Adiabatic limit and the slow motion of vortices in a Chern-Simons-Schroedinger system

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Abstract

For a nonlinear Chern-Simons-Schroedinger system on a Riemann surface, we prove a theorem describing its adiabatic approximation by a Hamiltonian system on a finite dimensional space, the moduli space of self-dual Ginzburg-Landau vortices, for values of the Higgs self-coupling constant λ close to the self-dual (Bogomolny) value of 1. The viability of the approximation scheme depends upon the fact that self-dual vortices form a symplectic submanifold of the phase space (modulo gauge invariance). The theorem provides a rigorous description of slow vortex dynamics in the self-dual limit.

1 Introduction and statement of results

In this article we study vortex dynamics in the Chern-Simons-Schroedinger system (1.1) introduced by Manton (1997). This is a gauge theoretic generalization of the two dimensional nonlinear Schroedinger equation whose static soliton solutions are Ginzburg-Landau, (also known as abelian Higgs), vortices (Jaffe and Taubes 1982). The article is organized as follows. We start by writing down the equations, and giving necessary background including a discussion of the self-dual vortices in §1.4. We then state our main result, theorem 1.5.2, which describes the adiabatic approximation of vortex motion in the self-dual limit. This is proved in §2 following a strategy explained in the context of a simple model problem in §1.6. The proof uses some specialized identities related to the self-dual (or Bogomolny) structure, presented in §3 (which may be read separately). Various subsidiary facts and lemmas are given in the appendix.

1.1 The equations

The dependent variables are a complex field $\Phi(t, x)$, an electric field $E = E_j dx^j$ and a magnetic field $B(t, x)$, all defined for $(t, x) \in \mathbb{R} \times \Sigma$ where Σ is a two dimensional spatial domain, taken to be a Riemann surface with metric $g_{jk}dx^j dx^k$, area form $d\mu_g$ and complex structure $J : T^*\Sigma \to T^*\Sigma$ (where $j, k, \ldots$ take values in $\{1, 2\}$ and we use the summation convention). The equations are

$$\begin{aligned}
&E_j + \frac{\partial B}{\partial x^j} = -J_j^k \langle i\Phi, D_k\Phi\rangle \\
&i(\frac{\partial}{\partial t} - iA_0)\Phi = -\Delta_A\Phi - \frac{\lambda}{2}(1 - |\Phi|^2)\Phi \\
&B = \frac{1}{2}(1 - |\Phi|^2).
\end{aligned} \tag{1.1}$$

The electric and magnetic field can be combined to give the space-time electromagnetic field

$$F_{\mu\nu}dx^\mu \wedge dx^\nu = E_j dt \wedge dx^j + Bd\mu_g.$$

This two form is obtained as the commutator of the space-time covariant derivative

$$D = (D_0, D_1, D_2) = (\frac{\partial}{\partial t} - iA_0, D_1, D_2)$$

which mediates the coupling in (1.1):

$$[D_\mu, D_\nu]\Phi = -iF_{\mu\nu}\Phi, \quad \text{where } F_{0k} = E_k, \text{ and } \frac{1}{2}F_{jk}dx^j dx^k = Bd\mu_g. \tag{1.2}$$

(Greek indices run through $0, 1, 2$ and Latin indices through $1, 2$ only. Bold face is used to indicate the spatial part of a vector or one-form etc., except in §3 where time does not appear at all.)

We now describe this set-up briefly in geometrical terms. Assume given a one dimensional complex vector bundle $L \to \Sigma$, with a real inner product h locally of the form $\langle a, b\rangle = h\Re\bar{a}b$, and corresponding norm $|a|^2 = \langle a, a\rangle$; if we employ a unitary frame over some chart then $\langle a, b\rangle = \Re\bar{a}b$. We are then solving for an S^1 connection on the bundle $\mathbb{L} \equiv \mathbb{R} \times L \to \mathbb{R} \times \Sigma$, with associated covariant derivative D, and a section Φ of $\mathbb{L}$. To be more explicit, fix a smooth connection on L determined by a covariant derivative operator ∇, so that the spatial part of D, which will be written $\mathbf{D}$, takes the form $D_j = \nabla_j - iA_j$ for a real 1-form $\mathbf{A} = A_j dx^j \in \Omega^1_{\mathbb{R}}(\Sigma)$; here ∇ is independent of time. (It is generally not possible to choose ∇ to be flat, and it will have a curvature, determined by a function b such that $[\nabla_j, \nabla_k]\Phi dx^j dx^k = -ibd\mu_g\Phi$; it is always possible to choose $b = \text{const.}$, and we will do this throughout.) In any case, with this procedure the space of connections on L can be identified with the space of real one-forms. Then at each time $t \in \mathbb{R}$ we are solving for a section $\Phi(t)$ of L, a 1-form $\mathbf{A}(t) = A_1(t)dx^1 + A_2(t)dx^2$ on Σ, and a real valued function $A_0(t)$ on Σ. The electric field is given by

$$E_j = \frac{\partial A_j}{\partial t} - \frac{\partial A_0}{\partial x^j}$$

and the magnetic field by

$$B d\mu_g = b d\mu_g + \mathbf{d}A.$$

(Here, and elsewhere, we write $\mathbf{d}$ in bold face when it is necessary to indicate that only the spatial part is taken.) The 2-form $-iE_j dt \wedge dx^j - iBd\mu_g$ is the curvature associated to the space-time covariant derivative D, as in (1.2). For the case $\Sigma = \mathbb{R}^2$, the system was proposed by Manton (1997), who derived it as the Euler-Lagrange equation for the Lagrangian (1.8).

Notation 1.1.1 *We shall always consider conformal co-ordinate systems on Σ in which the metric is of the form $g = e^{2\rho}\big((dx^1)^2 + (dx^2)^2\big)$ and the volume element is then $e^{2\rho}dx^1 \wedge dx^2$. On functions the Hodge operator acts as $*f = f d\mu_g = f e^{2\rho} dx^1 \wedge dx^2$ and $*^2 = 1$, so that $*d\omega = e^{-2\rho}(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2})$ for 1-forms ω. On 1-forms $*(\omega_1 dx^1 + \omega_2 dx^2) = \omega_1 dx^2 - \omega_2 dx^1$, which is just the negative of the complex structure J, represented in conformal co-ordinates by the anti-symmetric tensor J_i^j with $J_2^1 = -1, J_1^2 = +1$, the other components being zero. Correspondingly we decompose a one-form as $\omega = \omega^{(1,0)}dz + \omega^{(0,1)}d\bar{z}$; in particular for the derivative $df = \partial f dz + \bar{\partial} f d\bar{z}$, with $\bar{\partial} f = \frac{1}{2}(\frac{\partial f}{\partial x^1} + i\frac{\partial f}{\partial x^2})$, and*

$$\mathbf{D}\Phi = D^{(1,0)}\Phi + D^{(0,1)}\Phi = \partial_{\mathbf{A}}\Phi dz + \bar{\partial}_{\mathbf{A}}\Phi d\bar{z},$$

with $\bar{\partial}_{\mathbf{A}}\Phi = \frac{1}{2}\Big((\nabla_1 - iA_1) + i(\nabla_2 - iA_2)\Big)\Phi$ etc.; see §3. For a 1-form $\mathbf{A}$ we write the co-differential $\mathbf{d}^*\mathbf{A} = -div\,\mathbf{A}$, with $div\,\mathbf{A} = e^{-2\rho}(\frac{\partial A_1}{\partial x^1} + \frac{\partial A_2}{\partial x^2})$, and the Laplacian on real functions is $\Delta f = e^{-2\rho}\frac{\partial^2 f}{\partial x^i \partial x^i}$, (with the summation convention), and on sections of L the covariant Laplacian is $-\Delta_{\mathbf{A}}\Phi = e^{-2\rho}(D_1^2 + D_2^2)\Phi$ when a unitary frame is used. The operators $div, *d, \Delta$ (resp. $\Delta_{\mathbf{A}}$) all depend on g (resp. g, h), but this is not indicated as g, h are fixed, and similarly dependence of constants in estimates on (Σ, g) and h will be suppressed throughout the article.

Notation 1.1.2 *We are dealing with sections of smooth vector bundles V over Σ with an inner product $\langle\cdot,\cdot\rangle$ induced from the Riemannian metric g and the metric h on L in the standard way; since g, h are fixed throughout they will not be indicated. Thus, for example,*

$$|\mathbf{D}\Phi|^2 = e^{-2\rho}\Big(\langle D_1\Phi, D_1\Phi\rangle + \langle D_2\Phi, D_2\Phi\rangle\Big).$$

We write $\Omega^0(V)$ for the smooth sections of V and $\Omega^p(V)$ for the smooth p-forms taking values in V. We will make use of the Sobolev spaces $H^s(V)$ of sections of V whose coefficient functions (in any frame over any open set $\Omega \subset \Sigma$) lie in the standard Sobolev space $H^s(\Omega)$; the corresponding Sobolev space of V-valued p-forms is denoted $H^s(\Omega^p(V))$. In §1 and §2 we shall generally omit explicit reference to the vector bundle, since this is usually clear, and write H^s in place of $H^s(V)$ etc. (and $\|\cdot\|_{H^s}$ for the corresponding norms). However if it is necessary to emphasize that time is fixed, and the norm is taken over Σ, we shall write $H^s(\Sigma)$.

Further notational conventions are given in the appendix and in §3, particularly in relation to the complex structure (see also the textbook [12, §9.1] for a treatment of the background material).

1.2 Existence theory for the Cauchy problem

Inherent to the system (1.1) is the property of *gauge invariance*: let $\chi(t,x)$ be a smooth real valued function, then (A,Φ) is a smooth solution if and only if $(d\chi + A, \Phi e^{i\chi})$ is. This introduces a large degeneracy to the solution space which may be removed by a choice of gauge in various ways. We will adopt here the following gauge condition which involves the time derivatives $\dot{\mathbf{A}}, \dot{\Phi}$, of $\mathbf{A}, \Phi$:

$$\operatorname{div}\dot{\mathbf{A}} - \langle i\Phi, \dot{\Phi}\rangle \equiv e^{-2\rho}(\partial_1\dot{A}_1 + \partial_2\dot{A}_2) - \langle i\Phi, \dot{\Phi}\rangle = 0. \tag{1.3}$$

We make this choice because it allows a convenient description of the complex and symplectic structures on the moduli space of vortices (see remark 1.4.3 and §3), and also is useful in the derivation of energy estimates for the time derivatives (see §2.2 and §2.3). In this gauge global existence can be stated as follows:

Theorem 1.2.1 (Global existence in gauge (1.3)**)** *Consider the Cauchy problem for* (1.1) *with initial data* $\Phi(0) \in H^2(\Sigma)$ *and* $\mathbf{A}(0) \in H^1(\Sigma)$. *There exists a global solution satisfying* (1.3) *and the estimate*

$$|\Phi(t)|_{H^2(\Sigma)} \leq ce^{\alpha e^{\beta t}} \tag{1.4}$$

for some positive constants c, α, β *depending only on* (Σ, g), *the equations, and the initial data. The solution has regularity* $\Phi \in C\big([0,\infty); H^2(\Sigma)\big) \cap C^1\big([0,\infty); L^2(\Sigma)\big)$ *and* $\mathbf{A} \in C^1\big([0,\infty); H^1(\Sigma)\big)$. *If the initial data are smooth, then the solution is also smooth.*

It is explained in appendix A.3 how to derive this theorem from the global existence result of [10], which is stated in another gauge. Bounds of the type (1.4) were derived in [8] for the cubic nonlinear Schroedinger equation on $\mathbb{R}^2$, by means of the inequality

$$|u|_{L^\infty} \leq C[1 + \sqrt{\ln(1 + \|u\|_{H^2})}], \tag{1.5}$$

valid for $u \in H^2(\mathbb{R}^2)$ and with $C = C(\|u\|_{H^1})$. The proof of global regularity for (1.1) depends on a covariant version of this inequality (given in lemma A.11), and a careful treatment of various commutator terms $[D_\mu, D_\nu]$ which indicates that they have a comparable strength to the cubic nonlinear term.

In conclusion, theorem 1.2.1 provides a global solution which is a continuous curve in the space $\mathcal{H}_2$ where for $s \in \mathbb{R}$ we define

$$\mathcal{H}_s \equiv \{(\mathbf{A}, \Phi) \in H^{s-1}(\Sigma) \times H^s(\Sigma)\}, \tag{1.6}$$

with the corresponding norm $\|\cdot\|_{\mathcal{H}_s}$. From now on we will consider only $(\mathbf{A}, \Phi)$ which lie (at a given time) in the space $\mathcal{H}_2$. The gauge group at fixed time is given by

$$\mathcal{G} \equiv \{g \in H^2(\Sigma; S^1)\} \tag{1.7}$$

and acts on $\mathcal{H}_2$ according to $g \cdot (\mathbf{A}, \Phi) = (\mathbf{A} + g^{-1}dg, \Phi g)$. (Restricting to the set where Φ is not identically zero the action is free and gives a principal $\mathcal{G}-$bundle structure. The gauge condition (1.3) can be then regarded as giving a connection - i.e. a family of horizontal subspaces - on this bundle.)

1.3 Variational and Hamiltonian formulation

The equations (1.1) can be derived formally as the Euler-Lagrange equations associated to the functional

$$S(A,\Phi) = \frac{1}{2}\int_{\mathbb{R}\times\Sigma} -A\wedge F + \big(\langle i\Phi, D_0\Phi\rangle + A_0 + 2v_\lambda(A,\Phi)\big)dtd\mu_g, \tag{1.8}$$

where

$$v_\lambda(\mathbf{A},\Phi) = \frac{1}{2}\Big(B^2 + |\mathbf{D}\Phi|^2 + \frac{\lambda}{4}(1-|\Phi|^2)^2\Big) \tag{1.9}$$

is the density of the Ginzburg-Landau static energy. (The parameter λ is a positive real numbers). Although S is not manifestly gauge invariant it changes by an exact form under gauge transformation, and the Euler-Lagrange equations (1.1) are gauge invariant. Vortices are critical points of the static energy

$$\mathcal{V}_\lambda(\mathbf{A},\Phi) = \int_\Sigma v_\lambda(\mathbf{A},\Phi)d\mu_g,$$

as will be discussed further in the next section.

To see that the system (1.1) is Hamiltonian, observe that there is a complex structure on the phase space $\mathcal{H}_2$ given by $\mathbb{J} : (\dot{\mathbf{A}},\dot{\Phi}) = (-J\dot{\mathbf{A}}, i\dot{\Phi})$ which allows the introduction of a symplectic structure $\Omega(v,w) = \langle \mathbb{J}v, w\rangle$ where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the L^2 inner product. Using this symplectic form the system (1.1), in temporal gauge $A_0 = 0$, is a Hamiltonian flow generated by the Hamiltonian functional $\mathcal{V}_\lambda(\mathbf{A},\Phi)$, which was just defined. (A short calculation reveals that the third equation of (1.1) is preserved by the evolution, and as such is really only a condition on the initial data. It will be referred to as the constraint equation.)

