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Review

Analysis of the adiabatic limit for solitons in classical field theory

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We discuss the approximation of classical field theories by reduced systems of differential equations on the space of equilibria (the adiabatic limit). Various examples in which the approximation provides a useful description of the low-energy dynamics of solitons are discussed, including the sine-Gordon equation, the Yang–Mills–Higgs equations and the Chern–Simons–Schrödinger system. Particular emphasis is given to theorems on the validity of such approximations and proofs are given in some model cases.

Keywords: classical field theory; adiabatic limit; solitons

1. Introduction

The equations of classical field theory are typically systems of partial differential equations which can be written as evolutionary dynamical systems that define a well-posed initial-value problem. Even when this initial-value problem is well understood from the analytical perspective, it is not necessarily easy to make contact with the phenomenology described by the field theory, particularly in strongly nonlinear situations. Therefore, it is of interest to obtain a simpler description of the dynamics in various limiting regimes of particular physical interest. In this paper, we will discuss one such regime, the adiabatic limit, with particular reference to soliton dynamics, in which it corresponds to the energy being close to minimum. Here, the word *soliton* is used for a spatially localized, finite-energy, time-independent solution of the equations, while the word *adiabatic* is intended to suggest the approximation of an evolution by a slow motion through some space of equilibria. We will explain how this notion of adiabatic approximation enables one to formulate and prove theorems which provide a rigorous description of the low-energy dynamics of solitons. The fact that an adiabatic approximation could be used to provide an effective and practical description of the low-energy dynamics of solitons, in rather complicated systems of equations, arose in the work of Manton (1982). In that article, it was proposed that the Yang–Mills–Higgs equations could be approximated by the geodesic motion on the moduli space of monopoles, i.e.

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the space of gauge equivalence classes of minimizers of the static energy. For this reason, the adiabatic approximation, as used to study soliton dynamics, is often referred to as the moduli space approximation. It will become apparent in §§1*c* and 2 that the adiabatic approximation is equivalent, under rescaling, to the problem of motion under a strong constraining potential. As a general reference for solitons in classical field theory, and physical applications of the adiabatic approximation, we refer to Manton & Sutcliffe (2004).

In the remainder of this section, we discuss some examples and formulate some theorems on the adiabatic limit. In §2, we formulate corresponding theorems for certain model problems and provide proofs which are quite close to those that work for the more complicated systems such as the Yang–Mills–Higgs equations. In §3, we briefly mention some refinements of the analysis and directions for further work. We start in §1*a* by discussing three examples informally, before explaining more carefully the structural features relevant to our work in §1*b*, and then formulating some theorems precisely in §1*c*.

(*a*) *Some examples*

We now start to discuss three examples. The first of these has been chosen on account of its simplicity. The second and third illustrate how the approximation can be used in different settings.

(i) *Example 1: the sine-Gordon equation*

The simplest example from field theory is the sine-Gordon equation

$$\frac{\partial^2\theta}{\partial t^2} - \frac{\partial^2\theta}{\partial x^2} + \sin\theta = 0. \tag{1.1}$$

We work with boundary conditions at infinity $\theta(-\infty)=0$ and $\theta(+\infty)=2\pi$. As is explained in §1*b*(iv), under these conditions the only equilibria (static solutions) are the solitons $\theta_K(x-X_0)=4\arctan \mathrm{e}^{x-X_0}$, which are completely determined by their centre $X_0=\mathbb{R}$. Thus the moduli space of solitons $\mathcal{M}_{\mathrm{SG}}$ can be identified with the real line $\mathbb{R}$, and the adiabatic approximation consists of giving a dynamical system on $\mathbb{R}$ which approximates (1.1). It is perhaps to be expected, in view of translation invariance, that this system is just $\ddot{X}_0=0$, and this is indeed the case; see theorem 1.1 for a precise statement, which is proved in §2*b*.

(ii) *Example 2: the Yang–Mills–Higgs equations*

Yang–Mills theory is a nonlinear variant of Maxwell's electromagnetism in which the field strength is described by a two-form $F_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$, which (locally) takes values in a Lie algebra, in our case $su(2)$. The Yang–Mills–Higgs equations on $\mathbb{R}^{1+3}=\mathbb{R}\times\mathbb{R}^3$ can be expressed explicitly as a system of equations for an $su(2)$-valued one-form $A=A_0\,\mathrm{d}t+A_1\,\mathrm{d}x^1+A_2\,\mathrm{d}x^2+A_3\,\mathrm{d}x^3$ and an $su(2)$-valued function $\Phi(t,x)$. The field can be derived from the one-form A according to

$$F_{\mu\nu}=\frac{\partial A_\nu}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\nu}+[A_\mu,A_\nu], \tag{1.2}$$

where $[\cdot, \cdot]$ means matrix commutation, and we write

$$E_j = F_{0j} = \partial_t A_j - \partial_j A_0 + [A_0, A_j]. \tag{1.3}$$

The equations are

$$\begin{gathered} D_j E_j = [\Phi, D_0 \Phi], \\ D_0 E_j + D_j F_{kj} = -[\Phi, D_k \Phi] \quad \text{and} \\ D_0^2 \Phi - D_j^2 \Phi = 0. \end{gathered} \tag{1.4}$$

Here, we use standard relativistic notation in which Greek indices μ, ν, etc. take values in $\{0, 1, 2, 3\}$ while Roman indices take values in $\{1, 2, 3\}$, and the summation convention is used. In geometrical terms, we are solving for an $SU(2)$ connection A on a two-dimensional complex vector bundle $E \approx \mathbb{C}^2 \times \mathbb{R}^{1+3}$ coupled to Φ, a section of the three-dimensional real vector bundle associated with E via the adjoint representation of $SU(2)$ on its Lie algebra $su(2)$. (The section Φ is called the Higgs field.) The differential operator

$$D_\mu = \partial_\mu + [A_\mu, \cdot] \tag{1.5}$$

is the covariant derivative determined by the connection A, acting on $su(2)$-valued sections. An important property of the equations is *gauge invariance*: let $g(t, x)$ be a smooth SU(2)-valued function, then (A, Φ) is a smooth solution of equations (1.4) if and only if $(g\mathrm{d}g^{-1} + gAg^{-1}, g\Phi g^{-1})$ is a smooth solution. This gauge invariance can be factored out by imposing additional conditions, for example in temporal gauge it is required that $A_0 = 0$.

There are static solutions of (1.4) with $A_0 = 0$, which minimize the energy functional (1.12), under appropriate boundary conditions at infinity. They come in families $\mathcal{S}_k$ indexed by an integer $k \in \mathbb{Z}$ of topological origin, as described in §1*b*(iv). These families are infinite dimensional, but on dividing $\mathcal{S}_k$ by the action of the gauge transformations one obtains a finite-dimensional manifold $\mathcal{M}_k$, known as the charge k monopole moduli space. This space also inherits a Riemannian metric from the L^2 inner product, and it was suggested by Manton (1982) that the geodesics with respect to this metric should provide a good finite-dimensional approximation to (1.4) in the low-energy limit. Theorem 1.4, which asserts the validity of this on long, but finite, time intervals, was proved by Stuart (1994*b*). In this example, as well as the previous one, the space of solitons is an isotropic submanifold of the phase space. In fact, the approximation can be used in other settings, as will be illustrated by §1*a*(iii), in which the space of solitons is a symplectic submanifold.

(iii) *Example 3: the Chern–Simons–Schrödinger system*

The Chern–Simons–Schrödinger system, introduced by Manton (1997), is a gauge theoretic generalization of the two-dimensional nonlinear Schrödinger equation whose static soliton solutions are Abelian Higgs, or Ginzburg–Landau, vortices (Jaffe & Taubes 1982). The dependent variables are a complex field $\Phi(t, x)$, an electric field $E = E_j\, \mathrm{d}x^j$ and a magnetic field $B(t, x)$, all defined for $(t, x) \in \mathbb{R} \times \Sigma$, where Σ is a two-dimensional spatial domain taken to be a Riemann surface with metric $g_{jk}\, \mathrm{d}x^j\, \mathrm{d}x^k$, area $\mathrm{d}\mu_g$ and complex structure

$J: T^*\Sigma \to T^*\Sigma$ (where j, k, … take values in $\{1, 2\}$). The equations are

$$\begin{aligned} E + \mathrm{d}B &= -J\langle i\Phi, \boldsymbol{D}\Phi\rangle, \\ \mathrm{i}(\partial_t - \mathrm{i}A_0)\Phi &= -\Delta_A\Phi - \frac{\lambda}{2}(\tau - |\Phi|^2)\Phi \quad \text{and} \\ B &= \frac{1}{2}(\tau - |\Phi|^2). \end{aligned} \tag{1.6}$$

The coupling in (1.6) involves the space–time covariant derivative

$$D = (D_0, D_1, D_2) = (\partial_t - \mathrm{i}A_0, D_1, D_2),$$

whose spatial part is written $\boldsymbol{D} = (D_1, D_2)$. The commutator determines the electric field E_i and the magnetic field B in the usual way

$$[D_\mu, D_\nu]\Phi = -\mathrm{i}F_{\mu\nu}\Phi, \quad \text{where} \quad F_{0i} = E_i \quad \text{and} \quad \frac{1}{2}F_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k = B\,\mathrm{d}\mu_g. \tag{1.7}$$

In geometrical terms, we fix a one-dimensional complex vector bundle $L \to \Sigma$, on which is given an inner product and norm $|a|^2 = \langle a, a\rangle$. We are then solving for an S^1 connection on the bundle $\mathbb{L} \equiv \mathbb{R} \times L \to \mathbb{R} \times \Sigma$, with associated covariant derivative D, and a section Φ of $\mathbb{L}$. To be more explicit, fix a connection on L determined by a covariant derivative operator ∇, so that the spatial part of D takes the form $D_j = \nabla_j - \mathrm{i}A_j$ for a real one-form $A_j\,\mathrm{d}x^j$; here ∇ is independent of time. (It is generally not possible to choose ∇ to be flat and it will have a curvature determined by a function b such that $[\nabla_j, \nabla_k]\Phi\,\mathrm{d}x^j\,\mathrm{d}x^k = -\mathrm{i}b\,\mathrm{d}\mu_g\Phi$; it is always possible to choose $b = \text{const.}$) Then at each time $t \in \mathbb{R}$, we are solving for a section $\Phi(t)$ of L, a one-form $A(t) = A_1(t)\mathrm{d}x^1 + A_2(t)\mathrm{d}x^2$ on Σ, and a real-valued function $A_0(t)$ on Σ. The electric field is given by

$$E_j = \partial_t A_j - \partial_j A_0$$

and the magnetic field by

$$B\,\mathrm{d}\mu_g = b\,\mathrm{d}\mu_g + \mathrm{d}A.$$

The two-form $-\mathrm{i}E_j\mathrm{d}t \wedge \mathrm{d}x^j - \mathrm{i}B\mathrm{d}\mu_g$ is the curvature associated with the space–time covariant derivative $D = (\partial_t - \mathrm{i}A_0, \nabla - \mathrm{i}A)$. For the case $\Sigma = \mathbb{R}^2$, this system was proposed by Manton (1997), who derived it as the Euler–Lagrange equation for the Lagrangian (1.13). A global existence result was proved by Demoulini (2007). As for example 2, an important property of the system (1.6) is *gauge invariance*: let $\chi(t, x)$ be a smooth real-valued function, then (A, Φ) is a smooth solution if and only if $(\mathrm{d}\chi + A, \Phi\mathrm{e}^{\mathrm{i}\chi})$ is a smooth solution.

In this case, there are soliton solutions called Abelian Higgs, or Ginzburg–Landau, vortices. There is a special case, $\lambda = 1$, in which the adiabatic approximation is particularly powerful because the space of vortices is then unusually large—large enough that the motion on it can provide information on the dynamical interaction of several vortices. We call this the self-dual, or Bogomoln'yi, case. As discussed in §1b(iv), after dividing out by the gauge group, we obtain, for $\lambda=1$, finite-dimensional moduli spaces of self-dual vortices. These moduli spaces can be identified with $\mathrm{Sym}^N(\Sigma)$, the symmetric N-fold product of the spatial domain Σ. The solitons lie in the phase space as a symplectic submanifold, and the moduli spaces $\mathrm{Sym}^N(\Sigma)$ inherit a symplectic structure. For $\lambda=1$, the system (1.6) can be approximated by a first-order Hamiltonian system on $\mathrm{Sym}^N(\Sigma)$ (see theorem 1.6). For a discussion of phenomenological aspects of vortex dynamics in this system, see Manton (1997), Romao & Speight (2004) and Krusch & Sutcliffe (2006).

(*b*) *Solitons and classical field equations*

Classical field equations have certain structural features which are crucial for the developments under discussion: they possess both a variational formulation and a Hamiltonian formulation (possibly with constraints) and are usually Lorentz covariant. (This last feature is not necessarily relevant to problems arising from condensed matter physics, an example of which is the Chern–Simons–Schrödinger system, §1*a*(iii).) A more specialized feature which we exploit here is the Bogomoln'yi structure, which ensures the existence of relatively large spaces of equilibria on which to approximate the dynamics. We now discuss these features as a preparation for some precise formulations of approximation theorems.

