Effective dynamics for solitons in the nonlinear Klein Gordon Maxwell system and the Lorentz force law

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Abstract

We consider the nonlinear Klein Gordon Maxwell system derived from the Lagrangian $\int\big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\langle(\partial-ie\mathbb{A})_\mu\phi,(\partial-ie\mathbb{A})^\mu\phi\rangle-\mathcal{V}(\phi)-e\mathbb{A}^\mu\mathbb{J}^B_\mu\big)$ on four dimensional Minkowski space-time, where ϕ is a complex scalar field and $F_{\mu\nu}=\partial_\mu\mathbb{A}_\nu-\partial_\nu\mathbb{A}_\mu$ is the electromagnetic field. For appropriate nonlinear potentials $\mathcal{V}$, the system admits soliton solutions which are gauge invariant generalizations of the non-topological solitons introduced and studied by T.D. Lee and collaborators for pure complex scalar fields. In this article we develop a rigorous dynamical perturbation theory for these solitons in the small e limit, where e is the electromagnetic coupling constant. The main theorems assert the long time stability of the solitons with respect to perturbation by an external electromagnetic field produced by the background current $\mathbb{J}^B$, and compute their effective dynamics to $O(e)$. The effective dynamical equation is the equation of motion for a relativistic particle acted on by the Lorentz force law familiar from classical electrodynamics. The theorems are valid in a scaling regime in which the external electromagnetic fields are $O(1)$, but vary slowly over space-time scales of $O(\frac{1}{\delta})$, and $\delta=e^{1-k}$ for $k\in\big(0,\frac{1}{2}\big)$ as $e\to 0$. We work entirely in the energy norm, and the approximation is controlled in this norm for times of $O(\frac{1}{e})$.

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1 Statement of results

1.1 Introduction

In this article, we are interested in the effective dynamics of a class of solitary wave, or soliton, solutions to the nonlinear Klein-Gordon-Maxwell (nl-KGM) equations, in the presence of an external electromagnetic field. In this introduction we start by writing down the equations and giving a heuristic statement of, and motivation for, our results in §1.1.3 and §1.1.4. Then, in §1.2 and §1.3, we provide the necessary background for a precise formulation of the main results - theorems 10 and 12 - which appear in §1.4. These theorems are proved in the subsequent sections; a list of notation appears in §1.1.5 to facilitate reading of the article.

1.1.1 The equations

We study the following system of equations, called the nonlinear Klein-Gordon-Maxwell system, or (nl-KGM) system, which describe the interaction of a complex scalar field ϕ with an electromagnetic field $F_{\mu\nu}$ in the presence of an external space-time current $\mathbb{J}^B$:

$$\begin{aligned} &\partial^\mu F_{\mu\nu} = e\langle i\phi, \mathbb{D}_\nu \phi\rangle + e\mathbb{J}^B_\nu \\ &\mathbb{D}_\mu \mathbb{D}^\mu \phi + \mathcal{V}'(\phi) = 0. \end{aligned} \tag{1}$$

Here ϕ is a complex function on Minkowski space-time $\mathbb{R}^{1+3}$, and $\mathbb{D}_\mu = \partial_\mu - ie\mathbb{A}_\mu$ is the covariant derivative associated to an electromagnetic potential $\mathbb{A}_\mu dx^\mu = \mathbb{A}_0 dt + \mathbb{A}_j dx^j$ with associated field $F_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu$. (The operator $\mathbb{D}$ determines an S^1 connection over $\mathbb{R}^{1+3}$ whose curvature is $-iF$.) We use standard relativistic notation in which $\{x^\mu\}_{\mu=0}^{\mu=3}$ are co-ordinates, with greek indices running over $\{0,1,2,3\}$, $x^0 = t$ is the time co-ordinate, and $\{x^j\}_{j=1}^3$ are space co-ordinates with latin indices running over $\{1,2,3\}$; the Minkowski metric is

$$\eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2,$$

and is used to raise/lower indices in the usual way. When the spatial part of a space time vector or 1-form is considered separately bold face will often be used e.g. $\mathbf{x} = (x^1, x^2, x^3)$ for clarity. We refer to e as the (electromagnetic) coupling constant: for the purposes of this article it is a small positive parameter. The current four-vector is of the form

$$\mathbb{J}^B = \mathbb{J}^{B,\nu}\partial_\nu = \rho_B \partial_t + j^k_B \partial_k$$

and is conserved, i.e.

$$\partial_t \rho_B + \operatorname{div} \mathbf{j}_B = 0.$$

The quantity ρ_B is called the (background) charge density, while $\mathbf{j}_B$ is referred to as the (background spatial) current density. Throughout the paper we make the following hypotheses on the nonlinear potential function $\mathcal{V}$:

(H1) Phase invariance: there exists $G : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{V}(\phi) = G(|\phi|)$.

(H2) Positive mass: $\mathcal{V}(\phi) = \frac{m^2}{2}|\phi|^2 + \mathcal{V}_1(\phi)$ where $m > 0$ and $\mathcal{V}_1(\phi) = -U(|\phi|)$ is smooth with $U(0) = U'(0) = U''(0) = 0$.

(H3) Sub-criticality: the third derivative $D^{(3)}\mathcal{V}_1 = \mathcal{V}_1'''$ satisfies a growth condition $|\mathcal{V}_1'''(\phi)| \le c(1 + |\phi|^{p-3})$, for some $p \in (3,6)$. The significance of 6 is that it is the critical Sobolev exponent for the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$.

The function $\mathcal{V}$ is subject to a number of additional more specialized hypotheses, which we detail in §1.3.2, in particular to ensure existence and uniqueness of solitons solutions with the properties described in §1.3.

1.1.2 Solitons

The research in this paper is built upon the existence results for solitons in semi-linear wave equations given in [3],[4] and [22]. These solitons are time-periodic solutions of the nonlinear Klein-Gordon equation

$$\partial_\mu \partial^\mu \phi + \mathcal{V}'(\phi) = 0,$$

which is obtained by putting $e = 0$ in (1) (i.e. when there is no electromagnetic coupling), and are of the form

$$\phi(t, \mathbf{x}) = e^{i\omega t} f_\omega(\mathbf{x}).$$

T.D. Lee emphasized that solutions of this type, which he called non-topological solitons, provide a way of circumventing the Derrick-Pohozaev non-existence results on static solitons in scalar field theories; see [15, Chapter 7] for a discussion of their properties from the physical point of view.

It is proved in the references [3],[4] and [22] that, for certain potentials $\mathcal{V}$, solutions of this form exist with f_ω positive and radial. Also under further conditions these solutions are known to be essentially unique ([18]) and dynamically stable ([10, 24]); see §1.3 and the appendices for further details. For non-zero values of the coupling constant e solutions to (1) of this type have been constructed in [2, 1] directly, using a spherically symmetric ansatz, and perturbatively in [16, 17] for small e using the $e = 0$ case as a starting point. For small e it is possible to use the information on stability for $e = 0$ from [24] to prove modulational stability of the solitons and their Lorentz boosts, see §1.3.4 and [17] for details. Much of the same information for the $e = 0$ case will also be used in the present article to study the stability of the solitons when subjected to external (background) electromagnetic fields.

1.1.3 Informal statement of results on interaction of solitons with electromagnetic field

Our main concern in this article is to understand the interaction of the solitons just described, with an external electromagnetic field produced by the space-time current $\mathbb{J}^B$. In order to be able prove theorems giving precise information on the effect of this field on the soliton, we study (1) in a regime determined by two small parameters:

- The electromagnetic coupling constant $e = o(1)$.
- The external electric and magnetic fields, $\mathbf{E}^\delta_{ext}$ and $\mathbf{B}^\delta_{ext}$, vary over scales which are $O(\frac{1}{\delta})$, where $\delta = o(1)$. Thus the small parameter δ is the ratio of the size of the soliton to the length scale over which the external field varies.

The following is an informal version of our main theorems:

> *The system* (1) *has solutions which are close, in energy norm, to solitons of the type described above and which, in an appropriate scaling regime, move according to the Lorentz force equation:*
>
> $$\frac{d}{dt}\left(\gamma \mathbb{M}_S \mathbf{u}\right) = e\mathbb{Q}_S\left(\mathbf{E}^\delta_{ext} + \mathbf{u}\times\mathbf{B}^\delta_{\mathbf{ext}}\right), \tag{2}$$
>
> *where the effective mass* $\mathbb{M}_S$ *and charge* $\mathbb{Q}_S$ *of the soliton are as in* (60) *and* (61). *The scaling* $\delta = e^{1-k}$ *for* $k \in \left(0, \frac{1}{2}\right)$ *ensures that this holds for time intervals of length* $\frac{T_0}{e}$ *as* $e \to 0$.

The precise formulation is in the two theorems stated in §1.4.

1.1.4 Motivation and related work

Our interest in this problem stems from the classical, but ongoing, controversy surrounding the classical equation of motion for a point charge in an external electromagnetic field. The difficulty arises in attempts to account for the "back reaction" of the charge's own field on itself. Attempts to derive an equation of motion lead to modifications of the Lorentz force law (2) , most notably the Lorentz-Dirac equation ([21, Equation 9.1]). This equation is third order in time, and is difficult to interpret consistently without some further constraint on the type of solution allowed, due to the occurence of runaway solutions and violations of causality, (see [6] and [7, Chapter 28]). Recent discussions of this problem have been given in [9] and the books [21, 28]. One natural and well established approach to the problem of making sense of the back reaction is to start with a well-posed system of equations in which the point charge is explicitly replaced by a smooth bounded charge distribution, the Abraham model, or one of its generalizations like the Lorentz model, for example. One can then derive an equation of motion for the charge as an expansion, valid when the size of the charge distribution is small (compared with typical length scales set by the external fields), and show that this agrees with the Lorentz-Dirac equation at a certain order of approximation - see [14]. In this setting it turns out, however, that *at the same order* the Lorentz-Dirac equation can be approximated by a more conventional equation of motion which seems to be free of interpretational difficulties, (see [21, Equation 9.10], where the name Landau-Lifshitz equation is suggested for this effective equation of motion. The Landau-Lifshitz equation, which is second order in time, can be obtained formally from the Lorentz-Dirac equation by substituting for the third derivatives the expression obtained by differentiating the ordinary Lorentz force law (2) once in time.)

Our aim in studying soliton motion in the (nl-KGM) system is to attempt a similar analysis using a solitonic model for the particle (in place of the Abraham or Lorentz model). Our model has the virtue of being, in a very natural way, a Lorentz invariant system which is well posed (and so free of causality problems). Unfortunately the calculations required even just to derive the equation of motion for the soliton to $O(e)$ (i.e. the Lorentz force equation (2)) are long, and further work will be required to calculate additional corrections which may be compared with the Lorentz-Dirac equation in appropriate regimes. To achieve this, the starting point would be the equation of motion (116) for the soliton parameters derived from modulation theory. In §4 this equation is computed to highest order (i.e. to $O(e)$), and shown to give the Lorentz force law. A computation to the next order should give the Landau-Lifshitz equation ([21, Equation 9.10]). However it seems that some renormalization of the soliton mass and charge will have to be taken into account in this computation, and it is possible a refinement of the ansatz (62) will be needed to achieve this. It is to be hoped that at least in some simple cases such as one dimensional motion of the soliton in an electric field $\mathbf{E}^\delta_{ext} = (0,0,E(\delta t, \delta x))$ of fixed direction it will be possible to carry this through, and make a comparison with the corresponding specialization of the Landau-Lifshitz equation ([21, Equation 9.11]).

A corresponding theorem to our main result was proved for solitons in interaction with gravitational fields in the articles [25, 26]. The system treated there (Einstein's equation coupled to a nonlinear klein-Gordon equation) is in many ways more difficult than the one studied here (for example it is quasi-linear). Correspondingly, it is possible to carry out a more general analysis for the Klein-Gordon-Maxwell system under consideration here: in particular we emphasize that in the present article we are able to work entirely with the energy norm throughout (whereas for the Einstein system it was necessary to work with much stronger norms). There have also been theorems proved on effective dynamics for solitons moving under a potential in the nonlinear Schroedinger equation, see [12, 5, 27].

1.1.5 Notation

The following is a list of notations for important objects, with the section in which they are first introduced, for reference.

- $L^p(\mathbb{R}^3)$ is the Lebesgue space of (equivalence classes of) measurable functions with norm $\|f\|_{L^p} = \int_{\mathbb{R}^3} |f|^p \, dx < \infty$, and $H^k(\mathbb{R}^3)$ is the Sobolev space of (equivalence classes of) measurable functions with norm $\|f\|_{H^k} = \sum_{|\alpha|=0}^k \|\partial^\alpha f\|_{L^2} < \infty$, where ∂^α means the weak partial derivative determined by the multi-index α. We say $f \in H^k_{loc}$ if $\chi f \in H^k$ for every smooth, compactly supported χ, and

$$\dot{H}^1 = \left\{ f \in H^1_{loc} \cap L^6 : \|\nabla f\|_{L^2} = \|f\|_{\dot{H}^1} < \infty \right\}. \tag{3}$$

  Further we define H^k_r to be the intersection of H^k and the space of radial functions, i.e. functions of $|x|$, and similarly define L^p_r and $\dot{H}^1_r$.

- Electromagnetic potential $\mathbb{A}_\mu dx^\mu = \mathbb{A}_0 dt + \mathbb{A}_j dx^j$, electromagnetic field $F_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu$, and covariant derivative $\mathbb{D}_\mu = \partial_\mu - ie\mathbb{A}_\mu$: §1.1.1.

- Complex scalar (soliton) field ϕ and its self-interaction potential $\mathcal{V}(\phi) = G(|\phi|)$ subject to hypotheses (H1)-(H3): §1.1.1. Additional hypotheses (SOL), (KER) and (POS): §1.3.2 and §1.3.4.

- (nl-KGM) is the nonlinear Klein-Gordon-Maxwell system: (1) and §1.4.

- $\Psi = (\phi, \psi, \mathbb{A}_i, \mathbb{E}_i)$ is the dependent variable in the Hamiltonian formulation: §1.4 (and §1.3.1 for zero external current case).

- e electromagnetic coupling constant, δ external field scaling parameter are both small: §1.1.3 and §1.2.2.
- Scaled external electromagnetic potentials a_μ^δ, and electric and magnetic fields $\mathbf{E}_{ext}^\delta$ and $\mathbf{B}_{ext}^\delta$ induced by external current $(\rho_B^\delta, \mathbf{j}_B^\delta)$: §1.2.2.
- $\Psi_{ext}^\delta = (0, 0, \mathbf{a}^\delta, \mathbf{E}_{ext}^\delta)$ represents the external field in the Hamiltonian formulation, and $\Psi_{ext}^{\delta,\chi} = (0, 0, \mathbf{a}^{\delta,\chi}, \mathbf{E}_{ext}^\delta)$, its gauge transform by χ: §1.4 and §2.1.
- f_ω, $f_{\omega,e}$ are the soliton profile functions in (respectively) the $e = 0$ case and for non-zero e, while $\alpha_{\omega,e}$ is the A_0 component of the electromagnetic potential for soliton solutions: §1.3.
- $\Psi_{S,e}$ is the set of Lorentz transformed soliton solutions in Hamiltonian formalism, or $\Psi_{SC,e}$ in Coulomb gauge: §1.1.2 and §1.3. Gauge transformation to Coulomb gauge generated by ζ: §1.3.4.
- $\lambda = (\lambda_{-1}, \lambda_0, \lambda_1, \dots, \lambda_6) = (\omega, \theta, \boldsymbol{\xi}, \mathbf{u})$ are parameters for Lorentz transformed solitons: §1.3.
- $\gamma, P_{\mathbf{u}}, Q_{\mathbf{u}}, \Theta, \Theta_c, \mathbf{Z}, V_0(\lambda), N_\lambda, \tilde{\Xi}$ and ζ: §1.3.4.
- $\widetilde{O}_{stab} \in \mathbb{R}^8$ is the stable region of soliton parameter space, where Grillakis-Shatah-Strauss stability condition (39) holds and $\tilde{\Xi}$ is positive on the symplectic normal subspace N_λ: §1.3.4.
- $(\mathcal{H}_0, \Omega_0)$, $(\mathcal{H}, \Omega)$ and $\|\Psi\|_{\mathcal{H}}$ and $\|\Psi\|_{\mathcal{H}_s}^2$ are the symplectic phase spaces and norms:§1.3.1 and §1.2.1.
- W, K and Ξ are quadratic forms used in stability analysis: §2.3.
- $\widetilde{W}$ quantity like W but inluding certain nonlinear interaction $\tilde{H}$ parts of the Hamiltonian: §5.
- T_{loc}, T_0, T_1: §1.2.1,§1.4.1 and §2.2, respectively.
- $\widetilde{\partial_\lambda}$: §2.2.

1.2 The External Electromagnetic Field and Scaling

The external electromagnetic field $F_{\mu\nu}^{ext}$ is induced by the space-time current $\mathbb{J}^B = \rho_B \partial_t + \mathbf{j}_B^k \partial_k$ according to Maxwell's equations, i.e. the first equation of (1) with ϕ set equal to zero. Introducing an external electromagnetic potential, written in lower case symbols, $a_\mu dx^\mu = a_0 dt + a_j dx^j$, such that $F_{\mu\nu}^{ext} = \partial_\mu a_\nu - \partial_\nu a_\mu$, and imposing the Coulomb condition $\nabla \cdot \mathbf{a} = 0$, these equations can be written:

$$\begin{aligned} -\triangle a_0 &= e\rho_B, \\ \Box \mathbf{a} &= \nabla \partial_t a_0 - e\mathbf{j}_B. \end{aligned} \tag{4}$$

Here ρ_B is the background charge density, $\mathbf{j}_B$ is the background current density. The associated electric field, $\mathbf{E}_{ext}$, and magnetic field, $\mathbf{B}_{ext}$, are given by

$$\mathbf{E}_{ext} = \frac{\partial \mathbf{a}}{\partial t} - \nabla a_0, \tag{5}$$

$$\mathbf{B}_{ext} = \nabla \times \mathbf{a}. \tag{6}$$

We shall make the following assumptions on the external field:

(BG) The external electromagnetic potentials are smooth and satisfy:

$$\max_{\substack{|\alpha|=j \\ \mu=0,1,2,3}} \left\| \nabla_{t,x}^\alpha a_\mu \right\|_{L^\infty(\mathbb{R}^{1+3})} = L_j < \infty, \tag{7}$$

(using multi-index notation $\nabla_{t,x}^\alpha$ for arbitrary partial derivatives of order $|\alpha|$.)