1.4 Self-dual vortices and dynamics in the limit $\lambda \to 1$.

The system (1.1) admits *soliton* solutions, called abelian Higgs, or Ginzburg-Landau, vortices, which are energy minimizing critical points of the static energy functional $\mathcal{V}_\lambda(\mathbf{A},\Phi)$. We now discuss these solutions and their uses in understanding the dynamical system (1.1) via the adiabatic approximation. There is a special case, $\lambda = 1$, in which the adiabatic approximation is particularly powerful because the space of vortices is then unusually large - large enough that the motion on it can provide information on the dynamical interaction of several vortices. We call this the self-dual, or Bogomolny, case, and the corresponding solutions are called self-dual vortices. Now for such a solution, $(\mathbf{A}, \Phi)$, with a given value of the topological integer N, (the degree of L), the field Φ will have N zeros, counted with multiplicity. Each of these zeros can be thought of as the centre of a vortex. Thus the static solitons can be thought of as a nonlinear superposition of N vortices which do not interact. This was first fully understood in the case that Σ is the upper half plane with canonical metric, when the equations were solved exactly by Witten (1977) by reducing them to the Liouville equation. In general it is still possible to make a reduction to a nonlinear elliptic equation of Kazdan-Warner type, whose solutions can be completely parametrized although not explicitly given. Following this, Taubes proved an existence theorem when Σ is the Euclidean plane (Jaffe and Taubes 1982), and Bradlow (1988) did likewise for Σ a compact Riemann surface, proving the following:

Theorem 1.4.1 (Existence of vortices on a surface,[6]) *If the area of a closed Riemann surface $|\Sigma|$ is such that $|\Sigma| > 4\pi N$ the Bogomolny bound is saturated: in fact the minimum value πN of $\mathcal{V}$, where*

$$\begin{aligned} &\mathcal{V} : \mathcal{H}_2 \to \mathbb{R} \\ &\mathcal{V}(\mathbf{A}, \Phi) \equiv \mathcal{V}_1(\mathbf{A}, \Phi) = \frac{1}{2}\int_\Sigma \left(B^2 + |\mathbf{D}\Phi|^2 + \frac{1}{4}(1 - |\Phi|^2)^2 \right) \, d\mu_g, \end{aligned} \tag{1.10}$$

is achieved on a set $\mathcal{S}_N \subset \mathcal{H}_2$ of pairs $(\mathbf{A}, \Phi)$ which solve the **Bogomolny***, or* **self-dual vortex***, equations:*

$$\bar{\partial}_{\mathbf{A}}\Phi = 0, \qquad B - \frac{1}{2}(1 - |\Phi|^2) = 0.$$

These minimizers will be referred to as the self-dual vortices, or just vortices. The quotient of $\mathcal{S}_N$ by the gauge group $\mathcal{G}$ can be identified with $Sym^N(\Sigma)$, the symmetric N-fold product of Σ, via the mapping which takes Φ to the set of its zeros.

Remark 1.4.2 (Interaction and stability of vortices) The physical interpretation of theorem 1.4.1 is that for $\lambda = 1$ the vortices do not interact; see [16] for a discussion of this, and some related conjectures, and [25] for some stability theorems.

Remark 1.4.3 (Bogomolny structure and Bogomolny operator) The structural feature of $\mathcal{V}$ which makes theorem 1.4.1 possible was identified by Bogomolny in [5]. In this instance it amounts to the fact that if we introduce the Bogomolny operator $\mathcal{B}$ to be the nonlinear operator which maps $(\mathbf{A}, \Phi) \mapsto \big(B - \frac{1}{2}(1 - |\Phi|^2), \bar{\partial}_{\mathbf{A}}\Phi\big)$ then

$$\mathcal{V} = \frac{1}{2}\int |\mathcal{B}(\mathbf{A}, \Phi)|^2 d\mu_g + \pi N$$

(see §3 for more information in this regard). Also see [6] for higher dimensional versions of this decomposition, and [11] for generalizations to solutions with non-vanishing electric field.

Remark 1.4.4 (Geometry of moduli space) Quotient spaces of the type arising in theorem 1.4.1 are usually known as moduli spaces: in this case we define the moduli space $\mathcal{M}_N$ to be the space of gauge equivalence classes of self-dual vortices, so that $\mathcal{M}_N \equiv \mathrm{Sym}^N(\Sigma)$. We call the space $\mathcal{S}_N$ the vortex space and proj $: \mathcal{S}_N \to \mathcal{M}_N$ the natural projection which takes $(\mathbf{A}, \Phi)$ to its gauge equivalence class $[(\mathbf{A}, \Phi)]$. The space $\mathcal{M}_N$ inherits both a metric (induced from the L^2 metric) and a symplectic structure and is a Kaehler manifold (see [7]). Explicitly, we can identify the tangent space to $\mathcal{M}_N$ with solutions $(\dot{\mathbf{A}}, \dot{\Phi})$ of the linearized Bogomolny equations which also satisfy the condition (1.3). The complex structure and symplectic structure on $\mathcal{M}_N$ are then given by restricting the formulas given in the previous section to such $(\dot{\mathbf{A}}, \dot{\Phi})$, and consequently we will use the same notation, $\mathbb{J}$ and Ω, for these objects. The existence of this complex structure on $\mathcal{M}_N$ can be seen very clearly in the formulas in §3, in which complex notation is used to combine the linearized Bogomolny equations with (1.3) into a manifestly complex linear operator $\mathcal{D}_\psi$, for $\psi = (\mathbf{A}, \Phi) \in \mathcal{S}_N$. This can all be summarized by saying that we have an identification

$$T_{[\psi]}\mathcal{M}_N \approx \mathrm{Ker}\,\mathcal{D}_\psi \equiv \{(\dot{\mathbf{A}}, \dot{\Phi}) : DB_\psi[\dot{\mathbf{A}}, \dot{\Phi}] = 0, \ \text{ and (1.3) holds}\}. \tag{1.11}$$

1.5 Statement of the adiabatic limit theorem

In order to define the adiabatic limit system, we now define a Hamiltonian function $\mathcal{M}_N \to \mathbb{R}$ by restricting the energy $\mathcal{V}_\lambda$ to the space of vortices, and observing that by gauge invariance this actually gives a smooth function on the quotient space $\mathcal{M}_N$. The corresponding Hamiltonian flow determines the slow motion of vortices for λ close to 1:

For $\epsilon = |\lambda - 1|$ sufficiently small, the system (1.1) *can be approximated, for times of order $\frac{1}{\epsilon}$, by the Hamiltonian flow on the phase space $\mathcal{M}_N = Sym^N(\Sigma)$ associated to the Hamiltonian function $\mathcal{V}_\lambda|_{\mathcal{M}_N}$ via the symplectic form Ω.*

We now move towards a precise formulation of this in theorem 1.5.2. Since we are interested in the regime in which $|\lambda - 1| \ll 1$ it is useful to introduce a large parameter

$$\mu = \frac{1}{|\lambda - 1|} \tag{1.12}$$

and let also, for $\lambda \neq 1$,

$$\sigma = \frac{\lambda - 1}{|\lambda - 1|} = \pm 1 \tag{1.13}$$

(also defining $\sigma = 0$ for $\lambda = 1$ where necessary). We rescale time by $\tau = \frac{t}{\mu}$, and A_0 similarly, leading to the following *rescaled equations*:

$$\begin{aligned}
&\frac{\partial A_1}{\partial \tau} = \mu\big(- \partial_1 B - \langle i\Phi, D_2\Phi\rangle\big) + \frac{\partial A_0}{\partial x^1},\\
&\frac{\partial A_2}{\partial \tau} = \mu\big(- \partial_2 B + \langle i\Phi, D_1\Phi\rangle\big) + \frac{\partial A_0}{\partial x^2},\\
&i(\frac{\partial}{\partial \tau} - iA_0)\Phi = \mu(-\Delta_A\Phi - \frac{1}{2}(1 - |\Phi|^2)\Phi) - \frac{\sigma}{2}(1 - |\Phi|^2)\Phi.
\end{aligned} \tag{1.14}$$

It is also natural to separate the energy $\mathcal{V}_\lambda$ into the (main) self-dual piece $\mathcal{V} = \mathcal{V}_1$, and a perturbation term proportional $\lambda - 1$. Under the rescaling just introduced, the energy rescales by a factor μ, leading us to consider the Hamiltonian $H = \mu\mathcal{V} + U$, where $\mathcal{V} \equiv \mathcal{V}_1$ is as in (1.10), and the energy correction away from the self-dual, or Bogomolny, regime is given by

$$U(\Phi) = \frac{\sigma}{8}\int_\Sigma (1 - |\Phi|^2)^2 \; d\mu_g. \tag{1.15}$$

The rescaled equations (1.14) can be written as a Hamiltonian evolution for $\psi = (\mathbf{A}, \Phi)$ in the form

$$\mathbb{J}\frac{\partial \psi}{\partial \tau} = \mu\mathcal{V}' + U' + \mathbb{J}(dA_0, iA_0\Phi) \tag{1.16}$$

where $\mathbb{J}$ is the complex structure introduced at the end of §1.3,

$$\mathbb{J}(\dot{A}_1 dx^1 + \dot{A}_2 dx^2, \dot{\Phi}) = (-\dot{A}_2 dx^1 + \dot{A}_1 dx^2, i\dot{\Phi}) \tag{1.17}$$

with $\dot{\mathbf{A}} = \frac{\partial \mathbf{A}}{\partial \tau}$ $\dot{\Phi} = \frac{\partial \Phi}{\partial \tau}$.

Remark 1.5.1 (Explicit formulation of adiabatic limit system) We now write the equations for the adiabatic limit system in an explicit way which will be useful later. The function U is clearly gauge invariant and defines by restriction a smooth function u on $\mathcal{M}_N$. Now recall (1.11): under this identification, the gradient of the function u on $\mathcal{M}_N$ at $[\Psi_S]$ is identified with $\mathbb{P}_{\Psi_S} U'$, where $\mathbb{P}_{\Psi_S}$ is the orthogonal projector onto $\operatorname{Ker} \mathcal{D}_{\Psi_S}$ (see lemma 3.3.2). The Hamiltonian differential equations for u are then equivalent to

$$\mathbb{J} \frac{\partial \Psi_S}{\partial \tau} = \mathbb{P}_{\Psi_S} U'. \tag{1.18}$$

Given an initial value $\Psi_S(0) = \psi_0 \in \mathcal{S}_N$, this equation has a unique solution $\tau \mapsto \Psi_S(\tau) \in \mathcal{S}_N$ which satisfies the gauge condition (1.3).

Main Theorem 1.5.2 (Adiabatic limit) *Let Ψ_μ be the smooth solution of* (1.16)*, satisfying the gauge condition* (1.3)*, with smooth initial data $\Psi_\mu(0)$, such that*

(i) $\lim_{\mu \to +\infty} \|\Psi_\mu(0) - \psi_0\|_{\mathcal{H}_2} = 0$*, for some smooth $\psi_0 \in \mathcal{S}_N$, and*

(ii) $\sup_{\mu \geq 1} \|\Psi_\mu(0)\|_{\mathcal{H}_2} + \|\dot{\Psi}_\mu(0)\|_{H_1} \leq K < \infty$.

Then there exists $\tau_* > 0$, independent of $\mu \geq 1$, such that for $s < 2$,

$$\lim_{\mu \to \infty} \sup_{[-\tau_*, \tau_*]} \left\| \Psi_\mu(\tau) - \Psi_S(\tau) \right\|_{\mathcal{H}_s} = 0 \tag{1.19}$$

where $\tau \mapsto \Psi_S(\tau) \in \mathcal{S}_N$ is a curve in the vortex space $\mathcal{S}_N$, also satisfying (1.3)*, which is the unique solution of* (1.18) *with initial data $\Psi_S(0) = \psi_0$. The projection onto the moduli space $\mathcal{M}_N$:*

$$\tau \mapsto \left[\Psi_S(\tau)\right] \in \mathcal{M}_N,$$

is the unique solution of the Hamiltonian system on $(Sym^N(\Sigma), \Omega)$ associated to the Hamiltonian u defined in remark 1.5.1, with initial value $[\psi_0] \in \mathcal{M}_N$.

This theorem in proved in §2, employing a strategy which is explained in §1.6, following discussion of a very simple model problem. Some of the novel features which arise in the implementation of this strategy for (1.14) are highlighted at the begininng of §2.

The approximation of the dynamical system (1.14) by a dynamical system through a space of equilibria (in this case the self-dual vortices, which are the equilibria for $\lambda = 1$)

is referred to as an adiabatic limit or approximation. It was suggested in [18], following earlier conjectures of the same author on vortex and monopole dynamics in second order Lorentz invariant systems discussed in [19]. Proofs of the validity of the approximation in the case of second order dynamics were given in [26, 27]; the strategy for the proof here, however, is different from that adopted in those references - see the discussion in §1.6. A review of adiabatic limit problems is given in [29], mostly directed towards infinite dimensional natural Lagrangian systems of the type appearing in classical field theory. (Natural Lagrangian systems are those derivable from Lagrangians of the classical "kinetic energy minus potential energy" form). For some physical consequences of the approximation for vortex dynamics in the system (1.1), see also [15, 18, 22].

1.6 A simple model problem and discussion of methodology

We consider here a simple two-dimensional example in order to exhibit as clearly as possible the phenomenon under study, and the strategy which will be employed in the proof of theorem 1.5.2. (It is the basic strategy taken in [23] for finite dimensional natural Lagrangian systems, here adapted to the case of infinite dimensions and to take advantage of the Bogomolny structure.) For real numbers β and $\mu \gg 1$, we consider a linear first order Hamiltonian system for $z(\tau) = (z^1(\tau), z^2(\tau)) \in \mathbb{C}^2$:

Theorem 1.6.1 *For each $\mu \gg 1$, let $\tau \mapsto Z_\mu(\tau) \in \mathbb{C}^2$ be the solution of*

$$\begin{aligned} \dot{z}^1 &= i(z^1 + \beta z^2) \\ \dot{z}^2 &= i(\beta z^1 + \mu z^2), \end{aligned} \tag{1.20}$$

with initial data satisfying $|(Z_\mu^1(0), Z_\mu^2(0)) - (\gamma, 0)| = O(\mu^{-1})$ as $\mu \to +\infty$, for some fixed $\gamma \in \mathbb{C}$. Then

$$\lim_{\mu \to +\infty} \max_{\tau \in \mathbb{R}} |Z_\mu(\tau) - (\gamma e^{i\tau}, 0)| = 0. \tag{1.21}$$

Remark 1.6.2 The system (1.20) is Hamiltonian with the standard symplectic structure on $\mathbb{C}^2$ and with Hamiltonian function $\mu\mathcal{V} + U$ with $\mathcal{V}(z) = \frac{1}{2}\bar{z}^2 z^2$ and

$$U(z) = \frac{1}{2}\bar{z}^1 z^1 + \beta(\bar{z}^1 z^2 + \bar{z}^2 z^1).$$

Thus $\mathcal{V}$ acts as a constraining potential for $\mu \to +\infty$, forcing the solution onto the set $\mathcal{S} = \mathbb{C} \times \{0\} \subset \mathbb{C}^2$ where $z^2 = 0$. Projecting the system to $\mathcal{S}$ gives, formally,

$$i\dot{z}^1 + z^1 = 0. \tag{1.22}$$

The theorem asserts that (1.22) indeed governs the behaviour of the limit of appropriate sequences of solutions to (1.20).