(i) *Variational formulation*

The equations we study are all Euler–Lagrange equations, i.e. can be written in the form of the condition of vanishing derivative,

$$DS = 0, \tag{1.8}$$

for some action functional S, depending upon the fields and their partial derivatives up to some order (usually up to first order). The variational formulation of a field theory as in (1.8) is often called a Lagrangian formulation, and it is then referred to as a Lagrangian system. This is not only an appealing unifying principle in field theory, but also a useful analytical device. In particular, at the static level, many of the soliton solutions of interest are solutions of (1.8), which minimize some energy functional $\mathcal{V}$ that can be derived from the Lagrangian S. The direct variational method can then be used to prove the existence of solutions and derive information important for stability analysis. (It should be said, however, that more specific, often somewhat ad hoc, methods are needed for a really good detailed understanding of the properties of the solitons.) Regarding time-dependent problems, while the variational method does not seem to be useful in the analysis of general solutions of the Cauchy problem, it can be useful, for example, in the construction of time-periodic solutions; see Demoulini & Stuart (2000) for a relevant example.

An important class of systems is that in which S takes the form $S=\int \mathcal{L}\,\mathrm{d}t$ with $\mathcal{L}: TM\to\mathbb{R}$ a function on the tangent bundle of a Riemannian manifold (M, g) that takes the familiar form 'kinetic energy' minus 'potential energy',

$$\mathcal{L} = T - V = \frac{1}{2}g(\dot{\Psi},\dot{\Psi}) - V(\Psi), \tag{1.9}$$

where $\tau \to \Psi(\tau)$ is a curve in M with velocity $\dot{\Psi}(\tau) \in T_{\Psi(\tau)}M$. Lagrangian systems of this type will be referred to as *natural Lagrangian systems* (following the terminology of §19 in Arnold (1989)).

Example 1: the sine-Gordon equation. Equation (1.1) arises formally as the Euler–Lagrange equation associated with the functional

$$S(\theta) = \int \left(\frac{1}{2} \left(\theta_t^2 - \theta_x^2 \right) - (1 - \cos \theta) \right) \mathrm{d}x \, \mathrm{d}t. \tag{1.10}$$

This action has the form of a natural Lagrangian system, in which the kinetic energy is defined with the L^2 metric and the potential energy is

$$V(\theta) = \int \left(\frac{1}{2} \theta_x^2 + (1 - \cos \theta) \right) \mathrm{d}x.$$

Example 2: the Yang–Mills–Higgs equations. Equations (1.4) can be derived from the action

$$S(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^{1+3}} ((|D_0 \Phi|^2 + |E|^2) - 2v(A, \Phi)) \mathrm{d}x \, \mathrm{d}t, \tag{1.11}$$

where $v(A, \Phi)$ is the density of the static energy

$$\mathcal{V}(A, \Phi) = \int_{\mathbb{R}^3} v(A, \Phi) \mathrm{d}x = \int_{\mathbb{R}^3} \left(\frac{1}{2} \langle D_j \Phi, D^j \Phi \rangle + \frac{1}{4} \langle F_{jk}, F^{jk} \rangle \right) \mathrm{d}x \tag{1.12}$$

and the Killing inner product $\langle A, B \rangle = -(1/2) \mathrm{tr}\, AB$ is used. The first equation of (1.4), which is obtained by variation of A_0, plays the role of a constraint in the sense that if it holds for the initial data then the remaining two equations imply its validity at later times. This variational formulation of (1.4) makes it apparent that, ignoring the constraint equation, (1.4) is in fact an infinite-dimensional natural Lagrangian system. To be precise, recall that it is always possible to apply a gauge transformation such that $A_0 = 0$. The second and third equations of (1.4) are then the Euler–Lagrange equations of the action S just given, with A_0 set equal to 0, and this action is indeed of the form (1.9), the kinetic energy being determined by the L^2 norm, and with $\mathcal{V}$ playing the role of potential energy.

Example 3: Chern–Simons–Schrödinger system. Equations (1.6) can be derived formally as the Euler–Lagrange equations associated with the functional

$$S(A, \Phi) = \frac{1}{2} \int_{\mathbb{R} \times \Sigma} A \wedge F + (\langle \mathrm{i}\Phi, D_0 \Phi \rangle + A_0 + 2v_{\lambda,\tau}(A, \Phi)) \mathrm{d}t \, \mathrm{d}\mu_g, \tag{1.13}$$

where

$$v_{\lambda,\tau}(A, \Phi) = \frac{1}{2} \left(B^2 + g^{jk} \langle D_j \Phi, D_k \Phi \rangle + \frac{\lambda}{4} (|\Phi|^2 - \tau)^2 \right) \tag{1.14}$$

is the density of the Ginzburg–Landau static energy. (The parameters λ and τ are positive real numbers.) Although S is not manifestly gauge invariant it changes by an exact form under gauge transformation, and the Euler–Lagrange equations (1.6) are gauge invariant. Vortices are critical points of the static energy

$$\mathcal{V}_{\lambda,\tau}(A, \Phi) = \int_{\Sigma} v_{\lambda,\tau}(A, \Phi) \mathrm{d}\mu_g,$$

as will be discussed further below.

(ii) *Hamiltonian formulation*

Recall that given a symplectic manifold $(\mathcal{M}, \omega)$, one can associate to any differentiable function $H: \mathcal{M} \rightarrow \mathbb{R}$ a vector field X_{H} such that

$$\omega(X_{\mathrm{H}}, v) = \mathrm{d}H(v),$$

for all vector fields v. The flow generated by this Hamiltonian vector field X_{H} is called the Hamiltonian flow associated with H on the phase space $\mathcal{M}$. Natural Lagrangian system (1.9) can be reformulated as Hamiltonian systems, with phase space the cotangent bundle $\mathcal{M} = T{*}M$, via the Legendre transformation (Arnold 1989). Many classical field equations admit such a Hamiltonian formulation on a phase space which is a cotangent bundle. However, there are interesting equations which are Hamiltonian equations on a phase space that is not necessarily a cotangent bundle. Schrödinger's equation, and its variants, provides examples of this type. But, in addition, equations arising from the Chern–Simons action and various generalizations also give rise to more complicated examples with interesting soliton solutions. In particular, for (1.6), there is a complex structure on the phase space given by $\mathbb{J} : (\dot{A}, \dot{\Phi}) = (-J\dot{A}, \mathrm{i}\dot{\Phi})$, which allows the introduction of a symplectic structure $\Omega(v, w) = \langle \mathbb{J}v, w \rangle$, where $\langle \cdot, \cdot \rangle$ is the L^2 inner product. Using this symplectic form, the system (1.6), in temporal gauge $A_0 = 0$, is a Hamiltonian flow generated by the Hamiltonian functional $\mathcal{V}_{\lambda,\tau}(A, \Phi)$, which was defined immediately following (1.14). (Note that the third equation of (1.6) is preserved by the evolution and as such is really only a condition on the initial data. It will be referred to as the constraint equation.)

(iii) *Lorentz covariance*

The fundamental equations of classical field theory are required, by the principles of relativity, to be Lorentz covariant and often constitute a system of nonlinear wave equations of the form

$$(\partial_t^2 - \Delta)\mathcal{U} = F(\mathcal{U}, \partial\mathcal{U}, \ldots). \tag{1.15}$$

This is a semi-linear hyperbolic system of equations. Both of the first two examples discussed above fall into this class of equations. (In the general relativistic context, this situation is modified to require general covariance, and the equations form quasi-linear hyperbolic systems.) On the other hand, there are many systems of interest in condensed matter physics which are not Lorentz covariant, in particular the third example introduced above. Correspondingly, the system (1.6) is not hyperbolic, but can be thought of as a pair of coupled nonlinear Schrödinger equations, as can be seen by applying a gauge transformation so that A is divergence free (Coulomb gauge). In the proofs we provide for certain model problems in §2, we use methods which are, in principle, capable of adaptation to treat partial differential equations like (1.15) or systems of nonlinear Schrödinger equations.

(iv) *Bogomoln'yi structure and moduli spaces of solitons*

As explained previously, the solitons of interest to us are critical points of an energy functional $\mathcal{V}$. There are certain cases, in which the study of the adiabatic limit is of particular efficacy, in which this functional possesses a special form known as Bogomoln'yi, or self-dual, structure. This means it is possible to write

$$\mathcal{V}(\psi) = \int |G(\psi)|^2 + \#(\psi),$$

where $\#(\psi)$ is a number, determined either by the topological type of the configuration or possibly by the boundary conditions. In this circumstance, the minimizers, with $\#(\psi)$ fixed, will be given by solutions of $G(\psi)=0$, at least *if such solutions exist*—in this case, the Bogomoln'yi bound is said to be saturated (i.e. achieved) by these solutions. It may well be that there are no solutions of $G(\psi)=0$ with the given topological type (or given boundary conditions), in which case the bound is not saturated. The relevance of this structure to the adiabatic limit is that when the Bogomoln'yi bound is saturated, experience indicates that there is typically a large space of solitons, and projecting the dynamics onto this space may capture many of the essential features of the full dynamics. We now investigate the Bogomoln'yi structure in each of the three examples previously mentioned.

Example 1: sine-Gordon solitons. In this case, the Bogomoln'yi structure amounts to the simple observation that the potential energy V can be written

$$V(\theta) = \int \left(\frac{1}{2}\theta_x^2 + (1-\cos\theta)\right)\mathrm{d}x = \int \left(\frac{1}{2}\left(\theta_x \mp 2\sin\frac{\theta}{2}\right)^2 \mp 4\partial_x\cos\frac{\theta}{2}\right)\mathrm{d}x.$$

Working with asymptotic boundary conditions $\theta(-\infty)=0$, $\theta(+\infty)=2\pi$, and choosing the upper sign in this identity, we deduce that $V(\theta)\geq 8$, with equality attained precisely at any one of the soliton, or kink, solutions

$$\theta_K(x-X_0) = 4\arctan \mathrm{e}^{x-X_0}, \quad X_0 \in \mathbb{R}. \tag{1.16}$$

These are all solutions of the first-order equation $\theta_x = 2\sin(\theta/2)$; furthermore, any finite-energy solution satisfying the above boundary conditions equals $\theta_K(x-X_0)$ for some X_0. Let H^s denote the standard Sobolev space of functions whose derivatives up to order $s\in\mathbb{N}$ are square integrable, with the standard norm, and let H^s_{loc} be the corresponding local Sobolev space. We introduce the affine space $\mathcal{A}_1 = \theta_K + H^1(\mathbb{R})$ as an appropriate space within which to work; any $\theta \in H^1_{\mathrm{loc}}$ such that $V(\theta)<\infty$ satisfying the above asymptotic boundary conditions lies in $\mathcal{A}_1$ and vice versa. Thus in this case the moduli space of solitons $\mathcal{M}_{\mathrm{SG}}$ is just the real line $\mathbb{R}$ and we have an embedding $E_K : \mathcal{M}_{\mathrm{SG}} \to \mathcal{A}_1$ which maps $X_0\in\mathbb{R}$ to $E_K(X_0) \equiv \theta_K(\cdot - X_0) \in \mathcal{A}_1$. This embedding induces, from L^2, a metric on $\mathcal{M}_{\mathrm{SG}}$ which is easily computed to be just $8\,\mathrm{d}X_0^2$. The point $X_0 \in \mathcal{M}_{SG} \approx \mathbb{R}$ represents the soliton centred at X_0 and the induced metric is invariant under translation.

Example 2: Bogomolny-Prasad-Sommerfield (BPS) monopoles in the Yang–Mills–Higgs equations. The static Yang–Mills–Higgs energy (1.12) provides a more interesting example of Bogomoln'yi structure. We impose the asymptotic boundary condition $\lim_{|x|\to+\infty}|\phi(x)|=1$, so that restricted to a large sphere $\phi/|\phi|$ defines a map $S^2 \to S^2$ of winding number $k\in\mathbb{Z}$ (for suitable ϕ). Using the fact, proved by Groisser (1984), that this number is equal to the integral $(1/2\pi)\int D\Phi\wedge F$, which is a well-defined integer as long as $\mathcal{V}(A,\Phi)<\infty$, implies that the energy can be rewritten as

$$\mathcal{V}(A,\Phi) = \frac{1}{2}\int_{\boldsymbol{R}^3} |{}^*F \mp D\Phi|^2 \pm 4\pi k. \tag{1.17}$$

This structure arises as a dimensional reduction of the well-known self-duality structure possessed by the four-dimensional Yang–Mills functional. The minimizers of $\mathcal{V}$ with negative winding number k will therefore be solutions of the *Bogomoln'yi* equations

$$^*F = -D\Phi \quad \text{or} \quad F_{ln} = \epsilon_{lmn} D_m \Phi, \tag{1.18}$$

at least if such solutions exist; the space of all minimizers will then be written $\mathcal{S}_k$. An individual minimizer is referred to as a BPS monopole, or just a monopole, of charge k, and will also satisfy the usual second-order Euler–Lagrange equations corresponding to the static energy functional $\mathcal{V}$, and as such is gauge equivalent to a smooth solution and locally gauge equivalent to a real analytic one (Jaffe & Taubes 1982). A precise description of the space of monopoles $\mathcal{S}_k$ can be obtained by integrable systems methods of twistor theory: let $\mathcal{M}_k$ be the moduli space of gauge equivalence classes of monopoles. There is a circle bundle $\tilde{\mathcal{M}}_k$ over $\mathcal{M}_k$ which is slightly easier to describe: it was proved by Donaldson (1984) that $\tilde{\mathcal{M}}_k$ can be identified, via a diffeomorphism, with the space of degree k rational maps f satisfying $f(\infty)=0$. (The fibre of the bundle $\tilde{\mathcal{M}}_k \to \mathcal{M}_k$ just corresponds to the action of the gauge transformation determined by the Higgs field itself.) The moduli space $\tilde{\mathcal{M}}_k$ is thus a smooth $4k$-dimensional manifold, and, furthermore, it inherits a complete Riemannian metric from the L^2 inner product in the original infinite-dimensional space. (Here, it is necessary to take account of the fact that $\tilde{\mathcal{M}}_k$ is obtained as a quotient space, dividing out by the group of gauge transformations—the length of a tangent vector to $\tilde{\mathcal{M}}_k$ is the minimum L^2 norm of the various gauge equivalent representatives.) This metric has special properties: it is hyperkähler and Ricci flat, which enabled Atiyah & Hitchin (1988) to determine it explicitly for $k=2$ and calculate many properties of the geodesic flow.