It might appear that these assumptions are restrictive: in particular, the assumption that $\|\mathbf{a}\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^+)} < \infty$ precludes the consideration of a constant magnetic field. However, since we shall scale so that the external electric and magnetic fields do not change appreciably over the spread of the soliton, which is exponentially localized, these conditions could probably be relaxed with some further work. A more important restriction in our study appears to arise in the consideration of the scaling of the the external field, which we discuss below, after presenting results on local well-posedness for the (nl-KGM) system in the presence of an external field.

1.2.1 The Cauchy problem for (nl-KGM) in an external field

Throughout this article we make use of local well-posedness of the (nl-KGM) system in the energy norm. In the case that there is no external field and $\mathcal{V} \equiv 0$ this was proved in [13]. In this section we give conditions under which this is true in the more general situation of (1) considered here. Since our assumptions on the external field do not require finite energy it is convenient to subtract off the external field. Thus assume given an external electromagnetic potential $a_0 dt + a_j dx^j$ as above, in Coulomb gauge $\nabla \cdot \mathbf{a} = 0$, which solves the inhomogeneous Maxwell equations (4) and verifies (7). Write the electromagnetic potential appearing in (1) as $\mathbb{A}_\mu = a_\mu + A_\mu$. Then, requiring the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$, as is always possible, (1) is equivalent to the following system:

$$\begin{aligned} \dot{\phi} =& \psi + ie(a_0 + A_0)\phi, \\ \dot{\psi} =& \Delta\phi - 2ie(\mathbf{A} + \mathbf{a}) \cdot \nabla\phi - e^2 |\mathbf{A} + \mathbf{a}|^2 \phi - \mathcal{V}'(\phi) + ie(a_0 + A_0)\psi, \\ \Box\mathbf{A} =& \langle ie\phi, (\nabla - ie\mathbf{A})\phi\rangle + \nabla\dot{A}_0 - e^2 |\phi|^2 \mathbf{a}, \\ -\triangle A_0 =& \langle ie\phi, \psi\rangle, \end{aligned} \tag{8}$$

where $\mathbf{A} = (A_1, A_2, A_3)$ is the spatial part of $A = \mathbb{A} - a$. We solve this system in the energy space $\mathcal{H} \equiv H^1 \times L^2 \times \dot{H}^1 \times L^2$, which is endowed with the energy norm $\|\Psi\|_{\mathcal{H}} = \|(\phi, \psi, \mathbf{A}, \mathbf{E})\|_{H^1\times L^2\times\dot{H}^1\times L^2}$; see §1.1.5 for notation on standard norms. We also define corresponding higher energy norms indexed by $s \in \mathbb{N}$ by

$$\|(\phi, \psi, \mathbf{A}, \mathbf{E})\|_{\mathcal{H}_s}^2 \equiv \sum_{|\alpha|=0}^{s-1} \|\nabla_x^\alpha(\phi, \psi, \mathbf{A}, \mathbf{E})\|_{\mathcal{H}}^2, \tag{9}$$

with corresponding space denoted $\mathcal{H}_s$. We say that the Cauchy problem for (8) is locally well posed in $\mathcal{H}$ if the following two conditions hold:

(WP1) given initial data $\big(\phi(0), \psi(0), \mathbf{A}(0), \dot{\mathbf{A}}(0)\big) \in \mathcal{H}$ in Coulomb gauge (i.e. $\operatorname{div}\mathbf{A}(0) = 0$, $\operatorname{div}\dot{\mathbf{A}}(0) = 0$), satisfying

$$\Big\|\big(\phi(0), \psi(0), \mathbf{A}(0), \dot{\mathbf{A}}(0)\big)\Big\|_{\mathcal{H}} \le k_0 \tag{10}$$

there exists $T_{loc} = T_{loc}(k_0) > 0$ and a unique solution $\big((\phi(t), \psi(t), \mathbf{A}(t), \dot{\mathbf{A}}(t)\big)$ such that

$$\begin{aligned} &\big(\phi(t), \psi(t), \mathbf{A}(t), \dot{\mathbf{A}}(t)\big) \in C([0, T_{loc}); \mathcal{H}), \\ &\int_0^{T_{loc}} (\|\Box A\|_{L^2} + \|\Box\phi\|_{L^2})\, dt < \infty. \end{aligned}$$

(WP2) the solution is continuous with respect to the initial data in that, for another set of initial data $\big(\phi_1(0), \psi_1(0), \mathbf{A}_1(0), \dot{\mathbf{A}}_1(0)\big)$, which are close in $\mathcal{H}$, and also satisfy (10), and the Coulomb gauge conditions, the following holds on the common domain of definition $[0, T_{loc}]$, for some constant $c > 0$:

$$\max_{[0,T_{loc}]}\Big\|\big(\phi - \phi_1, \psi - \psi_1, \mathbf{A} - \mathbf{A}_1, \dot{\mathbf{A}} - \dot{\mathbf{A}}_1\big)\Big\|_{\mathcal{H}} \le c\Big\|\big(\phi(0) - \phi_1(0), \psi(0) - \psi_1(0), \mathbf{A}(0) - \mathbf{A}_1(0), \dot{\mathbf{A}}(0) - \dot{\mathbf{A}}_1(0)\big)\Big\|_{\mathcal{H}}.$$

As remarked above, in the absence of the external field, and with $\mathcal{V} \equiv 0$ the validity of (WP1)-(WP2) was proved in [13]. The general case was addressed in the thesis [17] where it was shown, using in addition Strichartz inequalities from [11, 23], that (WP1)-(WP2) hold if $\mathcal{V}$ is a smooth sub-critical nonlinearity:

Proposition 1 *Suppose $\mathcal{V}$ is smooth and that there exists a positive number $\kappa \in (0,4)$ such that, for all ϕ, φ,*

$$|\mathcal{V}'(\phi) - \mathcal{V}'(\varphi)| \leq C\,|\phi - \varphi| \left(1 + |\phi|^{4-\kappa} + |\varphi|^{4-\kappa}\right) \tag{11}$$

and that $\mathcal{V}'(0) = 0$. Assume that the external potential is smooth and verifies (7) for every non-negative integer j. Then the Cauchy problem for (8) is well-posed in the sense of (WP1) and (WP2). Further, if the initial data lie in $\mathcal{H}_s$ for some $s \geq 2$ then the solution exists for all time, and remains in $\mathcal{H}_s$, and is smooth if the initial data are smooth.

Remark 2 The Coulomb condition leaves a residual gauge invariance by functions $\chi(t, \mathbf{x})$ which are harmonic in $\mathbf{x}$. (These are either constant or unbounded.) In particular the system (8) is invariant under the transformation $a_\mu \mapsto a_\mu + \partial_\mu \chi$, $(\phi, \psi) \mapsto e^{ie\chi}(\phi, \psi)$ if $\chi = \alpha_0(t) + \alpha_j(t)x^j$ is linear in $\mathbf{x}$ and smooth in t. In this case the map $(\phi, \psi) \mapsto e^{i\chi}(\phi, \psi) = (\tilde{\phi}, \tilde{\psi})$ is Lipschitz on $H^1 \times L^2$. It follows that proposition 1 remains valid if the external potential is obtained from one satisfying (7) by gauge transformation by $\chi = \alpha_0(t) + \alpha_j(t)x^j$.

Remark 3 Notice that when the nonlinearity is determined by a smooth function $\mathcal{V}$ whose third derivative satisfies:

$$\left| D^{(3)}\mathcal{V}(\phi) \right| \leq c(1 + |\phi|^{3-\kappa}), \quad \text{for all } \phi \tag{12}$$

for some $c > 0, 0 < \kappa < 3$ the conditions of proposition 1 hold, and the Cauchy problem is well-posed. This assumption is also sufficient to estimate the nonlinear terms in the perturbation theory developed in §2,§3 and §5 of this article. Introduce $\mathcal{F}(\phi) = \mathcal{V}'(\phi) - m^2\phi = \mathcal{V}_1'(\phi) = \beta(|\phi|)\phi$ as the nonlinear part of $\mathcal{V}'(\phi)$, with $\mathcal{V}$ as in the introduction. Then (12) implies the inequality

$$|\mathcal{F}'(f + v) - \mathcal{F}'(f)| \leq c(1 + |f|^3)(|v| + |v|^4) \quad \text{where } c \text{ is a positive constant,} \tag{13}$$

which is convenient for our use. In fact, for our purposes it would be sufficient to make the following slightly more general assumption on $\mathcal{F}$:

$$\text{For all } f > 0 \text{ and for any } v,\ |\mathcal{F}'(f + v) - \mathcal{F}'(f)| \leq c(f^{r-1} + f^3)(|v| + |v|^4), \tag{14}$$

where r, c are positive constants, see [16]. Of course given a smooth potential $\mathcal{V}$ satisfying (12), let $\mathcal{F}$ be as just defined, then (14) will also hold with $r = 1$.

1.2.2 Scaling the external fields

As already mentioned, we require that the external electric and magnetic fields are approximately constant over the soliton. To ensure this, we introduce a scaled version of the external fields. Thus, we have

$$a_0^\delta(t, \mathbf{x}) = \frac{1}{\delta} a_0(\delta t, \delta \mathbf{x}), \qquad \mathbf{a}^\delta(t, \mathbf{x}) = \frac{1}{\delta}\mathbf{a}(\delta t, \delta \mathbf{x}), \tag{15}$$

with the scaled external electric and magnetic fields given by:

$$\mathbf{E}^\delta_{ext} = \mathbf{E}_{ext}(\delta t, \delta \mathbf{x}), \qquad \mathbf{B}^\delta_{ext} = \mathbf{B}_{ext}(\delta t, \delta \mathbf{x}). \tag{16}$$

Clearly these fields correspond to the following rescaled charge and current densities:

$$\rho^\delta_B(t, x) = \delta \rho_B(\delta t, \delta x), \qquad \mathbf{j}^\delta_B(t, x) = \delta \mathbf{j}^\delta_B(\delta t, \delta x). \tag{17}$$

Henceforth, we shall almost always refer exclusively to the scaled fields. It remains to choose the length scale, $\frac{1}{\delta}$, over which the external fields change: this is determined by the analysis in §5 which bounds the deviation of the solution from the modulated soliton. This analysis seems to require two main conditions on the scaling of δ and e:

- From lemma 21, it seems that we need

$$\lim_{e\to 0}\frac{e}{\delta}=0, \tag{18}$$

to bound the effect of the scaled external electromagnetic potential.

- Treatment of the last term in (137), seems to suggest that we need

$$\lim_{e\to 0}\frac{\delta^2}{e}=0. \tag{19}$$

This condition is used to ensure the deviation from the Lorentz force law is small for times of order $\frac{1}{e}$.

We will consider the limit $e \to 0$ with

$$\delta = e^{1-k} \tag{20}$$

for some constant $k \in \left(0, \frac{1}{2}\right)$, so that both of these conditions hold. It remains to be seen what are the optimal conditions for scaling e, δ under which the results of this paper hold.

1.3 Non-Topological Solitons

We now discuss existence and stability properties of non-topological solitons as solutions of (nl-KGM) in the absence of external fields. This means we are here concerned with the (nl-KGM) system with $\rho_B = 0 = \mathbf{j}_B$. We first discuss the Hamiltonian formulation of (nl-KGM), since that gives the appropriate context in which to introduce non-topological solitons.

1.3.1 Hamiltonian formalism

It is useful to present the Hamiltonian formalism for the (nl-KGM), not least because it will give us a language which we shall use in proving the existence and long-time stability of the non-topological solutions. Indeed, as we shall see, from the Hamiltonian point of view, non-topological solitons are relative equilibria, and recognizing this fact leads to the identification of the appropriate quantities with which to work.

In order to define the phase space we recall the standard function spaces defined in §1.1.5.

To start with, consider the nonlinear wave equation in isolation

$$\Box\phi + \mathcal{V}'(\phi) = 0. \tag{21}$$

This can be written as a Hamiltonian system on the phase space $\mathcal{H}_0 \equiv \{(\phi, \psi) \in H^1 \times L^2\}$, with symplectic form $\Omega_0((\phi', \psi'), (\dot\phi, \dot\psi)) = \int \langle \phi', \dot\psi\rangle - \langle \psi', \dot\phi\rangle \, dx$, and Hamiltonian

$$H_0(\phi,\psi) = \frac{1}{2}\int |\nabla\phi|^2 + 2\mathcal{V}(\phi). \tag{22}$$

The corresponding Hamiltonian evolution equations, equivalent to (21), are :

$$\partial_t \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \triangle\phi - \mathcal{V}'(\phi) \end{pmatrix}. \tag{23}$$

Next, for (nl-KGM), introduce the phase space

$$\mathcal{H} \equiv \{\Psi = (\phi, \psi, \mathbf{A}, \mathbf{E}) \in H^1 \times L^2 \times \dot H^1 \times L^2\}, \tag{24}$$

which is endowed with the norm $\|\Psi\|_{\mathcal{H}} = \|(\phi,\psi,\mathbf{A},\mathbf{E})\|_{H^1\times L^2\times \dot H^1 \times L^2}$ and the (densely defined, weak) symplectic form

$$\Omega(\Psi', \dot\Psi) = \int \langle\phi', \dot\psi\rangle - \langle\psi', \dot\phi\rangle + \mathbf{A}'\cdot\dot{\mathbf{E}} - \mathbf{E}'\cdot\dot{\mathbf{A}} dx, \tag{25}$$

where $\Psi' = (\phi', \psi', \mathbf{A}', \mathbf{E}')$ and similarly for $\dot{\Psi}$. The (nl-KGM) equations with $\rho_B = 0 = \mathbf{j}_B$ arise formally as the Hamiltonian flow on $\mathcal{H}$ associated to the Hamiltonian

$$H(\phi, \psi, \mathbf{A}, \mathbf{E}) = \frac{1}{2} \int \left(|\mathbf{E}|^2 + |\nabla \times \mathbf{A}|^2 + |\psi|^2 + |\nabla_{\mathbf{A}} \phi|^2 + 2\mathcal{V}(\phi) \right), \tag{26}$$

and subject to the constraint:

$$\mathcal{C}_0 \equiv \operatorname{div} \mathbf{E} - \langle ie\phi, \psi \rangle = 0. \tag{27}$$

Here $\nabla_{\mathbf{A}} \phi$ is the covariant derivative of ϕ given by $\nabla_{\mathbf{A}} \phi = \nabla \phi - ie\mathbf{A}\phi$ and $\mathbf{A}$ is the spatial part of the gauge field. The equations of motion for the augmented Hamiltonian $H_1 = H - \int A_0 \mathcal{C}_0$ are:

$$\partial_t \begin{pmatrix} \phi \\ \psi \\ \mathbf{A}_i \\ \mathbf{E}_i \end{pmatrix} = \begin{pmatrix} \psi + ieA_0 \phi \\ \triangle_{\mathbf{A}} \phi - \mathcal{V}'(\phi) + ieA_0 \psi \\ \mathbf{E}_i + \nabla_i A_0 \\ \triangle \mathbf{A}_i - \nabla_i (\operatorname{div} \mathbf{A}) + \langle ie\phi, \nabla_{\mathbf{A}} \phi \rangle \end{pmatrix} \tag{28}$$

where the “Lagrange multiplier” A_0 is identifiable with the temporal part of the gauge field, $\triangle_{\mathbf{A}} \phi = \triangle \phi - 2ie\mathbf{A} \cdot \nabla \phi - ie \operatorname{div} \mathbf{A} \phi + e^2 |\mathbf{A}|^2 \phi$, $i = 1...3$, and we have not yet imposed any gauge condition.