Proof The solution with initial data $z(0) = (z^1(0), z^2(0))$ is given by:

$$
\begin{aligned}
z^1(\tau) &= \frac{\beta}{\beta(\lambda_+ - \lambda_-)} \left[\left((1-\lambda_-)e^{i\lambda_+\tau} - (1-\lambda_+)e^{i\lambda_-\tau}\right) z^1(0) + \beta\left(e^{i\lambda_+\tau} - e^{i\lambda_-\tau}\right) z^2(0)\right] \\
z^2(\tau) &= \frac{-1}{\beta(\lambda_+ - \lambda_-)} \left[(1-\lambda_+)(1-\lambda_-)(e^{i\lambda_+\tau} - e^{i\lambda_-\tau}) z^1(0)\right] \\
&\qquad\qquad + \frac{-\beta}{\beta(\lambda_+ - \lambda_-)} \left[\left((1-\lambda_+)e^{i\lambda_+\tau} - (1-\lambda_-)e^{i\lambda_-\tau}\right) z^2(0)\right].
\end{aligned}
$$

Here the $\lambda_\pm$ are the characteristic values of the system:

$$
\lambda_\pm = \frac{1}{2}(1+\mu)\left[1 \pm \left(1 - \frac{4(\mu-\beta^2)}{(1+\mu)^2}\right)^{\frac{1}{2}}\right],
$$

which satisfy, by the binomial expansion,

$$
|\lambda_+ - \mu| = O(1), \quad |\lambda_- - 1| = O(\mu^{-1}).
$$

as $\mu \to \infty$. From this, and the fact that $\lambda_\pm \in \mathbb{R}$ for large μ so that $|e^{i\lambda_\pm\tau}| = 1$, the behaviour in (1.21) follows for the solutions $Z_\mu(\tau)$ with initial data as described. □

Remark 1.6.3 In this example the exact solutions indicate that while $Z^2_\mu \to 0$, the time derivatives $\dot{Z}^2_\mu$ are bounded, but cannot generally be expected to have limit zero.

In the absence of explicit formulae for $Z_\mu(\tau)$, it is still possible to prove results like theorem 1.6.1, either

(i) by explicit perturbative construction of solutions to the full system, using solutions of the restricted system as a starting point, or

(ii) by obtaining uniform bounds for the $Z_\mu(\tau)$ which allow the extraction of convergent subsequences, and then identifying the unique limit of all such subsequences as the corresponding solution of the restricted system with Hamiltonian $U\big|_{\mathcal{S}}$.

In the present article we will adopt the second strategy in our proof of theorem 1.5.2 (although it would be possible to use the first strategy, as in [26]). To make the structure of the proof transparent, it is useful to consider in some detail how to execute the second strategy to prove a variant of theorem 1.6.1:

Theorem 1.6.4 (Weaker version of theorem 1.6.1) *In the situation of 1.6.1*

$$\lim_{\mu\to+\infty} \max_{a<\tau<b} |Z_\mu(\tau) - (\gamma e^{i\tau}, 0)| = 0, \tag{1.23}$$

for every bounded interval $[a,b] \subset \mathbb{R}$.

Remark 1.6.5 Although weaker than theorem 1.6.1, the proof of theorem 1.6.4 that we give generalizes to the infinite dimensional problem (1.1), (1.14), in which the explicit solutions corresponding to those used in the proof of theorem 1.6.1 are of course not available.

Proof

- Differentiation of the equations (1.20) in time gives the identical system $\zeta = \dot{z}$. Use the energy identity:

  $$\mu\mathcal{V}(\zeta(\tau)) + U(\zeta(\tau)) = \mu\mathcal{V}(\zeta(0)) + U(\zeta(0)),$$

  together with identical estimate for $z(\tau)$, to deduce (using Cauchy-Schwarz) that the solutions Z_μ of theorem 1.6.1 satisfy $|Z_\mu(\tau)|+|\dot{Z}_\mu(\tau)| \leq C$, with C independent of $\mu \gg 1$.

- By the previous item, deduce that the family of functions $\tau \mapsto Z_\mu(\tau)$ is uniformly (in $\mu \gg 1$) bounded and equicontinuous, and so the Arzela-Ascoli theorem implies *subsequential* convergence $Z_{\mu_j} \to Z$ in $C(I)$ for any bounded interval $I \subset \mathbb{R}$.

- The energy estimate implies that, for large μ there exists $C > 0$, independent of μ, such that $\mu\bar{Z}^2 Z^2 \leq C$. It follows that $Z^2_\mu \to 0$ along any convergent subsequence. Now consider the integrated form of the first equation of (1.20) (i.e. project the system onto $\mathcal{S} = \mathbb{C} \times \{0\} \subset \mathbb{C}^2$ where $z^2 = 0$). Taking the limit $\mu_j \to \infty$, it follows that the limit $Z = (Z^1, Z^2)$ of any convergent subsequence satisfies $Z^1(\tau) = i\int_0^\tau Z^1(\tau')d\tau'$ and $Z^1(0) = \gamma$. This integral equation has unique solution $Z^1(\tau) = \gamma e^{i\tau}$, and hence the C_{loc} limit of any convergent subsequence is $(\gamma e^{i\tau}, 0)$. It follows that Z_μ converges to this limit in C_{loc} without restriction to subsequences. This proves theorem 1.6.4. (In view of remark 1.6.3 we should not expect this convergence to be in C^1_{loc}.) $\square$

The general situation to which theorem (1.6.4), and its proof, potentially generalize is the following: on a phase space $\mathcal{H}$ we consider the integral curves $Z_\mu(\tau)$ for a Hamiltonian $\mu\mathcal{V}+U$ for large μ ("the full system"). Under the assumption that $\mathcal{S} = \{z \in \mathcal{H} : \min \mathcal{V} = \mathcal{V}(z)\}$ is a symplectic submanifold of $\mathcal{H}$, we can consider the "restricted system" on $\mathcal{S}$ determined by the Hamiltonian $U\big|_{\mathcal{S}}$, and try to prove that this Hamiltonian system can be used to describe the limiting behaviour of $Z_\mu(\tau)$ as $\mu \to +\infty$. An infinite dimensional example of this situation is provided by the Chern-Simons-Schroedinger system (1.14): in the next section we will provide a proof of theorem 1.5.2 employing the same strategy to that used in the proof of theorem 1.6.4 just given.

2 Uniform bounds and proof of the main theorem

In this section we prove our main result, theorem 1.5.2, along the lines suggested by the discussion of the simple model problem in the last section. The crucial stage is the proof of the main estimate, theorem 2.3.1, which asserts the existence of a time interval, *independent of* μ, on which the solution $\psi = (\mathbf{A}, \Phi)$ is uniformly bounded in $\mathcal{H}_2$, and its time derivative is uniformly bounded in H^1 as $\mu \to +\infty$. Given this bound, theorem 1.5.2 can be deduced using a variant of the Lions-Aubin lemma, and a careful analysis of the $\mu \to +\infty$ limit of (1.14). Before obtaining the uniform bound, we collect some identities used in the proof. Some more specialized identities related to the self-dual structure are collected separately in §3, and referred to as needed. Specifically, we draw the reader's attention to the following two uses made of these more specialized identities:

(i) Differentiation in time gives rise to an equation (2.27) for $\zeta = \dot{\psi}$ in which the dominant term (as $\mu \to +\infty$) involves $\overline{L}_\psi$, the Hessian of $\mathcal{V}$ defined in (2.36). It is shown in §3 that this operator takes the special form

$$\overline{L}_\psi = \mathcal{D}_\psi^* \mathcal{D}_\psi + O(|\mathcal{B}|), \tag{2.24}$$

with $\mathcal{D}_\psi$ complex linear (see (3.58)), and $\mathcal{B}$ as in remark 1.4.3. Observing that the L^2 norm is exactly preserved for equations of the form $\mathbb{J}\dot{\zeta} = \mathcal{D}_\psi^* \mathcal{D}_\psi \zeta$, it is easy to believe that the stated structure of $\overline{L}_\psi$ is useful in the derivation of μ-independent bounds for (2.27), (for initial data as in the theorem); this indeed turns out to be the case - see the proof of theorem 2.3.1.

(ii) After obtaining a convergent subsequence of solutions of (1.16) it is necessary to take the limit of the equation itself along the subsequence $\mu = \mu_j \to +\infty$. For this purpose it is very convenient to be able to eradicate the term $\mu\mathcal{V}'$ on the right hand side, since this is clearly hard to control for large μ: this can be done by applying a projection operator $\mathbb{P}_\mu$ whose existence close to the set of self-dual vortices is assured by the Bogomolny structure: see lemmas 3.3.1 and 3.3.2. (In geometrical terms there is a foliation of the phase space $\mathcal{H}_2$, and the range of $\mathbb{P}_\mu$ is the tangent space to the leaves of this foliation, after dividing out by the action of the gauge group using (1.3).)

Although our final conclusions are in terms of the standard Sobolev norms based on the fixed connection ∇, it will be convenient to obtain bounds for the corresponding Sobolev norms defined at each fixed time with respect to the connection $\mathbf{D} = \nabla - i\mathbf{A}$, see (A.2). These can be related to the standard norms by (A.3)-(A.5).

2.1 The evolution equations and associated identities

In addition to the rescaled equation (1.16) for $\psi = (\mathbf{A}, \Phi)$:

$$\mathbb{J}\frac{\partial\psi}{\partial\tau} = \mu\mathcal{V}' + U' + \mathbb{J}(dA_0, iA_0\Phi),$$

we will use the differentiated equation for $\zeta = \dot\psi \equiv \frac{\partial\psi}{\partial\tau}$. To write this down we need the linearization of the operator $\mathcal{V}'(\psi)$, i.e. the second order linear differential operator L_ψ obtained by differentiation of the map $\psi \mapsto \mathcal{V}'(\psi)$:

$$L_\psi = D\mathcal{V}'(\psi),$$

or equivalently, $\langle\zeta, L_\psi\zeta\rangle_{L^2} = \frac{d^2}{ds^2}\mathcal{V}(\psi + s\zeta)|_{s=0}$. Explicitly, with $\zeta = (\dot{\mathbf{A}}, \dot\Phi)$, we have

$$\langle\zeta, L_\psi\zeta\rangle_{L^2} = \int\Big(|d\dot{\mathbf{A}}|^2 + |D\dot\Phi|^2 + |\Phi|^2|\dot{\mathbf{A}}|^2 - 2\langle\mathbf{D}\Phi, i\dot{\mathbf{A}}\dot\Phi\rangle - 2\langle\mathbf{D}\dot\Phi, i\dot{\mathbf{A}}\Phi\rangle + \langle\Phi,\dot\Phi\rangle^2 - \frac{1}{2}(1-|\Phi|^2)|\dot\Phi|^2\Big)d\mu_g. \tag{2.25}$$

Remark 2.1.1 There is a slightly simpler version of this formula, given in (2.36) below, when ζ is restricted by the gauge condition (1.3). Furthermore in §3 it is shown that the self-dual structure provides a useful way of rewriting this formula as in (2.24), in terms of the complex structure defined in (1.17), and using the complex one-form $\dot\alpha dz$, where $\dot\alpha = \frac{\dot A_1 - i\dot A_2}{2}$, in place of the real one-form $\dot A_1 dx^1 + \dot A_2 dx^2$, see (3.56). Since this is used only at one point in the proof - in lemma 2.3.8 - this formulation is presented separately in §3, and referred to only as needed.

The linearization of U' is the linear operator $K_\psi = DU'(\psi)$, given by

$$K_\psi = (\dot{\mathbf{A}}, \dot\Phi) \mapsto \left(0, \frac{\sigma}{2}(\tau - |\Phi|^2)\dot\Phi + \sigma\langle\Phi, \dot\Phi\rangle\Phi\right), \tag{2.26}$$

with σ defined in (1.13). Given these definitions, the chain rule implies that, if ψ is a smooth solution of (1.16), then $\zeta(\tau) = \dot\psi(\tau)$ solves

$$\mathbb{J}\frac{\partial\zeta}{\partial\tau} = \mu L_\psi\zeta + K_\psi\zeta + \mathbb{J}\frac{\partial}{\partial\tau}(dA_0, iA_0\Phi). \tag{2.27}$$

We also need identities for the evolution of the Bogomolny operator $\mathcal{B}$ defined in remark 1.4.3 and discussed in more detail in §3. The first component is preserved

$$\frac{\partial}{\partial\tau}\left((B - \frac{1}{2}(1 - |\Phi|^2)\right) = e^{-2\rho}(\partial_1\dot A_2 - \partial_2\dot A_1) + \langle\Phi, \dot\Phi\rangle = 0, \tag{2.28}$$

as a consequence of (1.14). We will require that the initial data are such that $B - \frac{1}{2}(1 - |\Phi|^2) = 0$ initially, and hence for all times. The second component of the Bogomolny operator $\mathcal{B}$ will be denoted

$$\eta = \bar\partial_{\mathbf{A}}\Phi = \frac{1}{2}(D_1 + iD_2)\Phi, \tag{2.29}$$

(see §3), and we have the following identity:

$$i(\partial_\tau - iA_0)\eta = \mu(-4\bar\partial_{\mathbf{A}}(e^{-2\rho}\partial_{\mathbf{A}}\eta) + |\Phi|^2\eta) - \frac{\sigma}{2}\bar\partial_{\mathbf{A}}\left((1 - |\Phi|^2)\Phi\right). \tag{2.30}$$

(To verify this identity: substitute $\Delta_{\mathbf{A}}\Phi = 4e^{-2\rho}\partial_{\mathbf{A}}\bar\partial_{\mathbf{A}}\Phi - B\Phi$ into the third line of (1.14) and then apply $\bar\partial_{\mathbf{A}}$ to the resulting equation and use the identity $(E_1 + iE_2)\Phi = -2\mu|\Phi|^2\bar\partial_{\mathbf{A}}\Phi$ which follows from the first two lines of (1.14).)

Of course, the energy

$$\mathcal{E}(\tau) = \mu\mathcal{V}(\psi(\tau)) + U(\psi(\tau)) = \mathcal{E}_0 > 0 \tag{2.31}$$

is independent of time τ for regular solutions, as is the L^2 norm

$$\|\Phi(\tau)\|_{L^2} = L > 0. \tag{2.32}$$

2.2 Choice of gauge condition and related estimates

The divergence of E can be calculated to be:

$$\begin{aligned}\operatorname{div} E &= e^{-2\rho}(\partial_1 E_1 + \partial_2 E_2)\\ &= \mu\Big((-\Delta B - e^{-2\rho}\partial_1\langle i\Phi, D_2\Phi\rangle + e^{-2\rho}\partial_2\langle i\Phi, D_1\Phi\rangle\Big)\\ &= \mu(4e^{-2\rho}|\eta|^2) + \langle i\Phi, (\frac{\partial}{\partial t} - iA_0)\Phi\rangle - \sigma B|\Phi|^2.\end{aligned}$$

In the last line we have used $B = \frac{1}{2}(1-|\Phi|^2)$, so that $\Delta B = -\langle\Phi, \Delta_{\mathbf{A}}\Phi\rangle - e^{-2\rho}(|D_1\Phi|^2 + |D_2\Phi|^2)$, the equation for Φ and the definition of η in (2.29). Under the gauge condition (1.3) we get the following equation for A_0:

$$(-\Delta + |\Phi|^2)A_0 = 4\mu e^{-2\rho}|\eta|^2 - \sigma B|\Phi|^2. \tag{2.33}$$

Lemma 2.2.1 (Estimates for A_0) *Assume $\tau \mapsto \psi(\tau) = (\mathbf{A}(\tau), \Phi(\tau))$, is a smooth solution, of* (1.16) *which satisfies the gauge condition* (1.3)*,* (2.31) *and* (2.32)*. Then for all $r < \infty$, there exists $c_0(\mathcal{E}_0, L, r) > 0$ such that,*

$$\|A_0(\tau)\|_{L^r} \leq \ c_0(\mathcal{E}_0, L, r) \tag{2.34}$$

and there exists $c_0(\mathcal{E}_0, L) > 0$ such that

$$\|A_0(\tau)\|_{H^2} \leq \ c_0(\mathcal{E}_0, L)(1 + \mu\|\bar\partial_{\mathbf{A}}\Phi(\tau)\|_{L^\infty}). \tag{2.35}$$

Remark 2.2.2 This shows that in the original system (before rescaling) the time component of the potential A_0 is $O(|\lambda - 1|)$ in the gauge defined by (1.3).