Example 3: self-dual vortices. Ginzburg–Landau vortices are critical points of the static energy $\mathcal{V}_{\lambda,\tau}(A,\Phi) = \int v_{\lambda,\tau}(A,\Phi)\mathrm{d}\mu_g$, where the energy density was defined in (1.14). It turns out that the case $\lambda=1$ is special: as in the previous example, the functional is then a dimensional reduction of the four-dimensional Yang–Mills functional. An indication of this is given by the existence of a Bogomoln'yi decomposition formula

$$\mathcal{V}_{1,\tau}(A,\Phi) = \pi\tau N + |D_A^{(0,1)}\Phi|^2_{L^2} + \frac{1}{2}|B_A - \frac{1}{2}(\tau - |\Phi|^2)|^2_{L^2},$$

where, using a complex coordinate $z=x^1+\mathrm{i}x^2$, in which the metric has the form $g=\mathrm{e}^{2\rho}((\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2)$, we have $D_A^{(0,1)}\Phi = \frac{1}{2}((\nabla_1 - \mathrm{i}A_1) + \mathrm{i}(\nabla_2 - \mathrm{i}A_2))\Phi\,\mathrm{d}\bar{z}$. The associated Bogomoln'yi equations are then

$$D_A^{(0,1)}\Phi = 0, \qquad B_A - \frac{1}{2}(\tau - |\Phi|^2) = 0. \tag{1.19}$$

For a solution (A, Φ) of these equations with a given value of the topological integer N, the field Φ will typically have N zeros, each of which can be thought of as the centre of a vortex. Thus the static solitons can be thought of as a nonlinear superposition of N vortices which do not interact. The phrase 'self-dual vortex' is often used in the special case $\lambda=1$ when static multi-vortices exist. Equations (1.19) were solved exactly by Witten (1977) in the case that Σ is the upper half-plane with canonical metric, by reducing them to the Liouville equation. Following this, Taubes proved an existence theorem when Σ is the Euclidean plane (Jaffe & Taubes 1982), and Bradlow (1988) did likewise for Σ a compact Riemann surface, by means of a reduction to a nonlinear elliptic equation of

Kazdan–Warner type. Bradlow proved that if the area of the surface $|\Sigma|$ is such that $\tau|\Sigma|>4\pi N$, then the Bogomoln'yi bound is saturated: in fact, the minimum value $\pi\tau N$ of $\mathcal{V}_{1,\tau}$ is achieved on a set whose quotient by the gauge group can be identified with $\mathrm{Sym}^N(\Sigma)$, the symmetric N-fold product of Σ. Thus the moduli space of self-dual vortices is $\mathrm{Sym}^N(\Sigma)$; it inherits both a metric (induced from the L^2 metric, as for monopoles) and a symplectic structure and is a Kähler manifold. Finally, we mention that there are other systems for which the vortices are the static solutions: see, for example, Stuart (1994*a*) and Demoulini & Stuart (1997).

(*c*) *Formulation of some theorems on adiabatic limits*

Example 1: slow motion of sine-Gordon solitons. As a first, phenomenologically rather trivial, example consider solutions $\theta^\epsilon(t,x)$ to (1.1) with smooth initial data of the form (for each $\epsilon>0$)

$$\theta^\epsilon(0,x)=\theta_K(x)+\tilde\theta_0(x;\epsilon),\qquad \partial_t\theta^\epsilon(0,x)=-\epsilon u_0\theta'_K(x)+\tilde v_0(x;\epsilon), \tag{1.20}$$

with $||(\tilde\theta_0,\tilde v_0)||_{H^2\times H^1}=O(\epsilon^2)$ as $\epsilon\to 0$. The moduli space approximation in this case amounts to restricting the Lagrangian (1.10) to the space of solitons $\mathcal{M}_{\mathrm{SG}}\subset\mathcal{A}_1$ described in the discussion following (1.16). The computation of this restricted Lagrangian just amounts to the computation of the metric induced from L^2, leading to the reduced Lagrangian which is just $8\dot X_0^2$. Since the Euler–Lagrange equation for this Lagrangian is just $\ddot X_0=0$, we find that the expected adiabatic limit description of the motion of sine-Gordon solitons is just uniform motion on a straight line: $X_0(\tau)=u_0\tau$. This is borne out by theorem 1.1.

Theorem 1.1. *Consider, for each $\epsilon>0$, the initial-value problem for(1.1) with smooth initial data (1.20) satisfying $||(\tilde\theta_0,\tilde v_0)||_{H^2\times H^1}=O(\epsilon^2)$ as $\epsilon\to 0$. Then for each $\epsilon>0$ there exists a unique global smooth solution $\theta^\epsilon(t,x)$ to(1.1) and (1.20). Furthermore, as $\epsilon\to 0$, the rescaled solutions $\theta^\epsilon((\tau/\epsilon),x)$ converge to $\theta_K(x-X_0(\tau))$ in the sense that*

$$\lim_{\epsilon\to 0}\max_{|\tau|\le\tau_*}\left\|\left(\theta^\epsilon\left(\frac{\tau}{\epsilon},\cdot\right)-\theta_K(\cdot-X_0(\tau))\right)\right\|H^1=0, \tag{1.21}$$

for every $\tau_*<\infty$.

Remark 1.2. Note that the behaviour under discussion is stable in the sense that any sequence of solutions whose initial data have the prescribed asymptotic behaviour as $\epsilon\to 0$ converges, after rescaling, to the same adiabatic limit. In this problem, applying Lorentz boosts by velocity ϵu_0 to a stationary kink gives very particular solution sequences having the stated limiting behaviour, but in order to see that this behaviour is stable, and so physically relevant, it is necessary to carry out some analysis as in the proof of theorem 1.1.

Remark 1.3. Note that in order to pick up the adiabatic motion in the limit, it is necessary to consider the rescaled functions $\theta^\epsilon(\tau/\epsilon,\cdot)$. This suggests the introduction of a *slow time* variable

$$\tau=\epsilon t. \tag{1.22}$$

Observe that $\theta(t,x)$ solves (1.1) if and only if $\Theta(\tau,\cdot)=\theta(\tau/\epsilon,\cdot)$ solves

$$\frac{\partial^2\Theta}{\partial\tau^2}+\mu V'(\Theta)=0, \tag{1.23}$$

where $\mu = \epsilon^{-2}$ is a large parameter. This problem is an example of a strongly constrained system: as $\mu \to +\infty$, the force $\mu V'$ acts to constrain the solution to the set of minimizers of V, i.e. to the moduli space $\mathcal{M}_{\mathrm{SG}}$.

The proof of theorem 1.1 given in §2*b* is carried out in the context of constrained systems, i.e. for solutions of (1.23) as $\mu \to \infty$. Prior to this, strongly constrained finite-dimensional systems are studied in §2*a*. The proofs given there are chosen to be adaptable to treat partial differential equations like the sine-Gordon equation, as well as the more phenomenologically interesting cases such as the Yang–Mills–Higgs and Chern–Simons–Schrödinger equations which we now discuss.

Example 2: the Yang–Mills–Higgs equations and motion on the moduli space of monopoles. In this case, the restriction of the action (1.11) to the space of monopoles gives a Lagrangian which is again a kinetic energy defined by means of the metric induced from L^2, which is discussed above. There is a technical issue here in that we are really interested in the action restricted to the moduli space of gauge equivalence classes of monopoles, $\mathcal{M}_k$, which is obtained as a quotient space of the space of monopoles $\mathcal{S}_k$ by the group of gauge transformations $\mathcal{G}$,

$$\mathcal{S}_k \to \mathcal{M}_k = \mathcal{S}_k/\mathcal{G} \quad \text{and,} \tag{1.24}$$

$$\Psi_0 \mapsto [\Psi_0]. \tag{1.25}$$

So, it is necessary to correctly factor out the gauge group in this reduction—this is explained by Stuart (1994*b*), where theorem 1.4 is proved.

Theorem 1.4. *Consider the initial-value problem for the Yang–Mills–Higgs equations (1.4) with initial data* $\Psi(0) = \Psi_0(0) \in \mathcal{S}_k$ *a monopole,* $\partial_t \Psi(0) = \epsilon v_0$ *with* $v_0 \in T_{\Psi_0(0)}\mathcal{S}_k$ *tangent to the space of monopoles at* $\Psi_0(0)$*. Then for* ϵ *sufficiently small, there exists* $\tau_* > 0$ *such that there is a smooth solution for* $|t| \le (\tau_*/\epsilon)$ *which is close in uniform norm to a monopole* $\Psi_0(\epsilon t) \in \mathcal{S}_k$ *such that* $\tau \to \gamma(\tau) = [\Psi_0(\tau)] \in \mathcal{M}_k$ *is the constant energy geodesic with initial conditions* $(\gamma(0), \dot{\gamma}(0)) = ([\Psi_0(0)], [v_0])$.

Remark 1.5. The proof employs energy estimates which actually lead to the approximation holding in certain integral norms that are similar to, but weaker than, the Sobolev norms H^{s}. They are based on a norm introduced by Taubes (1983) for a study of index theory for the Yang–Mills–Higgs functional. The validity of the approximation in uniform norm is then a consequence of its validity with respect to these integral norms.

Example 3: the Chern–Simons–Schrödinger system and first-order vortex dynamics. The system (1.6) is not a natural Lagrangian system, but is Hamiltonian as detailed in §1*b*(ii). Here, it is crucial that the space of vortices is a symplectic submanifold of the phase space, and that the moduli space $\mathrm{Sym}^N(\Sigma)$ inherits a symplectic form ω from Ω, the symplectic form defined in §1*b*(ii). We now define a function $h{:}\mathrm{Sym}^N(\Sigma) \to \mathbb{R}$ by restricting the energy $\mathcal{V}_{\lambda,\tau}$ to the space of vortices, and observing that by gauge invariance this actually gives a smooth function h on the quotient space $\mathrm{Sym}^N(\Sigma)$. It is the Hamiltonian flow of this function on the phase space $(\mathrm{Sym}^N(\Sigma), \omega)$ which determines the slow motion of vortices for λ close to 1.

Theorem 1.6. *For $\epsilon = |\lambda - 1|$ sufficiently small, the system (1.6) can be approximated, for times of order $1/\epsilon$, by the Hamiltonian flow on $(\mathrm{Sym}^N(\Sigma), \omega)$ associated with the Hamiltonian h, obtained by restriction of $\mathcal{V}_{\lambda,\tau}$ to the moduli space.*

The proof of this will appear in a future paper.

Remark 1.7. In the first two examples, which were natural Lagrangian systems, the space of solitons was an isotropic submanifold of the phase space and the adiabatic limit system was also natural Lagrangian. For the case of (1.6), it was crucial, rather, that the space $\mathrm{Sym}^N(\Sigma)$ inherits a symplectic form ω from its construction as a quotient. Thus the basic idea of the adiabatic approximation can be used in a variety of different settings; parabolic systems obtained from the gradient flow of (1.14) are another example (Demoulini & Stuart 1997; Strauss & Sigal 2006)

Remark 1.8. As discussed further in §2, there are two approaches to validating analytically the adiabatic approximation, based on compactness as in theorem 1.1 or by direct construction as in theorem 1.4.

Remark 1.9. These theorems leave open various interesting related questions regarding bound states and time-periodic solutions, scattering theory and singularity formation which are discussed in §3.

2. Proofs in some special cases

In §2*a*, we explain how to prove theorems analogous to those stated in §1*c* for some finite-dimensional model problems. Problems of this type have previously been treated by Rubin & Ungar (1957) and Ebin (1977). The first of these references treats finite-dimensional problems by means of a compactness argument—uniform estimates for the solution are obtained which allow passage to the limit, and then it is proved that the limit is a solution of the constrained system. In contrast, the second reference provides a direct construction of solutions which are close to a given solution of the constrained system; this was the line of attack adopted by Stuart (1994*a*,*b*) also, and is briefly explained in §2*a*(iii). Here, however, we adopt the compactness method, providing proofs which—with the use of an additional compactness device, the Lions–Aubin lemma—can be adapted to the infinite-dimensional setting required for the partial differential equations of classical field theory discussed previously. (In fact, the article of Ebin does treat infinite-dimensional problems, but was directed towards the problem of the incompressible limit in fluid mechanics, and the techniques there would require some modification to treat the type of problem under consideration here. An alternative approach to the incompressible limit was given by Klainerman & Majda (1981).) As an infinite-dimensional example, we then prove theorem 1.1 on the adiabatic approximation for the sine-Gordon equation in §2*b*.