1.3.2 Existence of non-topological solitons: the $e = 0$ case

The class of solitary wave solutions of interest is that of non-topological solitons discussed in [15, Chapter 7]. These are examples of a special type of solution to a Hamiltonian system with symmetry called relative equilibrium: this means that the time evolution is given by an orbit of a one parameter subgroup of the symmetry group. For (23) the Hamiltonian is invariant under the action of S^1 by phase rotation, as long as $\mathcal{V}(\phi) = G(|\phi|)$ is a function of $|\phi|$ only; the charge corresponding to this S^1 action is

$$Q(\phi, \psi) = \int \langle i\psi, \phi \rangle \, dx.$$

A relative equilibrium is then a solution of the form $(\phi, \psi) = e^{i\omega t}(f_\omega(x), i\omega f_\omega)$, where f_ω is a real valued function which satisfies an elliptic equation. These solutions are critical points of the functional $H_0 + \omega Q$, often called the *augmented Hamiltonian* in this context. We consider G of the form

$$G(f) = \frac{m^2}{2} f^2 - U(f), \quad \text{with} \quad U(f) = \int_0^{|f|} t\beta(t) dt.$$

then the equation satisfied by f_ω is $-\triangle f_\omega + (m^2 - \omega^2) f_\omega = \beta(f_\omega) f_\omega$. This equation typically has many solutions (see [3] and references therein), but we are only interested in positive, radially symmetric solutions because it is these which are dynamically stable: these are sometimes called the *ground state solitons*. Thus, crucial to our analysis is the following hypothesis on existence and uniqueness of the $e = 0$ ground state soliton:

(SOL) For $\omega^2 < m^2$, there exists a unique positive radial function $f_\omega \in H^4(\mathbb{R}^3)$ which solves

$$\left(-\triangle + m^2 - \omega^2 \right) f_\omega = \beta(f_\omega) f_\omega.$$

Theorem 4 *The existence part of* (SOL) *holds under the following conditions:*

$$U'(f) = -U'(-f) \quad \textit{and} \quad U' \in C^1(\mathbb{R}) \cap C^2((0, \infty)), \tag{29}$$

$$U'(0) = U''(0) = 0 \quad \textit{and} \quad \exists s \in (0,1) : \lim_{f \to 0} f^s U'''(f) = 0, \tag{30}$$

$$\exists \zeta > 0 : U(\zeta) > \frac{m^2 - \omega^2}{2} \zeta^2, \tag{31}$$

$$\lim_{f \to \infty} \frac{U'(f)}{f^5} = 0. \tag{32}$$

The uniqueness part of (SOL) *holds under the additional conditions:*

(U1) $\exists l_1 > 0 : 0 < f < l_1 \Longrightarrow U'(f) < (m^2 - \omega^2) f$
and $l_1 < f < \infty \Longrightarrow U'(f) > (m^2 - \omega^2) f$
and $U''(l_1) - (m^2 - \omega^2) > 0$,

and that

(U2) *For* $l_2 > l_1, \exists \lambda = \lambda(l_2) \in C[(l_1, \infty), \mathbb{R}^+]$
such that $2(m^2 - \omega^2) f + \lambda f U'(f) - (\lambda + 2) U'(f)$
is non-negative on $(0, l_2)$ *and non-positive on* (l_2, ∞).

Proof The existence part of this hypothesis was proved in [3] under the given conditions on the nonlinearity. It was shown in several articles (see for example, [18], where further references are given), that these solutions are unique under the given additional conditions. □

The following two operators, $L_\pm(\omega)$, which appear on linearizing (23) about the soliton solution, are crucial to an understanding of the stability and dynamical properties of the $e = 0$ soliton:

$$\begin{aligned} L_+(\omega) &= -\triangle + m^2 - \omega^2 - \beta(f_\omega) - \beta'(f_\omega) f_\omega, \\ L_-(\omega) &= -\triangle + m^2 - \omega^2 - \beta(f_\omega). \end{aligned} \tag{33}$$

We make the following hypothesis on $L_+(\omega)$:

(KER) The kernel of $L_+(\omega)$ is empty in $H^2_r(\mathbb{R}^3)$.

(Recall that H^s_r was defined as the space of radial Sobolev H^s functions, immediately following (3).)

Theorem 5 *The hypothesis* (KER) *is valid under the conditions* (U1)-(U2).

Proof See [18]: establishing (KER) is a crucial step in proving uniqueness of the positive function f_ω. □

The operators $L_\pm$ also determine stability properties of the soliton. For proving stability the following spectral assumption is used:

(S1) The subspace in which L_+ is strictly negative is one dimensional,

This assumption is valid for the ground state solitons f_ω obtained by the constrained minimisation technique of [3], because they are minimizers subject to a single constraint, see [24] (where a direct proof in the pure power case is also given).
Some additional more technical results on the solitons can be found in appendix A.1.

1.3.3 Existence of non-topological solitons: the general case

We now show that for small values of the coupling constant e the ground state solitons just discussed can be continued (via the implicit function theorem) to give soliton solutions of (28). The properties of the $e = 0$ soliton needed to achieve this were detailed already in §1.3.2. As shown in [1, 2] it is also possible to obtain soliton solutions for systems like (28) by variational techniques *applied within the class of radial functions*, but for present purposes we prefer to use the implicit function theorem so that we can carry over stability information from the $e = 0$ case, which seems to be hard to obtain otherwise.

Generalizing the class of non-topological solitons to the case of the gauge invariant system (28) leads us to search for solutions to (28) of the form

$$\begin{pmatrix} \phi \\ \psi \\ \mathbf{A} \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} Exp[i\omega t] f_{\omega,e} \\ Exp[i\omega t] i(\omega - e\alpha_{\omega,e}) f_{\omega,e} \\ 0 \\ -\nabla \alpha_{\omega,e} \end{pmatrix}, \tag{34}$$

where we have emphasized the dependence on the parameters ω and e; we will assume the functions $f_{\omega,e}$ and $\alpha_{\omega,e}$ to be radially symmetric. It can easily be checked that this gives a solution to (28) with $A_0 = \alpha_{\omega,e}$ as long as the functions $f_{\omega,e}$ and $\alpha_{\omega,e}$ satisfy

$$-\triangle \alpha_{\omega,e} + e^2 f_{\omega,e}^2 \alpha_{\omega,e} - e\omega f_{\omega,e}^2 = 0, \tag{35}$$

$$-\triangle f_{\omega,e} - U'(f_{\omega,e}) + (m^2 - (\omega + e\alpha_{\omega,e})^2) f_{\omega,e} = 0. \tag{36}$$

The first of these equations implies $\mathcal{C}_0 = 0$. It can readily be checked that if a gauge transformation is made to put the solution thus obtained into temporal gauge, $A_0 = 0$, then its time dependence amounts to the action of the one parameter group of gauge transformations $e^{i(\omega - e\alpha_{\omega,e})t}$, so that it is indeed a relative equilibrium solution as defined above.

Theorem 6 ([17]) *Assume that the hypotheses (SOL) and (KER) hold for ω_0 with ${\omega_0}^2 < m^2$. Then, there exists a neighbourhood U of ω_0 such that for $\omega \in U$, there is a number $e(\omega) > 0$ such that for $\omega \in U, |e| < e(\omega)$, there exists $f_{\omega,e} \in H_r^2(\mathbb{R}^3)$ such that*

$$-\triangle f_{\omega,e} + m^2 f_{\omega,e} - (\omega - e\alpha_{\omega,e})^2 f_{\omega,e} = \beta(f_{\omega,e}) f_{\omega,e}, \tag{37}$$

where $\alpha_{\omega,e} \in \dot{H}_r^1(\mathbb{R}^3)$ is a non-local function of $f_{\omega,e}$ uniquely determined by

$$-\triangle \alpha_{\omega,e} + e^2 f_{\omega,e}^2 \alpha_{\omega,e} = \omega e f_{\omega,e}^2. \tag{38}$$

In addition the map $\omega \mapsto f_{\omega,e}$ is C^2 from U to H_r^2.

1.3.4 Stability in the absence of an external field

The stability of the solutions to (23) of the form $e^{i\omega t} f_\omega(x)$ was first considered in [20, 10] where it was proved that the positive radial solution was stable, with respect to radially symmetric perturbations of the initial data, as long as

$$\partial_\omega \left(\omega \, \| f_\omega \|_{L^2}^2 \right) < 0. \tag{39}$$

It was also shown that the solutions are unstable when this quantity is positive. In [24] an alternative, modulational, approach to stability was adopted along the lines of [27], with the aim, both of generalizing previous stability results to prove stability of uniformly moving solutions with respect to arbitrary (non-symmetric) perturbations, and also of providing techniques which could provide useful information in dynamically non-trivial settings. The presence of external fields is an example of the latter circumstance, and so the analysis in this article is based on that in [24], which we will now summarize. It turns out that the condition (39) implies the strict positivity of the Hessian of the augmented Hamiltonian on the symplectic normal space to the space of solitons. To explain this properly in the generality needed it is necessary to consider the action of the Poincare (or inhomogeneous Lorentz) group

Action of the Poincare group on the solitons. The equations (28) are Poincare covariant. The action of the Poincare group on the radial soliton (34) gives a family of functions depending smoothly on eight parameters $\{\lambda_A\}_{A=-1}^6$, with

$$\lambda = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_6) = (\omega, \theta, \boldsymbol{\xi}, \mathbf{u}) \tag{40}$$

determining (respectively) the frequency, the phase, the centre and the velocity of the soliton. Explicitly:

$$\Psi_{S,e}(\mathbf{x};\lambda) = \begin{pmatrix} Exp[i\Theta](f_{\omega,e}(\mathbf{Z})) \\ Exp[i\Theta](i\gamma(\omega - e\alpha_{\omega,e}(\mathbf{Z}))f_{\omega,e}(\mathbf{Z}) - \gamma\mathbf{u}\cdot\nabla_{\mathbf{Z}} f_{\omega,e}(\mathbf{Z})) \\ -\gamma\mathbf{u}\alpha_{\omega,e}(\mathbf{Z}) \\ -(\frac{1}{\gamma}P_{\mathbf{u}} + \gamma Q_{\mathbf{u}})\nabla_{\mathbf{Z}}\alpha_{\omega,e}(\mathbf{Z}) \end{pmatrix}. \tag{41}$$

Here the projection operators $P_{\mathbf{u}} : \mathbb{R}^3 \to \mathbb{R}^3$ and $Q_{\mathbf{u}} : \mathbb{R}^3 \to \mathbb{R}^3$ are defined by $(P_{\mathbf{u}})_{ij} = \frac{u_i u_j}{|\mathbf{u}|^2}$ and $Q_{\mathbf{u}} = 1 - P_{\mathbf{u}}$, and

$$\mathbf{Z}(\mathbf{x},\lambda) = \gamma P_{\mathbf{u}}(\mathbf{x} - \boldsymbol{\xi}) + Q_{\mathbf{u}}(\mathbf{x} - \boldsymbol{\xi}), \tag{42}$$

$$\Theta(\mathbf{x},\lambda) = \theta - \omega\mathbf{u}\cdot\mathbf{Z}, \tag{43}$$

with $\gamma(\mathbf{u}) = \frac{1}{\sqrt{1-|\mathbf{u}|^2}}$. The parameters are required to lie in the set $\widetilde{O} \subset \mathbb{R}^8$ defined by

$$\widetilde{O} \equiv \{(\omega,\theta,\boldsymbol{\xi},\mathbf{u}) \subset \mathbb{R}^8 : |\mathbf{u}| < 1 \text{ and } \omega^2 < m^2\}. \tag{44}$$

The parameter range corresponding to stable solitons is

$$\widetilde{O}_{stab} \equiv \{(\omega,\theta,\boldsymbol{\xi},\mathbf{u}) \subset \widetilde{O} : \text{ condition (39) holds}\}. \tag{45}$$

The Poincare covariance of the equations of motion (28) implies that the solitons given by (41) form an eight parameter family of solutions $t \mapsto \Psi_{S,e}(\mathbf{x};\lambda(t))$ of (28) as long as $\frac{d}{dt}\lambda = V_0(\lambda)$, where V_0 is the vector field on $\widetilde{O}$ defined by

$$V_0(\lambda) \equiv \big(0, \frac{\omega}{\gamma}, \mathbf{u}, 0\big), \tag{46}$$

for $\lambda = (\omega,\theta,\boldsymbol{\xi},\mathbf{u})$.

The case of the nonlinear wave equation (23) can be obtained by putting $e = 0$ in the first two components of the formulae just given. Simplifying to this case we obtain an eight parameter family of functions,

$$\big(\phi_{S,0},\psi_{S,0}\big)(\mathbf{x};\lambda) \equiv e^{i\Theta}\big(f_\omega(\mathbf{Z}), (i\gamma\omega f_\omega(\mathbf{Z}) - \gamma\mathbf{u}\cdot\nabla_{\mathbf{Z}} f_\omega(\mathbf{Z}))\big) \tag{47}$$

such that

$$t \mapsto (\phi_{S,0},\psi_{S,0})(\mathbf{x};\lambda(t)),$$

solves (23), as long as $\frac{d}{dt}\lambda = V_0(\lambda)$, with V_0 as above.

Stability for $e = 0$ (nonlinear Klein-Gordon). The starting point for stability analysis is the observation that $(\phi_{S,0},\psi_{S,0})$ is a critical point of the *augmented Hamiltonian*

$$F_0(\phi,\psi;\lambda) = H_0(\phi,\psi) + u^i\Pi_i(\phi,\psi) + \frac{\omega}{\gamma}Q(\phi,\psi) \tag{48}$$

where H_0, Q are the functionals defined above, and Π_i are the momenta $\Pi_i(\phi,\psi) = \int\langle\psi,\partial_i\phi\rangle\,dx$. The Hessian of F_0 at $(\phi_{S,0},\psi_{S,0})$ is a quadratic form depending upon λ:

$$\tilde{\Xi}(\tilde\phi,\tilde\psi;\lambda) \equiv D^2F_0(\phi_{S,0},\psi_{S,0};\lambda)((\tilde\phi,\tilde\psi),(\tilde\phi,\tilde\psi)).$$

Introduce the subspace

$$N_\lambda \equiv \{(\tilde\phi,\tilde\psi) \in H^1\times L^2 : \Omega_0\big((\tilde\phi,\tilde\psi), \partial_\lambda(\phi_{S,0},\psi_{S,0})(\lambda)\big) = 0\} \tag{49}$$

then the following hypothesis is crucial for stability:

(POS) For each compact $\mathbf{K} \subset \widetilde{O}_{stab}$ $\exists\tau_* = \tau_*(\mathbf{K}) > 0$ such that
$\tilde{\Xi}(\tilde\phi,\tilde\psi;\lambda) \geq \tau_*\|(\tilde\phi,\tilde\psi)\|^2_{H^1\times L^2}$ for all $(\tilde\phi,\tilde\psi) \in N_\lambda$.

Remark 7 $\widetilde{O}_{stab}$ is the set of parameter values corresponding to stable solitons, which are obtained as Poincare transforms of solitons $e^{i\omega t}f_\omega$ with ω such that (39) holds.

Theorem 8 ([24]) *If the nonlinearity satisfies the conditions given in §1.3.2 then (POS) is true. Furthermore, solitons of* (23) *corresponding to frequencies* ω *such that* (39) *holds are modulationally stable with respect to small, arbitrary perturbation in energy norm. To be precise, consider the initial value problem for* (23) *with initial data close to a soliton* $\big(\phi_{S,0},\psi_{S,0}\big)(\cdot;\lambda(0)\big)$ *with* $\lambda(0)\in\widetilde{O}_{stab}$*, in the sense that*

$$\epsilon=\big\|\big(\phi(0,\cdot),\psi(0,\cdot)\big)-\big(\phi_{S,0},\psi_{S,0}\big)(\cdot;\lambda(0)\big)\big\|_{\mathcal{H}_0}$$

is sufficiently small. Then there exists a global solution which satisfies:

$$\sup_{t\in\mathbb{R}}\big\|\big(\phi(t,\cdot),\psi(t,\cdot)\big)-\big(\phi_{S,0},\psi_{S,0}\big)(\cdot;\lambda(t)\big)\big\|_{\mathcal{H}_0}\le c\epsilon, \tag{50}$$

for some C^1 *curve* $t\mapsto\lambda(t)\in\widetilde{O}_{stab}$.

Stability for small e (nonlinear Klein-Gordon-Maxwell). It was shown in [17], that stability holds also for solitons in (28) under the condition (39), for sufficiently small values of the electromagnetic coupling constant e. This was proved using the Coulomb condition, so we first write down the soliton solutions (41) in Coulomb gauge. (The Coulomb condition is not invariant under Lorentz boosts, therefore, it is necessary to perform a gauge transformation to move the Lorentz boosted solitons into the Coulomb gauge). The Lorentz boosted solitons $\Psi_{SC,e}$ in the Coulomb gauge have the form

$$\begin{pmatrix}\phi_{SC,e}(\mathbf{x})\\ \psi_{SC,e}(\mathbf{x})\\ \mathbf{A}_{SC,e}(\mathbf{x})\\ \mathbf{E}_{SC,e}(\mathbf{x})\end{pmatrix}=\begin{pmatrix}Exp[i\Theta_C](f_{\omega,e}(\mathbf{Z}))\\ Exp[i\Theta_C](i\gamma(\omega-e\alpha_{\omega,e}(\mathbf{Z}))f_{\omega,e}(\mathbf{Z})-\gamma\mathbf{u}.\nabla_{\mathbf{Z}}f_{\omega,e}(\mathbf{Z}))\\ -\gamma\mathbf{u}\alpha_{\omega,e}(\mathbf{Z})+\nabla\zeta\\ -(\frac{1}{\gamma}P_{\mathbf{u}}+\gamma Q_{\mathbf{u}})\nabla_{\mathbf{Z}}\alpha_{\omega,e}(\mathbf{Z})\end{pmatrix} \tag{51}$$

where $\Theta_C=\Theta+ie\zeta$, and $\zeta(x;\lambda)$ is a solution of

$$-\triangle\zeta=-\gamma\mathbf{u}\cdot\nabla\alpha_{\omega,e}(\mathbf{Z}). \tag{52}$$

It is a smooth function of x and also depends smoothly on λ; requiring that $\nabla\zeta\in L^p, p>3$ fixes it up to a constant. Some estimates for ζ are given in appendix A.1.2. The temporal part of the gauge field is given by

$$(A_{SC,e})_0=\gamma\alpha_{\omega,e}(Z)+\dot{\zeta}=\gamma\alpha_{\omega,e}(Z)+V_0(\lambda)\cdot\partial_\lambda\zeta.$$

Theorem 9 ([17]) *In the situation of the previous theorem the solitons* (51) *of* (28) *corresponding to frequencies* ω *such that* (39) *holds are, for sufficiently small* $|e|$*, modulationally stable in Coulomb gauge with respect to small, arbitrary perturbation of the initial data in energy norm* $\|\cdot\|_{\mathcal{H}}$ *defined in* (24)*. The stability is in the same sense as in the previous theorem, see [17] for full details.*

1.4 The main theorems

We now explain and state our main results on the interaction of the solitons of §1.3 with the scaled external electromagnetic field of §1.2. We write the total electromagnetic potential as $\mathbb{A}=\mathbb{A}_\mu dx^\mu$ (as described in §1.1.1) with corresponding electric field $\mathbb{E}_j=\partial_t\mathbb{A}_j-\partial_j\mathbb{A}_0$. The potential $\mathbb{A}$ will be formed from three constituents:

1. the external field, produced by a background charge ρ_B^δ and current $\mathbf{j}_B^\delta$, and scaled as described in §1.2,

2. the soliton contribution, as described in §1.3 but with parameters $\lambda(t)$ varying in a dynamically determined way,

3. an additional component produced by interaction of the initial data with the two previous components. This component is not explicitly given, and must be estimated.