Proof The crucial point here is the μ independence of the bounds. The second inequality follows from standard elliptic theory once the first is established. By (2.33) it is possible to write $A_0 = A_0^+ + \hat{A}_0$ where $(-\Delta + |\Phi|^2)A_0^+ = 4\mu e^{-2\rho}|\eta|^2$, so that $A_0^+ \geq 0$ by the maximum principle, and $(-\Delta + |\Phi|^2)\hat{A}_0 = -\sigma B|\Phi|^2$. The bounds stated in the lemma will follow by the triangle inequality once they are proved for A_0^+, since they are immediate for $\hat{A}_0$. Now integrating the equation for A_0^+ implies that $\||\Phi|^2 A_0^+\|_{L^1} = \int_\Sigma |\Phi|^2 A_0^+ d\mu_g \leq C(\mathcal{E}_0, L)$ since $A_0^+ \geq 0$; this bound is independent of $\mu \gg 1$ on account of (2.31). The standard elliptic theory for $-\Delta u = f \in L^1$ now gives the L^r estimates for A_0^+ and hence the lemma. $\square$

Lemma 2.2.3 (Estimates for $\dot{\mathbf{A}}$) *Let $\zeta = (\dot{\mathbf{A}}, \dot{\Phi})$ satisfy the gauge condition* (1.3)*, as well as the linearized constraint equation* (2.28)*. Then there exists a constant $c_1 > 0$ such that $\|\dot{\mathbf{A}}\|_{H^1} \leq c_1 \|\Phi\dot{\Phi}\|_{L^2}$, and more generally, for any $1 < p < \infty$, there exists a constant $c_1(p) > 0$ such that $\|\dot{\mathbf{A}}\|_{W^{1,p}} \leq c_1 \|\Phi\dot{\Phi}\|_{L^p}$. In particular these estimates hold for a smooth solution, $\tau \mapsto \psi(\tau) = (\mathbf{A}(\tau), \Phi(\tau))$, of* (1.16) *which satisfies the gauge condition* (1.3).

Proof These are the standard estimates for the Hodge system, proved by using the Hodge decomposition to reduce to the Calderon-Zygmund estimate for the Laplacian. □

On the subspace of $\zeta = (\dot{\mathbf{A}}, \dot{\Phi})$ satisfying the gauge condition (1.3), the operator L_ψ has a simpler form: $L_\psi \zeta = \overline{L}_\psi \zeta$, where $\overline{L}_\psi$ is the operator defined by

$$\langle \zeta, \overline{L}_\psi \zeta \rangle_{L^2} = \int \Big(|d\dot{\mathbf{A}}|^2 + |\mathrm{div}\, \dot{\mathbf{A}}|^2 + |\mathbf{D}\dot{\Phi}|^2 + |\Phi|^2 (|\dot{\mathbf{A}}|^2 + |\dot{\Phi}|^2) - 4\langle \mathbf{D}\Phi, i\dot{\mathbf{A}}\dot{\Phi}\rangle - \frac{1}{2}(1 - |\Phi|^2)|\dot{\Phi}|^2 \Big) d\mu_g. \tag{2.36}$$

Lemma 2.2.4 (The Hessian) *Let $\psi = (\mathbf{A}, \Phi)$ be smooth. Then the second order differential operator $\overline{L}_\psi$ is a self-adjoint operator with domain H^2, and there exist numbers c_2, c_3 such that*

$$\langle \zeta, \overline{L}_\psi \zeta \rangle_{L^2} \geq c_2 \|\zeta\|^2_{H^1_A} - c_3 \|\zeta\|^2_{L^2}.$$

The numbers c_2, c_3 depend only on the numbers L and $\mathcal{E}_0$, defined as in (2.31),(2.32).

Proof First of all, observe that

$$\int \Big(|d\dot{\mathbf{A}}|^2 + |\mathrm{div}\, \dot{\mathbf{A}}|^2 + |\mathbf{D}\dot{\Phi}|^2 + |\Phi|^2 (|\dot{\mathbf{A}}|^2 + |\dot{\Phi}|^2) \Big) d\mu_g \geq c(\mathcal{E}_0, L) \|(\dot{\mathbf{A}}, \dot{\Phi})\|^2_{H^1_A}.$$

This can be proved by a straightforward contradiction argument that is very similar to the proof of lemma 3.2.2 given below, so the details will be omitted. Next, to deduce the stated result, just bound the final two terms in (2.36) using the Holder inequality with $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$, the interpolation inequality in lemma A.9 and Cauchy-Schwarz. □

Corollary 2.2.5 *Assume given a smooth solution, $\tau \mapsto \psi(\tau) = (\mathbf{A}(\tau), \Phi(\tau))$, of* (1.16) *which satisfies the gauge condition* (1.3), (2.31) *and* (2.32)*. Then the quantity*

$$\mathcal{E}_1(\tau) = \frac{1}{2} \langle \zeta(\tau), (L_\psi + \mu^{-1} K_\psi) \zeta(\tau) \rangle_{L^2}, \tag{2.37}$$

where $\psi = \psi(\tau)$, satisfies for $\mu \geq 1$

$$\mathcal{E}_1 \;\geq\; c_4\|\zeta\|^2_{H^1_A} - c_5\|\zeta\|^2_{L^2}$$

with c_4, c_5 depending only on $\mathcal{E}_0, L$.

2.3 The main estimate

We say that a smooth solution, $\tau \mapsto \psi(\tau) = (A(\tau), \Phi(\tau))$, of (1.16) satisfies conditions (AE) and (AI), if the following conditions hold:

(AE) There exists positive numbers $\mathcal{E}_0, L$ such that $\|\Phi(\tau)\|_{L^2} = L$ and $\mathcal{E}(\tau) = \mathcal{E}_0$, for all times $\tau \in \mathbb{R}$, where $\mathcal{E}(\tau)$ is the energy (2.31). (Recall that both these quantities are independent of τ.)

(AI) The initial data are such that $\|\psi(0)\|_{\mathcal{H}_2} + \|\dot\psi(0)\|_{H^1} \leq K < \infty$.

Theorem 2.3.1 *For $\mu \geq 1$ let $\tau \mapsto \psi(\tau)$ be a smooth solution of* (1.16) *satisfying conditions (AE) and (AI), for some fixed numbers $K, L, \mathcal{E}_0$. There exist numbers $\tau_* > 0$ and $M_* > 0$, independent of μ, such that*

$$\max_{|\tau|\leq\tau_*}\Big|\big(\psi(\tau), \frac{\partial}{\partial\tau}\psi(\tau)\big)\Big|_{\mathcal{H}_2\times H^1} \;\leq\; M_*. \tag{2.38}$$

Beginning of proof of theorem 2.3.1. By time reversal invariance it is sufficient to prove the bound for $0 \leq \tau \leq \tau_*$, for some $\tau_* > 0$ independent of μ. Let

$$\zeta(\tau) = \frac{\partial}{\partial\tau}\psi(\tau) = \dot\psi(\tau).$$

For any $M > \|\zeta(0)\|_{L^2}$ there exists a time $T(M, \mu) > 0$ such that

$$\sup_{0\leq\tau\leq T(M,\mu)} \|\zeta(\tau)\|_{L^2} \leq\; M. \tag{2.39}$$

We will prove that there exist positive numbers M_*, τ_*, independent of μ, such that $T(M_*, \mu) \geq \tau_*$, and hence $\sup_{0\leq\tau\leq\tau_*}\|\zeta(\tau)\|_{L^2} \leq\; M_*$. The proof proceeds by obtaining a series of μ-independent bounds, predicated upon (2.39), which imply boundedness of $\big(\psi(\tau), \dot\psi(\tau)\big)$ in the Hilbert space $\mathcal{H}_2$ defined in (1.6) for $0 \leq \tau \leq \tau_*$. These bounds are now stated in a sequence of lemmas, all of which refer to a smooth solution of (1.16),(1.3) which verifies (AE), (AI) and (2.39) for all τ under consideration.

Lemma 2.3.2 (Estimate for Φ in H^2) *There exists $C_1 = C_1(\mathcal{E}_0, L) > 0$, independent of μ, such that*

$$\|\Phi(\tau)\|_{H^2_A} \le C_1(1 + \|\zeta(\tau)\|_{L^2}) \le C_1(1+M).$$

Proof Using the third equation of (1.14) for Φ, we bound

$$\|\Delta_{\mathbf{A}}\Phi\|_{L^2} \le \|\dot{\Phi}\|_{L^2} + \|A_0\Phi\|_{L^2} + \frac{1}{2}\|\Phi(1-|\Phi|^2)\|_{L^2}.$$

Now, by lemma A.2.2, we can bound $\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^2} \le \|\Delta_{\mathbf{A}}\Phi\|_{L^2} + c(\mathcal{E}_0)\|\nabla_{\mathbf{A}}\Phi\|_{L^4}$, and hence, by lemma A.9 and Cauchy-Schwarz: $\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^2} \le 2\|\Delta_{\mathbf{A}}\Phi\|_{L^2} + c(\mathcal{E}_0, L)$. Therefore, using also lemma 2.2.1, we deduce the bound $\|\Phi(t)\|_{H^2} \le c(1 + \|\zeta(\tau)\|_{L^2}) \le c(1+M)$, for some $c = c(\mathcal{E}_0, L) > 0$, and the result follows. □

Corollary 2.3.3 $\exists C_2 = C_2(\mathcal{E}_0, L) > 0$ *such that,* $\|\Phi(\tau)\|_{L^\infty} \le C_2\big(1 + \sqrt{\ln(1+M)}\big)$.

Proof This follows from lemma A.11 and the previous lemma. □

Lemma 2.3.4 (Energy estimate for $\zeta = \dot{\psi}$) *There is a constant $C_3(\mathcal{E}_0, L) > 0$ such that,*

$$\left|\frac{d\mathcal{E}_1}{d\tau}\right| \le C_3(1 + \|\Phi\|^2_{L^\infty})\|\zeta\|^2_{H^1_A} + C_3\|\zeta\|^6_{L^2} + C_3\|\zeta\|^4_{L^2}. \tag{2.40}$$

where $\mathcal{E}_1$ is the quantity defined in (2.37).

Proof Compute $\frac{d}{dt}\mathcal{E}_1$, substitute from (2.27), and use the observation that

$$\langle \mathbb{J}\dot{\zeta}, (d\dot{A}_0, i\Phi\dot{A}_0)\rangle_{L^2} = 0, \tag{2.41}$$

by the constraint equation $B = \frac{1}{2}(1-|\Phi|^2)$ in (1.1), to obtain

$$\frac{d\mathcal{E}_1}{d\tau} = \langle i\dot{\Phi}, iA_0\dot{\Phi}\rangle_{L^2} + \frac{1}{2}\langle \zeta, [\frac{\partial}{\partial\tau}, L_\psi + \mu^{-1}K_\psi]\zeta\rangle_{L^2}.$$

To handle the second term, we make use of the following bounds (written schematically, i.e. suppressing indices and inner products which play no role):

$$\|\Phi\zeta^3\|_{L^1} \le \|\Phi\|_{L^\infty}\|\zeta\|_{L^2}\|\zeta\|^2_{L^4} \le c\|\Phi\|_{L^\infty}\|\zeta\|^2_{L^2}\|\zeta\|_{H^1_A}$$
$$\|\dot{\Phi}\dot{\mathbf{A}}\nabla_{\mathbf{A}}\dot{\Phi}\|_{L^1} \le \|\nabla_{\mathbf{A}}\dot{\Phi}\|_{L^2}\|\dot{\mathbf{A}}\|_{L^4}\|\dot{\Phi}\|_{L^4} \le c\|\Phi\|_{L^\infty}\|\zeta\|^{3/2}_{H^1_A}\|\zeta\|^{3/2}_{L^2}$$
$$\|\dot{\Phi}^2\nabla\dot{\mathbf{A}}\|_{L^1} \le \|\zeta\|^2_{L^4}\|\nabla\dot{\mathbf{A}}\|_{L^2} \le c\|\Phi\|_{L^\infty}\|\zeta\|^2_{L^2}\|\zeta\|_{H^1_A}.$$

All of these bounds follow directly from Holder's inequality, the interpolation inequality in lemma A.2.1, lemma 2.2.3 and the bound

$$\|\dot{\mathbf{A}}\|_{L^4} + \|\dot{\mathbf{A}}\|_{H^1} \leq c\|\Phi\|_{L^\infty}\|\dot{\Phi}\|_{L^2}.$$

It then follows, by inspection of the formulae for L_ψ, K_ψ in (2.25) and (2.26), that the second term in $\frac{d\mathcal{E}_1}{d\tau}$ can be bounded by a sum of terms of this type, and hence:

$$\Big|\big\langle\zeta,\ [\frac{\partial}{\partial\tau}\ ,\ L_\psi + \mu^{-1}K_\psi]\zeta\big\rangle_{L^2}\Big| \leq c(1 + \|\Phi\|_{L^\infty}^2)\|\zeta\|_{H_A^1}^2 + c\|\zeta\|_{L^2}^6 + c\|\zeta\|_{L^2}^4.$$

Also, we can bound

$$|\langle i\dot{\Phi}, iA_0\dot{\Phi}\rangle_{L^2}| \leq c\|A_0\|_{L^r}\|\dot{\Phi}\|_{L^{2r'}}^2 \leq c\|A_0\|_{L^r}\|\dot{\Phi}\|_{H_A^1}^2$$

where $r > 1$ and $1/r + 1/r' = 1$. Combining these with lemma 2.2.1, we obtain (2.40), completing the proof of the lemma. $\square$

Corollary 2.3.5 *There is a constant* $C_4 = C_4(\mathcal{E}_0, K, L, M) > 0$ *such that,* $\|\zeta(\tau)\|_{H_A^1} \leq C_4(1+\tau)$*, for all times* $\tau \in [0, T(M,\mu)]$.