(*a*) *Finite-dimensional natural Lagrangian systems*

A good starting point is the problem of strongly constrained motion in finite-dimensional natural Lagrangian systems, i.e. systems of ordinary differential equations which are the Euler–Lagrange equations for a Lagrangian of the familiar kinetic energy minus potential energy form,

$$\mathcal{L}[\psi] = \frac{1}{2}||\dot{\psi}||^2 - V_\mu(\psi), \tag{2.1}$$

where $\dot{\psi}(\tau) = \partial_\tau \psi(\tau)$ and $\psi : \mathbb{R} \to \mathbb{R}^N$ is a curve in $\mathbb{R}^N$. Euclidean space $\mathbb{R}^N$ is endowed with the standard Euclidean norm $||v||^2 = \langle v, v\rangle = v \cdot v$, and $V_\mu : \mathbb{R}^N \to \mathbb{R}$ is a family of smooth functions parametrized by $\mu \in \mathbb{R}$. The Euler–Lagrange equation associated with $\mathcal{L}$ is

$$\ddot{\psi} + V'_\mu(\psi) = 0. \tag{2.2}$$

(i) *A simple model problem (Ginzburg–Landau constraining potential)*

The problem of constrained motion arises, for example, with the family of potentials

$$V_\mu(\psi) = U(\psi) + \frac{\mu}{4}(1 - ||\psi||^2)^2, \tag{2.3}$$

in the limit $\mu \to +\infty$; in this case, (2.2) can be written as

$$\ddot{\psi} + U'(\psi) - \mu\psi(1 - ||\psi||^2) = 0, \tag{2.4}$$

where $\langle U'(\psi), v\rangle = DU(\psi)(v)\ \forall\ v \in \mathbb{R}^N$. It would seem reasonable, in view of energy conservation,

$$\frac{1}{2}||\dot{\psi}(\tau)||^2 + V_\mu(\psi(\tau)) = E_0 = \text{const.},$$

that solutions Ψ_μ, with energy bounded independent of μ, will be forced, as $\mu \to +\infty$, onto the set $S^{N-1} = \{\psi \in \mathbb{R}^N : ||\psi|| = 1\}$ i.e. $\Psi_\mu(\tau) \to \Psi(\tau) \in S^{N-1}$. Furthermore, it may be expected that $\Psi(\tau)$ will be a solution of the constrained system, i.e. the Euler–Lagrange equations characterizing critical points of (2.1) among functions $\psi : \mathbb{R} \to S^{N-1}$. The weak formulation of this condition is

$$\int (\langle \dot{\psi}, \dot{\zeta}\rangle - \langle U', \zeta\rangle)\mathrm{d}t = 0, \qquad \forall \zeta \in C_0^\infty(\mathbb{R}; \mathbb{R}^N) : \langle \zeta(t), \psi(t)\rangle = 0\ \forall\ t \in \mathbb{R}. \tag{2.5}$$

Alternatively, introducing the orthogonal projection operator

$$\mathbb{P}_\psi : v \to v - \frac{\psi \cdot v}{||\psi||^2}\psi = v - \psi \cdot v\psi, \quad \text{if } ||\psi|| = 1, \tag{2.6}$$

constrained critical points can be characterized by

$$\int \langle -\ddot{\psi} - U'(\psi), \mathbb{P}_\psi(\eta)\rangle \mathrm{d}t = 0, \qquad \forall \eta \in C_0^\infty(\mathbb{R}; \mathbb{R}^N), \tag{2.7}$$

which is just the weak formulation of $\mathbb{P}_\psi(\ddot{\psi} + U'(\psi)) = 0$ or equivalently

$$\ddot{\psi} + ||\dot{\psi}||^2\psi + \mathbb{P}_\psi U'(\psi) = 0. \tag{2.8}$$

The basic analytical ingredient for the proof which follows is just the following simple consequence of the Arzela–Ascoli theorem:

Lemma 2.1. *Given a sequence of C^1 functions $f_n : [-\tau_*, \tau_*] \to \mathbb{R}^N$ satisfying $\max_{|\tau| \le \tau_*}(||f_n(\tau)|| + ||\dot{f}_n(\tau)||) \le C$, there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ which converges in $C([-\tau_*, \tau_*]; \mathbb{R}^N)$ to a limit $f \in C([-\tau_*, \tau_*]; \mathbb{R}^N)$.*

Theorem 2.2. *Assume that U is a smooth bounded function on $\mathbb{R}^N$. For $\mu>0$, let $\Psi_\mu \in C^\infty(\mathbb{R}, \mathbb{R}^N)$ be the unique solution of (2.4) with initial values*

$$\Psi_\mu(0) = \Psi_0 \qquad \dot{\Psi}_\mu(0) = v_0, \tag{2.9}$$

which satisfy

$$||\Psi_0|| = 1, \qquad \Psi_0 \cdot v_0 = 0. \tag{2.10}$$

Then

$$\Psi_\mu \to \Psi \quad in\ C^1_{loc} \quad as\ \mu \to +\infty, \tag{2.11}$$

where $\Psi \in C^\infty(\mathbb{R}; S^{N-1})$ is the unique solution of (2.8) with initial data $\Psi(0) = \Psi_0$, $\Psi(0) = v_0$.

Proof. By the local existence theorem there exists, for every μ, a unique local smooth solution, $\Psi_\mu(\tau)$, to (2.4), (2.9) defined on some non-empty time interval $(-T_\mu, T_\mu)$. Now we derive some estimates for fixed μ, temporarily writing ψ in place of Ψ_μ to avoid a proliferation of symbols. Any solution of (2.4) and (2.9) satisfies the energy conservation law

$$\frac{1}{2}||\dot{\psi}(\tau)||^2 + U(\psi(\tau)) + \frac{\mu}{4}(1-||\psi||^2)^2 = \frac{1}{2}||v_0||^2 + U(\Psi_0) = E_0. \tag{2.12}$$

Since U is bounded, $|U(\psi)| \leq L$, this implies

$$\frac{1}{2}||\dot{\psi}||^2 + \frac{\mu}{4}(1-||\psi||^2)^2 \leq E_0 + L, \tag{2.13}$$

so that for $\mu>0$ we have the *a priori* estimate for the velocity $v = \dot{\psi}$,

$$||v(\tau)|| \leq \sqrt{2(E_0 + L)}. \tag{2.14}$$

Furthermore, there exists $\mu_* > 0$ such that for $\mu \geq \mu_*$ the solutions lie inside the set

$$\mathcal{N} \equiv \left\{\psi \in \mathbb{R}^N : \frac{5}{6} \leq ||\psi||^2 \leq \frac{7}{6}\right\}. \tag{2.15}$$

It follows that for all $\mu>0$ the solutions Ψ_μ can be extended for all time. Furthermore, the fact that the velocities $v_\mu = \dot{\Psi}_\mu$ are uniformly bounded by (2.14) implies, by lemma 2.1, subsequential convergence as $\mu \to +\infty$ to a limit $\Psi \in C(\mathbb{R}; \mathbb{R}^N)$, uniformly on closed bounded intervals. The energy identity (2.13) implies $||\Psi(\tau)|| = 1$ so that in fact $\Psi \in C(\mathbb{R}; S^{N-1})$. Since U is smooth there exists L_1 such that

$$\sup_{\psi \in \mathcal{N}} |U'(\psi)| \leq L_1. \tag{2.16}$$

The next step is to obtain uniform estimates for the derivatives. To this end, and again writing ψ in place of Ψ_μ, we decompose the acceleration $\dot{v}(\tau) = \ddot{\psi}(\tau)$, thus

$$\dot{v}(\tau) = \dot{v}^N(\tau) + \dot{v}^T(\tau) = \frac{\dot{v}(\tau)\cdot\psi(\tau)}{||\psi(\tau)||^2}\psi(\tau) + \mathbb{P}_{\psi(\tau)}(\dot{v}(\tau)). \tag{2.17}$$

The fact that ψ solves (2.4) implies immediately that $\dot{v}^T = -\mathbb{P}_{\psi(\tau)}(U'(\psi(\tau)))$ is bounded,

$$||\dot{v}^{\mathrm{T}}|| \leq L_1. \tag{2.18}$$

Thus to deduce C^1 convergence from lemma 2.1 it suffices to estimate $\dot{v}^N(\tau)$. To achieve this it is convenient to define $\phi = v\cdot\psi$, and observe that since

$\dot{\phi} = \dot{v}\cdot\psi + ||v||^2$ we can estimate $\dot{v}^N$ in terms of $\dot{\phi}$ by means of

$$\dot{v}^N = \frac{\dot{\phi} - ||v||^2}{||\psi||^2}\psi. \tag{2.19}$$

To be precise, assuming that $\psi(\tau) \in \mathcal{N}$ and (2.16) holds, there exists $c = c(E_0, L, L_1)$ such that

$$||\dot{v}(\tau)|| \leq c(1 + |\dot{\phi}(\tau)|). \tag{2.20}$$

Now a calculation gives the following equation for ϕ:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + \mu(6||\psi||^2 - 4)\right)\phi = -D^2U(\psi)(v, \psi) - 3v\cdot U'(\psi), \tag{2.21}$$

suggesting the introduction of the quantity

$$E^N = \frac{1}{2}(\dot{\phi}^2 + \mu(6||\psi||^2 - 4)\phi^2), \tag{2.22}$$

as a measure of the magnitude of the normal oscillations. Indeed, for $\psi \in \mathcal{N}$,

$$E^N \geq \frac{1}{2}(\dot{\phi}^2 + \mu\phi^2), \tag{2.23}$$

which implies $||\dot{v}|| \leq c(1 + \sqrt{E^N})$ by (2.20). Differentiate with respect to τ and substitute from (2.21) to deduce

$$\frac{\mathrm{d}}{\mathrm{d}\tau}E^N(\tau) = \dot{\phi}(-D^2U(\psi)(v, \psi) - 3v\cdot U'(\psi)) + 6\mu\phi^3, \tag{2.24}$$

and hence, since $\mu\phi^2 \leq 2E^N$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}E^N\right| \leq C(E^N + \sqrt{E^N}) \leq C(1 + E^N), \tag{2.25}$$

for some C independent of μ. Now, for the assumed initial data (2.9), $\phi(0) = 0$ and $|\dot{\phi}(0)|$ is bounded in terms of E_0 and L_1, and hence there exists $C > 0$, independent of μ, such that

$$E^N(0) \leq C. \tag{2.26}$$

The Gronwall inequality applied to (2.25) and (2.26) implies that

$$E^N(\tau) \leq C_1\mathrm{e}^{C_2\tau}, \tag{2.27}$$

with both constants independent of μ. Given this, it follows from (2.18) and (2.19) that the acceleration $\dot{v}_\mu = \ddot{\Psi}_\mu$ of the solution Ψ_μ is bounded uniformly in $\mu > \mu_*$, as is $\sqrt{\mu}\phi_\mu$, where $\phi_\mu = v_\mu \cdot \Psi_\mu$. It follows from lemma 2.1 that, along a subsequence μ_k, the Ψ_μ converge in $C^1(\mathbb{R}; \mathbb{R}^N)$ to $\Psi \in C^1(\mathbb{R}; S^{N-1})$ while the normal component of the velocity converges to 0 (since $\phi_\mu = v_\mu \cdot \Psi_\mu$ converges to 0).

To identify the limit Ψ, observe that since Ψ_μ solves (2.4), and $\mathbb{P}_\psi(\psi) = 0$, we have

$$\int\langle\partial_\tau(\mathbb{P}_{\Psi_\mu}(\zeta)), \dot{\Psi}_\mu\rangle\mathrm{d}\tau = \int\langle\mathbb{P}_{\Psi_\mu}(\zeta), U'(\Psi_\mu)\rangle\mathrm{d}\tau, \tag{2.28}$$

for all $\zeta \in C_0^\infty(\mathbb{R};\mathbb{R}^N)$. Calculate

$$[\partial_\tau, \mathbb{P}_{\Psi_\mu}]\zeta = \frac{-\dot{\Psi}_\mu \cdot \zeta \Psi_\mu - \Psi_\mu \cdot \zeta \dot{\Psi}_\mu}{||\Psi_\mu||^2} + 2\frac{(\Psi_\mu \cdot \zeta)(\Psi_\mu \cdot \dot{\Psi}_\mu)\Psi_\mu}{||\Psi_\mu||^4}.$$

Using this and the fact that $\langle \Psi_\mu, \dot{\Psi}_\mu \rangle \to \langle \Psi, \dot{\Psi} \rangle = 0$ (since $||\Psi(\tau)|| = 1$), we deduce that, along the subsequence,

$$\langle [\partial_\tau, \mathbb{P}_{\Psi_\mu}]\zeta, \dot{\Psi}_\mu \rangle \to -||\dot{\Psi}||^2 \Psi \cdot \zeta,$$

and hence that

$$\int (\langle \dot{\zeta}, \dot{\Psi} \rangle - \langle \zeta, ||\dot{\Psi}||^2 \Psi \rangle) \mathrm{d}\tau = \int \langle \zeta, \mathbb{P}_\Psi(U'(\Psi)) \rangle \mathrm{d}\tau,$$

which is the weak formulation of (2.8). Since (2.8) has a unique smooth solution with initial values as in (2.9), we deduce that the Ψ_μ converge in C^1 to Ψ *without taking subsequences* (since any subsequence has a subsequence which converges to the same limit Ψ). ■

There are various ways in which this example can be generalized, for example:

— by allowing several constraints, $G^j(\psi) = g^j$, so that the motion is constrained to a submanifold of co-dimension larger than one, as in §2*a*(ii), and
— by considering the case that ψ takes values in an infinite-dimensional vector space, as in the PDEs of classical field theory described in §§1 and 2*b*, or even by allowing ψ to be a function taking values in a manifold as in the σ-model (wave map) problem (see §3*c* and Haskins & Speight 2003).