Similarly, the solitonic field will be made up of a component which is the moving soliton, and a remainder produced by interactions, which must be estimated.

It is convenient to write the (nl-KGM) equations in first order form. Including the scaled background current density, the equations read:

$$\partial_t \begin{pmatrix} \phi \\ \psi \\ \mathbb{A}_i \\ \mathbb{E}_i \end{pmatrix} = \begin{pmatrix} \psi + ie\mathbb{A}_0\phi \\ \triangle_{\mathbb{A}}\phi - \mathcal{V}'(\phi) + ie\mathbb{A}_0\psi \\ \mathbb{E}_i + \nabla_i\mathbb{A}_0 \\ \triangle\mathbb{A}_i + \langle ie\phi, \nabla_{\mathbb{A}}\phi\rangle - e\mathbf{j}_B^\delta \end{pmatrix}, \tag{53}$$

with the Coulomb gauge condition imposed. These equations are to be solved with the Gauss law

$$\operatorname{div}\mathbb{E} - \langle ie\phi, \psi\rangle = \rho_B^\delta, \tag{54}$$

as a constraint. We shall abbreviate a general solution by making use of the following definition:

$$\Psi = (\phi, \psi, \mathbb{A}_i, \mathbb{E}_i)\,, \tag{55}$$

with $i \in \{1, 2, 3\}$.

Using this Hamiltonian formulation with Ψ as dynamical variable we write the external field

$$\Psi_{ext}^\delta = (0, 0, \mathbf{a}^\delta, \mathbf{E}_{ext}^\delta).$$

It will be convenient also to have the freedom of applying a gauge transformation $\chi(t, x)$ to this:

$$\Psi_{ext}^{\delta,\chi} = (0, 0, \mathbf{a}^{\delta,\chi}, \mathbf{E}_{ext}^\delta),$$

with $\mathbf{a}^{\delta,\chi} = \mathbf{a}^\delta + \mathbf{d}\chi$. The aim is now to construct a solution Ψ to (53) consisting of $\Psi_{ext}^{\delta,\chi}$ with a soliton $\Psi_{SC,e}(\lambda)$ superimposed. We choose the gauge transformation χ so that the transformed external electromagnetic potentials vanish along the world-line of the soliton $\mathbf{x} = \boldsymbol{\xi}(t)$; in particular, at $t = 0$ we will choose

$$\chi(0, \mathbf{x}) = \chi_0(\mathbf{x}) = -(\mathbf{x} - \boldsymbol{\xi}(0)) \cdot \mathbf{a}^\delta(0, \boldsymbol{\xi}(0))$$

so that $\mathbf{a}^{\delta,\chi_0}(0, \mathbf{x}) = \mathbf{a}^\delta(0, \mathbf{x}) - \mathbf{a}^\delta(0, \boldsymbol{\xi}(0))$.

1.4.1 Stability in the presence of an external field

The following theorem asserts the long time stability, under the influence of an external field, of stable solitons to (53). Recall that the stable solitons are those parametrised by $\lambda \in \widetilde{O}_{stab}$, so that (39) and hence (POS) hold, and they are stable by theorems 8 and 9 in the absence of an external field.

Theorem 10 *Assume that the nonlinearity satisfies the hypotheses (H1)-(H3), and also is such that the hypotheses (SOL), (KER) and (POS) in §1.3.2 hold. In addition, assume that the external field satisfies the assumptions in §1.2. Suppose further that the scaling parameters satisfy* $\delta^2 = o(e), e = o(1)$ *and* $e = o(\delta)$.

(i) Consider initial data of the form

$$\Psi(0) = (\phi_{SC,e}(\lambda(0)), \psi_{SC,e}(\lambda(0)), \mathbb{A}_i(0), \mathbb{E}_i(0))$$

where $\lambda(0) \in \widetilde{O}_{stab}$ *corresponds to a stable soliton (which verifies (POS)). It follows that, if* e *is sufficiently small and*

$$\left\|\Psi(0) - \Psi_{ext}^{\delta,\chi_0}(0) - \Psi_{SC,e}(\lambda(0))\right\|_{\mathcal{H}}^2 = o(e), \tag{56}$$

there exists

- *a positive number $T_0 > 0$, independent of e,*
- *a C^1 gauge transformation $\chi(t, \mathbf{x})$ defined in (63), linear in $\mathbf{x}$ at each time t, satisfying $\chi(0, \mathbf{x}) = \chi_0(\mathbf{x})$*
- *a curve $\lambda(t) \in C^1([0, \frac{T_0}{|e|}], \widetilde{O}_{stab})$, and*
- *a distributional solution $\Psi(t)$ of (53),*

such that

$$\Psi(t) - \Psi^{\delta,\chi}_{ext}(t) \in C([0, \frac{T_0}{|e|}]; \mathcal{H})$$

and

$$\sup_{t\in[0,\frac{T_0}{|e|}]} \left\|\Psi(t) - \Psi^{\delta,\chi}_{ext}(t) - \Psi_{SC,e}(\lambda(t))\right\|^2_{\mathcal{H}} = o(e), \tag{57}$$

Furthermore, $\lambda(t)$ satisfies a system of ordinary differential equations given by (116) with $|\partial_t\lambda - V_0(\lambda)| = O(e)$. The time component of the potential $\mathbb{A}_0$ is determined by the Coulomb condition and the Gauss law, (71), and has properties detailed in §2.

(ii) More generally, the same conclusions hold for initial data sufficiently close to a stable soliton in an appropriate sense: see §2.4.3 for a precise statement.

This theorem is proved in §2.

Remark 11 As explained in 1.3.2, if the nonlinear potential satisfies (29)-(32), $U(1)$, $U(2)$, $S(1)$ above in addition to (H1)-(H3) then the conditions (SOL), (KER) and (POS) all hold.

1.4.2 Motion in the presence of an external field: the Lorentz force

The previous theorem provides ordinary differential equations (116) which determine the evolution of the soliton parameters. A detailed investigation of these equations allows us to deduce an equation of motion for the soliton, which is expected to be the Lorentz force law for a moving charge, at least to highest order in e. As remarked earlier, if the analysis were carried out explicitly to higher order in e, corrections would be expected to appear, in particular due to the back reaction of the soliton's electromagnetic field on itself. However, these are not expected to appear in the $O(e)$ force law, and the following theorem validates this:

Theorem 12 *Assume the hypotheses and conclusions of theorem 10 hold, and let $\lambda = (\omega, \theta, \boldsymbol{\xi}, \mathbf{u})$ be the parameters of the soliton $\Psi_{SC,e}(\lambda)$. Then, on the interval $[0, \frac{T_0}{|e|}]$, the centre and velocity of the soliton evolve according to the equations:*

$$\frac{d}{dt}\boldsymbol{\xi} = \mathbf{u} + o(e) \tag{58}$$

$$\frac{d}{dt}\left(\mathbb{M}_S\gamma(\mathbf{u})\mathbf{u}\right) = e\mathbb{Q}_S\left(\mathbf{E}^{\delta}_{ext}(t,\boldsymbol{\xi}) + \mathbf{u}\times\mathbf{B}^{\delta}_{ext}(t,\boldsymbol{\xi})\right) + o(e), \tag{59}$$

where the mass of the soliton, $\mathbb{M}_S$, is given by

$$\mathbb{M}_S = \frac{1}{3}\left\|\nabla f_\omega\right\|^2_{L^2} + \omega^2\left\|f_\omega\right\|^2_{L^2}, \tag{60}$$

and the charge of the soliton is given by

$$\mathbb{Q}_S = \int(\omega - e\alpha)\, f^2_{\omega,e}. \tag{61}$$

This theorem is proved in §4.

Remark 13 Observe that, since we have scaled the external field so that $\mathbf{E}^{\delta}_{ext}$ and $\mathbf{B}^{\delta}_{ext}$ are independent of e, the soliton undergoes $O(1)$ motion on the time interval $[0, \frac{T_0}{|e|}]$ according to the Lorentz force law.

2 Stability: proof of theorem 10

In this section we explain the proof of theorem 10, making use of results which are proved separately in §3 and §5. Throughout this section the hypotheses of theorem 10 are understood to hold without explicit mention. Also we may assume, without loss of generality, that the solution is smooth in the course of the following calculations: since finite energy solutions can be approximated by smooth ones by $(WP2)$ in §1.2.1, and all the bounds we use depend only on the energy norm, this implies the result for finite energy initial data as in theorem 10.

2.1 Beginning of proof of theorem 10

2.1.1 Ansatz for the solution

We make an ansatz for a solution $\Psi(t) = \Psi^{\delta,\chi}_{ext}(t) + \Psi_{SC,e}(\lambda(t)) + Perturbation$, which is close to a soliton with time varying (modulating) parameters $\lambda(t)$, in the background external field $\Psi^{\delta,\chi}_{ext}(t)$. Explicitly the ansatz reads:

$$\begin{pmatrix} \phi(t,\mathbf{x}) \\ \psi(t,\mathbf{x}) \\ \mathbb{A}_\mu(t,\mathbf{x}) \\ \mathbb{E}_j(t,\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \phi_{SC,e}\left(\lambda(t)\right) + Exp[i\Theta_C]v \\ \psi_{SC,e}\left(\lambda(t)\right) + Exp[i\Theta_C]w \\ \left(A_{SC,e}\right)_\mu\left(\lambda(t)\right) + a^{\delta,\chi}_\mu + \tilde{A}_\mu \\ \left(E_{SC,e}\right)_j\left(\lambda(t)\right) + \left(E^{\delta}_{ext}\right)_j + \tilde{E}_j \end{pmatrix}. \tag{62}$$

Notice that we have included here an ansatz for the temporal part of the potential $\mathbb{A}_0$. Since we have imposed the Coulomb gauge throughout, it follows that $\operatorname{div}\tilde{\mathbf{A}} = 0$. The choice of the gauge transformation χ is:

$$\chi(t,\mathbf{x}) = -(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{a}^\delta(t,\boldsymbol{\xi}) - \int_0^t a_0^\delta(s,\boldsymbol{\xi}(s)) + \dot{\boldsymbol{\xi}}(s)\cdot\mathbf{a}^\delta(s,\boldsymbol{\xi})ds. \tag{63}$$

This is chosen so that the gauge transformed external potentials vanish along the world line of the soliton:

$$\begin{aligned} a^{\delta,\chi}_\mu &= a^\delta_\mu + \partial_\mu\chi, \\ a^{\delta,\chi}_\mu\big(t,\boldsymbol{\xi}(t)\big) &= 0. \end{aligned} \tag{64}$$

These imply

$$\begin{aligned} a^{\delta,\chi}_0(t,x) &= a^\delta_0(t,x) - a^\delta_0(t,\boldsymbol{\xi}(t)) - (x-\boldsymbol{\xi}(t))\cdot\dot{\mathbf{a}}^\delta(t,\boldsymbol{\xi}(t)), \\ \mathbf{a}^{\delta,\chi}(t,x) &= \mathbf{a}^\delta(t,x) - \mathbf{a}^\delta(t,\boldsymbol{\xi}(t)), \end{aligned} \tag{65}$$

exhibiting the claimed vanishing of $a^{\delta,\chi}_\mu$ along the soliton's world line. This allows certain quantities to be proved to be bounded in the course of the proof. Notice that χ is linear (and so harmonic) in $\mathbf{x}$, and so preserves the Coulomb condition (see remark 2).

There is clearly a redundancy in our ansatz, in that $\lambda(t)$ is so far completely undetermined. The appropriate choice of $\lambda(t)$ is dictated by the requirement that the solution be close to a soliton determined by the parameters $\lambda(t)$, i.e. by the requirement that we have good bounds for field perturbation $(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})$. This is carried out in §3, with the main results summarized next in §2.2. First we write explicitly the equations for the $(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})$, and give some bounds for the inhomogeneous terms in these equations.

2.1.2 Equations for the perturbations of the fields

$$\partial_t v + i\left(\omega\gamma + h\right)v = w + \jmath_1, \tag{66}$$

$$\partial_t w + i(\omega\gamma+h)w = -M_\lambda v + \jmath_2 + \mathcal{N}, \tag{67}$$

$$\partial_t\tilde{\mathbf{A}} = \tilde{\mathbf{E}} + \jmath_3, \tag{68}$$

$$\partial_t \tilde{\mathbf{E}} = \Delta \tilde{\mathbf{A}} + \jmath_4, \tag{69}$$

where the inhomogeneous terms $h, \jmath_1, \dots, \jmath_4$ and $\mathcal{N}$ are defined in §2.1.3, and M_λ is the operator

$$M_\lambda v = (-\triangle_x + m^2 + \gamma^2\omega^2|u|^2)v + 2i\omega\gamma\mathbf{u}\cdot\nabla_x v - \beta(f_\omega)v - f_\omega\beta'(f_\omega)\Re v. \tag{70}$$

The last two terms have been chosen to depend on the $e = 0$ profile function f_ω, rather than $f_{\omega,e}$, so that it is possible to make direct use of the stability assumption (POS) in §1.3.4. (This choice is reflected in the expression for the inhomogeneous term $\mathcal{N}$ in (76) and its corresponding estimate in (95)).

In addition to these evolution equations, the fields are constrained to satisfy the Gauss law (27), which takes the form:

$$\operatorname{div}\tilde{\mathbf{E}} = -\triangle\tilde{A}_0 = e\left\langle iExp[-i\Theta_C]\phi_{SC,e}, w\right\rangle + e\left\langle iv, Exp[-i\Theta_C]\psi_{SC,e} + w\right\rangle. \tag{71}$$

Under finite energy assumptions this equation has a unique solution with $\tilde{A}_0 \in \dot{H}^1$; this defines uniquely $\tilde{A}_0$ as a nonlocal function of v, w, λ at each time. Estimates for $\tilde{A}_0$ are given in lemma 39.

2.1.3 Inhomogeneous terms in the field perturbation equations (66)-(69)

The following quantity appears in both (66) and (67):

$$h = \dot{\Theta}_c - \omega\gamma - e(A_{SC,e})_0 - ea_0^{\delta,\chi} - e\tilde{A}_0. \tag{72}$$

The inhomogeneous term in (66) is $\jmath_1 = \jmath_1^I + \jmath_1^{II} + \jmath_1^0$, where

$$\jmath_1^I = -(\dot{\lambda} - V_0(\lambda))\cdot e^{-i\Theta_c}\partial_\lambda\phi_{SC,e} \tag{73}$$
$$\jmath_1^{II} = iea_0^{\delta,\chi} f_{\omega,e} \tag{74}$$
$$\jmath_1^0 = ie\tilde{A}_0 f_{\omega,e}. \tag{75}$$

The inhomogeneous terms in (67) are

$$\begin{aligned}\mathcal{N}(f_{\omega,e}, f_\omega, v) &= \beta(|f_{\omega,e}+v|)(f_{\omega,e}+v) - \beta(|f_{\omega,e}|)f_{\omega,e} \\ &\quad -\beta(|f_\omega|)v - f_\omega\beta'(|f_\omega|)\Re v,\end{aligned} \tag{76}$$

and $\jmath_2 = \jmath_2^I + \jmath_2^{II} + \jmath_2^{III} + \jmath_2^{IV} + \jmath_2^0$ where

$$\jmath_2^I = -(\dot{\lambda} - V_0(\lambda))\cdot e^{-i\Theta_c}\partial_\lambda\psi_{SC,e}$$
$$\jmath_2^{II} = e\mathbf{R}f_{\omega,e} + iea_0^{\delta,\chi}e^{-i\Theta_c}\psi_{SC,e}, \tag{77}$$
$$\jmath_2^{III} = e\mathbf{R}v + \mathbf{S}v \tag{78}$$
$$\jmath_2^{IV} = e^{-i\Theta_c}\left[\Delta_{\mathbb{A}}(e^{i\Theta_c}v) + (\Delta_{\mathbb{A}} - \Delta_{A_{SC,e}})\phi_{SC,e}\right] - \jmath_2^{II} - \jmath_2^{III} \tag{79}$$
$$\jmath_2^0 = ie\tilde{A}_0 e^{-i\Theta_c}\psi_{SC,e}. \tag{80}$$

Here, the operators $\mathbf{R}, \mathbf{S}$ are given by

$$\begin{aligned}\mathbf{R}v &= 2i\left(\mathbf{a}^{\delta,\chi}\right).(i\gamma\left(\omega - e\alpha_{\omega,e}\right)\mathbf{u} - \nabla)v - e\left|\mathbf{a}^{\delta,\chi}\right|^2 v, \\ \mathbf{S}v &= 2ie\alpha_{\omega,e}\gamma\mathbf{u}\cdot\nabla v + ie\gamma\left(\mathbf{u}\cdot\nabla\alpha_{\omega,e}\right)v + 2e\gamma^2|\mathbf{u}|^2\omega\alpha_{\omega,e}v - e^2\left(\gamma\alpha_{\omega,e}|\mathbf{u}|\right)^2 v.\end{aligned} \tag{81}$$

(In verifying these formulas, it is helpful to note that by the exact solutions in §1.3.4

$$e^{-i\Theta_c}(\nabla - ieA_{SC,e} - ie\mathbf{a}^{\delta,\chi})e^{i\Theta_c}v = \nabla v - i(\gamma(\omega - e\alpha_{\omega,e})\mathbf{u} + e\mathbf{a}^{\delta,\chi})v,$$

and a similar formula for the second derivatives.)