Lemma 2.3.6 (Estimate for $\eta = \bar{\partial}_{\mathbf{A}}\Phi$) *There exists* $C_5 = C_5(\mathcal{E}_0) > 0$ *such that, at each time* τ,

$$\mu\|\eta\|_{H_A^2} \leq C\big(\|\dot{\Phi}\|_{H_A^1} + \|\dot{\mathbf{A}}\|_{L^2}^2 + \|\Phi\|_{L^\infty}^2\big). \tag{2.42}$$

Proof From the equation (2.30) for η, and using the interpolation inequality in lemma A.9, the elliptic term

$$\mathcal{L}_{(\mathbf{A},\Phi)}\eta \equiv (-4\bar{\partial}_{\mathbf{A}}(e^{-2\rho}\partial_{\mathbf{A}}\eta) + |\Phi|^2\eta)$$

satisfies, for some $c = c(\mathcal{E}_0) > 0$,

$$\mu\|\mathcal{L}_{(\mathbf{A},\Phi)}\eta\|_{L^2} \leq \|\dot{\Phi}\|_{H_A^1} + \|\Phi\|_{L^\infty}\|\dot{\mathbf{A}}\|_{L^2} + c\|A_0\|_{L^4}(1 + \|\eta\|_{H^1}^{1/2}) + c\|\Phi\|_{L^\infty}^2. \tag{2.43}$$

We next see that (2.42) follows from the usual elliptic regularity estimate. Firstly, observe that associated to the operator $\mathcal{L}_{(\mathbf{A},\Phi)}$ is the quadratic form

$$Q_{(\mathbf{A},\Phi)}(\eta) \ = \ \langle\eta, \mathcal{L}_{(\mathbf{A},\Phi)}\eta\rangle_{L^2(\Sigma)} = \ \int_\Sigma \big(4|\partial_{\mathbf{A}}\eta|^2 e^{-4\rho} + |\Phi|^2|\eta|^2 e^{-2\rho}\big)\ d\mu_g,$$

which is bounded below by $c\|\eta\|^2_{H^1_A}$ where $c = c(\mathcal{E}_0, L) > 0$ by lemma 3.2.2. It follows that $\|\eta\|_{H^1_A} \leq c\|\mathcal{L}_{(\mathbf{A},\Phi)}\eta\|_{L^2}$, a result which can be strengthened by the following

Claim: $\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\eta\|_{L^2} \leq c\|\mathcal{L}_{(\mathbf{A},\Phi)}\eta\|_{L^2}$ where $c = c(\mathcal{E}_0, L) > 0$.

By the Garding inequality

$$\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\eta\|_{L^2} \leq \|\mathcal{L}_{(\mathbf{A},\Phi)}\eta\|_{L^2} + c(\mathcal{E}_0, L)(\|\nabla_{\mathbf{A}}\eta\|_{L^4} + \|\eta\|_{H^1_A}).$$

Finally, using the interpolation inequality (A.9) and the Cauchy-Schwarz inequality, we deduce the inequality claimed.

□

Corollary 2.3.7 *There is a constant* $C_6 = C_6(\mathcal{E}_0, K, L, M) > 0$ *such that,* $\mu\|\bar{\partial}_{\mathbf{A}}\Phi(\tau)\|_{L^\infty} \leq C_6(1+\tau)$.

Lemma 2.3.8 (Closing the argument: estimate for ζ in L^2) *There is a constant* $C_7(\mathcal{E}_0, L, M)$ *such that* $\zeta = \frac{\partial\psi}{\partial\tau}$ *satisfies*

$$\|\zeta(\tau)\|^2_{L^2} \leq \|\zeta(0)\|^2_{L^2} e^{C_7\int_0^\tau(\|\mu\bar{\partial}_{\mathbf{A}}\Phi(s)\|_{L^\infty} + \|\Phi(s)\|^2_{L^\infty})ds}.$$

Proof Compute, using (2.27), that

$$\frac{d}{d\tau}\|\zeta(\tau)\|^2_{L^2} = 2\langle\mathbb{J}\zeta, (\mu L_\psi + K_\psi)\zeta\rangle$$

since (by the gauge condition) $\langle\zeta, (d\dot{A}_0, i\Phi\dot{A}_0)\rangle_{L^2} = 0$, and $\langle\zeta,\ (0, iA_0\dot{\Phi})\rangle_{L^2} = 0$ (using $\langle i\dot{\Phi}, \dot{\Phi}\rangle = 0$ pointwise). By corollary 3.2.1 and the formula for K_ψ, there exists $C_7 = C_7(\mathcal{E}_0, L) > 0$ such that

$$\left|\frac{d}{d\tau}\|\zeta(\tau)\|^2_{L^2}\right| \leq C_7(\mu\|\bar{\partial}_A\Phi(\tau)\|_{L^\infty} + \|\Phi(\tau)\|^2_{L^\infty})\|\zeta(\tau)\|^2_{L^2}$$

and so the stated inequality follows by the Gronwall lemma. □

Completion of proof of theorem 2.3.1. The previous lemma allows us to validate the claim that (2.39), and thus all the bounds in lemmas 2.3.2-2.3.8, in fact hold on a μ-independent interval $[0, \tau_*]$, thus closing the argument. Indeed, by corollaries 2.3.3 and 2.3.7 we have $\mu\|\bar{\partial}_A\Phi(\tau)\|_{L^\infty} + \|\Phi(\tau)\|^2_{L^\infty} \leq C_8(1+\tau)$ for some $C_8 = C_8(\mathcal{E}_0, L, M)$. Now let τ_*, M_* be such that

$$\|\zeta(0)\|^2_{L^2} e^{C_7C_8(\tau_* + \tau_*^2/2)} \leq M_*^2.$$

(This is always possible for $M_* > \|\zeta(0)\|_{L^2}$ and τ_* small.) Then it follows that (2.39) holds with $T(M_*, \mu) \geq \tau_*$, and that the bounds given in lemma 2.3.2 through corollary 2.3.7 hold on the interval $[0, \tau_*]$. In particular, using (A.3),(A.4) we can deduce that $\big(\psi(\tau), \dot{\psi}(\tau)\big)$ is bounded in the (τ-independent) norm $\mathcal{H}_2 \times H^1$ as claimed. □

2.4 Proof of theorem 1.5.2

There are three stages to the proof:

- Deduce, from the uniform bounds of theorem 2.3.1 and the compactness lemma 2.4.1, that for any sequence $\mu_j \to +\infty$, there exists a subsequence along which the Ψ_{μ_j} converge.
- Identify the limit of these convergent subsequences.
- Deduce, from the uniqueness of the limit just identified, that the Ψ_μ do in fact converge as $\mu \to +\infty$ (without restriction to subsequences).

The first stage of the proof depends upon the following version of the Lions-Aubin compactness lemma (see [17, lemma 10.4]), which is proved by a modification of the standard proof of the usual Ascoli-Arzela theorem:

Lemma 2.4.1 *Assume that (V, h) is a smooth vector bundle with inner product, over a compact Riemannian manifold (Σ, g), which is endowed with a smooth unitary connection ∇ and corresponding Sobolev norms $\|\cdot\|_{H^s}$ on the space of sections defined as in [20]. Assume that l, s are positive numbers with $l < s$. Assume $f_n(\tau)$ is a sequence of smooth time-dependent sections of V which satisfy*

$$\max_{|\tau| \leq \tau_*} \big(\|f_n(\tau)\|_{H^s} + \|\dot{f}_n(\tau)\|_{H^l}\big) \leq C.$$

Then there exists a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ which converges to a limiting time-dependent section $f \in C([-\tau_*, \tau_*]; H^s(V))$, in the sense that, $\max_{|\tau| \leq \tau_*} \|(f_n(\tau, \cdot) - f(\tau, \cdot))\|_{H^r} \to 0$, *for every $r < s$.*

Applying this we infer immediately the existence of a subsequence $\mu_j \to +\infty$ along which the solutions $\Psi_{\mu_j} = (\mathbf{A}^{\mu_j}, \Phi^{\mu_j})$ converge to a limit $\Psi_S(\tau)$ in the sense that

$$\lim_{\mu_j \to \infty} \sup_{[-\tau_*, \tau_*]} \big\|\Psi_{\mu_j}(\tau) - \Psi_S(\tau)\big\|_{\mathcal{H}_r} = 0, \tag{2.44}$$

for $r < 2$. It follows from corollary (2.3.7), that

$$\lim_{\mu\to+\infty} \sup_{[-\tau_*,\tau_*]} \|\bar{\partial}_{\mathbf{A}^\mu}\Phi^\mu\|_{L^\infty} = 0,$$

and since the other Bogomolny equation $B = \frac{1}{2}(1-|\Phi|^2)$ is satisfied as a constraint, we deduce by theorem 1.4.1, that $\Psi_S(\tau) \in \mathcal{S}_N$, i.e. the limit $\Psi_S(\tau)$ is a self-dual vortex for each $\tau \in [-\tau_*, \tau_*]$. In addition, by (2.38) we have

$$\|\Psi_\mu(\tau_1) - \Psi_\mu(\tau_2)\|_{H^1} \leq M_*|\tau_1 - \tau_2|$$

so that, by (2.44), the limit Ψ_S will also satisfy

$$\|\Psi_S(\tau_1) - \Psi_S(\tau_2)\|_{H^{r'}} \leq c|\tau_1 - \tau_2|$$

for $r' < 1$, i.e. the limit is Lipschitz, and in particular lies in $W^{1,\infty}([-\tau_*, \tau_*]; L^2)$.

For the second stage, we need to identify the limiting curve $\tau \mapsto \Psi_S(\tau) \in \mathcal{S}_N$ as that described in remark 1.5.1. It is clear, from the conditions on the initial data in the statement of theorem 1.5.2, that $\Psi_S(0) = \psi_0 \in \mathcal{S}_N$, and so it remains to deduce the ordinary differential equation (1.18) which then determines the curve completely. To do this it is necessary to take the limit of (1.16):

$$\mathbb{J}\frac{\partial\Psi_\mu}{\partial\tau} = \mu\mathcal{V}' + U' + \mathbb{J}(dA_0^\mu, iA_0^\mu\Phi^\mu) \tag{2.45}$$

as $\mu \to \infty$. The first term on the right hand side is the most evidently problematic. However, since the limiting motion is constrained to the vortex space $\mathcal{S}_N$, it is only necessary to take a limit projected onto the tangent space $T_{\Psi_S}\mathcal{S}_N$. To this end, it is actually most convenient to introduce $\mathbb{P}_\mu(\tau) = \mathbb{P}_{\Psi_\mu(\tau)}$ the spectral projection operator onto $\operatorname{Ker}\mathcal{D}_{\Psi_\mu(\tau)} = \operatorname{Ker}\mathcal{D}^*_{\Psi_\mu(\tau)}\mathcal{D}_{\Psi_\mu(\tau)}$, discussed in lemma 3.3.2. By the final statement of lemma 3.3.2, and the convergence of Ψ_{μ_j} in (2.44), we know that $\mathbb{P}_\mu(\tau)$ converge, in the $L^2 \to L^2$ operator norm, to the operator $\mathbb{P}_{\Psi_S(\tau)}$, which is the spectral projection operator onto $\operatorname{Ker}\mathcal{D}_{\Psi_S(\tau)} = \operatorname{Ker}\mathcal{D}^*_{\Psi_S(\tau)}\mathcal{D}_{\Psi_S(\tau)}$. (This latter operator is also the orthogonal L^2 projector onto the tangent space $T_{\Psi_S}\mathcal{S}_N$ (subject to the gauge condition (1.3)). Apply the operator $\mathbb{P}_\mu(\tau)$ to the equation (1.16), to obtain:

$$\mathbb{P}_\mu(\tau)\mathbb{J}\frac{\partial\Psi_\mu}{\partial\tau} = \mathbb{P}_\mu(\tau)U'(\Psi_\mu(\tau)), \tag{2.46}$$

since $\mathbb{J}(dA_0, iA_0\Phi_\mu)$ and $\mathcal{V}'(\Psi_\mu)$ are both in the kernel of $\mathbb{P}_\mu$ by lemma 3.3.2. We can now identify the limit of the right hand side as $\mathbb{P}_{\Psi_S(\tau)}U'(\Psi_S(\tau))$ at each τ, and the convergence is strong in $L^2(\Sigma)$, by (2.44) and the above mentioned convergence of $\mathbb{P}_\mu(\tau)$. For the left hand side it is necessary to consider the limit of the derivatives $\frac{\partial\Psi_\mu}{\partial\tau}$. Noting that these are bounded in e.g. $L^2([-\tau_*,\tau_*];L^2(\Sigma))$, we may assume (by restricting to a further subsequence if necessary), the weak in L^2 subsequential convergence to a limit which is the weak time derivative of Ψ_S:

$$\langle\tilde{f},\frac{\partial\Psi_{\mu_j}}{\partial\tau}\rangle_{L^2([-\tau_*,\tau_*];L^2(\Sigma))}\to\langle\tilde{f},\frac{\partial\Psi_S}{\partial\tau}\rangle_{L^2([-\tau_*,\tau_*];L^2(\Sigma))},$$

for every $\tilde{f}\in L^2([-\tau_*,\tau_*];L^2(\Sigma))$. Now to identify the limit along a convergent subsequence $\mu_j\to+\infty$, consider the projection operator $\mathbb{P}_{\Psi_S(\tau)}$. Choosing $\tilde{f}(\tau,\cdot)=\mathbb{P}_{\Psi_S(\tau)}(f(\tau,\cdot))$, and using the symmetry of $\mathbb{P}_{\Psi_S(\tau)}$ this implies that

$$\begin{aligned}\int_{-\tau_*}^{+\tau_*}\langle f,\mathbb{P}_{\Psi_S(\tau)}\mathbb{J}\frac{\partial\Psi_{\mu_j}}{\partial\tau}\rangle_{L^2(\Sigma)}d\tau &= \int_{-\tau_*}^{+\tau_*}\langle\mathbb{P}_{\Psi_S(\tau)}f,\,\mathbb{J}\frac{\partial\Psi_{\mu_j}}{\partial\tau}\rangle_{L^2(\Sigma)}d\tau\\ \to\int_{-\tau_*}^{+\tau_*}\langle\mathbb{P}_{\Psi_S(\tau)}f,\,\mathbb{J}\frac{\partial\Psi_S}{\partial\tau}\rangle_{L^2(\Sigma)}d\tau &= \int_{-\tau_*}^{+\tau_*}\langle f,\,\mathbb{P}_{\Psi_S(\tau)}\mathbb{J}\frac{\partial\Psi_S}{\partial\tau}\rangle_{L^2(\Sigma)}d\tau,\end{aligned}$$

for any $f\in L^2([-\tau_*,\tau_*];L^2(\Sigma))$. On the other hand, by the above mentioned convergence of $\mathbb{P}_\mu(\tau)$ to $\mathbb{P}_{\Psi_S(\tau)}$ and the bounded convergence theorem we have

$$\int_{-\tau_*}^{+\tau_*}\left[\langle\mathbb{P}_{\mu_j}(\tau)f,\mathbb{J}\frac{\partial\Psi_{\mu_j}}{\partial\tau}\rangle_{L^2(\Sigma)}d\tau-\langle\mathbb{P}_{\Psi_S(\tau)}f,\mathbb{J}\frac{\partial\Psi_{\mu_j}}{\partial\tau}\rangle_{L^2(\Sigma)}\right]d\tau\to 0,$$

on account of the bound (2.38). Therefore, we have in the limit:

$$\int_{-\tau_*}^{+\tau_*}\langle f,\mathbb{P}_{\Psi_S(\tau)}\mathbb{J}\frac{\partial\Psi_S}{\partial\tau}\rangle_{L^2(\Sigma)}d\tau=\int_{-\tau_*}^{+\tau_*}\langle f,\mathbb{P}_{\Psi_S(\tau)}U'(\Psi_S(\tau))\rangle_{L^2(\Sigma)}d\tau, \tag{2.47}$$

for any $f\in L^2([-\tau_*,\tau_*];L^2(\Sigma))$. But since the limit is known by the above to be in $W^{1,\infty}([-\tau_*,\tau_*];L^2)$, it is differentiable (with respect to τ, as a map into L^2) almost everywhere (the standard result extends to Hilbert space-valued functions, see, e.g., [2, prop. 6.41]); the derivative lies in the tangent space $T_{\Psi_S}\mathcal{S}_N$, which is the range of the projector $\mathbb{P}_{\Psi_S(\tau)}$. Consequently (2.47) implies that $\tau\mapsto\Psi_S(\tau)$ is a solution of (1.18), with equality holding in L^2 for almost every τ. But this in turn implies that $\tau\mapsto\Psi_S(\tau)$ is actually continuously differentiable into L^2, and we have a classical solution of (1.18).