To conclude this section, we shall briefly mention the possibility of increasing the dimension of the domain. A problem of this type was considered by Bethuel *et al.* (1993), where theorem 2.3 was proved.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected and simply connected, open set with smooth boundary $\partial\Omega$ on which is given a smooth function $g : \partial\Omega \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$ having zero degree, which is the restriction of a smooth function $\tilde{g} : \bar{\Omega} \to S^1$. Let $H_g^1(\Omega) = \tilde{g} + H_0^1(\Omega)$ be the complex-valued H^1 functions with boundary values g. Then solutions $\Psi_\mu \in H_g^1 \cap C^\infty(\Omega;\mathbb{C})$ of*

$$-\Delta \Psi_\mu = \mu \Psi_\mu (1 - |\Psi_\mu|^2),$$

which minimize the energy $\int (|\nabla\psi|^2 + (\mu/2)(1 - |\psi|^2)^2)$, converge in $C^{1,\alpha}$, $\alpha \in (0,1)$, as $\mu \to \infty$, to $\Psi \in H_g^1 \cap C^\infty(\Omega; S^1)$, the solution of

$$-\Delta \Psi = |\nabla \Psi|^2 \Psi,$$

which minimizes the energy $\int |\nabla\psi|^2$.

In this theorem, it is crucial that the boundary value g has zero degree, so that it does indeed admit a smooth S^1-valued extension to $\bar{\Omega}$: without this assumption there would be no putative limit function $\Psi \in C^\infty(\Omega; S^1)$, extending g, to which the sequence Ψ_μ might converge. The description of the asymptotic behaviour of Ψ_μ in the case of non-zero degree involves the emergence of rescaled Ginzburg–Landau vortices at locations determined by a renormalized energy, and has given rise to a very large literature starting with Bethuel *et al.* (1994). Since the focus of the present survey is on dynamical aspects of adiabatic limits, we will not attempt to describe the many interesting results in this area.

(ii) *Natural Lagrangian systems with vector constraints*

In order to generalize theorem 2.2, so as to allow for several constraints, we consider the Lagrangian of the form (2.1), with $V_\mu = U + \mu\mathcal{V}$, where the constraining potential is of the form $\mathcal{V}(\psi) = ||G(\psi)||^2 = \sum_{j=1}^{s} G_j(\psi)^2$ for some vector-valued function $G(\psi) = (G_1(\psi), \ldots, G_s(\psi)) \in \mathbb{R}^s$. We introduce the following assumptions (which are given at some length so as to fix notation for the subsequent discussion).

— *G1*. There is an open set $\mathcal{O}$ on which G is a smooth submersion, $G : \mathcal{O} \to \mathbb{R}^s$, whose level sets $S_g = \{\psi \in \mathcal{O} : G(\psi) = g\}$ are leaves of a smooth foliation of $\mathcal{O}$, and $\min_{\psi \in \mathcal{O}} \mathcal{V}(\psi) = 0 = \{\mathcal{V}(\psi_0) : \psi_0 \in S_0\}$.

— *G2*. There are open sets $\mathcal{O}_1 \subset \mathbb{R}^{N-s}$ and $\mathcal{O}_2 \subset \mathbb{R}^s$ and a diffeomorphism $\tilde{\Psi} : \mathcal{O}_1 \times \mathcal{O}_2 \to \mathcal{O}$ so that every $\psi \in \mathcal{O}$ can be written uniquely as $\psi = \tilde{\Psi}(\sigma, g)$ with $\sigma \in \mathcal{O}_1, g \in \mathcal{O}_2$ and $G(\tilde{\Psi}(\sigma, g)) = g$. Furthermore, there is a corresponding orthogonal decomposition of the tangent space $T_\psi\mathcal{O} \approx \mathbb{R}^N$ as

$$T_\psi\mathcal{O} = T_\psi S_{G(\psi)} \oplus N_\psi S_{G(\psi)} = \tilde{\mathbb{P}}_\psi(\mathbb{R}^N) \oplus \tilde{\mathbb{Q}}_\psi(\mathbb{R}^N),$$

with corresponding orthogonal projections $\tilde{\mathbb{P}}_\psi, \tilde{\mathbb{Q}}_\psi$ satisfying $\tilde{\mathbb{P}}_\psi \oplus \tilde{\mathbb{Q}}_\psi = 1$, and which map, respectively, onto the tangent and normal spaces to the leaves, i.e. at $\psi = \tilde{\Psi}(\sigma, g)$,

$$\tilde{\mathbb{P}}_\psi(\mathbb{R}^N) = \text{Ker } DG(\psi) = D_1\tilde{\Psi}(\sigma, g)(\mathbb{R}^{N-s}) \quad \text{and}$$

$$\tilde{\mathbb{Q}}_\psi(\mathbb{R}^N) = DG(\psi)^*(\mathbb{R}^s)$$

where $DG(\psi)^*$ means the adjoint operator $\mathbb{R}^s \to T_\psi\mathcal{O}$, defined using the Euclidean inner products to identify the vector spaces with their duals.

— *G3*. There exists $m_* > 0$ such that

$$D^2\mathcal{V}(\psi_0)(n, n) \geq m_*||n||^2,$$

for every $\psi_0 \in S_0$ and $n \in N_{\psi_0}S_0$. Then for $\psi = \psi_0 \in S_0$, the decomposition in (G2) reduces to

$$T_{\psi_0}\mathcal{O} = \text{Ker } L_{\psi_0} \oplus L_{\psi_0}(\mathbb{R}^N),$$

where L_ψ is the symmetric linear operator $\mathbb{R}^N \to \mathbb{R}^N$ determined by $\langle v, L_\psi w\rangle = D^2\mathcal{V}(\psi)(v, w)$. Furthermore, for ψ in some set $\mathcal{N}$ close to S_0, there are $N - s$ eigenvalues which are less than (say) $m_*/4$ and s eigenvalues greater than (say) $3m_*/4$, and there are corresponding orthogonal spectral projections $\mathbb{P}_\psi$ and $\mathbb{Q}_\psi$ such that

$$T_\psi\mathcal{O} = \mathbb{P}_\psi(\mathbb{R}^N) \oplus \mathbb{Q}_\psi(\mathbb{R}^N).$$

Remark 2.4. The assumed form of the constraining potential $\mathcal{V}$ is chosen to match the Bogomoln'yi form of the potentials in the field theoretic cases of interest discussed in §1*b*(iv). More general forms for $\mathcal{V}$ could be handled with the same methods.

Remark 2.5. Regarding (G3), observe that the fact that 0 is a minimum value of $\mathcal{V}$ implies that $\mathcal{V}'(\psi_0)=0$ for all $\psi_0=\tilde{\Psi}(\sigma,0)\in S_0$, and hence, by the chain rule, that $L_\psi w=0\,\forall\, w\in T_{\psi_0}S_0=D_1\tilde{\Psi}(\sigma,0)(\mathbb{R}^{N-s})$, i.e. the kernel of the Hessian contains the tangent space to S_0. The condition of positivity of the Hessian on the normal space to S_0 is the crucial defining condition of a Morse–Bott critical submanifold.

We now introduce the constrained system. The minimum value, 0, of the constraining potential $\mathcal{V}$ is attained on the set S_0, and so we consider critical points of (2.1) among curves $\tau\mapsto\psi(\tau)\in S_0\subset\mathbb{R}^N$, i.e. those for which

$$\begin{gathered}\int(\langle\dot{\psi},\dot{\zeta}\rangle-\langle U',\zeta\rangle)\mathrm{d}\tau=0,\\ \forall\zeta\in C_0^\infty(\mathbb{R};\mathbb{R}^N):\zeta(\tau)\in T_{\psi(\tau)}S_0=\mathbb{P}_{\psi(\tau)}(\mathbb{R}^N).\end{gathered}\tag{2.29}$$

An alternative, and more familiar, way to write this condition is

$$\frac{D}{D\tau}\dot{\psi}+\mathbb{P}_\psi U'=0,\tag{2.30}$$

where $D/D\tau=\mathbb{P}_\psi(\mathrm{d}/\mathrm{d}\tau)$ is the covariant derivative along ψ determined by the metric on S_0 induced from the ambient Euclidean structure.

Theorem 2.6. *Assume that $U,G_1,\dots,G_s$ are smooth bounded functions on $\mathbb{R}^N$ and the properties (G1)–(G3) hold. For $\mu>0$, let $\Psi_\mu\in C^\infty(\mathbb{R},\mathbb{R}^N)$ be the unique solution of (2.2), with $V_\mu=U+\mu\mathcal{V}=U+\mu||G||^2$, and with initial values*

$$\Psi_\mu(0)=\Psi_0\in S_0\qquad\dot{\Psi}_\mu(0)=v_0\in T_{\Psi_0}S_0.\tag{2.31}$$

Then there exists a non-empty time interval, $[-\tau_*,\tau_*]$, such that

$$\Psi_\mu\to\Psi\quad\text{in } C^1([-\tau_*,\tau_*];\mathbb{R}^N)\quad\text{as }\mu\to+\infty,\tag{2.32}$$

where $\Psi\in C^\infty([-\tau_*,\tau_*];S_0)$ is the unique solution of (2.30) with initial data $\Psi(0)=\Psi_0$, $\dot{\Psi}(0)=v_0$.

Remark 2.7. Small modifications of the proof below show that this behaviour is stable, i.e. holds for solution sequences whose initial data converge rapidly to (2.31).

Proof. The Euler–Lagrange equation (2.2) can be written more explicitly as

$$\ddot{\psi}+U'(\psi)+\mu\mathcal{V}'(\psi)=0.\tag{2.33}$$

By the local existence theorem, this equation has, for every μ and initial data (2.31), a local smooth solution Ψ_μ defined on some time interval $(-T_\mu,T_\mu)$ which satisfies the energy conservation law $E(\tau)=H(\Psi_\mu(\tau),\dot{\Psi}_\mu(\tau))=E(0)$, where

$$H(\psi,\dot{\psi})=\frac{1}{2}||\dot{\psi}||^2+U(\psi)+\mu\mathcal{V}(\psi).\tag{2.34}$$

The initial data (2.31) are such that the energy $E(0)=E_0$ is finite and independent of μ. Since U is bounded, by say L, and $\mathcal{V}\geq 0$, it follows that for $\mu>0$ the velocities $v_\mu=\dot{\Psi}_\mu$ are uniformly bounded as in (2.14). Now note that there exists $\tau_*>0$, independent of μ, such that all solutions $\Psi_\mu(\tau)$ remain inside a fixed compact subset $\mathcal{N}\subset\mathcal{O}$ containing Ψ_0, so that restricting to $|\tau|\le\tau_*$ we can make use of the properties (G1)–(G3). Also, since the $U, G_1, \dots, G_s$ are smooth, we may assume that all their derivatives are bounded (by constants depending on $\mathcal{N}$). Therefore, the solution can be continued, and subsequentially the Ψ_μ converge to a limit $\Psi\in C([-\tau_*,\tau_*];\mathbb{R}^N)$, uniformly on closed bounded intervals. (This convergence will be improved to C^1 below.) The energy identity (2.34) implies $G(\Psi(\tau))=0$ so that in fact $\Psi\in C([-\tau_*,\tau_*];S_0)$.

To prove C^1 convergence via lemma 2.1, it is sufficient to prove that the accelerations $\dot v_\mu=\ddot\Psi_\mu$ are bounded independent of μ. We now derive these estimates, temporarily writing ψ, v in place of Ψ_μ, v_μ for clarity. As in the model problem, the 'normal' and 'tangential' components are estimated separately. However, an additional complication here is that there are two different orthogonal decompositions into normal and tangential components, provided by (G2) and (G3), respectively, namely

$$\dot v=\mathbb{P}_\psi(\dot v)+\mathbb{Q}_\psi(\dot v)\quad\text{and}$$
$$\dot v=\tilde{\mathbb{P}}_\psi(\dot v)+\tilde{\mathbb{Q}}_\psi(\dot v).$$

(In the model problem considered above, these decompositions coincide). It turns out that it is sufficient (and most convenient) to estimate $\tilde{\mathbb{P}}_\psi(\dot v)$ and $\mathbb{Q}_\psi(\dot v)$ in order to bound $\dot v$ itself.

To estimate the first of these, observe that if ψ solves (2.33), then

$$||\tilde{\mathbb{P}}_{\psi(\tau)}(\dot v(\tau))||=||\tilde{\mathbb{P}}_{\psi(\tau)}(U'(\psi(\tau)))||\le c(\mathcal{N}). \tag{2.35}$$

For the estimation of the normal velocity, consider first the differentiated equation satisfied by $v=\dot\psi$, i.e.

$$\ddot v+K_\psi v+\mu L_\psi v=0, \tag{2.36}$$

where K_ψ bears the same relation to U as L_ψ does to $\mathcal{V}$, i.e. $\langle v, K_\psi w\rangle=D^2U(\psi)(v,w)$. Now introduce $w=\mathbb{Q}_\psi(v)$; given (2.35), this quantity is sufficient to bound $\dot v$.

Claim. *If $\psi(\tau)$ lies in a compact set $\mathcal{N}$ which is sufficiently close to S_0, then there exists a constant $c=c(E_0,L,\mathcal{N})$ such that $||\dot v||\le c(1+||\dot w||)$.*

To prove this claim, observe that the projection operators $\mathbb{Q}_\psi$ and $\tilde{\mathbb{Q}}_\psi$ are continuous functions of ψ which coincide on S_0, and hence making $\mathcal{N}$ close to S_0 we may assume that $||(\tilde{\mathbb{Q}}_\psi-\mathbb{Q}_\psi)\dot v||\le(1/2)||\dot v||$. From this, we deduce

$$||\dot v||\le||\tilde{\mathbb{P}}_\psi\dot v||+||\tilde{\mathbb{Q}}_\psi\dot v||\le||\tilde{\mathbb{P}}_\psi\dot v||+||\mathbb{Q}_\psi\dot v||+\frac{1}{2}||\dot v||, \tag{2.37}$$

so that $||\dot v||\le 2(||\tilde{\mathbb{P}}_\psi\dot v||+||\mathbb{Q}_\psi\dot v||)$. Now differentiation gives $\dot w=\mathbb{Q}_\psi(\dot v)+[\partial_\tau,\mathbb{Q}_\psi]v$ and one can check (e.g. using the Riesz contour integral formula for $\mathbb{Q}_\psi$; Riesz & Sz.-Nagy 1990) that $||[\partial_\tau,\mathbb{Q}_\psi]||\le c||v||$; here we write $||\cdot||$ for the operator norm. Using the bound (2.14) and substituting back into (2.37) gives the claim.