The inhomogeneous term in (68) is $\jmath_3 = \jmath_3^I + \jmath_3^0$ where

$$\begin{aligned} \jmath_3^I &= -(\dot\lambda - V_0(\lambda)) \cdot \partial_\lambda A_{SC,e}, \\ \jmath_3^0 &= \nabla \tilde A_0. \end{aligned} \tag{82}$$

and in (69) we have $\jmath_4 = \jmath_4^I + \jmath_4^{II} + \jmath_4^{III} + \jmath_4^{IV} + \jmath_4^0$, with $\jmath_4^0 = 0$ and

$$\jmath_4^I = -(\dot\lambda - V_0(\lambda)) \cdot \partial_\lambda E_{SC,e}, \tag{83}$$

$$\jmath_4^{II} = -e^2 |f_{\omega,e}|^2 \mathbf{a}^{\delta,\chi} \tag{84}$$

$$\jmath_4^{III} = e\langle i e^{i\Theta_c} v, \nabla_{\mathbb{A}}(\phi_{SC,e} + e^{i\Theta_c} v)\rangle - e^2 |f_{\omega,e}|^2 \tilde{\mathbf{A}}, \tag{85}$$

$$\jmath_4^{IV} = e\langle i\phi_{SC,e}, \nabla_{\mathbb{A}}(e^{i\Theta_c} v)\rangle. \tag{86}$$

To clarify the structure of these terms it is helpful to insert the ansatz (62) into the Hamiltonian (26) and write $H - \int \mathcal{V} = \sum_{n=0}^4 \hat H^{(n)}$, where $\hat H^{(n)}$ has homogeneity n in $(v, \tilde A)$. (The terms of degree larger than two arise solely from $\frac12 \int |\nabla_{\mathbb{A}} \phi|^2$.) Then the pieces of $\jmath_2$, (resp. $\jmath_4$), which are of degree $n \in \{1, 2, 3\}$ in $(v, \tilde A)$ arise, respectively, as the Frechet derivatives $-D_v \hat H^{(n+1)}$, (resp. $-D_{\tilde A} \hat H^{(n+1)}$). The nonlinear potential $\mathcal{V}$ only appears through $M_\lambda v$ and $\mathcal{N}$ in (67). With this understood we now introduce notation for the various terms arising in (67) and (69), organized according to their homogeneity. Let $\hat H = \sum_{n=2}^4 \hat H^{(n)}$, then in (67) the corresponding terms are

$$-D_v \hat H = -M_\lambda^0 v + \jmath_2^{III} + \jmath_2^{IV},$$

where

$$M_\lambda^0 v = (-\triangle_x + \gamma^2\omega^2 |u|^2) v + 2i\omega\gamma \mathbf{u} \cdot \nabla_x v = -e^{-i\Theta} \triangle\big(e^{i\Theta} v\big) \tag{87}$$

with Θ the soliton phase factor in (43). Notice that the operator M_λ^0 consists of those terms in (70), which do not arise from the $\mathcal{V}$ term in the energy, because we have so far excluded this term in our expansion (which is of $H - \int \mathcal{V}$). However, it is convenient to put back in the quadratic parts of the Taylor expansion of $\mathcal{V}$, but expanded around f_ω (the $e = 0$ soliton), so as to obtain the M_λ operator which appears in (67). Thus we let

$$\tilde H = \hat H + \frac12 D^2 \mathcal{V}(f_\omega)(v, v) = \hat H + \frac12 \Big[m^2 |v|^2 - \beta(f_\omega)|v|^2 - f_\omega \beta'(f_\omega)(\Re v)^2\Big],$$

so that, using the same notation for the homogeneous components of $\tilde H$ as for $\hat H$, we have $\tilde H^{(n)} = \hat H^{(n)}$ for $n > 2$ and $\tilde H^{(2)} - \hat H^{(2)} = \frac12 D^{\mathcal{V}}(f_\omega)(v, v)$. In (69) the corresponding terms are

$$-D_{\tilde A} \tilde H = \Delta \tilde{\mathbf{A}} + \jmath_4^{III} + \jmath_4^{IV}.$$

To write these terms explicitly we introduce a multilinear notation as follows.

$$(-D_v \tilde H, -D_{\tilde A} \tilde H) = \mathbb{B}^{(1)}(v, \tilde{\mathbf{A}}) + \mathbb{B}^{(2)}(v, \tilde{\mathbf{A}}) + \mathbb{B}^{(3)}(v, \tilde{\mathbf{A}}), \tag{88}$$

where $\mathbb{B}^{(n)}(v, \tilde{\mathbf{A}})$ is a homogeneous degree n function of $(v, \tilde{\mathbf{A}})$, as indicated by the superscript. We will define $\mathbb{B}^{(1)} : H^1(\mathbb{R}^3; \mathbb{C}) \oplus \dot H^1(\mathbb{R}^3; \mathbb{R}^3) \mapsto H^{-1}(\mathbb{R}^3; \mathbb{C}) \oplus \dot H^{-1}(\mathbb{R}^3; \mathbb{R}^3)$, where by H^{-1} (resp. $\dot H^{-1}$) we mean the dual space of H^1 (resp. $\dot H^1$.) Explicitly:

$$\mathbb{B}^{(1)}(v, \tilde{\mathbf{A}}) = (\mathbb{B}_{11} v + \mathbb{B}_{12} \tilde{\mathbf{A}}, \mathbb{B}_{21} v + \mathbb{B}_{22} \tilde{\mathbf{A}}), \tag{89}$$

where

$$\mathbb{B}_{11} v = -M_\lambda v + e\mathbf{R} v + \mathbf{S} v, \tag{90}$$

and the operators $\mathbf{R}$ and $\mathbf{S}$ are as just defined. Next

$$\begin{aligned} \mathbb{B}_{12} \tilde{\mathbf{A}} &= -2e f_{\omega,e} \left(\gamma\left(\omega - e\alpha_{\omega,e}\right)\mathbf{u} + e\mathbf{a}^{\delta,\chi}\right) \cdot \tilde{\mathbf{A}} - 2ie\tilde{\mathbf{A}} \cdot \nabla f_{\omega,e}, \\ \mathbb{B}_{21}\, v &= -2e f_{\omega,e} \left(\gamma\left(\omega - e\alpha_{\omega,e}\right)\mathbf{u} + e\mathbf{a}^{\delta,\chi}\right) \Re v + \langle iev, \nabla f_{\omega,e}\rangle + \langle ie f_{\omega,e}, \nabla v\rangle, \end{aligned}$$

and finally, $\mathbb{B}_{22}\tilde{\mathbf{A}} = \triangle\tilde{\mathbf{A}} - e^2 f_{\omega,e}\tilde{\mathbf{A}}$. Since $\operatorname{div}\tilde{\mathbf{A}} = 0$ integration by parts yields $\langle\tilde{\mathbf{A}}, \mathbb{B}_{21}\, v\rangle_{L^2} = \langle v, \mathbb{B}_{12}\,\tilde{\mathbf{A}}\rangle_{L^2}$, and

$$\begin{aligned}\tilde{H}^{(2)} &= -\frac{1}{2}\left\langle (v,\tilde{\mathbf{A}}), \mathbb{B}^{(1)}(v,\tilde{\mathbf{A}})\right\rangle_{L^2}, \\ &= \frac{1}{2}\int \Big[|\nabla\tilde{\mathbf{A}}|^2 + e^2|f_{\omega,e}\tilde{\mathbf{A}}|^2 + |\nabla v - i(\gamma\,(\omega - e\alpha_{\omega,e})\,\mathbf{u} + e\mathbf{a}^{\delta,\chi})v|^2 \\ &\qquad + \frac{1}{2}\big(m^2|v|^2 - \beta(f_\omega)|v|^2 - f_\omega\beta'(f_\omega)(\Re v)^2\big) \\ &\qquad - 2\langle ie\tilde{\mathbf{A}}f_{\omega,e}, \nabla v - i(\gamma\,(\omega - e\alpha_{\omega,e})\,\mathbf{u} + e\mathbf{a}^{\delta,\chi})v\rangle \\ &\qquad\quad - 2\langle ie\tilde{\mathbf{A}}v, \nabla f_{\omega,e} - i(\gamma\,(\omega - e\alpha_{\omega,e})\,\mathbf{u} + e\mathbf{a}^{\delta,\chi})f_{\omega,e}\rangle\Big]\,dx.\end{aligned} \tag{91}$$

Next, the quadratic terms in the equations can be expressed in terms of a rank three symmetric tensor

$$\mathbb{B}^{(2)} : \Big(H^1\left(\mathbb{R}^3;\mathbb{C}\right) \oplus \dot{H}^1\left(\mathbb{R}^3;\mathbb{R}^3\right)\Big)^2 \mapsto H^{-1}\left(\mathbb{R}^3;\mathbb{C}\right) \oplus \dot{H}^{-1}\left(\mathbb{R}^3;\mathbb{R}^3\right)$$

which is given explicitly by

$$\mathbb{B}^{(2)}(v,\tilde{\mathbf{A}}) = \begin{pmatrix} \mathbb{B}_{111}[v,v] + \mathbb{B}_{112}[v,\tilde{\mathbf{A}}] + \mathbb{B}_{121}[\tilde{\mathbf{A}},v] + \mathbb{B}_{122}[\tilde{\mathbf{A}},\tilde{\mathbf{A}}], \\ \mathbb{B}_{211}[v,v] + \mathbb{B}_{212}[v,\tilde{\mathbf{A}}] + \mathbb{B}_{221}[\tilde{\mathbf{A}},v] + \mathbb{B}_{222}[\tilde{\mathbf{A}},\tilde{\mathbf{A}}] \end{pmatrix},$$

where $\mathbb{B}_{111} = \mathbb{B}_{222} = 0$, and

$$\begin{aligned}\mathbb{B}_{112}[v,\tilde{\mathbf{A}}] &= -ev\left(\gamma\,(\omega - e\alpha_{\omega,e})\,\mathbf{u} + e(\mathbf{a}^{\delta,\chi})\right)\cdot\tilde{\mathbf{A}} - ie\nabla v\cdot\tilde{\mathbf{A}}, \\ \mathbb{B}_{121}[\tilde{\mathbf{A}},v] &= -ev\left(\gamma\,(\omega - e\alpha_{\omega,e})\,\mathbf{u} + e(\mathbf{a}^{\delta,\chi})\right)\cdot\tilde{\mathbf{A}} - ie\nabla v\cdot\tilde{\mathbf{A}}, \\ \mathbb{B}_{122}[\tilde{\mathbf{A}},\tilde{\mathbf{A}}] &= -e^2 f_{\omega,e}|\tilde{\mathbf{A}}|^2,\end{aligned}$$

and

$$\mathbb{B}_{211}[v,v] = -e\left(\gamma\,(\omega - e\alpha_{\omega,e})\,\mathbf{u} + e(\mathbf{a}^{\delta,\chi})\right)|v|^2 + \langle iev, \nabla v\rangle\,,$$

along with

$$\mathbb{B}_{221}[\tilde{\mathbf{A}},v] = \mathbb{B}_{212}[v,\tilde{\mathbf{A}}] = -e^2\,\langle f_{\omega,e}, v\rangle\,\tilde{\mathbf{A}}.$$

These terms are obtained by differentiation of the cubic part of the expanded Hamiltonian, which is

$$\begin{aligned}\tilde{H}^{(3)} &= -\frac{1}{2}\left\langle (v,\tilde{\mathbf{A}}), \mathbb{B}^{(2)}(v,\tilde{\mathbf{A}})\right\rangle_{L^2} \\ &= \langle\nabla v - i\gamma\mathbf{u}(\omega - e\alpha_{\omega,e})v - ie\mathbf{a}^{\delta,\chi}v, -ie\tilde{\mathbf{A}}v\rangle_{L^2} + e^2\langle\tilde{\mathbf{A}}f_{\omega,e}, \tilde{\mathbf{A}}v\rangle_{L^2}.\end{aligned}$$

Finally the cubic terms in the equations arise by differentiation of the quartic part of the Hamiltonian

$$\tilde{H}^{(4)} = -\frac{1}{2}\left\langle (v,\tilde{\mathbf{A}}), \mathbb{B}^{(3)}(v,\tilde{\mathbf{A}})\right\rangle_{L^2} = \frac{e^2}{2}\int |\tilde{\mathbf{A}}|^2|v|^2,$$

and are determined by a rank four tensor,

$$\mathbb{B}^{(3)} : \Big(H^1\left(\mathbb{R}^3;\mathbb{C}\right) \oplus \dot{H}^1\left(\mathbb{R}^3;\mathbb{R}^3\right)\Big)^3 \mapsto H^{-1}\left(\mathbb{R}^3;\mathbb{C}\right) \oplus \dot{H}^{-1}\left(\mathbb{R}^3;\mathbb{R}^3\right)$$

which, using an identical notation to the rank three case, has as its only non-zero entries

$$\mathbb{B}_{1122} = \frac{-e^2}{3} \tag{92}$$

and the other entries obtained by permuting the indices.

2.1.4 Some bounds for the inhomogeneous terms

We record here some simple bounds for the quantities defined above:

- $\|\jmath_1^{II}\|_{L^p} + \|\jmath_2^{II}\|_{L^p} = O(e)$ and $\|\jmath_4^{II}\|_{L^p} = O(e^2)$ for every $p \in [1, \infty]$ by (187),(188),
- $\|hf_{\omega,e}\|_{L^p} = O(e + |\dot\lambda - V_0| + e|\tilde A_0|_{L^q})$, for any $q > 3$, which can be read off from (72), using results from appendices A.1.2 A.2.2 and A.2.1, and the assumptions on the applied fields. $\tilde A_0$ can be bounded in L^q, $q > 3$ by appendix A.2.2.
- It is possible to write $h = h_1 - e\tilde A_0$ with $\|\nabla h_1\|_{L^\infty} = O(e + |\dot\lambda - V_0|)$ and $\nabla \tilde A_0$ bounded in L^p, $p \in (3/2, 3]$, by appendix A.2.2.

Finally, consider $\mathcal{N}$: by lemma 35 we can write

$$\begin{aligned}\mathcal{N}(f_{\omega,e}, f_\omega, v) &= \beta(|f_{\omega,e} + v|)(f_{\omega,e} + v) - \beta(|f_{\omega,e}|)f_{\omega,e} \\ &\quad -\beta(|f_{\omega,e}|)v - f_{\omega,e}\beta'(|f_{\omega,e}|)\Re v + O(e^2|v|) \\ &= \mathcal{N}(f_{\omega,e}, f_{\omega,e}, v) + O(e^2|v|)\end{aligned} \tag{93}$$

Using the condition (12), or more generally (13), and the fundamental theorem of calculus, we can estimate

$$|\mathcal{N}(f, f, v)| \le c\left(1 + |f|^3\right)\left(|v|^2 + |v|^5\right), \tag{94}$$

for any f. Therefore, choosing $f = f_{\omega,e}$, which is bounded, and using (93) we have

$$|\mathcal{N}(f_{\omega,e}, f_\omega, v)| \le c_1\left(|v|^2 + |v|^5\right) + c_2 e^2|v|. \tag{95}$$

2.2 Results from modulation theory

The assumptions on the nonlinearity under which we are working ensure that the Cauchy problem for (53) is locally well-posed in the sense of $(WP1)$ and $(WP2)$, see §1.2.1. Since so far χ is unknown (since $\lambda(t)$ and hence $\boldsymbol{\xi}(t)$ are not yet determined) we cannot solve directly for $\Psi = (\phi, \psi, \mathbb{A}_j, \mathbb{E}_j)$ in the background potential $a_\mu^{\delta,\chi}$. Instead we exploit gauge invariance and solve for

$$\hat\Psi = \left(\hat\phi, \hat\psi, \hat{\mathbb{A}}_j, \mathbb{E}_j\right) \equiv \left(e^{-ie\chi}\phi, e^{-ie\chi}\psi, \mathbb{A}_j - \partial_j\chi, \mathbb{E}_j\right) = e^{-ie\chi}\cdot\Psi \tag{96}$$

in the potential a_μ^δ, which is known. (Since $\chi(t, \mathbf{x})$ is harmonic in $\mathbf{x}$ this gauge transformation preserves both the equations (53) and the Coulomb gauge condition (see remark 2)). By proposition 1 on local well-posedness, there exists a time $T_{loc} > 0$ and unique solution to (28)) with

$$\left(\hat\Psi - \Psi_{ext}^\delta\right) \in C([0, T_{loc}]; \mathcal{H}), \tag{97}$$

with initial data

$$\hat\Psi(0) = \left(e^{-ie\chi_0}\phi(0), e^{-ie\chi_0}\psi(0), \mathbb{A}_j(0) - \partial_j\chi_0, \mathbb{E}_j(0)\right). \tag{98}$$

Once $\lambda(t) = (\omega(t), \theta(t), \boldsymbol{\xi}(t), \mathbf{u}(t))$, and hence $\chi(t)$, is determined, then $\Psi(t)$ is obtained from $\hat\Psi(t)$ by the above relation. As remarked previously, by proposition 1 these solutions can be approximated in energy norm by smooth solutions evolving in any of the spaces $\mathcal{H}_s$ of (9) (after subtracting off the background field). Thus, although the statement and proof of theorem 10 involve only the energy norm, it is permissible to assume smoothness of the solutions throughout the proof.