Finally for the third stage: we have now identified the limit as a solution of the limiting Hamiltonian system specified using remark 1.5.1. Choosing smooth co-ordinates on $\mathcal{M}_N$ as in [28] we see that this is a smooth finite dimensional Hamiltonian system, and as such its solutions (for given initial data) are unique. Therefore all subsequences have the same limit, and so we can assert full convergence without resort to subsequences.

3 Equations and identities related to the self-dual structure

> **Notation change:** *In this section time does not appear at all, and so the boldface* $\mathbf{A}$ *for the spatial component is not used: i.e. in this section only,* A *refers to the spatial part of the connection,* $A = A_1 dx^1 + A_2 dx^2$.

Ginzburg-Landau vortices are critical points of the static Ginzburg Landau energy functional $\mathcal{V}_\lambda = \int_\Sigma v_\lambda(A,\Phi)d\mu_g$ introduced following (1.9). The *coupling* constant $\lambda > 0$ is central to the theory of critical points of the Ginzburg-Landau functional and the value $\lambda = 1$ is special as in this case the functional admits the *Bogomolny decomposition* introduced in remark 1.4.3. This allows for a detailed understanding of the critical points not available for general values of λ, and the theory of critical points for such general values is incomplete. (There is, however, a substantial literature on the asymptotic behaviour of critical points in the $\lambda \to +\infty$ limit, starting with [4]; see [24] and references therein.) This decomposition of $\mathcal{V} \equiv \mathcal{V}_1$ has proved to be very useful not only for the analysis of critical points, but also for the associated time-dependent equations of vortex motion. For our purposes we need in particular to derive a special form for the operator L_ψ associated to the Hessian of $\mathcal{V}$, see (3.57).

3.1 Complex structure

To discuss the Bogomolny structure in detail it is useful to use a complex formulation so we introduce the complex co-ordinate $z = x^1 + ix^2$ for the complex structure J on Σ. Using this, there is a decomposition of the complex 1-forms $\Omega^1_{\mathbb{C}} = \Omega^{1,0} \oplus \Omega^{0,1}$ into the $\pm i$ eigenspaces of J, see notation 1.1.1. Let $\Omega^p(L)$ be the space of p-forms taking values in the bundle L: then for $p = 1$ there is a similar decomposition,

$$\Omega^1(L) = \Omega^{1,0}(L) \oplus \Omega^{0,1}(L).$$

Applying this decomposition to $D\Phi \in \Omega^1(L)$ we are led to introduce the operator $D^{0,1}$ given by

$$D^{0,1}\Phi = \frac{1}{2}\Big((\nabla_1 - iA_1) + i(\nabla_2 - iA_2)\Big)\Phi d\bar{z} \ = \ \bar{\partial}_A\Phi d\bar{z}.$$

For *real* 1-forms $A_1dx^1 + A_2dx^2 \in \Omega^1_{\mathbb{R}}$ this decomposition reads

$$A_1dx^1 + A_2dx^2 = \alpha dz + \bar{\alpha}d\bar{z},$$

where $\alpha = \frac{A_1 - iA_2}{2}$, and the map $A \mapsto \alpha$ (resp. $A \mapsto \bar{\alpha}$) is an $\mathbb{R}$-linear isomorphism from $\Omega^1_{\mathbb{R}}$ to $\Omega^{1,0}$ (resp. $\Omega^{0,1}$), and $\|A\|^2_{L^2} = 4\int \bar{\alpha}\alpha e^{-2\rho}d\mu_g$. With this α notation we can write

$$\bar{\partial}_A\Phi = \frac{\partial\Phi}{\partial\bar{z}} - i\bar{\alpha}\Phi.$$

3.2 The Hessian

The Bogomolny decomposition amounts to the observation that, with $\lambda = 1$,

$$\mathcal{V}(A,\Phi) \equiv \mathcal{V}_1(A,\Phi) = \frac{1}{2}\int_\Sigma \Big(4|\bar{\partial}_A\Phi|^2 e^{-2\rho} \ + \ (B - \frac{1}{2}\left(1 - |\Phi|^2\right))^2\Big) \, d\mu_g \ + \ \pi N$$

where $N = degL$. If the following first order equations, called the Bogomolny equations,

$$\begin{aligned} \bar{\partial}_A\Phi &= 0, \\ B - \frac{1}{2}(1 - |\Phi|^2) &= 0 \end{aligned} \tag{3.48}$$

have solutions in a given class, they will automatically minimize $\mathcal{V}$ within that class.

We introduce the nonlinear Bogomolny operator associated to this decomposition,

$$\begin{aligned} \mathcal{B} \ : \ \Omega^1_{\mathbb{R}} \oplus \Omega^0(L) &\longrightarrow \Omega^0_{\mathbb{R}} \oplus \Omega^{0,1}(L) \\ (A,\Phi) &\mapsto \left(B - \frac{1}{2}(1 - |\Phi|^2) \ , \ \bar{\partial}_A\phi\right). \end{aligned}$$

Using the norm $\|(\beta,\eta)\|^2_{L^2} \ = \ \int(|\beta|^2 + 4e^{-2\rho}|\eta|^2)d\mu_g$ induced from the metric on the target space, we see that $\mathcal{V}(A,\Phi) = \frac{1}{2}\|\mathcal{B}(A,\Phi)\|^2_{L^2} + \pi N$ as in remark 1.4.3; see [6]. The derivative of $\mathcal{B}$ at $\psi = (A,\Phi)$ is the map $D\mathcal{B}_\psi : \Omega^1_{\mathbb{R}} \oplus \Omega^0(L) \longrightarrow \Omega^0_{\mathbb{R}} \oplus \Omega^{0,1}(L)$ given by

$$(\dot{A},\dot{\Phi}) \ \mapsto \ (*d\dot{A} + \langle\Phi,\dot{\Phi}\rangle \, , \, \bar{\partial}_A\dot{\Phi} - i\dot{\bar{\alpha}}\Phi) \tag{3.49}$$

where $\alpha = \frac{A_1 - iA_2}{2}$ and $\dot\alpha = \frac{\dot A_1 - i\dot A_2}{2}$. Using this complex notation allows a simple unified formulation, which takes account of the *gauge condition* (1.3): this condition is the real part of

$$4e^{-2\rho}\bar\partial\dot\alpha - i\Phi\bar{\dot\Phi} \;=\; 0, \tag{3.50}$$

while the imaginary part of this expression is just the condition $*d\dot A + (\Phi, \dot\Phi) \;=\; 0$, appearing in the linearized Bogomolny equations. This suggests the introduction of the operators

$$\begin{aligned} \mathcal{D}_\psi &: \big(\Omega^{1,0} \oplus \Omega^0(L)\big) \longrightarrow \big(\Omega^0_{\mathbb{C}} \oplus \Omega^{0,1}(L)\big) \\ \mathcal{D}^*_\psi &: \big(\Omega^0_{\mathbb{C}} \oplus \Omega^{0,1}(L)\big) \longrightarrow \big(\Omega^{1,0} \oplus \Omega^0(L)\big) \end{aligned} \tag{3.51}$$

given by

$$\begin{aligned} \mathcal{D}_\psi(\dot\alpha, \dot\Phi) &= (4e^{-2\rho}\bar\partial\dot\alpha - i\Phi\bar{\dot\Phi},\ \bar\partial_A\dot\Phi - i\bar{\dot\alpha}\Phi) \\ \mathcal{D}^*_\psi(\beta, \eta) &= (-\partial\beta - i\Phi\bar\eta,\ -4e^{-2\rho}\partial_A\eta - i\Phi\bar\beta). \end{aligned} \tag{3.52}$$

We use the real inner product associated to the L^2 norms induced from the metric as above, i.e.:

$$\begin{aligned} \Big\langle(\dot\alpha, \dot\Phi), (\alpha', \Phi')\Big\rangle_{L^2} &= \int \big(4e^{-2\rho}\Re\bar{\dot\alpha}\alpha' + \Re\bar{\dot\Phi}\Phi'\big)\, d\mu_g \qquad \text{on } \Omega^{1,0} \oplus \Omega^0(L) \\ \Big\langle(\beta, \eta), (\beta', \eta')\Big\rangle_{L^2} &= \int \big(\Re\bar\beta\beta' + 4e^{-2\rho}\Re\bar\eta\eta'\big)\, d\mu_g \qquad \text{on } \Omega^0_{\mathbb{C}} \oplus \Omega^{0,1}(L)\ . \end{aligned}$$

Integrating by parts we deduce that

$$\Big\langle\mathcal{D}_\psi(\dot\alpha, \dot\Phi), (\beta, \eta)\Big\rangle_{L^2} = \Big\langle(\dot\alpha, \dot\Phi), \mathcal{D}^*_\psi(\beta, \eta)\Big\rangle_{L^2}$$

so that $\mathcal{D}^*_\psi$ is the L^2 adjoint of $\mathcal{D}_\psi$ and

$$\begin{aligned} \mathcal{D}^*_\psi\mathcal{D}_\psi(\dot\alpha, \dot\Phi) = \Big(&- \partial(4e^{-2\rho}\bar\partial\dot\alpha - i\Phi\bar{\dot\Phi}) - i\Phi(\overline{\bar\partial_A\dot\Phi} + i\dot\alpha\bar\Phi)\ , \\ &- 4e^{-2\rho}\partial_A(\bar\partial_A\dot\Phi - i\bar{\dot\alpha}\Phi) - i\Phi(4e^{-2\rho}\partial\bar{\dot\alpha} + i\bar\Phi\dot\Phi)\Big) \\ = \;\Big(&- \partial(4e^{-2\rho}\bar\partial\dot\alpha) + i(\partial_A\Phi)\bar{\dot\Phi} + |\Phi|^2\dot\alpha\ ,\ -4e^{-2\rho}\partial_A\bar\partial_A\dot\Phi + |\Phi|^2\dot\Phi + i4e^{-2\rho}\bar{\dot\alpha}\partial_A\Phi\Big). \end{aligned} \tag{3.53}$$

We compare this expression with the operator defined in (2.36):

$$\overline{L}_\psi \;:\; \big(\Omega^{1,0} \oplus \Omega^0(L)\big) \longrightarrow \big(\Omega^{1,0} \oplus \Omega^0(L)\big) \tag{3.54}$$

which defines the Hessian of $\mathcal{V}$ on the subspace on which the gauge condition (1.3) is satisfied, i.e.,

$$\langle\dot\psi,\overline{L}_\psi\dot\psi\rangle_{L^2}=D^2\mathcal{V}_\psi(\dot\psi,\dot\psi)=\frac{d^2}{d\epsilon^2}|_{\epsilon=0}\mathcal{V}(\psi+\epsilon\dot\psi), \tag{3.55}$$

for $\dot\psi=(\dot A,\dot\Phi)$ satisfying (1.3). Using mixed real/complex notation for A/α, (2.36) implies the following formula:

$$\overline{L}_\psi=\Big(-4\partial(e^{-2\rho}\bar\partial\dot\alpha)+|\Phi|^2\dot\alpha-(i\dot\Phi,D_1\Phi)+i(i\dot\Phi,D_2\Phi)\ \ ,\\ -\Delta_A\dot\Phi-\frac{1}{2}(1-3|\Phi|^2)\dot\Phi+2ie^{-2\rho}\dot A\cdot D\Phi\Big). \tag{3.56}$$

Calculate $\dot A\cdot D\Phi=2\dot\alpha\bar\partial_A\Phi+2\bar{\dot\alpha}\partial_A\Phi$ and $-(i\dot\Phi,D_1\Phi)+i(i\dot\Phi,D_2\Phi)=i\bar{\dot\Phi}\partial_A\Phi-i\dot\Phi\overline{\bar\partial_A\Phi}$, from which it follows that

$$(\overline{L}_\psi-\mathcal{D}_\psi^*\mathcal{D}_\psi)\dot\psi=\begin{pmatrix}-i\dot\Phi\overline{\bar\partial_A\Phi}\\ \left(B-\frac{1}{2}(1-|\Phi|^2)\right)\dot\Phi+4ie^{-2\rho}\dot\alpha\bar\partial_A\Phi\end{pmatrix}. \tag{3.57}$$

(Incidentally, observing that

$$\mathcal{B}(A+\dot A,\Phi+\dot\Phi)=\mathcal{B}(A,\Phi)+\mathcal{D}_\psi\dot\psi+\left(\frac{1}{2}|\dot\Phi|^2,-i\dot\alpha\dot\Phi\right),$$

with $\dot\psi=(\dot A,\dot\Phi)$ satisfying (1.3), the identity (3.57) can also be read off from the quadratic part of the Taylor expansion for $\mathcal{V}(A+\dot A,\Phi+\dot\Phi)$:

$$\begin{aligned}\frac{1}{2}\langle\dot\psi,\overline{L}_\psi\dot\psi\rangle_{L^2}&=\frac{1}{2}|\mathcal{D}_\psi\dot\psi|^2_{L^2}+\left\langle\mathcal{B}(\psi)\ ,\left(\frac{1}{2}|\dot\Phi|^2,-i\dot{\bar\alpha}\dot\Phi\right)\right\rangle\\ &=\frac{1}{2}|\mathcal{D}_\psi\dot\psi|^2_{L^2}+\int_\Sigma\left(\frac{1}{2}(B-\frac{1}{2}(1-|\Phi|^2))|\dot\Phi|^2+4e^{-2\rho}\langle\bar\partial_A\Phi,-i\dot{\bar\alpha}\dot\Phi\rangle\right)\,d\mu_g,\end{aligned}$$

using the inner product on $\Omega^{1,0}\oplus\Omega^0(L)$ defined above.)