Thus it remains to estimate w. Applying $\mathbb{Q}_\psi$ to (2.36) gives

$$\ddot w+K_\psi w+\mu L_\psi w=[\partial_\tau^2,\mathbb{Q}_\psi]v+[K_\psi,\mathbb{Q}_\psi]v, \tag{2.38}$$

since $[\mathbb{Q}_\psi, L_\psi]=0$ (because spectral resolution projectors always commute with the original operator). This suggests the introduction of the quantity

$$E^N = \frac{1}{2}(\|\dot{w}\|^2 + \langle w, (K_\psi + \mu L_\psi)w\rangle), \tag{2.39}$$

since by (G3) and $w \in \mathbb{Q}_\psi(\mathbb{R}^N)$ we have

$$E^N \geq \frac{1}{2}\left(\|\dot{w}\|^2 + \frac{\mu m_*}{2}\|w\|^2\right), \tag{2.40}$$

for sufficiently large μ. Together with the claim above, this implies

$$\|\dot{v}\| \leq c(1 + \sqrt{E^N}). \tag{2.41}$$

The energy identity gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}E^N = \langle \dot{w}, [\partial_\tau^2, \mathbb{Q}_\psi]v + [K_\psi, \mathbb{Q}_\psi]v\rangle + \frac{1}{2}\langle w, [\partial_\tau, K_\psi + \mu L_\psi]w\rangle.$$

As noted above, $\|[\partial_\tau, \mathbb{Q}_\psi]\| \leq c\|v\|$, and similarly $\|[\partial_\tau^2, \mathbb{Q}_\psi]\| \leq c(\|v\|^2 + \|\dot{v}\|)$, so that there exists $c = c(E_0, L, \mathcal{N})$ such that

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}E^N\right| \leq c\|\dot{w}\|(\|\dot{v}\| + 1) + (1+\mu)\|w\|^2. \tag{2.42}$$

And so, since these estimates apply to the solutions Ψ_μ, (2.41) and a simple application of the Gronwall inequality give the following bound for $\dot{v}_\mu = \ddot{\Psi}_\mu$ along the subsequence:

$$\|\dot{v}_\mu(\tau)\| \leq C_1 \mathrm{e}^{C_2\tau}, \tag{2.43}$$

for $|\tau| \leq \tau_*$, with both constants depending only on $E_0, L, \mathcal{N}$ (and independent of μ). It then follows from lemma 2.1 that, along the subsequence, the Ψ_μ converge in $C^1([-\tau_*, \tau_*]; \mathbb{R}^N)$ to some limit $\Psi \in C^1([-\tau_*, \tau_*]; S_0)$. The remainder of the argument can be completed as in the model problem. ■

(iii) *A direct constructive approach*

An alternative approach to the problem of adiabatic motion and strongly constrained systems is to ask whether, given a solution to the limit (constrained) system, it is possible to construct a nearby solution to the original system (for large values of the constraining parameter μ)? We sketch a proof of a theorem which answers this question affirmatively in the context of the natural Lagrangian systems discussed in theorem 2.6 in §2*a*(ii). Theorem 1.4 from Stuart (1994*b*) is also based on this type of direct constructive approach, although the compactness approach, discussed in §2*a*(ii), could equally well be used for the problems in that article and Stuart (1994*a*).

Assume given, as a starting point, a solution to the constrained system (2.30) which can be written

$$\Psi(\tau) = \tilde{\Psi}(\sigma_0(\tau), 0), \tag{2.44}$$

with initial conditions as in (2.9). Here, we are using the same notation as in §2*a*(ii) so that $\tilde{\Psi}(\sigma, 0) \in S_0$, $\forall \sigma \in \mathcal{O}$ by (G2). We search for a solution of (2.33) in the form

$$\Psi_\mu(\tau) = \tilde{\Psi}(\sigma(\tau), 0) + \zeta, \tag{2.45}$$

with $\sigma(\tau)$ determined by the requirement that

$$\mathbb{P}_{\sigma(\tau)}(\zeta(\tau)) = 0, \tag{2.46}$$

where $\mathbb{P}_{\sigma(\tau)} = \mathbb{P}_{\tilde{\Psi}(\sigma(\tau),0)}$ is the projection onto the tangent space to S_0. (The condition (2.46) would hold if $\tilde{\Psi}(\sigma(\tau), 0)$ were the nearest point on S_0 to $\Psi_\mu(\tau)$.)

Theorem 2.8. *For every solution to (2.30) in the form (2.44), there exists $\tau_* > 0$ and a solution to (2.33) which can be written in the form (2.45) and (2.46) on the time interval $[-\tau_*, \tau_*]$, where*

$$\sup_{|\tau|\le\tau_*} [\mu||\dot{\zeta}(\tau)||^2 + \mu^2||\zeta(\tau)||^2 + \sqrt{\mu}(||\sigma(\tau) - \sigma_0(\tau)|| + ||\dot{\sigma}(\tau) - \dot{\sigma}_0(\tau)||)] \le c, \tag{2.47}$$

with c independent of μ.

The idea of the proof is to rewrite (2.33) as an equivalent coupled system of equations for $\zeta(\tau)$ and $\sigma(\tau)$ by requiring that (2.45) and (2.46) hold for each τ. A careful treatment of the energy identity then yields (2.47). Substitution gives the following equation for ζ:

$$\ddot{\zeta} + \mu L_\sigma \zeta = -(\ddot{\sigma} \cdot D_1 + \dot{\sigma} \otimes \dot{\sigma} \cdot D_1^2)\tilde{\Psi} - \mu j_\sigma(\zeta) - U'(\tilde{\Psi} + \zeta), \tag{2.48}$$

where, abusing notation slightly, we write $L_\sigma = L_{\tilde{\Psi}(\sigma,0)}$ and $j_\sigma(\zeta) = \mathcal{V}'(\tilde{\Psi} + \zeta) - L_\sigma\zeta$. Differentiate (2.46) twice and use the fact that $\mathbb{P}_\sigma L_\sigma = 0$ to deduce

$$\mathbb{P}_\sigma\left[-(\ddot{\sigma} \cdot D_1 + \dot{\sigma} \otimes \dot{\sigma} \cdot D_1^2)\tilde{\Psi} - \mu \mathrm{j}_\sigma(\zeta) - U'(\tilde{\Psi} + \zeta)\right] + [\partial_\tau^2, \mathbb{P}_\sigma]\zeta = 0, \tag{2.49}$$

which, for large μ, is a small perturbation of (2.30) when (2.47) holds. This means that (2.48) and (2.49) potentially provide a scheme in which (2.47) can be proved to hold on a finite time interval for appropriate initial data. To carry out this, choose, for simplicity, initial data $\zeta = 0 = \dot{\zeta}$ and $\sigma(0) = \sigma_0(0), \dot{\sigma}(0) = \dot{\sigma}_0(0)$, so that for each μ there exists $\tau_1(\mu) > 0$ such that (2.47) holds for some $c > 0$ on $[-\tau_1(\mu), \tau_1(\mu)]$ (by continuity). This information is then used to show that in fact $\tau_1(\mu)$ may be taken to be $\ge \tau_* > 0$, with τ_* independent of μ for large μ. The estimates for (2.48) necessary to achieve this can be obtained by consideration of the quantity

$$E^N = \frac{1}{2}(||\dot{\zeta}||^2 + \langle \zeta, (K_\sigma + \mu L_\sigma)\zeta \rangle), \tag{2.50}$$

with $K_\sigma = K_{\tilde{\Psi}(\sigma,0)}$, as defined immediately following (2.36). The crucial estimate, from which theorem 2.8 follows quickly, is $|E^N| \le c/\mu$. To obtain this, differentiate (2.50) to get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} E^N = &\langle \dot{\zeta}, -(\ddot{\sigma} \cdot D_1 + \dot{\sigma} \otimes \dot{\sigma} \cdot D_1^2)\tilde{\Psi} - \mu j_\sigma(\zeta) - (U'(\tilde{\Psi} + \zeta) - K_\sigma \zeta) \rangle \\ &+ \frac{1}{2}\langle \zeta, [\partial_\tau, K_\psi + \mu L_\psi]\zeta \rangle. \end{aligned} \tag{2.51}$$

The idea is to write $E^N(T) = E^N(0) + \int_0^T \dot{E}^N \mathrm{d}T$, estimate $\dot{E}^N$ and apply the Gronwall inequality. The terms on the right-hand side, which are at least quadratic in $\zeta, \dot{\zeta}$, can be estimated in the obvious way. Some care is needed with the terms in the first line of (2.51), which are only linear in ζ, since if estimated naively they contribute $O(1)$, whereas it is necessary to bound them as $O(1/\mu)$ to establish (2.47); this can be done by means of a single integration by parts in time.

(*b*) *The sine-Gordon equation*

We will now prove theorem 1.1, showing that in the low-energy adiabatic limit the sine-Gordon equation (1.1) can be approximated by uniform motion along the moduli space $\mathcal{M}_{\mathrm{SG}}$ of all solitons. The proof of this theorem will be carried out using the rescaling (1.22), so that we are interested in solutions, for large μ, of

$$\frac{\partial^2\theta}{\partial\tau^2}+\mu\mathcal{V}'(\theta)=0, \tag{2.52}$$

where $\mathcal{V}(\theta)=V(\theta)-8=(1/2)\int(\theta_x-2\sin(\theta/2))^2\mathrm{d}x$ is the potential energy discussed in §1*b*(iv), shifted to have minimum 0. We consider smooth initial data obtained by rescaling (1.20) via (1.22), and using $\mu=\epsilon^{-2}$ as parameter in place of ϵ,

$$\begin{gathered}\theta(0,x)=\theta_K(x)+\hat{\theta}_0(x;\mu),\qquad \partial_\tau\theta(0,x)=-u_0\theta_K'(x)+\hat{v}_0(x;\mu)\quad\text{and}\\ \mu\|\hat{\theta}_0\|_{H^2}+\sqrt{\mu}\|\hat{v}_0\|_{H^1}=O(1),\quad\text{as }\mu\to\infty.\end{gathered} \tag{2.53}$$

Referring to the discussion in §§1*b*,*c*, we see that equation (2.52) is a natural Lagrangian system, an infinite-dimensional version of those studied in §2*a*(ii), with $U=0$, $\mu=\epsilon^{-2}\gg1$ and with infinite-dimensional vector constraint function $G:\mathcal{A}_1\to L^2(\mathbb{R})$, $\theta\mapsto G(\theta)=\theta_x-2\sin(\theta/2)$, defined on the configuration space $\mathcal{A}_1$ defined in §1*c*. In the next three paragraphs we develop a framework for the discussion similar to the properties (G1)–(G3) used in the finite-dimensional case.

Analogous to (G1), we have the following.

Lemma 2.9. *G is a smooth submersion $\mathcal{A}_1\to L^2(\mathbb{R})$ whose level sets $G^{-1}(g)$, for $g\in G(\mathcal{A}_1)$, define a foliation with one-dimensional leaves, whose tangent spaces at θ are spanned by a positive function $\beta_\theta\in\operatorname{Ker}DG(\theta)$ with $\|\beta_\theta\|_{L^2}=1$, and the mapping $\theta\mapsto\beta_\theta$ is continuous from $\mathcal{A}_1$ to H^1.*

Proof. Since $\sup_x|\theta(x)|\le c\|\theta\|_{H^1}$, the smoothness (and in fact real analyticity) of G follows immediately from that of $\sin(\theta/2)$. To prove that it is a submersion, it is sufficient to prove that the derivative

$$DG(\theta):w\mapsto w_x-\cos\frac{\theta}{2}w,$$

is surjective and has a one-dimensional kernel. That these statements are true follows from the fact that if $\theta\in\mathcal{A}_1$ then $\cos(\theta(x)/2)\to\mp1$ as $x\to\pm\infty$. This means that $w_x-\cos(\theta/2)w=0$ has a solution $w(x)=w(0)\exp(\int_0^x\cos(\theta/2))$, which is square integrable and gives a one-dimensional kernel; normalizing it by $\|\beta_\theta\|_{L^2}=1$ and requiring it to be positive determines β_θ uniquely. Surjectivity can be proved similarly by construction of a fundamental solution for $DG(\theta)$

$$K(x,\,y)=\begin{cases}\exp\left(-\int_x^y\cos(\theta/2)\right) & \text{if }\ 0<y<x,\\ -\exp\left(-\int_x^y\cos(\theta/2)\right) & \text{if }\ x<y<0,\\ 0 & \text{otherwise.}\end{cases} \tag{2.54}$$

The fact that $\theta\in\mathcal{A}_1$ implies that $\cos(\theta/2)$ has limit ∓ 1 as $x\to\pm\infty$. Using this it is straightforward to show from the generalized Young inequality that the operator $\mathbb{K}: f\mapsto\int K(x,\,y)f(y)\mathrm{d}y$ is bounded $L^2\to L^2$, and gives a solution to the equation $w_x-\cos(\theta/2)w=f$. This equation in turn implies boundedness of $\mathbb{K}$ from L^2 to H^1 so that $DG(\theta)$ is surjective as claimed. The final assertion follows from the implicit function theorem. ∎

To further pursue the analogy with §2*a*(ii) we introduce projection operators as in (G2),

$$\tilde{\mathbb{P}}_\theta(w)=\langle w,\beta_\theta\rangle\beta_\theta,\quad \tilde{\mathbb{Q}}_\theta(w)=w-\tilde{\mathbb{P}}_\theta(w). \tag{2.55}$$

Now β_θ is tangent to the leaves of the foliation and as such is orthogonal to $\mathcal{V}'(\theta)$, a fact which can be checked directly,

$$\langle\beta_\theta,(-\theta_{xx}+\sin\theta)\rangle_{L^2}=\int\left((\theta_x-2\sin\frac{\theta}{2})((\beta_\theta)_x-\cos\frac{\theta}{2}\beta_\theta)+(2\sin\frac{\theta}{2}\beta_\theta)_x\right)\mathrm{d}x=0. \tag{2.56}$$

This means that $\tilde{\mathbb{P}}_\theta(\mathcal{V}'(\theta))=0$, which will be used below to estimate the tangential component of $\ddot{\theta}$.

