to \phi$ *in* $C^1(B(x_0, \rho_0))$.

Proof Firstly we deduce from the $\epsilon-$ regularity lemma 15 that the absolutely continuous part of λ is given by $|\nabla\phi|^2$. Assume ν is sufficiently large that $\sqrt{\kappa_\nu}C_1C_2M < 1/4$, fix arbitrary $\epsilon \in (0, 1/(2(4C_1C_2)^2)$ and define the set

$$S_\epsilon = \cap_{r>0}\{x \in \Omega : \lambda(B(x, r)) \ge \epsilon/2\}.$$

Clearly S_ϵ is a finite set of cardinality less than $\frac{2\lambda(\Omega)}{\epsilon} \le c = c(M)$. Let $f_\nu = |\nabla\phi_\nu|^2$, and consider a point $x_0 \notin S_\epsilon$; there exists $r > 0$ such that $\lambda(B(x_0, 2r)) < \epsilon/2$. Take a cut-off function $\chi \in C_0^\infty(B(x_0, 2r)$ with $0 \le \chi \le 1$ on $B(x_0, 2r)$ and $\chi = 1$ on $B(x_0, r)$. The sequence f_ν is bounded in L^1 and weak* convergent in $\mathcal{M}_+(\Omega)$ to λ by (72), so that if ν is sufficiently large $|\int_\Omega f_\nu\chi - \lambda(\chi)| < \epsilon/2$, which implies

$$\int_{B(x_0,r)} f_\nu dx \le \int_\Omega f_\nu\chi \le \lambda(\chi) + \epsilon/2 < \lambda(B(x_0, 2r)) + \epsilon/2 < \epsilon.$$

It follows that $|\nabla\phi_\nu|_{L^2(B(x_0,r))} < \frac{1}{4C_1C_2}$ for such ν and so (66)–(67) hold.

Since $W^{2,P}$ for $P > 2$ is compactly embedded into C^1_{loc} we deduce that for small ρ_0, and possibly after redefinition on a set of measure zero,

$$\phi \in C^1\big(B(x_0, \rho_0)\big)$$

and that

$$\nabla\phi_\nu \longrightarrow \nabla\phi \qquad \text{uniformly in } \overline{B(x_0, \rho)} \tag{74}$$

$$\phi_\nu \longrightarrow \phi \qquad \text{uniformly in } \overline{B(x_0, \rho)}, \tag{75}$$

which completes the proof. □

We now prove, following [21].

Theorem 17 *In a neighbourhood of one of the concentration points s_j the sequence converges in C^1_{loc}, after rescaling, to a non constant finite energy harmonic map $\mathbb{R}^2 \to S^2$ (which has an extension to a non constant harmonic map $S^2 \to S^2$.)*

Proof It is immediate from Lemma 15 that if $|\nabla\phi_\nu|_{L^\infty(B(y,r))} \leq L < \infty$ for some $r > 0$ then $y \notin \{s_1, \ldots, s_N\}$ and convergence is strong in a neighbourhood of y. Consequently for each s_j, and each $\theta > 0$, there exists a sequence of points $x_\nu \to s_j$ such that

$$b_\nu = \max_{x\in\overline{B(s_j,\theta)}} |\nabla\phi_\nu(x)| = |\nabla\phi_\nu(x_\nu)| \to \infty.$$

Define $\tilde\phi_\nu(z) = \phi_\nu(x_i + z/b_\nu)$ then $\tilde\phi_\nu : B(0, b_\nu\theta) \to S^2$ satisfies

$$|\nabla\tilde\phi_\nu(0)| = 1 \tag{76}$$

$$|\nabla\tilde\phi_\nu(z)| \leq 1 \quad \text{for } |z| \leq b_\nu\theta \tag{77}$$

$$-\Delta\tilde\phi_\nu = \nabla\tilde\phi \times \tilde{\mathcal{J}}^\nu - b_\nu^{-2}(\mathbf{B} - \tilde\phi_\nu \cdot \mathbf{B}\tilde\phi_\nu) \tag{78}$$

$$\tilde{\mathcal{J}}^\nu = \tilde\phi_\nu \times \nabla\tilde\phi_\nu + \kappa_\nu\tilde\phi_\nu\epsilon_{ij}\partial_j\tilde u_\nu \tag{79}$$

$$-\Delta\tilde u_\nu = (j_{\tilde\phi_\nu} - b_\nu^{-2}\tilde\sigma_\nu), \tag{80}$$

where $\tilde u_\nu = u_\nu(x_\nu + z/b_\nu)$ and $\tilde\sigma_\nu(z) = \sigma(x_\nu + z/b_\nu)$. Notice that the radii of the balls on which these hold have limit $+\infty$. Therefore on any ball $B(0, R)$ the following hold for large ν,

$$\sup_{|z|\leq R} |\nabla\tilde\phi_\nu(z)| \leq 1$$

$$\int_{B(0,R)} |\nabla\tilde\phi_\nu|^2 + \kappa_\nu|\nabla\tilde u_\nu|^2 \leq M^2, \tag{81}$$

(the latter by conformal invariance of the Dirichlet integral). Therefore by a Cantor argument there exists a subsequence $\tilde\phi_\nu$ converging in C^0_{loc} to a limit ϕ. Now the system (78)–(80) is of the same form as (12)–(14) and so the bounds (66)–(67) proved in Lemma 15 hold on balls of sufficiently small size (by (77)). As a consequence of the compact embedding of $W^{2,P}$ into C^1_{loc} for $P > 2$ we have $\tilde\phi_\nu \to \phi$ in C^1_{loc} and hence ϕ solves

$$-\Delta\phi - \nabla\phi \times (\phi \times \nabla\phi) = 0$$

in the sense of distributions (since $|\kappa_\nu\nabla\tilde u_\nu|_{L^2(B(0,R))} \leq \sqrt{\kappa_\nu}M \to 0$). The condition $|\nabla\tilde\phi_\nu(0)| = 1$ implies $|\nabla\phi(0)| = 1$ which ensures that the limit is a non-constant harmonic map, smooth by Helein's theorem and of finite energy by Fatou's lemma. Consequently, as in [21], it has an extension to a smooth harmonic map $S^2 \to S^2$. □

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