We now state a theorem which asserts that it is possible to choose the soliton parameters $\lambda(t)$ in such a way that the quantity W defined in (103) is equivalent to the energy norm. This is achieved by choosing $\lambda(t)$ in such a way that the pair (v, w) satisfies some conditions which are equivalent to those in (49) (after adjusting the phase).

Theorem 14 *(a) Let $\hat{\Psi}$ be a solution to the Cauchy problem for (53) satisfying* (97) *with initial data* (98) *with* $\Psi(0)$ *as described in theorem 10. Then, for sufficiently small e, there exists $T_1 > 0$ and $\lambda \in C^1([0,T_1];\widetilde{O}_{stab})$ with the following properties. On the interval $[0,T_1]$ define $\Psi(t) = (\phi(t), \psi(t), \mathbb{A}_j(t), \mathbb{E}_j(t))$ by* (63) *and* (96)*. Then it is possible to write Ψ in the form* (62) *where v, w are constrained to satisfy*

$$\Omega_0\big((v,w), \widetilde{\partial_\lambda}(\phi_{S,0}, \psi_{S,0})\big) = 0, \tag{99}$$

where we define

$$\widetilde{\partial_\lambda}\phi_{S,0} = Exp[-i\,(\Theta)]\partial_\lambda \phi_{S,0}, \tag{100}$$

and likewise for $\widetilde{\partial_\lambda}\psi_{S,0}$. Furthermore, the function $t \mapsto \lambda(t)$ solves a system of differential equations (116)*. The condition* (99) *is equivalent to requiring $\big(\phi - \phi_{SC,e}, \psi - \psi_{SC,e}\big) \in N_\lambda$.*

(b) If e and $\|(v, w, \tilde{\mathbf{A}}, \tilde{\mathbf{E}})\|_{\mathcal{H}}$ are sufficiently small, then

$$|\dot{\lambda} - V_0(\lambda)| = O\left(e + \|(v, w, \tilde{\mathbf{A}}, \tilde{\mathbf{E}})\|^2_{\mathcal{H}}\right), \tag{101}$$

so that, in particular, if $\|(v, w, \tilde{\mathbf{A}}, \tilde{\mathbf{E}})\|^2_{\mathcal{H}} = O(e)$ then

$$|\dot{\lambda} - V_0(\lambda)| = O\left(e\right). \tag{102}$$

Proof This is a consequence of the lemmas in §3. □

2.3 The main growth estimate

As discussed in §1.3.4, the natural quantity for stability and perturbation analyses of the solitons (51) is the Hessian of the augmented Hamiltonian. Here we modify this quantity to take account of the phase shifts in (62), and discard terms which are formally $O(e)$, leading us to the introduction of the following quadratic form:

$$W(v, w, \tilde{\mathbf{A}}, \tilde{\mathbf{E}}; \lambda) = K + \Xi, \tag{103}$$

where

$$K(\tilde{\mathbf{A}}, \tilde{\mathbf{E}}; \lambda) = \frac{1}{2}\left(\|\tilde{\mathbf{E}}\|^2_{L^2} + \|\nabla \times \tilde{\mathbf{A}}\|^2_{L^2} + 2\langle \tilde{\mathbf{E}}, (\mathbf{u}\cdot\nabla)\,\tilde{\mathbf{A}}\rangle_{L^2}\right), \tag{104}$$

and

$$\Xi(v, w; \lambda) = \frac{1}{2}\left(\|w - i\gamma\omega v\|^2_{L^2} + \big\langle v, M_\lambda - \gamma^2\omega^2)v\big\rangle_{L^2} + 2\,\langle w, \mathbf{u}\cdot\nabla v\rangle_{L^2}\right), \tag{105}$$

where M_λ is as defined in (70).

Theorem 15 (Equivalence of W and energy norm) *Suppose that the nonlinearity is such that (H1)-(H3) and* (SOL), (KER) *and* (POS) *hold. Suppose further that λ lies in a compact subset, $\mathbf{K}$, of $\widetilde{O}_{stab}$. Then the quadratic form W just defined, is equivalent uniformly on $\mathbf{K}$ to $\|(v, w, \tilde{\mathbf{A}}, \tilde{\mathbf{E}})\|^2_{\mathcal{H}}$ provided that (v, w) satisfy the constraints* (99)*.*

Proof This is essentially theorem 2.7 in [24]. Since there is no coupling in W between (v, w) and $(\tilde{\mathbf{A}}, \tilde{\mathbf{E}})$, it is only necessary to show separately the equivalence of Ξ and K to the corresponding parts of $\|(v, w, \tilde{\mathbf{A}}, \tilde{\mathbf{E}})\|^2_{\mathcal{H}}$. For K this can be achieved by completing the square (since $\|\nabla \times \tilde{\mathbf{A}}\|_{L^2} = \|\tilde{\mathbf{A}}\|_{\dot{H}^1}$ by the Coulomb condition), while for Ξ it is an immediate consequence of (POS). □

Theorem 16 (Main growth estimate) *Assume given a solution to the Cauchy problem for (28) for which theorem 14 applies on an interval $[0, \frac{T_2}{|e|}]$ for some fixed positive T_2. Assume that $\lambda(t) \in \mathbf{K}$, a compact subset of $\widetilde{O}_{stab}$, so that by theorem 15 there exists $c_1 > 0$ such that,*

$$\frac{1}{c_1}W \le \|(v, w, \tilde{\mathbf{A}}, \tilde{\mathbf{E}})\|^2_{\mathcal{H}} \le c_1 W, \tag{106}$$

on $[0, \frac{T_2}{|e|}]$*. Assume further that there exist* $c_2 > 0, c_3 > 0$ *such that that* $\delta^2 \leq c_2|e|$ *and and* $W \leq c_3|e|$*, and that* $e = o(\delta)$*. It follows that, for sufficiently small* e*, there exists* $c_4 > 0$ *such that, on* $[0, \frac{T_2}{|e|}]$

$$W(t) \leq c_4(W(0) + e^2 + \delta^2) \exp(c_4|e|t). \tag{107}$$

Proof See §5. □

2.4 Completion of the proof of theorem 10

2.4.1 Local solution verifying constraints

For simplicity of exposition we first prove part (i) of the theorem, i.e. we consider initial data $\Psi(0)$ consisting of an exact soliton as in (51) determined by parameters $\lambda(0) = (\theta(0), \omega(0), \mathbf{u}(0), \boldsymbol{\xi}(0)) \in \widetilde{O}_{stab}$, with $\omega(0)$ satisfying the stability condition. On account of the applied fields there will be a non-trivial evolution starting from this initial value. Applying the local existence theorem 1, and theorem 14 as in §2.2, we deduce the existence a positive time $T_1 > 0$ such that on the interval $[0, T_1]$ there is a solution to the Cauchy problem which can be written as in (62) where $v(0) = 0 = w(0)$, and $\big(v(t), w(t)\big)$ satisfy the constraints (109) (or (99)), and $t \mapsto \lambda(t)$ solves (116). We may assume that $\lambda(t) \in \mathbf{K}$, a fixed compact subset of $\widetilde{O}_{stab}$, so that (106) holds.

2.4.2 Growth of the energy norm

Since we have a local solution satisfying the constraints (99) we can assume that the conclusions of theorem 15 hold. Furthermore, by continuity we may assume (making T_1 smaller if need be) that on this interval $W(t) \leq c_3|e|$, and (106) holds. Now apply the growth estimate in theorem 16:

$$W(t) \leq c_4(W(0) + e^2 + \delta^2) \exp(c_4|e|t),$$

to deduce by a standard continuation argument, since $W(0) = 0$ and $\delta^2 = o(e)$, that there exists an interval $[0, \frac{T_0}{|e|}]$, with $T_0 > 0$ fixed (independent of e, δ), on which

$$W(t) \leq c_5(e^2 + \delta^2) = o(e)$$

which completes the proof of theorem 10 for the case of exact soliton initial data - part (i) of theorem 10.

2.4.3 General initial data

Part (ii) of theorem 10 says that the behaviour described in part (i) also holds for nearby initial data: for a precise formulation it is necessary to consider the initial data for the gauge transform $\hat{\Psi}$:

Theorem 17 *Under the same assumptions as theorem 10, let* $\hat{\Psi}$ *be a solution to the Cauchy problem for (53) with* $(\hat{\Psi} - \Psi^\delta_{ext}) \in C(\mathbb{R}; \mathcal{H})$ *and initial data* $\hat{\Psi}(0) = (\hat{\phi}(0), \hat{\psi}(0), \hat{\mathbb{A}}_j(0), \hat{\mathbb{E}}_j(0))$ *having the following property. There exists* $\widetilde{\lambda} = (\widetilde{\theta}, \widetilde{\omega}, \widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\xi}}) \in \widetilde{O}_{stab}$ *such that if we define* $\widetilde{\chi}(\mathbf{x}) = -(\mathbf{x} - \widetilde{\boldsymbol{\xi}}) \cdot \mathbf{a}^\delta(0, \widetilde{\boldsymbol{\xi}})$*, then*

$$\kappa_0 \equiv \left\| e^{-ie\widetilde{\chi}} \cdot \hat{\Psi}(0) - \Psi^{\delta,\widetilde{\chi}}_{ext}(0) - \Psi_{SC,e}(\widetilde{\lambda}) \right\|_{\mathcal{H}} = o(e^{\frac{1}{2}}). \tag{108}$$

It follows that, if e *is sufficiently small there exists* $T_0 > 0$, $\chi(t, \mathbf{x})$ *and* $\lambda(t) \in C^1([0, \frac{T_0}{|e|}], \widetilde{O}_{stab})$*, all as in theorem 10, such that if* $\Psi(t)$ *is defined as in* (96) *it satisfies all the conclusions of part (i) of theorem 10.*

Proof It is only necessary to argue, as in the proof of lemma 18, that under the stated conditions there exists $\lambda(0) \in \widetilde{O}_{stab}$ with $|\lambda(0) - \widetilde{\lambda}| = o(e^{\frac{1}{2}})$ such that $\Psi(0) = (\phi(0), \psi(0), \mathbb{A}_j(0), \mathbb{E}_j(0)) \equiv e^{-ie\chi_0} \cdot \hat{\Psi}(0)$ can be written as

$$\Psi(0) = \Big(\phi_{SC,e}(\lambda(0)) + \tilde{\phi}(0), \psi_{SC,e}(\lambda(0)) + \tilde{\psi}(0), \mathbb{A}_i(0), \mathbb{E}_i(0)\Big),$$

with

$$\big(\tilde{\phi}(0), \tilde{\psi}(0)\big) \in N_{\lambda(0)}$$

where $N_{\lambda(0)}$ is the symplectic normal subspace, of codimension eight, defined in (49). This is a simple consequence of the implicit function theorem, as is lemma 18. There is only a slight modification required in that $\phi(0) = e^{-ie\chi_0}\hat{\phi}(0)$ depends on $\lambda(0)$, and so does $\psi(0)$, unlike the case considered in that lemma. However for small e this has no effect on the non-degeneracy condition required to apply the implicit function theorem. (Also the fact that χ_0 grows linearly in $\mathbf{x}$ can easily be handled using the exponential decay in $\mathbf{x}$ of $\phi_{SC,e}, \psi_{SC,e}$ and their derivatives.)

Now using $|\lambda(0) - \widetilde{\lambda}| = o(e^{\frac{1}{2}})$ we can deduce from (108) that $W(0) = o(e)$. Indeed for the electromagnetic components this is immediate since the gauge transformation leaves the electric field unchanged, and only shifts $\mathbb{A}_j$ by $\partial_j\chi_0$, and this shift is put onto the background potential (and so does not contribute to $W(0)$ since $\tilde{\mathbf{A}}$ is unchanged). The change of the electromagnetic components of the soliton induced by the change of $\widetilde{\lambda}$ to $\lambda(0)$ are easily estimated in energy norm as $O(|\widetilde{\lambda} - \lambda(0)|)$ by lemmas 33 and 34. For the other components we just use phase invariance to estimate, e.g.

$$\begin{aligned}
\|e^{-ie\chi_0}\hat{\phi}(0) - \phi_{SC,e}(\lambda(0))\|_{L^2} &= \|\hat{\phi}(0) - e^{ie\chi_0}\phi_{SC,e}(\lambda(0))\|_{L^2} \\
&\le \|\hat{\phi}(0) - e^{ie\widetilde{\chi}}\phi_{SC,e}(\widetilde{\lambda})\|_{L^2} + \|e^{ie\widetilde{\chi}}\phi_{SC,e}(\widetilde{\lambda}) - e^{ie\chi_0}\phi_{SC,e}(\lambda(0))\|_{L^2} \\
&\le \kappa_0 + O(|\lambda(0) - \widetilde{\lambda}|) = o(e^{\frac{1}{2}}).
\end{aligned}$$

From this point on, the argument can be completed as before: since $\big(\tilde{\phi}(0), \tilde{\psi}(0)\big) \in N_{\lambda(0)}$ is equivalent to the conditions (99), theorems 14 and 16 can be applied to produce a local solution satisfying the growth estimate in §2.4.2.

3 Modulation theory

In this section we state and prove some theorems which imply theorem 14, which is needed in the proof of the main results (theorems 10 and 12). The proofs are a direct application of the developments in [24], and so the presentation will be brief and reference made to [24, 16] for some of the calculations. The crucial point is that the conditions (99) are equivalent to a locally well-posed set of ordinary differential equations. Recall from (47) that, for $e = 0$, the soliton solutions are of the form $\big(\phi_{S,0}, \psi_{S,0}\big)(\mathbf{x}; \lambda) \equiv e^{i\Theta}\big(f_\omega(\mathbf{Z}), (i\gamma\omega f_\omega(\mathbf{Z}) - \gamma\mathbf{u}\cdot\nabla_{\mathbf{Z}} f_\omega(\mathbf{Z}))\big)$ with $\lambda(t)$ an integral curve of the vector field V_0. Explicitly, the conditions (99) read

$$\Big\langle v, \widetilde{\partial_{\lambda}}\psi_{S,0}(\lambda)\Big\rangle_{L^2} - \Big\langle w, \widetilde{\partial_{\lambda_A}}\phi_{S,0}(\lambda)\Big\rangle_{L^2} = 0 \tag{109}$$

for $A = -1, 0, ..., 6$.

In the next two subsections we state two lemmas which prove that these constraints can be enforced thoroughout a time interval:

- The first shows that by an appropriate choice of $\lambda(0)$, they can be assumed to hold in an open neighbourhood of the set of stable solitons in the phase space $\mathcal{H}$. This shows that the class of initial data considered in part (ii) of theorem 10 forms an open set containing the stable solitons.
- The second shows that an appropriate choice of $\partial_t\lambda$ implies that they are preserved for later times.

3.1 Preparation of the initial data

Lemma 18 *Suppose that there exists* $\widetilde{\lambda} = (\widetilde{\theta}, \widetilde{\omega}, \widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\xi}}) \in \widetilde{O}_{stab}$ *(so that* (39) *holds with* $\omega = \widetilde{\omega}$*). Then, there exists* $e(\widetilde{\lambda})$, $\kappa(\widetilde{\lambda}, e)$, *such that, if* $|e| < e(\widetilde{\lambda})$ *and*

$$\widetilde{\kappa_1} = \|\phi(0) - \phi_{SC,e}(\widetilde{\lambda})\|_{H^1} + \|\psi(0) - \psi_{SC,e}(\widetilde{\lambda})\|_{L^2} < \kappa, \tag{110}$$

there exists $\lambda(0)\in\widetilde{O}_{stab}$ *depending differentiably upon* $(\phi(0),\psi(0))$ *such that* $(v(0),w(0))$*, determined by the first two equations of* (62) *at* $t=0$*, satisfy the constraints* (109) *with* $\lambda=\lambda(0)$*. Furthermore there exists* $c_1>0$ *such that*

$$|\lambda(0)-\widetilde{\lambda}|+\|\phi(0)-\phi_{SC,e}(\lambda(0))\|_{H^1}+\|\psi(0)-\psi_{SC,e}(\lambda(0))\|_{L^2}<c_1\widetilde{\kappa}_1. \tag{111}$$

Proof The condition in (39) allows this to be deduced from the implicit function theorem, see [24, §2.3] or [16] for details. □

3.2 Modulation equations and constraints

Lemma 19 *Let* $\lambda(0)\in\widetilde{O}_{stab}$ *and* $(v(0),w(0))$ *be as given in the conclusions of lemma 18. Let* $\hat{\Psi}$ *be a solution to the Cauchy problem for (53) on the time interval* $[0,T_{loc}]$ *with regularity as in* (97)*, and such that*

$$\sup_{[0,T_{loc}]}\left\|\hat{\Psi}(t)-\Psi^{\delta}_{ext}(t)\right\|_{\mathcal{H}}<N_0. \tag{112}$$

Fix a compact subset $\mathbf{K}$ *of the stable parameter set* $\widetilde{O}_{stab}$*, which is the closure of an open neighbourhood of* $\lambda(0)$*. Then, there exists* $\kappa_2>0$ *and* $T_1>0$ *such that, if* $\|(v(0),w(0))\|_{H^1\oplus L^2}<\kappa_2$*, there exists* $\lambda(t)\in C^1([0,T_1];\mathbf{K})$ *such that the constraints (109) are satisfied for* $0\le t\le T_1$*, where* v,w *are as in* (62) *with* Ψ *obtained from* $\hat{\Psi}$ *via* (63) *and* (96)*. The function* $t\mapsto\lambda(t)$ *is a solution of a system of ordinary differential equations* (116)*.*

Proof The proof of this is essentially the same as [24, § 2.5]. For clarity it is divided into three stages.