To introduce the decomposition in (G3), consider the operator $L_\theta=-\partial_x^2+\cos\theta$ which satisfies $\int wL_\theta w\,\mathrm{d}x=D^2\mathcal{V}(\theta)(w,w)$. When $\theta(\cdot)=\theta_K(\cdot-X_0)$ is one of the soliton solutions, we write L_{X_0} for this operator; it has precisely one L^2 eigenvector, θ_k, corresponding to the eigenvalue 0, which arises due to translation invariance. The remaining spectrum is continuous spectrum filling the interval $[1,\,\infty)$. For $\theta\in\mathcal{A}_1$ sufficiently close to $\theta_K(\cdot-X_0)$, i.e. if $\|\theta-\theta_K(\cdot X_0)\|_{H^1}$ is sufficiently small, L_θ is a compact perturbation of L_{X_0}. Therefore, by Weyl's theorem, its spectrum has the same set of limit points as L_{X_0}, and, by results in ch. 7 of Kato (1980), has an isolated simple eigenvalue λ_θ close to 0; there may also be eigenvalues close to the bottom of the continuous spectrum. There exists a number $\delta_*>0$ such that if $\|\theta-\theta_K(\cdot-X_0)\|_{H^1}<\delta_*$, for some X_0, then $|\lambda_\theta<(1/4)|$ and the remainder of the spectrum is $>(3/4)$. Let ζ_θ be the positive, normalized eigenvector with eigenvalue λ_θ. The functions $\theta\mapsto\lambda_\theta$ (respectively, ζ_θ) are smooth from $\mathcal{A}_1$ to $\mathbb{R}$ (respectively, H^2). Introduce corresponding spectral projection operators on L^2,

$$\mathbb{P}_\theta(w)=\langle w,\zeta_\theta\rangle\zeta_\theta,\quad \mathbb{Q}_\theta(w)=w-\mathbb{P}_\theta(w). \tag{2.57}$$

Since $\beta_{\theta_K}=\zeta_{\theta_K}=\theta'_K$, it follows, from continuity, that

$$\|(\tilde{\mathbb{Q}}_\theta-\mathbb{Q}_\theta)\|\le\frac{1}{2}, \tag{2.58}$$

in operator norm, as long as $\|\theta-\theta_K(\cdot-X_0)\|_{H^1}<\delta_*$, with δ_* sufficiently small. Another immediate consequence is that for small δ_* there exists $c_4>0$ such that

$$\int wL_\theta w\,\mathrm{d}x\ge c_4\|w\|^2_{H^1}, \tag{2.59}$$

for all $w\in\mathbb{Q}_\theta(H^1(\mathbb{R}))$.

A proof of theorem 1.1 will now be given. The basic strategy is the same as in the finite-dimensional case, but a new compactness criterion is required on account of the infinite dimensionality of the problem. In place of the Arzela–Ascoli theorem, the following version of the Lions–Aubin lemma (Majda & Bertozzi 2001) will be used to deduce compactness.

Lemma 2.10. *Given positive numbers $l<s$ and a sequence of smooth functions $f_n(t,x)$ satisfying*

$$\max_{|\tau|\leq\tau_*}\left(||f_n(\tau)||_{H^s}+||\dot{f}_n(\tau)||_{H^l}\right)\leq C,$$

there exists $\{f_{n_j}\}_{j=1}^{\infty}$ which converges to a limit $f\in C([-\tau_*,\tau_*];H^s(\mathbb{R}))$, in the sense that if ρ is smooth and compactly supported, then $\max_{|\tau|\leq\tau_*}||\rho(\cdot)\times(f_n(\tau,\cdot)-f(\tau,\cdot))||_{H^r}\to 0$ *for every $r\in(l,s)$.*

Proof of theorem 1.1.

(i) The initial-value problem for (2.52) with smooth initial data has a unique smooth global solution which satisfies energy conservation $E(\tau)=E(0)$, where

$$E(\tau)=\frac{1}{2}\int\dot{\theta}(\tau)^2\mathrm{d}x+\mu\mathcal{V}(\theta(\tau)).$$

Furthermore, the solution $\theta(\tau)=\Theta_\mu(\tau)$ with initial data (2.53) remains close to the soliton moduli space for all time as $\mu\to+\infty$: there exists $X_\mu(\tau)\in\mathbb{R}$ and $c>0$ such that, for every $\tau\in\mathbb{R}$,

$$||\theta(\tau,\cdot)-\theta_K(\cdot-X_\mu(\tau))||_{H^1}\leq c\mu^{-(1/2)}. \tag{2.60}$$

These facts are proved by Henry *et al.* (1982); an alternative proof, more similar to the methods being discussed here, follows as a simplification of the work by Stuart (2001). Let μ be sufficiently large that (2.58) and (2.59) hold.

(ii) Let $v=\dot{\theta}$ and observe that it solves the equation

$$\ddot{v}+\mu L_\theta v=0. \tag{2.61}$$

Apply the tangential projection operator $\tilde{\mathbb{P}}_\theta$ to (2.52) to deduce $\tilde{\mathbb{P}}_\theta(\dot{v})=0$. Therefore, $\dot{v}=\tilde{\mathbb{Q}}_\theta(\dot{v})$, so that $||\dot{v}||_{L^2}\leq||\mathbb{Q}_\theta(\dot{v})||_{L^2}+||\tilde{\mathbb{Q}}_\theta(\dot{v})-\mathbb{Q}_\theta(\dot{v})||_{L^2}$, and

$$||\dot{v}||_{L^2}\leq 2||\mathbb{Q}_\theta(\dot{v})||_{L^2}, \tag{2.62}$$

for sufficiently large μ by (2.58) and (2.60). Define $w=\mathbb{Q}_\theta(v)$, then

$$\ddot{w}+\mu L_\theta w=\left[\partial_\tau^2,\mathbb{Q}_\theta\right]v. \tag{2.63}$$

Now, by the paragraph preceding (2.57), $|\dot{\lambda}_\theta|+||\dot{\zeta}_\theta||_{L^2}\leq c||\dot{\theta}||_{L^2}$ and hence (2.62) implies $||\dot{v}||_{L^2}\leq c(1+||\dot{w}||_{L^2})$, with c depending only on the energy, and similarly $||\dot{w}||_{L^2}\leq c(1+||\dot{v}||_{L^2})$. Introduce $E^N(\tau)=(1/2)\int(\dot{w}^2+\mu wL_\theta w)\mathrm{d}x$ as a measure of the normal oscillations. Then for initial data as in (2.53), we claim that $E^N(0)=O(1)$ as $\mu\to+\infty$. To see this, first note that (2.52) and (2.53) imply that $||\partial_\tau^2\theta(0,\cdot)||_{L^2}=O(1)$ and hence $||\dot{w}||_{L^2}=O(1)$. Next, write $w=Q_\theta(\dot{\theta})=Q_\theta(\hat{v}_0)+(Q_\theta-Q_{\theta_K})(-u_0\theta_K')$ when $t=0$ (since $Q_{\theta_K}(\theta_K')=0$). Then note that (2.53) implies that $||\theta(0,\cdot)-\theta_K||_{H^2}=O(\mu^{-1})$

so that $||\zeta_\theta - \zeta_{\theta_K}||_{H^2} = O(\mu^{-1})$ also; therefore, since $||\hat{v}_0||_{H^1} = O(\mu^{-1/2})$, we deduce $||w(0,\cdot)||_{H^1} = O(\mu^{-1/2})$, and hence $E^N(0) = O(1)$ as claimed.

(iii) We need the identity $(\mathrm{d}/\mathrm{d}\tau)E^N = \langle \dot{w}, [\partial_\tau^2, \mathbb{Q}_\theta]v\rangle + (1/2)\mu\langle w, [\partial_\tau, L_\theta]w\rangle$ and the lower bound (which follows immediately from (2.59))

$$E^N(\tau) \geq c_5\big(||\dot{w}(\tau)||_{L^2}^2 + \mu||w(\tau)||_{H^1}^2\big). \tag{2.64}$$

Now, by the smoothness properties preceding (2.57), we have $|\ddot{\lambda}_\theta| \leq c\big(\|\ddot{\theta}\|_{L^2} + \|\dot{\theta}\|_{L^2}^2\big)$ and, using also (2.59), we deduce $\|\ddot{\zeta}_\theta\|_{L^2} \leq c\big(\|\ddot{\theta}\|_{L^2} + \|\dot{\theta}\|_{L^2}^2 + \|\dot{\theta}\|_{L^4}^2\big)$ Therefore, using $||f||_{L^4} \leq c||f||_{H^1}^{1/2}||f||_{L^2}^{1/2}$, there exists $c>0$, independent of $\mu>1$, such that

$$||[\partial_\tau^2, \mathbb{Q}_\theta]v||_{L^2} \leq c(1 + \sqrt{E^N}),$$

and hence

$$E^N(\tau) \leq E^N(0) + c\int_0^\tau (1 + E^N(s))\mathrm{d}s,$$

with c depending on the energy and independent of μ. As a consequence, on any time interval $[-\tau_*, \tau_*]$ the solutions $(\Theta_\mu, v_\mu = \dot{\Theta}_\mu)$ satisfy the following (μ independent) bounds:

$$\max_{|\tau|\leq\tau_*}(||\dot{v}_\mu(\tau)||_{L^2} + ||v_\mu(\tau)||_{H^1} + \sqrt{\mu}||w_\mu||_{H^1}) \leq c, \tag{2.65}$$

where $w_\mu = \mathbb{Q}_{\Theta_\mu}(v_\mu)$. Now let $f_\mu = \Theta_\mu - \Theta_\mu(0)$ so that $\dot{f}_\mu = \dot{\Theta}_\mu = v_\mu$. From these we can bound $\ddot{f}_\mu = \ddot{\Theta}_\mu$ in L^2, $\dot{f}_\mu$ in H^1 and hence f_μ in H^1 by a constant independent of μ, but depending upon τ_*. Given this, equation (2.52) implies a bound for $\partial_x^2\Theta_\mu = \partial_x^2(f_\mu + \Theta_\mu(0))$, and hence

$$\max_{|\tau|\leq\tau_*}(||\dot{f}_\mu(\tau)||_{H^1} + ||f_\mu(\tau)||_{H^2}) \leq c, \tag{2.66}$$

with c independent of μ.

(iv) Now applying lemma 2.10, we extract a subsequence $\{\mu_j\}$ along which there is convergence to a limit $f \in C([-\tau_*, \tau_*]; H^2(\mathbb{R}))$, for every $r<2$ in the sense that if $\rho = \rho(x)$ is smooth and compactly supported, then $\max_{|\tau|\leq\tau_*} ||\rho(f_{\mu_j}(\tau) - f(\tau))||_{H^r} \to 0$ and similarly $\dot{f}_{\mu_j}$ converge to $\dot{f} \in C([-\tau_*, \tau_*]; H^1(\mathbb{R}))$ in a corresponding sense with $r<1$. Define

$$\Theta \equiv \Theta_\mu(0) + f,$$

then $(\Theta, \partial_x\Theta, \dot{\Theta}) \in C([-\tau_*, \tau_*]; \mathcal{A}_1 \times H^1 \times H^1)$, and $G(\Theta) = 0$,

$$\text{i.e.}\quad \partial_x\Theta = 2\sin\frac{\Theta}{2},$$

so that there exists $X_0(\tau)$ such that $\Theta(\tau,\cdot) = \theta_K(\cdot - X_0(\tau))$ at each time τ. It follows from the regularity just asserted for Θ, and the positivity of θ_K', that the function $\tau \mapsto X_0(\tau)$ is in fact C^1. Furthermore, comparing with (2.60), we see that $X_\mu(\tau) \to X_0(\tau)$ uniformly and

$$\lim_{\mu_j\to\infty} \max_{|\tau|\le\tau_*} \|(\Theta_{\mu_j}(\tau,\cdot) - \theta_K(\cdot - X_0(\tau)))\|_{H^1} = 0. \tag{2.67}$$

(v) It follows from (2.56) that $\langle \beta_{\Theta_\mu}, \partial_\tau^2 \Theta_\mu \rangle_{L^2} \to 0$ as $\mu \to \infty$ along the subsequence. This implies also that, along (μ_j),

$$\langle \theta'_K(\bullet - X_0), \quad \partial_\tau^2 \Theta_\mu \rangle_{L^2} \to 0,$$

since by lemma 2.9 and the previous item $\beta_{\Theta_\mu} \to \theta'_K(\bullet - X_0)$ strongly in H^1. Note also that

$$\langle \theta''_K(\bullet - X_0), \quad \partial_\tau \Theta_\mu \rangle_{L^2} \to 0,$$

since $\langle \theta''_K, \theta'_K \rangle_{L^2} = 0$ and $\partial_\tau \Theta_\mu$ converges to $-\dot{X}_0 \theta'_K(\cdot - X_0)$ weakly in L^2, along the subsequence (μ_j). Now

$$\langle \theta'_K(\cdot - X_0(\tau)), \partial_\tau \Theta_\mu(\tau) \rangle_{L^2} \Big|_\alpha^\beta = \int_\alpha^\beta \theta'_K(\cdot - X_0(\tau)), \partial_\tau^2 \Theta_\mu \rangle_{L^2}$$
$$- \dot{X}_0 \langle \theta''_K(\cdot - X_0(\tau)), \partial_\tau \Theta_\mu \rangle_{L^2} \mathrm{d}\tau.$$

Taking the limit as $\mu_j \to \infty$, we deduce that $\dot{X}_0(\beta) = \dot{X}_0(\alpha)$ for all α, β so that X_0 is indeed the solution of $\ddot{X}_0 = 0$ with the initial data $X_0(0) = 0$, $\dot{X}_0(0) = u_0$ as expected. Since the solution $X_0(\tau) = u_0 \tau$ is unique, it follows that the Θ_μ converge to the same limit along every convergent subsequence, and hence there is convergence, without restriction to subsequences, completing the proof. ∎

3. Further developments and directions for future work

In this section, we briefly discuss various directions in which the previous work either has been, or could be, pursued and strengthened.