Corollary 3.2.1 *Let $\mathbb{J}$ denote the complex structure defined in* (1.17). *There exists a number $c>0$, independent of $\psi=(\alpha,\Phi)$ and $\zeta=\dot\psi=(\dot\alpha,\dot\Phi)\in\Omega^{1,0}\oplus\Omega^0(L)$, such that*

$$|\langle\mathbb{J}\zeta,L_\psi\zeta\rangle_{L^2}|\ \le c\ |\mathcal{B}(\psi)|_{L^\infty}|\zeta|^2_{L^2}$$

Proof By (3.57) $|\langle\mathbb{J}\zeta,L_\psi\zeta\rangle_{L^2}-\langle\mathcal{D}_\psi\mathbb{J}\zeta,\mathcal{D}_\psi\zeta\rangle_{L^2}|\le|\mathcal{B}(\psi)|_{L^\infty}|\zeta|^2_{L^2}$. Now the complex structure $\mathbb{J}$ written in complex notation, i.e. acting on $\Omega^{1,0}\oplus\Omega^0_(L)$, is given by $\mathbb{J}(\dot\alpha,\dot\Phi)=(-i\dot\alpha,i\dot\Phi)$. Correspondingly, on $\Omega^0_{\mathbb{C}}\oplus\Omega^{0,1}(L)$ we introduce the complex structure $\mathbb{J}'(\beta,\eta)=(i\beta,-i\eta)$. Then, by observation

$$\mathcal{D}_\psi\mathbb{J}\zeta=-\mathbb{J}'\mathcal{D}_\psi\zeta. \tag{3.58}$$

Therefore, writing $w = \mathcal{D}_\psi \zeta$, we have $\langle \mathcal{D}_\psi \mathbb{J}\zeta, \mathcal{D}_\psi \zeta\rangle_{L^2} = \langle -\mathbb{J}' w, w\rangle_{L^2} = 0$ by skew-symmetry, and the result follows. $\square$

Lemma 3.2.2 *Assume there are positive numbers $L, \mathcal{E}_0$ such that $|\Phi|_{L^2} = L$, and $\mathcal{V}_\lambda(A,\Phi) = \mathcal{E}_0$ and $\lambda > 0$. Then the quadratic forms*

$$\tilde{Q}_\Phi(\beta) = \int_\Sigma 4|\partial\beta|^2 e^{-2\rho} + |\Phi|^2|\beta|^2 d\mu_g \quad on \quad \oplus\,\Omega^0_{\mathbb{C}}\, and$$
$$Q_{(A,\Phi)}(\eta) = \int_\Sigma 4e^{-4\rho}|\partial_A\eta|^2 + e^{-2\rho}|\Phi|^2|\eta|^2 d\mu_g \quad on \quad \Omega^{0,1}(L)$$

are strictly positive, and in fact bounded below by (respectively) $C\|\beta\|^2_{H^1}$ and $C\|\eta\|^2_{H^1_A}$ where C is a positive number depending only upon the numbers $L, \mathcal{E}_0$.

Proof We will present the proof for the quadratic form $Q_{(A,\Phi)}(\eta)$ as the other is similar but easier. Clearly $Q_{(A,\Phi)}(\eta) \geq 0$ and in fact $Q_{(A,\Phi)}(\eta) = 0$ if and only if $\eta \equiv 0$ on Σ (because if $\partial_A\eta \equiv 0$ then η has isolated zeros (as in [16], sec. 3.5); if $\Phi\eta \equiv 0$ then $\eta \equiv 0$ since $\Phi = 0$ *a.e.* contradicts $\int_\Sigma |\Phi|^2 = L > 0$. Furthermore, we show that $Q_{(A,\Phi)}(\eta) \geq c|\eta|^2_{L^2}$ for a constant c; to be precise there exists $c = c(L, \mathcal{E}_0)$ such that

$$Q_{(A,\Phi)}(\eta) \geq c, \text{ for all } \eta \text{ such that } \|\eta\|_{L^2} = 1. \tag{3.59}$$

We will prove this by contradiction. First we obtain some bounds. By gauge invariance we are free to assume that the Coulomb gauge condition $\operatorname{div} A = 0$ holds. With this gauge condition, we have the bound $\|A\|_{H^1} \leq c(\mathcal{E}_0)$ and so A is bounded in every L^p space. Now use $\|\partial\eta\|_{L^p} \leq \|\partial_A\eta\|_{L^p} + \|A\eta\|_{L^p}$ to deduce that

$$\|\partial\eta\|^2_{L^p} \leq C(1 + Q_{(A,\Phi)}(\eta))$$

for every $p < 2$, by Holder's inequality. This in turn implies, by the L^p estimate for the inhomogeneous Cauchy-Riemann system, that η is bounded similarly in L^4, and so since A is also we can bound $\partial\eta$ in L^2 and hence η in H^1. Finally, since A and η are bounded similarly in L^4, this imples that $\|\eta\|^2_{H^1_A} \leq C(1 + Q_{(A,\Phi)}(\eta))$, with C depending only upon $\mathcal{E}_0, L$. To conclude, in Coulomb gauge the A, Φ, η are all bounded in H^1 in terms of $L, \mathcal{E}_0, Q_{(A,\Phi)}(\eta)$.

The contradiction argument now starts: assume (3.59) fails. Then, by the bounds just obtained and the Banach-Alaoglu and Rellich theorems, there is a sequence $(A_\nu, \Phi_\nu, \eta_\nu)$

with

$$\|A_\nu\|_{H^1} + \|\nabla\Phi_\nu\|_{L^2} \leq K(\mathcal{E}_0, L),$$

$\|\Phi_\nu\|_{L^2} = L$ and $\|\eta_\nu\|_{L^2} = 1$, such that

$$\begin{aligned}
&Q_{(A_\nu,\Phi_\nu)}(\eta_\nu) \longrightarrow 0\\
&A_\nu \longrightarrow A \;\text{ weakly in } H^1\\
&\Phi_\nu \longrightarrow \Phi \;\text{ weakly in } H^1 \text{ and strongly in } L^p \text{ for any } p<\infty\\
&\eta_\nu \longrightarrow \eta \;\text{ weakly in } H^1 \text{ and strongly in } L^p.
\end{aligned}$$

This implies that $|\Phi|_{L^2} = L > 0$, $Q_{(A,\Phi)}(\eta) = 0$ which implies as above that $\Phi = 0$ *a.e.* and contradicts as above that $|\Phi|_{L^2}$ is constant. This leads to

$$Q_{(A,\Phi)}(\eta) \geq c_1|\eta|^2_{L^2} \text{ where } c_1 = c_1(L,\mathcal{E}_0).$$

Finally just apply the bound above for $\|D\eta\|_{L^2}$ to improve this up to the H^1_A lower bound claimed. □

3.3 The Bogomolny foliation

We introduce a foliation associated to the Bogomolny operator, which we regard as a map between the following Hilbert spaces:

$$\begin{aligned}
\mathcal{B} \;:\; H^1\big(\Omega^1_{\mathbb{R}} \oplus \Omega^0(L)\big) &\longrightarrow L^2\big(\Omega^0_{\mathbb{R}} \oplus \Omega^{0,1}(L)\big),\\
(A,\Phi) &\mapsto \left(B - \frac{1}{2}(1-|\Phi|^2)\;,\;\bar\partial_A\phi\right).
\end{aligned}$$

With this choice of norms $\mathcal{B}$ is a smooth function. The next result shows that it is a submersion if the energy is close to the minimum value:

Lemma 3.3.1 *There exists $\theta_* > 0$ such that $\|\bar\partial_A\Phi\|_{L^2} < \theta_*$ implies that $Ker\mathcal{D}^*_\Psi = \{0\}$, and $Ker\mathcal{D}_\Psi$ is $2N$ dimensional (where $N = degL$).*

Proof $\mathcal{D}^*(\beta,\eta) = 0$ is equivalent to

$$\begin{aligned}
&-\partial\beta - i\Phi\bar\eta = 0\\
&-4e^{-2\rho}\partial_A\eta - i\Phi\bar\beta = 0.
\end{aligned}$$

Apply the operations $4\bar{\partial}$ to the first and $4\bar{\partial}_A$ to the second of these equations to deduce that

$$
\begin{aligned}
&-4e^{-2\rho}\bar{\partial}\partial\beta + |\Phi|^2\beta - 4ie^{-2\rho}\bar{\partial}_A\Phi\bar{\eta} = 0 \\
&-4\bar{\partial}_A(e^{-2\rho}\bar{\partial}_A\eta) + |\Phi|^2\eta - i(\bar{\partial}_A\Phi)\beta = 0
\end{aligned}
$$

The first two terms of these two equations are respectively the Euler- Lagrange operators associated to the quadratic forms $\tilde{Q}_\Phi(\beta)$ and $Q_{A,\Phi}(\eta)$ studied in the previous lemma. Then we get the estimates

$$
\begin{aligned}
\tilde{Q}_\Phi(\beta) &\leq c|\bar{\partial}_A\Phi|_{L^2}|\beta|_{L^4}|\eta|_{L^4} \\
Q_{A,\Phi}(\eta) &\leq c|\bar{\partial}_A\Phi|_{L^2}|\beta|_{L^4}|\eta|_{L^4}
\end{aligned}
$$

which implies the result, since $\tilde{Q}_\Phi(\beta) \ \geq \ c|\beta|^2_{H^1}$ and $Q_{A,\Phi}(\eta) \ \geq \ c|\eta|^2_{H^1_A}$. $\square$

The natural geometrical context for the results of this section will now be explained. Define $\mathcal{O}_* \equiv \{(A,\Phi) \in H^1(\Omega^1_{\mathbb{R}} \oplus \Omega^0(L)) : \|\bar{\partial}_A\Phi\|_{L^2} < \theta_*\}$ which is an open set containing $\{\psi = (A,\Phi) : \mathcal{B}(\psi) = 0\} \subset H^1(\Omega^1_{\mathbb{R}} \oplus \Omega^0(L))$. Furthermore, the previous lemma implies that $\mathcal{D}_\psi : (\Omega^{1,0} \oplus \Omega^0(L)) \longrightarrow (\Omega^0_{\mathbb{C}} \oplus \Omega^{0,1}(L))$ is surjective for $\psi \in \mathcal{O}_*$. By the discussion in the paragraph preceding (3.51), this implies that $D\mathcal{B}_\psi : \Omega^1_{\mathbb{R}} \oplus \Omega^0(L) \longrightarrow \Omega^0_{\mathbb{R}} \oplus \Omega^{0,1}(L)$ is also surjective for $\psi \in \mathcal{O}_*$, and hence the level sets of $\mathcal{B}$ form a foliation of $\mathcal{O}_*$ whose leaves have tangent space equal to $\operatorname{Ker} D\mathcal{B}_\psi$ by [1, §3.5 and §4.4]. The intersection of this tangent space with $\mathcal{SL}_\psi = \{(\dot{A},\dot{\Phi}) : (\dot{A},\dot{\Phi})$ satisfies (1.3)$\}$ is $\operatorname{Ker} \mathcal{D}_\Psi$.

Lemma 3.3.2 *Assume $\psi \in (\Omega^{1,0} \oplus \Omega^0_{\mathbb{C}}(L)) \cap \mathcal{O}_*$. The operators $\mathcal{D}^*_\psi\mathcal{D}_\psi$ defined in* (3.53) *are self-adjoint operators on L^2, with domain H^2, with $2N$-dimensional kernel equal to $Ker\mathcal{D}_\psi$, and:*

$$\|\mathcal{D}^*_\psi\mathcal{D}_\psi\zeta\|_{L^2} + \|\zeta\|_{L^2} \geq c\|\zeta\|_{H^2}. \tag{3.60}$$

Let $\mathbb{P}_\psi$ be the orthogonal spectral projector onto $Ker\mathcal{D}^*_\psi\mathcal{D}_\psi = Ker\mathcal{D}_\psi$. Then $\mathbb{P}_\psi(\mathcal{V}'(\psi)) = 0$ and $\mathbb{P}_\psi(\mathbb{J}(d\chi, i\chi\Phi_\mu)) = 0$ for any smooth real valued function χ. Finally, if also $\psi^{(j)} \in (\Omega^{1,0} \oplus \Omega^0_{\mathbb{C}}(L)) \cap \mathcal{O}_*$, and $\sup_j \|\psi^{(j)}\|_{\mathcal{H}_2} < \infty$ and $\lim_{j\to+\infty} \|\psi^{(j)} - \psi\|_{\mathcal{H}_r} = 0$, for all $r < 2$, the corresponding projectors $\mathbb{P}_{\psi^{(j)}}$ converge to $\mathbb{P}_\psi$ in $L^2 \to L^2$ operator norm.

Proof The first assertion and the bound (3.60) follow from lemma 3.3.1 and standard elliptic theory. The next statement follows by noting that if $n \in \operatorname{Ker}\mathcal{D}_\psi$, then differentiation of $\mathcal{V}(\psi) = \frac{1}{2}\int |\mathcal{B}(\psi)|^2 d\mu_g + \pi N$ yields

$$\langle n, \mathcal{V}'(\psi)\rangle_{L^2} = \frac{d}{ds}\bigg|_{s=0} \mathcal{V}(\psi + sn) = \langle \mathcal{B}(\psi), D\mathcal{B}_\psi(n)\rangle_{L^2} = 0$$

since $\operatorname{Ker}\mathcal{D}_\psi \subset \operatorname{Ker} D\mathcal{B}_\psi$ by the discussion preceding (3.51). Next, $n \in \operatorname{Ker}\mathcal{D}_\psi$ implies that $\mathbb{P}_\psi(\mathbb{J}(d\chi, i\chi\Phi_\mu)) = 0$ since integration by parts reduces this to the fact that n solves the first component of $D\mathcal{B}_\psi n = 0$ in (3.49).