3.2.1 Beginning of proof of lemma 19

Equations (66) and (67) define a linear operator $\widetilde{\mathcal{M}}_\lambda$ in an obvious way:

$$\widetilde{\mathcal{M}}_\lambda(v,w)=\left(-\partial_t v-i\omega\gamma v+w,-\partial_t w-i\omega\gamma w-M_\lambda v\right). \tag{113}$$

and let $\widetilde{\mathcal{M}}^*_\lambda$ be the formal $L^2(dxdt)$ adjoint of this operator. Then, by [24, § 2.5], there exists an 8×8 matrix D_{AB} such that

$$\widetilde{\mathcal{M}}^*_\lambda(-\widetilde{\partial_{\lambda_A}}\psi_{S,0},\widetilde{\partial_{\lambda_A}}\phi_{S,0})=\sum_B D_{AB}(-\widetilde{\partial_{\lambda_B}}\psi_{S,0},\widetilde{\partial_{\lambda_B}}\phi_{S,0})+(\tilde{\mathbf{I}}^1_A,\tilde{\mathbf{I}}^2_A). \tag{114}$$

where the inhomogeneous terms $\tilde{\mathbf{I}}^j_A$ are proportional to $\dot{\lambda}-V_0(\lambda)$:

$$\tilde{\mathbf{I}}^j_A=\tilde{I}^j_{AB}(\dot{\lambda}-V_0(\lambda))_B$$

with $\tilde{I}^j_{AB}$ smooth functions of x, which are exponentially decreasing as $|x|\to\infty$; the precise formulae, which are unimportant here, can be found in [24, § 2.5]. A simple integration by parts then shows that the constraints in (109) are satisfied on an interval containing the initial time, if they hold at that initial time and if the following is true

$$\begin{aligned}&\langle-\widetilde{\partial_{\lambda_A}}\psi_{S,0},j_1\rangle_{L^2}+\langle\widetilde{\partial_{\lambda_A}}\phi_{S,0},j_2+\mathcal{N}\rangle_{L^2}\\&\quad+\langle\tilde{\mathbf{I}}^1_A-ih\widetilde{\partial_{\lambda_A}}\psi_{S,0},v\rangle_{L^2}+\langle\tilde{\mathbf{I}}^2_A+ih\widetilde{\partial_{\lambda_A}}\phi_{S,0},w\rangle_{L^2}=0,\end{aligned} \tag{115}$$

for all $A=-1,0,\dots 6$, and at each time in the interval. A calculation as in [24], which is reviewed in the next stage of the proof in §3.2.2, shows that these latter conditions are equivalent to the following system of differential equations

$$\left(\mathbb{M}(e)_{AB}+\beth_{AB}(v,w,\lambda)\right)\left(\dot{\lambda}-V_0(\lambda)\right)_B=\mathbf{F}_A(e,\Psi^{\delta}_{ext},\Psi,\lambda), \tag{116}$$

where $\mathbb{M}(e)_{AB}$ is defined in (117), $\beth_{AB}$ is defined in (118), $\mathbf{F}_A$ is given by (122) and where the indices $A,B\in\{-1,0,1,...,6\}$, and we sum over the repeated index B.

3.2.2 Explicit computation of the modulational equation (116)

We write out explicitly the various terms in the conditions (115). The first thing to note is that the overall expression is affine in $(\dot{\lambda} - V_0(\lambda))$ so we divide into the *inertial* terms, which are proportional to this quantity (and give rise to the left hand side of (116)), and the remaining *force* terms, which give rise to the right hand side of (116). The dominant contribution to the inertial terms arises from $\jmath_1^I, \jmath_2^I$, while that to the force terms arises from $\jmath_1^{II}, \jmath_2^{II}$.

To describe the inertial terms we need the following matrix, which, to highest order, describes the mass of the soliton:

$$\mathbb{M}_{AB}(e) = \left\langle \widetilde{\partial_{\lambda_A}} \psi_{S,0}, e^{-i\Theta_c} \partial_{\lambda_B} \phi_{SC,e} \right\rangle_{L^2} - \left\langle \widetilde{\partial_{\lambda_A}} \phi_{S,0}, e^{-i\Theta_c} \partial_{\lambda_B} \psi_{SC,e} \right\rangle_{L^2}. \tag{117}$$

Then the dominant inertial term is

$$\langle -\widetilde{\partial_{\lambda_A}} \psi_{S,0}, \jmath_1^I \rangle_{L^2} + \langle \widetilde{\partial_{\lambda_A}} \phi_{S,0}, \jmath_2^I \rangle_{L^2} = \mathbb{M}_{AB}(e)(\partial_t \lambda - V_0(\lambda))_B.$$

Next, we have the following matrices, which may be thought of as corrections - owing to the presence of the perturbations v and w - to the "inertia" matrix above :

$$\beth_{AB} = \left\langle v, \left(\tilde{I}^1_{AB} - i\partial_{\lambda_B} \Theta_c \widetilde{\partial_{\lambda_A}} \psi_{S,0} \right) \right\rangle_{L^2} - \left\langle w, \left(\tilde{I}^2_{AB} + i\partial_{\lambda_B} \Theta_C \widetilde{\partial_{\lambda_A}} \phi_{S,0} \right) \right\rangle_{L^2}. \tag{118}$$

We now present the abbreviations for the force terms appearing in the modulational equation. Firstly, we have what is effectively the Lorentz force term.

$$\begin{aligned} \mathbf{F}^L_A &= \langle \widetilde{\partial_{\lambda_A}} \psi_{S,0}, \jmath_1^{II} \rangle_{L^2} - \langle \widetilde{\partial_{\lambda_A}} \phi_{S,0}, \jmath_2^{II} \rangle_{L^2} \\ &= \left\langle \widetilde{\partial_{\lambda_A}} \psi_{S,0}, iea_0^{\delta,\chi} f_{\omega,e} \right\rangle_{L^2} \\ &\quad - \left\langle \widetilde{\partial_{\lambda_A}} \phi_{S,0}, iea_0^{\delta,\chi} \left(i\gamma(\omega - e\alpha_{\omega,e}) - \mathbf{u}\cdot\nabla \right) f_{\omega,e} + e\mathbf{R} f_{\omega,e} \right\rangle_{L^2}. \end{aligned} \tag{119}$$

We also have a force $\mathbf{F}^n_A + \mathbf{F}^p_A$ due to the nonlinear interactions, where

$$\mathbf{F}^n_A = -\left\langle \widetilde{\partial_{\lambda_A}} \phi_{S,0}, \mathcal{N} \right\rangle_{L^2}, \tag{120}$$

$$\begin{aligned} \mathbf{F}^p_A = &\left\langle \widetilde{\partial_{\lambda_A}} \psi_{S,0}, \jmath_1^0 + ie\left(\gamma\alpha_{\omega,e} + a_0^{\delta,\chi} + \tilde{A}_0 \right) v \right\rangle_{L^2} \\ &- \left\langle \widetilde{\partial_{\lambda_A}} \phi_{S,0}, \jmath_2^{III} + \jmath_2^{IV} + \jmath_2^0 + ie\left(\gamma\alpha_{\omega,e} + a_0^{\delta,\chi} + \tilde{A}_0 \right) w \right\rangle_{L^2}. \end{aligned} \tag{121}$$

We abbreviate the total force as follows:

$$\mathbf{F}_A = \mathbf{F}^L_A + \mathbf{F}^n_A + \mathbf{F}^p_A. \tag{122}$$

Bound for the inertia matrix. It follows from the definition of $\beth_{AB}$ that

$$|\beth_{AB}| = O\left(\|(v, w, \tilde{\mathbf{A}}, \tilde{\mathbf{E}})\|_{\mathcal{H}} \right). \tag{123}$$

Bounds for the forces. Firstly, the main force term can be bounded as

$$\mathbf{F}^L_A = O(e), \tag{124}$$

because of (187), (188) and (154). For some values of A there are better bounds:

$$\mathbf{F}^L_0 = O(e^3). \tag{125}$$

Referring to (119), and using lemmas A.1.3 and 34, we deduce that

$$\begin{aligned} \mathbf{F}_0^L &= O(e^3) - \langle if_\omega, e\mathbf{R}f_\omega\rangle_{L^2} \\ &\quad + \langle \widetilde{\partial_\theta}\psi_{S,0}, iea_0^{\delta,\chi} f_\omega\rangle_{L^2} - \langle \widetilde{\partial_\theta}\phi_{S,0}, iea_0^{\delta,\chi}(i\omega\gamma f_\omega - \mathbf{u}\cdot\nabla f_\omega)\rangle_{L^2}. \end{aligned}$$

By the reality of f_ω and the Coulomb condition the last three terms vanish, proving the bound (125). Also, for $A = 3+j$ we have an improvement:

$$\mathbf{F}_{3+j}^L = O(e^2 + e\delta). \tag{126}$$

To establish this, we first argue as above that

$$\begin{aligned} \mathbf{F}_{3+j}^L &= O(e^3) - \langle \widetilde{\partial_{u^j}}\phi_{S,0}, e\mathbf{R}f_\omega\rangle_{L^2} \\ &\quad + \langle \widetilde{\partial_{u^j}}\psi_{S,0}, iea_0^{\delta,\chi} f_\omega\rangle_{L^2} - \langle \widetilde{\partial_{u^j}}\phi_{S,0}, iea_0^{\delta,\chi}(i\omega\gamma f_\omega - \mathbf{u}\cdot\nabla f_\omega)\rangle_{L^2}. \end{aligned}$$

Now referring to the formulae in A.1.4 we see that $\widetilde{\partial_{u^j}}\phi_{S,0} = even + iodd$, while $\widetilde{\partial_{u^j}}\psi_{S,0} = odd + ieven$ where *even* (resp. *odd*) means a real valued function which is even (resp. odd) as a function of $\mathbf{Z}$. The bound asserted then follows by inspection and use of lemma 37.

Next, (95) implies, by (150)), (151) , (154) and by the Hölder and Sobolev inequalities, that

$$|\mathbf{F}_A^n| = O\left(e^2\|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}}\right) + O\left(\|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}}^2 + \|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}}^5\right).$$

Finally

$$|\mathbf{F}_A^p| = O\left(e\|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}} + e\|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}}^2 + e^2\|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}}^3\right). \tag{127}$$

This is obtained directly from the formula above by means of the Sobolev and Hölder inequalities and using the bounds in §A.2.1 and §A.2.2.

3.2.3 Completion of proof of lemma 19

The matrix $\mathbb{M}(e)_{AB}$ is invertible for small e on account of the stability condition (39) and lemma (35). Also the matrix $\daleth_{AB}$ is small when (v,w) is small, so that in this case the system of evolution equations (116) can be manipulated - as in the proof of theorem 2.6 in [24] - to form a system of equations of the form

$$\dot{\lambda} = V_0(\lambda) + V_1(e, \Psi_{ext}^\delta, \hat{\Psi}, \lambda).$$

This is almost a locally well-posed system of ordinary differential equations - there is a slight modification of the standard proof from [24] required: $\hat{\Psi}$ is known to exist already, but (v,w), determined as in the statement, depend on $\lambda(t)$ through the gauge transformation (63), which is nonlocal in the $\boldsymbol{\xi}$ component of λ, and so V_1 is similarly nonlocal. To allow for this it is necessary to augment λ by the nonlocal quantity appearing in (63), which is in fact $\chi(t,\boldsymbol{\xi})$. Call $\Lambda = (\lambda, \chi(t,\boldsymbol{\xi}(t)))$, then there is a locally well-posed system of ordinary differential equations of the form $\dot{\Lambda} = W(\Lambda), e, \Psi_{ext}^\delta, \hat{\Psi})$, allowing the proof of lemma 19 to be completed in the same way in [24]. □

3.3 A bound for $\dot{\lambda}$

Lemma 20 *In the situation of the previous lemma,*

$$|\dot{\lambda} - V_0(\lambda)| = O\left(e + \|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}}^2 + e\|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}}\right)$$

in the limit of e going to zero.

Proof The function $\lambda(t)$ is obtained as a solution of the modulation equations (116). Referring to the bounds for the inertial matrix and forces in §3.2.2, it is immediate that for $e, \|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}}$ sufficiently small the bound claimed holds. □

4 The Lorentz force law: proof of theorem 12

The starting point is (116). Define

$$\mathbb{M}_{AB}(0) = \left\langle \widetilde{\partial_{\lambda_A}} \psi_{S,0}, \widetilde{\partial_{\lambda_B}} \phi_{S,0} \right\rangle_{L^2} - \left\langle \widetilde{\partial_{\lambda_A}} \phi_{S,0}, \widetilde{\partial_{\lambda_B}} \psi_{S,0} \right\rangle_{L^2}, \tag{128}$$

and observe that by lemmas 35 and 34 $\mathbb{M}_{AB}(e) - \mathbb{M}_{AB}(0) = O(e^2)$. Using this, and referring to the decomposition of $\mathbf{F}_A$ in equation (122), and the associated bounds following it, we infer that

$$\Big(\mathbb{M}(0)_{AB} + O\big(e^2 + \widetilde{W}^{\frac{1}{2}}\big)\Big)\big(\dot{\lambda} - V_0\big)_B = \mathbf{F}^L_A + O\big(e\widetilde{W}^{\frac{1}{2}} + \widetilde{W}\big), \tag{129}$$

where $\mathbf{F}^L_A$ is as in (119). Since the right hand side is known, up to the stated error term, it is now just a matter of calculation to obtain explicit forms for the left hand side of these equations, and thence to deduce theorem 12. The calculation is done in [24, §A.7], using a set of functions defined in §A.1.4 which are convenient linear combinations of the $\widetilde{\partial_{\lambda_A}}(\phi_{S,0}, \psi_{S,0})$. We now record the conclusions.

Using (102), the $A = 0$ component of (129) reads:

$$\partial_\omega(\omega \|f_\omega\|^2_{L^2})\dot{\omega} = \mathbf{F}^L_0 + O(e^2) + O(\widetilde{W}),$$

with a formula for $\mathbf{F}^L_0$ given in (119) which indicates that $\mathbf{F}^L_0 = O(e^3)$ (see §3.2.2), and all together:

$$\partial_\omega(\omega \|f_\omega\|^2_{L^2})\dot{\omega} = O(e^2) + O(\widetilde{W}). \tag{130}$$

Similarly, the bound (126) for $\mathbf{F}^L_{3+j}$ implies the following equation for the centre of the soliton:

$$\dot{\boldsymbol{\xi}} = \mathbf{u} + O(e^2) + O(\widetilde{W}) + O(e\delta). \tag{131}$$

Next, using (130) and (102), the $A = i \in \{1, 2, 3\}$ component of (129) reads

$$\partial_t\Big[\big(\frac{1}{3}\|\nabla f_\omega\|^2_{L^2} + \omega^2 \|f_\omega\|^2_{L^2}\big)\gamma \mathbf{u}^i\Big] = \mathbf{F}^L_i + O(\widetilde{W}) + O(e^2), \tag{132}$$

again with $\mathbf{F}^L_i$ given in (119) as:

$$\begin{aligned}\mathbf{F}^L_i = &\left\langle \widetilde{\partial_{\xi_i}} \psi_{S,0}, iea_0^{\delta,\chi} f_{\omega,e} \right\rangle_{L^2} \\ &- \left\langle \widetilde{\partial_{\xi_i}} \phi_{S,0}, iea_0^{\delta,\chi}\left(i\gamma(\omega - e\alpha_{\omega,e}) - \mathbf{u}\cdot\nabla\right) f_{\omega,e} + e\mathbf{R} f_{\omega,e} \right\rangle_{L^2},\end{aligned} \tag{133}$$

where the operator $\mathbf{R}$ is defined in (81). Here, on the left hand side, $\|f_\omega\|^2_{L^2} = \int f_\omega(\mathbf{Z})^2 d^3\mathbf{Z}$ and by the Lorentz transformation (42) $d^3\mathbf{Z} = \gamma d^3\mathbf{x}$. The inner products on the right hand side are in $L^2(d^3\mathbf{x})$. It remains to simplify this expression for $\mathbf{F}^L_i$: firstly,

$$\begin{aligned}&\left\langle \widetilde{\partial_{\xi_i}} \psi_{S,0}, iea_0^{\delta,\chi} f_{\omega,e} \right\rangle_{L^2} - \left\langle \widetilde{\partial_{\xi_i}} \phi_{S,0}, iea_0^{\delta,\chi}\big(i\gamma(\omega - e\alpha_{\omega,e}) - \mathbf{u}\cdot\nabla\big) f_{\omega,e} \right\rangle_{L^2} \\ &\qquad = \left\langle \widetilde{\partial_{\xi_i}} \psi_{S,0}, iea_0^{\delta,\chi} f_\omega \right\rangle_{L^2} - \left\langle \widetilde{\partial_{\xi_i}} \phi_{S,0}, iea_0^{\delta,\chi}(i\gamma\omega - \mathbf{u}\cdot\nabla) f_\omega \right\rangle_{L^2} + O(e^3)\end{aligned}$$

by lemma 35,

$$= \left\langle (i\gamma\omega - \mathbf{u}\cdot\nabla) f_\omega, ie\nabla a_0^{\delta,\chi} f_\omega \right\rangle_{L^2} + O(e^3)$$

by integration by parts,

$$= e\omega \|f_\omega\|^2_{L^2}\Big[\nabla_i a_0^\delta(t,\boldsymbol{\xi}) - \dot{\mathbf{a}}^\delta(t,\boldsymbol{\xi}) - \mathbf{u}\cdot\nabla\mathbf{a}^\delta(t,\boldsymbol{\xi})\Big] + O(e\delta + e^3),$$

by (65) and lemma 38. (Again, $\|f_\omega\|_{L^2}^2 = \int f_\omega(\mathbf{Z})^2 d^3\mathbf{Z}$.) But also, referring to (81),

$$\Big\langle -\widetilde{\partial_{\xi^j}}\phi_{S,0}, e\mathbf{R}f_{\omega,e}\Big\rangle_{L^2} = \gamma\omega e\int f_\omega^2(\mathbf{Z})\nabla\mathbf{u}\cdot\mathbf{a}^\delta(t,\mathbf{x})dx,$$
$$= \omega e\|f_\omega\|_{L^2}^2\mathbf{u}_l\nabla_j\cdot a_l^\delta(t,\boldsymbol{\xi}) + O(e\delta),$$

again using lemma 38. Adding together these contributions, we end up with

$$\mathbf{F}^L = e\omega\|f_\omega\|_{L^2}^2\Big(\nabla a_0^\delta - (\partial_t\mathbf{a}^\delta) + \mathbf{u}\times\big(\nabla\times\mathbf{a}^\delta\big)\Big)(t,\boldsymbol{\xi}) + O(e^3+e\delta),$$

which is the required form of the Lorentz force law, as given in theorem 12, once we note that $\int \omega f_\omega^2 = \int(\omega - e\alpha)\, f_{\omega,e}^2 + O(e^2)$. □