(a) *Scattering theory*

An immediate question raised by the formulation of, for example, theorem 1.4, is whether there is a longer time scale on which the approximation is valid. This is a natural question mathematically which is also of interest phenomenologically, e.g. for the description of soliton scattering. Scattering is a process in which two, or more, solitons, initially well separated, move towards one another and interact for a time before moving apart again (usually); the interaction is generally hoped to have a negligible effect except over a finite time interval.

For the case of BPS monopoles (§1 *a*(ii) in §1), there is a quite explicit description of the scattering of two monopoles, at the level of the moduli space approximation, which is given by Atiyah & Hitchin (1988). It is desirable to extend theorem 1.4 to provide a rigorous description of monopole scattering on an infinite time interval: this would be achieved by the construction of solutions to (1.4), which are close for all time to the monopole scattering processes given by the geodesics given by Atiyah & Hitchin (1988). This is open, as is a more basic problem, to prove asymptotic stability of a single BPS monopole. Regarding this question, it has been proved by Stuart (1999*a*) that the BPS monopoles are uniformly stable in a certain norm, similar to but weaker than the H^1 norm. For the case of a single monopole, this implies that for initial data close to a monopole the solution at later times is uniformly close to some translate and gauge transformation of that monopole. To strengthen this assertion to prove asymptotic stability amounts to showing that the solution actually converges to a monopole as $t \to \infty$, in some topology.

Once asymptotic stability of a single monopole is proved, it would be interesting to understand the asymptotic behaviour as $t \to \infty$ when several monopoles are present in some appropriate class of initial data: does the solution converge to an approximate superposition of monopoles as $t \to \infty$ in some norm? Results of this type are in principle known for certain integrable equations, although precise statements are not easy to come by (see Eckhaus & Schuur 1983; Cheng *et al.* 1999). Some progress has been made towards developing methods allowing a more general treatment of such problems, starting with Soffer & Weinstein (1990), Buslaev & Perelman (1993) and more recently Cuccagna (2003), Rodnianski *et al.* (2003), Perelman (2004) and Buslaev *et al.* (2007).

(*b*) *Bound states and time-periodic solutions*

We now discuss questions related to the existence of periodic solutions representing bound states of solitons. These arise if the adiabatic limit system has periodic solutions representing such bound states. An approximation theorem like theorem 1.4 would then imply that there is a corresponding solution of the full system which is close to the bound state on some time interval. However, there is no guarantee this would be close for all time, or even on a time interval long compared to the period of the bound state, and it is clearly necessary to refine the analysis carried out hitherto to seriously address the issues of existence, persistence and stability of periodic and quasi-periodic motions. The existence of periodic orbits in finite-dimensional adiabatic limit problems is treated by Uhlenbeck (1995) and Malchiodi (2001). As a specific infinite-dimensional example, the Abelian Higgs model is a system of hyperbolic nonlinear wave equations for which a moduli space approximation has been proved to be valid (Stuart 1994*a*). At the moduli space level there exist time-periodic solutions, which have been proved to persist in the full system in certain cases (Stuart 1999*b*). The Chern–Simons–Schrödinger system will also admit periodic solutions of a similar type. Work is currently underway to apply Hamiltonian techniques associated with KAM theory and Nekhoroshev estimates to understand the stability of such solutions, and then extend this understanding to quasi-periodic solutions.

(*c*) *Singularities*

In certain models, the limiting moduli space dynamics is singular (Speight 2003; Bizon *et al.* 2004) and it is natural to investigate to what extent this is a reflection of singular behaviour of the original system. This circumstance arises in particular for systems having a scale invariant static energy $\mathcal{V}$, such as the Yang–Mills equations on $\mathbb{R}^{1+4}$ and the σ-model (wave map) problem on $\mathbb{R}^{1+2}$. In certain cases, the L^2-induced metric on the moduli space is incomplete, and there exist geodesics which cease to exist after a finite time. These geodesics correspond to a finite time collapse of the soliton by rescaling. However, the very fact of this singular collapse means the question of validity of the moduli space approximation is a subtle one. For the case of the equivariant σ-model, it has recently been proved by Rodnianski & Sterbenz (2006) that soliton collapse does occur in certain cases. However, as had been observed numerically by Bizon *et al.* (2004), the asymptotics at the blow-up point is different (by a logarithmic term) from the self-similar collapse suggested by naive application of the moduli space approximation.

References

Arnold, V. I. 1989 *Mathematical methods of classical mechanics*. New York, NY: Springer.

Atiyah, M. F. & Hitchin, N. 1988 *Geometry and dynamics of magnetic monopoles*. Princeton, NJ: Princeton University Press.

Bethuel, F., Brezis, H. & Helein, F. 1993 Asymptotics for the minimization of a Ginzburg–Landau functional. *Calc. Var. Partial Differ. Eqns* **1**, 123–148. (doi:10.1007/BF01191614)

Bethuel, F., Brezis, H. & Helein, F. 1994 *Ginburg–Landau vortices*. Boston, MA: Birkhauser.

Bizon, P., Ovchinnikov, Y. & Sigal, I. M. 2004 Collapse of an instanton. *Nonlinearity* **17**, 1179–1191. (doi:10.1088/0951-7715/17/4/003)

Bradlow, S. 1988 Vortices in holomorphic line bundles and closed Kähler manifolds. *Commun. Math. Phys.* **118**, 635–640. (doi:10.1007/BF01221112)

Buslaev, V. S. & Perelman, G. S. 1993 Scattering for the nonlinear Schrödinger equation: states that are close to a soliton. *St Petersburg Math. J.* **4**, 1111–1142.

Buslaev, V., Komech, A., Kopylova, E. & Stuart, D. 2007 On asymptotic stability of solitary waves in a nonlinear Schrödinger equation. arXiv:math-ph/0702013.

Cheng, P.-J., Venakides, S. & Zhou, X. 1999 Long-time asymptotics for the pure radiation solution of the sine-Gordon equation. *Commun. Partial Differ. Eqns* **24**, 1195–1262. (doi:10.1080/03605309908821464)

Cuccagna, S. 2003 On asymptotic stability of ground states of NLS. *Rev. Math. Phys.* **15**, 877–903. (doi:10.1142/S0129055X03001849)

Demoulini, S. 2007 Global existence for a nonlinear Schrödinger–Chern–Simons system on a surface. *Ann. de l'IHP* **24**, 207–225. (doi:10.1016/j.anihpc.2006.01.004)

Demoulini, S. & Stuart, D. 1997 Gradient flow of the superconducting Ginzburg–Landau functional on the plane. *Commun. Anal. Geom.* **5**, 121–198.

Demoulini, S. & Stuart, D. 2000 Variational approach to uniform rigid rotation of Ginzburg–Landau vortices. *Lett. Math. Phys.* **52**, 127–142. (doi:10.1023/A:1007621017448)

Donaldson, S. 1984 Nahm's equations and the classification of monopoles. *Commun. Math. Phys.* **96**, 387–407. (doi:10.1007/BF01214583)

Ebin, D. 1977 The motion of slightly compressible fluids viewed as a motion with strong constraining force. *Ann. Math.* **105**, 141–200. (doi:10.2307/1971029)

Eckhaus, W. & Schuur, P. 1983 The emergence of solitons of the Korteweg–de Vries equation from arbitrary initial conditions. *Math. Methods Appl. Sci.* **5**, 97–116. (doi:10.1002/mma.1670050108)

Groisser, D. 1984 Integrality of the monopole number. *Commun. Math. Phys.* **93**, 367–378. (doi:10.1007/BF01258535)

Haskins, M. & Speight, J. M. 2003 The geodesic approximation for lump dynamics and coercivity of the Hessian for harmonic maps. *J. Math. Phys.* **44**, 3470–3494. (doi:10.1063/1.1586480)

Henry, D., Perez, J. & Wreszinski, W. 1982 Stability theory for solitary wave solutions of scalar field equations. *Commun. Math. Phys.* **85**, 351–361. (doi:10.1007/BF01208719)

Jaffe, A. & Taubes, C. 1982 *Vortices and monopoles*. Boston, MA: Birkhauser.

Kato, T. 1980 *Perturbation theory for linear operators*. New York, NY: Springer.

Klainerman, S. & Majda, A. 1981 Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Commun. Pure Appl. Math.* **34**, 481–524. (doi:10.1002/cpa.3160340405)

Krusch, S. & Sutcliffe, P. 2006 Schrödinger–Chern–Simons vortex dynamics. *Nonlinearity* **19**, 1515–1534. (doi:10.1088/0951-7715/19/7/003)

Majda, A. & Bertozzi, A. 2001 *Vorticity and incompressible fluid flow*. Cambridge, UK: Cambridge University Press.

Malchiodi, A. 2001 Adiabatic limits of closed orbits for some Newtonian systems in $\mathbb{R}^n$. *Asympt. Anal.* **25**, 149–181.

Manton, N. 1982 A remark on scattering of BPS monopoles. *Phys. Lett. B* **110**, 54–56. (doi:10.1016/0370-2693(82)90950-9)

Manton, N. 1997 First order vortex dynamics. *Ann. Phys.* **256**, 114–131. (doi:10.1006/aphy.1997.5672)

Manton, N. & Sutcliffe, P. 2004 *Topological solitons*. Cambridge, UK: Cambridge University Press.

Perelman, G. 2004 Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations. *Commun. Partial Differ. Eqns* **29**, 1051–1095. (doi:10.1081/PDE-200033754)

Riesz, F. & Sz.-Nagy, B. 1990 *Functional analysis*. New York, NY: Dover.

Rodnianski, I. & Sterbenz, J. 2006 On the formation of singularities in the critical $O(3)\sigma$-model. arXiv. 2006 mathAP/0605023

Rodnianski, I., Schlag, W. & Soffer, A. 2003 Asymptotic stability of N-soliton states of NLS. arXiv:math.AP/0309114.

Romao, N. M. & Speight, J. M. 2004 Slow Schrödinger dynamics of gauged vortices. *Nonlinearity* **17**, 1337–1355. (doi:10.1088/0951-7715/17/4/010)

Rubin, H. & Ungar, P. 1957 Motion under a strong constraining force. *Commun. Pure Appl. Math.* **10**, 65–87. (doi:10.1002/cpa.3160100103)

Soffer, A. & Weinstein, M. I. 1990 Multichannel nonlinear scattering for nonintegrable equations. *Commun. Math. Phys.* **133**, 119–146. (doi:10.1007/BF02096557)

Speight, J. M. 2003 The L^2 geometry of spaces of harmonic maps $S^2 \to S^2$ and $\mathbb{R}P^2 \to \mathbb{R}P^2$. *J. Geomet. Phys.* **47**, 343–368. (doi:10.1016/S0393-0440(02)00227-9)

Strauss, Y. & Sigal, I. M. 2006 Effective dynamics of a magnetic vortex in a local potential. *J. Nonlin. Sci.* **16**, 123–157. (doi:10.1007/s00332-004-0680-3)

Stuart, D. 1994*a* Dynamics of abelian Higgs vortices in the near Bogomolny regime. *Commun. Math. Phys.* **159**, 51–91. (doi:10.1007/BF02100485)

Stuart, D. 1994*b* The geodesic approximation for the Yang–Mills–Higgs equations. *Commun. Math. Phys.* **166**, 149–191. (doi:10.1007/BF02099305)

Stuart, D. 1999*a* Uniform stability of monopoles. *Calc. Var. Partial Differ. Eqns* **8**, 123–157. (doi:10.1007/s005260050120)

Stuart, D. 1999*b* Periodic solutions of the abelian Higgs model and rigid rotation of vortices. *Geomet. Funct. Anal.* **9**, 568–595. (doi:10.1007/s000390050096)

Stuart, D. M. A. 2001 Modulational approach to stability of non-topological solitons in semilinear wave equations. *J. Math. Pures Appl.* **80**, 51–53. (doi:10.1016/S0021-7824(00)01189-2)

Taubes, C. 1983 Stability in Yang–Mills theories. *Commun. Math. Phys.* **91**, 235–263. (doi:10.1007/BF01211160)

Uhlenbeck, K. 1995 Adiabatic limits and moduli spaces. *Notices AMS* **42**, 41–42.

Witten, E. 1977 Some exact multipseudoparticle solutions of classical Yang–Mills theory. *Phys. Rev. Lett.* **38**, 121–124. (doi:10.1103/PhysRevLett.38.121)