The final statement follows by [13, § IV.3], if it can be established that $T_j \equiv \mathcal{D}^*_{\psi^{(j)}}\mathcal{D}_{\psi^{(j)}}$ converges to $T \equiv \mathcal{D}^*_\psi\mathcal{D}_\psi$ in the generalized sense of Kato (see [13, §IV.2.6]), or equivalently in the norm resolvent sense:

$$\lim_{j\to\infty} \|(i+T)^{-1} - (i+T_j)^{-1}\|_{L^2\to L^2} = 0. \tag{3.61}$$

To verify this convergence, it is convenient first of all to verify it in Coulomb gauge. So let $\tilde\psi^{(j)} = (\tilde A^{(j)}, \tilde\Phi^{(j)}) = e^{i\chi_j}\cdot\psi^{(j)}$ and $\tilde\psi = (\tilde A, \tilde\Phi) = e^{i\chi}\cdot\psi$ be gauge transforms (as defined following (1.7)), such that $\operatorname{div}\tilde A^{(j)} = 0 = \operatorname{div}\tilde A$. The assumed properties of $\psi^{(j)}$ ensure that $\sup\|\chi_j\|_{H^2} < \infty$ and that $\lim\|\chi_j - \chi\|_{H^r} = 0$, $\forall\, r < 2$ so that also $\tilde\psi^{(j)} \to \tilde\psi$ in $\mathcal{H}_r$ for $r < 2$. Now observe that *in Coulomb gauge* the formula (3.53) does not involve any derivatives of the connection one-form A at all. From this it is then immediate by inspection that (writing $\tilde T_j \equiv \mathcal{D}^*_{\tilde\psi^{(j)}}\mathcal{D}_{\tilde\psi^{(j)}}$, and $\tilde T \equiv \mathcal{D}^*_{\tilde\psi}\mathcal{D}_{\tilde\psi}$,)

$$\|(\tilde T - \tilde T_j)\zeta\|_{L^2} \le \delta_j\|\zeta\|_{H^2} \le c\delta_j(\|\zeta\|_{L^2} + \|\tilde T\zeta\|_{L^2}) \tag{3.62}$$

where $\delta_j \to 0$ as $j \to +\infty$. But this last fact implies (by [13, Theorems IV.2.24-25]) that $\tilde T_j$ converges to $\tilde T$ in the generalized sense, and hence in the resolvent sense:

$$\lim_{j\to\infty} \|(i+\tilde T)^{-1} - (i+\tilde T_j)^{-1}\|_{L^2\to L^2} = 0. \tag{3.63}$$

This would establish the convergence of the corresponding spectral projectors in Coulomb gauge. To go back to the original ψ_j it is just necessary to make use of the following gauge invariance property: on $\zeta = (\dot\alpha, \dot\Phi)$ the induced action of the gauge group is $g\bullet(\dot\alpha, \dot\Phi) = (\dot\alpha, g\dot\Phi)$ for any S^1 valued function g, and

$$\tilde T\left(e^{i\chi}\bullet\zeta\right) = e^{i\chi}\bullet(T\zeta)\,,$$

and similarly with T_j, χ_j replaced by T, χ. This gauge invariance property implies that $(i+T_j)^{-1} = e^{-i\chi_j}\circ(i+\tilde T_j)^{-1}\circ e^{i\chi_j}$ and $(i+T)^{-1} = e^{-i\chi}\circ(i+\tilde T)^{-1}\circ e^{i\chi}$, where by $\circ$

we mean operator composition, and $e^{i\chi}$ is shorthand for the operator $e^{i\chi}\bullet$ etc. Finally, using $\lim \|\chi_j - \chi\|_{H^r} = 0$, $\forall\, r < 2$ we see that (3.62) and (3.63) imply (3.61), completing the proof. $\square$

Appendix

A.1 Operators

To describe in detail the Laplacian operators which appear in the text, we assume Σ to be covered by an atlas of charts U_α on each of which is a local trivialisation of L determined by a choice of a local unitary frame. (A smooth section Φ of L then corresponds to a family of smooth functions $\Phi_\alpha : U_\alpha \to \mathbb{C}$ so that on $U_\alpha \cap U_\beta$ we have $\Phi_\alpha = e^{i\theta_{\alpha\beta}}\Phi_\beta$ with $e^{i\theta_{\alpha\beta}} : U_\alpha \cap U_\beta \to S^1$ smooth.) We assume given a smooth connection $\mathbf{D} = \nabla - i\mathbf{A}$ on L acting as a covariant derivative operator on sections of L. Working in such a chart, and suppressing the index α, the Laplacian on sections Φ of L is given by:

$$-\Delta_A \Phi = -\frac{1}{\sqrt{g}} D_j\big(g^{ij}\sqrt{g} D_i \Phi\big) = -e^{-2\rho}\big(D_i D_i \Phi\big). \tag{A.1}$$

This satisfies $\langle -\Delta_A \Phi, \Phi' \rangle_{L^2} = \frac{d}{d\epsilon}\frac{1}{2}|\mathbf{D}(\Phi + \epsilon \Phi')|^2_{L^2}|_{\epsilon=0}$.

Next we need the Laplacian on one-forms. Starting with $\mathbf{A} = A_1 dx^1 + A_2 dx^2 \in \Omega^1_{\mathbb{R}}$, the negative Laplacian is the Euler-Lagrange operator associated to the Dirichlet form $\frac{1}{2}\int(|\mathrm{div}\,\mathbf{A}|^2 + |d\mathbf{A}|^2)d\mu_g$ (with the norms inside the integral determined by g in the standard way). Transferring to complex form $\alpha = \frac{1}{2}(A_1 - iA_2) \in \Omega^{1,0}$, this Dirichlet form is just $I(\alpha) = 8\int e^{-4\rho}\overline{\bar\partial\alpha}\bar\partial\alpha\, d\mu_g$. The corresponding negative Laplacian $-\Delta^{1,0}$ is then defined by $\langle -\Delta^{1,0}\alpha, \beta\rangle_{L^2} = \frac{d}{d\epsilon}I(\alpha + \epsilon\beta)|_{\epsilon=0}$ where we use the induced inner product $\Omega^{1,0}$ as in §3. This leads to the following formula for the negative Laplacian $-\Delta^{1,0}$ on $\alpha \in \Omega^{1,0}$:

$$-\Delta^{1,0}\alpha = -4\partial(e^{-2\rho}\bar\partial\alpha),$$

which is precisely the operator appearing in §3. Similarly, on $\Omega^{0,1}(L)$ the negative Laplacian is

$$-\Delta^{0,1}_A\eta = -4\bar\partial_A(e^{-2\rho}\partial_A\eta),$$

which is the operator in (2.30).

A.2 Norms and inequalities

We define the Sobolev norms defined with the covariant derivative $\mathbf{D} = \nabla_{\mathbf{A}} = \nabla - i\mathbf{A}$. (We write $\nabla_{\mathbf{A}}$ in place of $\mathbf{D}$ for emphasis here.) The first Sobolev norm is defined by

$$|\Phi|^2_{H^1_{\mathbf{A}}} = \int_\Sigma \big(|\Phi|^2 + |\nabla_A \Phi|^2\big)\, d\mu_g. \tag{A.2}$$

In the above integral the inner products are the standard ones induced from h and g. The higher norms $H^2_{\mathbf{A}}, \dots$ are defined similarly, as are the $W^{k,p}_{\mathbf{A}}$ norms for integral k and any $p \in [1, \infty]$. The L^p norms of the higher covariant derivatives arising from the connections $\nabla_{\mathbf{A}}$ and ∇ are related as expressed schematically in the following:

$$\|\nabla\Phi\|_{L^p} \leq \|\nabla_{\mathbf{A}}\Phi\|_{L^p} + c\|\mathbf{A}\|_{L^\infty}\|\Phi\|_{L^p}, \tag{A.3}$$

$$\begin{aligned} \|\nabla\nabla\Phi\|_{L^p} \leq\ & \|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^p} + c\|\mathbf{A}\|_{L^\infty}\|\nabla_{\mathbf{A}}\Phi\|_{L^p} \\ & + c(1 + \|\nabla\mathbf{A}\ \Phi\|_{L^p} + \|\mathbf{A}\|^2_{L^\infty}\|\Phi\|_{L^p}), \end{aligned} \tag{A.4}$$

$$\begin{aligned} \|\nabla\nabla\nabla\Phi\|_{L^p} \leq\ & \|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^p} + c\|\mathbf{A}\|_{L^\infty}\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^p} \\ & + c(1 + \|\nabla\mathbf{A}\|_{L^\infty} + \|\mathbf{A}\|^2_{L^\infty})\|\nabla_{\mathbf{A}}\Phi\|_{L^p} \\ & + c\big(1 + \|\nabla^2\mathbf{A}\|_{L^q}\|\Phi\|_{L^r} + \|\mathbf{A}\|^3_{L^\infty}\|\Phi\|_{L^p}\big), \end{aligned} \tag{A.5}$$

where $q^{-1} + r^{-1} = p^{-1}$.

We now collect together some inequalities from [10].

The system of equations

$$B = f \qquad \operatorname{div}\mathbf{A} = g \tag{A.6}$$

(where as above div $: \Omega^1 \to \Omega^0$ is minus the adjoint of d) is a first order elliptic system which can be solved for $\mathbf{A}$ subject to the condition on $\int f d\mu_g$ dictated by an integer N, the degree of L. It can be rewritten

$$d\mathbf{A} = (f - b)d\mu_g \qquad \operatorname{div}\mathbf{A} = g \tag{A.7}$$

and solved via Hodge decomposition as long as the right hand sides have zero integral. There is a solution unique up to addition of harmonic 1-forms which satisfies $\|\mathbf{A}\|_{W^{1,p}} \leq c_p(1 + \|f\|_{L^p} + \|g\|_{L^p})$ for $p < \infty$.

Lemma A.2.1 (Covariant Sobolev and Gagliardo-Nirenberg inequalities) *For (Σ, g) as above and for $(\mathbf{A}, \Phi) \in (H^1 \times H^2_{\mathbf{A}})(\Sigma)$ then $\nabla_{\mathbf{A}}\Phi \in L^4(\Sigma)$ and*

$$\|\nabla_{\mathbf{A}}\Phi\|_{L^4} \leq c\|\nabla_{\mathbf{A}}\Phi\|_{H^1_{\mathbf{A}}} \tag{A.8}$$

and also for all $1 \leq p < \infty$, $H^2_{\mathbf{A}} \hookrightarrow W^{1,p}_{\mathbf{A}} \hookrightarrow L^\infty$ continuously on Σ. Also

$$\|\nabla_{\mathbf{A}}\Phi\|_{L^4} \leq c\|\nabla_{\mathbf{A}}\Phi\|^{1/2}_{L^2}\Big(\|\nabla_{\mathbf{A}}\Phi\|^{1/2}_{L^2} + \|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|^{1/2}_{L^2}\Big) \tag{A.9}$$

where c depends only on (Σ, g).

Lemma A.2.2 (Covariant version of the Garding inequality) *For $\Psi = (\mathbf{A}, \Phi)$ such that the norms on Σ appearing below are finite we have*

$$\|\nabla_{\mathbf{A}}\nabla_{\mathbf{A}}\Phi\|_{L^2} \leq \|\Delta_{\mathbf{A}}\Phi\|_{L^2} + c\|B\|^{1/2}_{L^\infty}\|\nabla_{\mathbf{A}}\Phi\|_{L^2} + c\|\Phi\|^{1/2}_{L^\infty}\|\nabla_{\mathbf{A}}\Phi\|^{1/2}_{L^2}\|\nabla B\|^{1/2}_{L^2} \tag{A.10}$$

where c is a number depending only on (Σ, g).

Lemma A.2.3 (Covariant version of the Brezis-Gallouet inequality) *If $\mathbf{A} \in H^1(\Sigma)$ and $\Phi \in H^2_{\mathbf{A}}(\Sigma)$ then*

$$\|\Phi\|_{L^\infty(\Sigma)} \leq c\Big(1 + \|\Phi\|_{H^1_{\mathbf{A}}}\sqrt{\ln(1 + \|\Phi\|_{H^2_{\mathbf{A}}})}\Big) \tag{A.11}$$

where c depends only on (Σ, g).

A.3 Global existence results and different choices of gauge

In this section we will summarize the existence theory for (1.1) from [3] and [10], and explain how theorem 1.2.1 can be deduced from it. Existence theory can be worked out using various gauge conditions, and a choice of gauge is usually made to facilitate the calculations. The simplest condition for the statement of the theorem, which also is convenient if we wish to make the Hamiltonian structure manifest - see §1.3, is the temporal gauge condition $A_0 = 0$; however, the regularity is stronger in Coulomb gauge $\operatorname{div}\mathbf{A} = 0$. We have the following statements.

Theorem A.3.1 (Global existence in temporal gauge) *Given data $\Phi(0) \in H^2(\Sigma)$ and $\mathbf{A}(0) \in H^1(\Sigma)$, there exists a global solution for the Cauchy problem for (1.1) satisfying $A_0 = 0$, with regularity $\Phi \in C\big([0,\infty); H^2(\Sigma)\big) \cap C^1\big([0,\infty); L^2(\Sigma)\big)$ and $\mathbf{A} \in$*

$C^1\big([0,\infty);H^1(\Sigma)\big)$. *Furthermore, it is the unique such solution satisfying* $A_0 = 0$ *and satisfies the estimate*

$$\|\Phi(t)\|_{H^2(\Sigma)} \le c e^{\alpha e^{\beta t}},$$

for some positive constants c, α, β *depending only on* (Σ, g)*, the equations, and the initial data.*

This can be derived from theorem 1.1 in [10], by applying a gauge transformation to put the solution obtained there into temporal gauge. To be precise the cited result gives a global solution $(a_0, \mathbf{a}, \phi)$ of the system (1.1) satisfying the parabolic gauge condition $a_0 = \operatorname{div} \mathbf{a}$, and the gauge invariant growth estimate

$$\|\phi\|_{H^2_a(\Sigma)}(t) \le c e^{\alpha e^{\beta t}}. \tag{A.12}$$

The solution satisfies $\phi \in C\big([0,\infty);H^2(\Sigma)\big) \cap C^1\big([0,\infty);L^2(\Sigma)\big)$, $\mathbf{a} \in C\big([0,\infty);H^1(\Sigma)\big)$ and $a_0 \in C\big([0,\infty);L^2(\Sigma)\big)$. Now define $\chi \in C^1\big([0,\infty);L^2(\Sigma)\big)$ by $\partial_t \chi + a_0 = 0$ and $\chi(0) = 0$. Define $(\Phi, A) = (\phi e^{it\chi}, a + d\chi)$: this gives a solution to (1.1) satisfying the properties asserted in theorem A.3.1. (Most of this can be read off immediately, except perhaps to verify that $\mathbf{A} \in C^1\big([0,\infty);H^1(\Sigma)\big)$, but this follows from the first equation in (1.1), using the fact that $A_0 = 0$ and the right hand side is continuous into L^2.)

An alternative approach to local existence is given in [3], where it is shown that, in Coulomb gauge, systems of the type (1.1) can be put in the form of an abstract evolution equation to which Kato's theory ([14]) applies. This yields the existence of a *local* solution denoted (A', Φ') with Φ' continuous into H^2 on a time interval of length determined by the H^2 norm of the initial data. But the estimate (A.12) above is gauge invariant, and allows continuation of the local solution to provide a global solution in Coulomb gauge with regularity $\Phi' \in C\big([0,\infty);H^2(\Sigma)\big) \cap C^1\big([0,\infty);L^2(\Sigma)\big)$ and $\mathbf{A}' \in C\big([0,\infty);H^3(\Sigma)\big) \cap C^1\big([0,\infty);H^1(\Sigma)\big)$ satisfying the Coulomb gauge condition $\operatorname{div} A' = 0$.

Finally, we explain how to obtain theorem 1.2.1 from these results. Given a solution $\mathbf{A}', \Phi'$ in Coulomb gauge, as just described, define $\chi(t,x)$ to be the solution of

$$(-\Delta + |\Phi'|^2)\dot{\chi} = \operatorname{div} \dot{\mathbf{A}}' - \langle i\Phi', \dot{\Phi}'\rangle = -\langle i\Phi', \dot{\Phi}'\rangle,$$

with $\chi(0,x) = 0$. Then it is easy to verify that $\mathbf{A} = \mathbf{A}' + d\chi, \Phi = \Phi' e^{i\chi}$ satisfies (1.3). Under the condition $\|\Phi(t)\|^2_{L^2(\Sigma)} = L > 0$ the solution exists and is unique at time t; this condition is natural because $\|\Phi(t)\|_{L^2(\Sigma)}$ is independent of time for solutions of (1.1). Now by the above mentioned Coulomb gauge regularity and the basic estimates for the Laplacian we deduce that $\chi \in C^1([0,\infty); H^2)$. This gives the global existence theorem in the gauge stated in theorem 1.2.1.

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