5 Proof of the main growth estimate

In this section we are concerned with the proof of theorem 16. In order to control W it is helpful to introduce a quantity $\widetilde{W}$ which allows us to take advantage of certain cancellations occuring in the energy identity to handle some of the nonlinear interaction terms which would otherwise be difficult to estimate directly. The direct nonlinear interactions between v and $\tilde{\mathbf{A}}$ arise from terms in the Hamiltonian obtained by expanding the expression $\frac{1}{2}\int|(\nabla - ie\mathbb{A})\phi|^2$ in terms of $v, \tilde{\mathbf{A}}$ by means of (62). (There are also indirect interactions mediated by $\tilde{A}_0$ via the Gauss law, but these are easier to estimate.) In §2.1.3 this expansion of $\frac{1}{2}\int|(\nabla - ie\mathbb{A})\phi|^2$ is carried out explicitly, and, including also the quadratic part of the Taylor expansion of the potential $\mathcal{V}$, leads to the introduction of the quantity:

$$\begin{aligned}\tilde{H}(v,\tilde{\mathbf{A}}) &= \sum_{n=2}^{4}\tilde{H}^{(n)} \\ &= -\frac{1}{2}\sum_{n=2}^{4}\Big\langle(v,\tilde{\mathbf{A}}), \mathbb{B}^{(n-1)}(v,\tilde{\mathbf{A}})\Big\rangle_{L^2},\end{aligned}$$

where the superscript n (resp. $n-1$) indicates the homogeneity in $v, \tilde{\mathbf{A}}$ of the term $\tilde{H}^{(n)}$ in the expanded Hamiltonian (resp. of the term $\mathbb{B}^{(n-1)}$ in the expanded evolution equations (67),(69)); see §2.1.3 for explicit expressions and explanations. Using these definitions we have an alternative form for the expanded evolution: equations (66),(68) can be written in the form

$$\partial_t\big(v,\tilde{\mathbf{A}}\big) = \big(w,\tilde{\mathbf{E}}\big) - \big(i(\gamma\omega+h)v,0\big) - \big(\partial_t\lambda - V_0(\lambda)\big)\cdot\big(\widetilde{\partial_\lambda}\phi_{SC,e},\partial_\lambda\mathbf{A}_{SC,e}\big) + \big(\jmath_1^0,\jmath_3^0\big) + \big(\Phi_{11},0\big), \qquad (134)$$

with $\Phi_{11} = \jmath_1^{II}$. The remaining two equations (67),(69) can be written:

$$\begin{aligned}\partial_t\big(w,\tilde{\mathbf{E}}\big) = \big(-D_v\tilde{H}, -D_{\tilde{A}}\tilde{H}\big) - \big(i(\gamma\omega+h)w,0\big)& \\ - \big(\partial_t\lambda - V_0(\lambda)\big)\cdot\big(\widetilde{\partial_\lambda}\psi_{SC,e},\partial_\lambda\mathbf{E}_{SC,e}\big) + \big(\jmath_2^0,0\big) + \big(\Phi_{21},\Phi_{22}\big),& \qquad (135)\end{aligned}$$

where h is defined in (72), and $\Phi_{21} = \jmath_2^{II} + \mathcal{N}$, and $\Phi_{22} = \jmath_4^{II}$ are given in terms of the inhomogeneous terms defined in §2.1.3; notice that the inhomogeneous terms $\jmath_2^{III}, \jmath_2^{IV}, \jmath_4^{III}, \jmath_4^{IV}$ are included in the first term on the right hand side of (135).

To study these equations it will turn out that the following quantity is useful:

$$\widetilde{W} = \frac{1}{2}\big\|w - i\gamma\omega v\big\|_{L^2}^2 - \frac{1}{2}\gamma^2\omega^2\big\|v\big\|_{L^2}^2 + \frac{1}{2}\big\|\tilde{\mathbf{E}}\big\|_{L^2}^2 + \Big\langle\big(w,\tilde{\mathbf{E}}\big),\mathbf{u}\cdot\nabla\big(v,\tilde{\mathbf{A}}\big)\Big\rangle_{L^2} + \tilde{H}(v,\tilde{\mathbf{A}}).$$

We can think of $\widetilde{W}$ as follows: it is formed by adding to the Hessian of the augmented Hamiltonian W those terms arising in the expanded Hamiltonian (when we input the perturbed solution ansatz (62)) which describe the interactions of the fields $(v, \tilde{\mathbf{A}})$ with themselves and with the external electromagnetic field. An important reason for introducing $\widetilde{W}$ is that the following two lemmas imply a long time bound for W, and hence a stability estimate in energy norm.

Lemma 21 *In the situation of theorem 15, so that*

- *λ lies in a compact subset, $\mathbf{K} \subset \widetilde{O}_{stab}$,*
- *(v, w) satisfy the constraints (99), and*
- *W is equivalent (uniformly on $\mathbf{K}$) to $\|(v, w, \tilde{\mathbf{A}}, \tilde{\mathbf{E}})\|^2_{\mathcal{H}}$,*

assume that $W < 1$, and that $e = o(1)$ and $e = o(\delta)$. Then, there exists a constant $c(\mathbf{K}) > 0$ such that, for all $\lambda \in \mathbf{K}$,

$$cW \leq \widetilde{W} \leq \frac{1}{c} W. \tag{136}$$

Proof Referring to the formulae in §2.1.3 for the $\tilde{H}^{(n)}$ which occur in the definition of $\tilde{H}$, it is a straghtforward consequence of the Hölder inequality that

$$W = \widetilde{W} + O(\frac{e}{\delta} W) + O(\frac{e^2}{\delta^2} W) + O(\frac{e^2}{\delta} W^{\frac{3}{2}}) + O(e^2 W) + O(e W^{\frac{3}{2}}) + O(e^2 W^2),$$

lemma 27 and the assumptions on the external field in §1.2. The lemma follows immediately. □

Notation 22 *In the following we write, $f = \frac{d}{dt}\left(O(A) + o(B)\right)$ if there exist C^1 functions g, h such that $f = \frac{d}{dt}(g + h)$ and $g = O(A)$ and $h = o(B)$.*

Lemma 23 *Assume the hypotheses of theorem 16. It follows that,*

$$\left|\frac{d}{dt}\widetilde{W}\right| = \frac{d}{dt}\left(O\left(e\widetilde{W}^{\frac{1}{2}}\right) + o(\widetilde{W})\right) + O\left(e^4 + (e + \frac{e^2}{\delta})\widetilde{W} + (e^2 + e\delta)\widetilde{W}^{\frac{1}{2}}\right), \tag{137}$$

in the limit of e and $\widetilde{W}$ going to zero.

Proof See §5.2. □

5.1 Proof of theorem 16, assuming lemma 23

Proof Integrating up equation (137), and using the Cauchy-Schwarz inequality, $2e\delta\widetilde{W}^{1/2} \leq +e\delta^2 + e\widetilde{W}$, we infer the existence of a constant $c > 0$ such that, for $t \in [0, T_2/e]$,

$$\left|\widetilde{W}(t) - \widetilde{W}(0)\right| \leq c\left(e^2 + \delta^2 + |e|\int_0^t \widetilde{W}(s)ds\right), \tag{138}$$

as long as $e = O(\delta)$. By Gronwall's inequality and lemma 23, for $|e|$ sufficiently small there exists a constant $c > 0$ such that, on $[0, T_2/e]$,

$$\widetilde{W}(t) \leq c\left(\widetilde{W}(0) + e^2 + \delta^2\right)\exp[c|e|t]. \tag{139}$$

By lemma 21, the result is proved. □

5.2 Proof of lemma 23

5.2.1 Beginning of proof of lemma 23

By the assumptions of theorem 16 we have a solution of equations (134),(135) satisfying the conclusions of theorems 14 and 15, so that the constraints (109) hold and $W = O(e)$. Then, by lemma 21 and theorem 15, there exists $c > 0$ such that

$$\frac{1}{c}\widetilde{W} \leq \|(v, w, \tilde{\mathbf{A}}, \tilde{\mathbf{E}})\|^2_{\mathcal{H}} \leq c\widetilde{W}.$$

Also since $W = O(e)$ the bound (102) holds, and will be used in the course of the proof. The estimate for $\widetilde{W}$ will be obtained as a consequence of the energy identity for (134),(135), so the next stage is to write that identity down and separate the terms out in a way that allows them to be usefully estimated.

5.2.2 The energy identity for (66)-(69)

$$\begin{aligned} \frac{d}{dt}\widetilde{W} &= \Big\langle \partial_t\big(w, \tilde{\mathbf{E}}\big), \big(w, \tilde{\mathbf{E}}\big) + \mathbf{u}\cdot\nabla\big(v, \tilde{\mathbf{A}}\big) - i\gamma\omega\big(v, 0\big)\Big\rangle_{L^2} \\ &- \Big\langle \partial_t\big(v, \tilde{\mathbf{A}}\big), \big(-D_v\tilde{H}, -D_{\tilde{A}}\tilde{H}\big) - \mathbf{u}\cdot\nabla\big(w, \tilde{\mathbf{E}}\big) + i\gamma\omega\big(w, 0\big)\Big\rangle_{L^2} \\ &+ \int \partial_t\tilde{h}(v, \tilde{\mathbf{A}})dx - \partial_t\big(\gamma\omega\big)\left\langle iv, w\right\rangle_{L^2} + \Big\langle \partial_t\mathbf{u}\cdot\nabla\big(v, \tilde{\mathbf{A}}\big), \big(w, \tilde{\mathbf{E}}\big)\Big\rangle_{L^2}. \end{aligned} \tag{140}$$

Here we have introduced a notation $\tilde{h}(v, \tilde{\mathbf{A}})$ for the integrand defining $\tilde{H}$, i.e.

$$\begin{aligned} \tilde{H}(v, \tilde{\mathbf{A}}) &= \int \tilde{h}(v, \tilde{\mathbf{A}})\, dx \\ &= -\frac{1}{2}\int \sum_{n=2}^{4} \Big\langle (v, \tilde{\mathbf{A}}), \mathbb{B}^{(n-1)}(v, \tilde{\mathbf{A}})\Big\rangle\, dx. \end{aligned} \tag{141}$$

Explicit expressions for the nonlinear operators $\mathbb{B}^{(n-1)}(v, \tilde{\mathbf{A}})$ show that they depend on t, x, and the $\partial_t\tilde{h}$ in the final line of (140) refers to differentiation with $(v, \tilde{\mathbf{A}})$ held fixed; similar conventions will be understood below.

Substituting for the time derivatives from (134) and (135), and noting the usual cancellations which occur in the derivation of the energy identity, we obtain the following expression:

$$\begin{aligned} \frac{d}{dt}\widetilde{W} &= Q_1 + Q_2 + Q_3 - \langle D_v\tilde{H}, ihv\rangle_{L^2} + \int (\partial_t + \mathbf{u}\cdot\nabla)\tilde{h}(v, \tilde{\mathbf{A}})\, dx \\ &- \langle iv, (\mathbf{u}\cdot\nabla h)w\rangle_{L^2} - \partial_t\big(\gamma\omega\big)\left\langle iv, w\right\rangle_{L^2} + \Big\langle \partial_t\mathbf{u}\cdot\nabla(v, \tilde{\mathbf{A}}), (w, \tilde{\mathbf{E}})\Big\rangle_{L^2}, \end{aligned} \tag{142}$$

where

$$\begin{aligned} Q_1 &= \Big\langle \big(\Phi_{21}, \Phi_{22}\big), \big(w, \tilde{\mathbf{E}}\big) + \mathbf{u}\cdot\nabla\big(v, \tilde{\mathbf{A}}\big) - i\gamma\omega\big(v, 0\big)\Big\rangle_{L^2} \\ &\quad - \Big\langle \big(\Phi_{11}, 0\big), \big(-D_v\tilde{H}, -D_{\tilde{A}}\tilde{H}\big) + \mathbf{u}\cdot\nabla\big(w, \tilde{\mathbf{E}}\big) - i\gamma\omega\big(w, 0\big)\Big\rangle_{L^2}, \end{aligned}$$

$$\begin{aligned} Q_2 &= \Big\langle \big(\jmath_2^0, 0\big), \big(w, \tilde{\mathbf{E}}\big) + \mathbf{u}\cdot\nabla\big(v, \tilde{\mathbf{A}}\big) - i\gamma\omega\big(v, 0\big)\Big\rangle_{L^2} \\ &\quad - \Big\langle \big(\jmath_1^0, \jmath_3^0\big), \big(-D_v\tilde{H}, -D_{\tilde{A}}\tilde{H}\big) - \mathbf{u}\cdot\nabla\big(w, \tilde{\mathbf{E}}\big) + i\gamma\omega\big(w, 0\big)\Big\rangle_{L^2} \end{aligned}$$

and $Q_3 = -\big(\partial_t\lambda - V_0(\lambda)\big)\cdot\tilde{Q}_3$, where

$$\begin{aligned} \tilde{Q}_3 = &\Big\langle \big(\widetilde{\partial_\lambda}\psi_{SC,e}, \partial_\lambda\mathbf{E}_{SC,e}\big), \big(w, \tilde{\mathbf{E}}\big) + \mathbf{u}\cdot\nabla\big(v, \tilde{\mathbf{A}}\big) - i\gamma\omega\big(v, 0\big)\Big\rangle_{L^2} \\ - &\Big\langle \big(\widetilde{\partial_\lambda}\phi_{SC,e}, \partial_\lambda\mathbf{A}_{SC,e}\big), \big(-D_v\tilde{H}, -D_{\tilde{A}}\tilde{H}\big) - \mathbf{u}\cdot\nabla\big(w, \tilde{\mathbf{E}}\big) + i\gamma\omega\big(w, 0\big)\Big\rangle_{L^2}. \end{aligned}$$

We control Q_1, Q_2, Q_3 in the next three subsections before completing the proof of lemma 23. In the course of estimating the various terms we will use bounds for $\mathcal{N}, h$ and the $\Phi's$ (which may be read off from those in §2.1.4), and the bounds for $\tilde{A}_0$ in §A.2.2.

5.2.3 Estimation of Q_1

The following proposition is the main result about Q_1 needed for the basic growth estimate:

Proposition 24 *In the situation of lemma 23*

$$Q_1 = \partial_t \left(o(\widetilde{W}) + O(e\widetilde{W}^{\frac{1}{2}}) \right) + O\left(e^4 + e\widetilde{W} + e^2\widetilde{W}^{\frac{1}{2}} + e\delta\widetilde{W}^{\frac{1}{2}} \right).$$

Proof Substituting from (134) and (135) we obtain:

$$\begin{aligned} Q_1 \quad = \quad & (\partial_t\lambda - V_0) \cdot \langle \partial_\lambda \mathbf{A}_{SC,e} \Phi_{22} \rangle_{L^2} \\ & + (\partial_t\lambda - V_0) \cdot \left[\left\langle (\widetilde{\partial_\lambda} \phi_{SC,e}, \Phi_{21} \right\rangle_{L^2} - \left\langle \widetilde{\partial_\lambda} \psi_{SC,e}, \Phi_{11}, \right\rangle_{L^2} \right] \\ & + \left\langle ie\tilde{A}_0(i\gamma(\omega - e\alpha_{\omega,e}) - \mathbf{u}\cdot\nabla) f_{\omega,e}, \Phi_{11} \right\rangle_{L^2} \\ & - \left\langle (ie\tilde{A}_0 f_{\omega,e}, \nabla\tilde{A}_0), (\Phi_{21}, \Phi_{22}) \right\rangle_{L^2} \\ & + \langle (\partial_t + \mathbf{u}\cdot\nabla)v, \Phi_{21} \rangle_{L^2} + \langle ihv, \Phi_{21} \rangle_{L^2} - \langle ihw, \Phi_{11} \rangle_{L^2} \\ & + \big\langle (\partial_t + \mathbf{u}\cdot\nabla)\tilde{\mathbf{A}}, \Phi_{22} \big\rangle_{L^2} - \big\langle (\partial_t + \mathbf{u}\cdot\nabla)w, \Phi_{11} \big\rangle_{L^2} \end{aligned} \tag{143}$$

since $\Phi_{12} = 0$.

Estimation of the first line in Q_1 The first line of Q_1 is easily seen to be small, since $\Phi_{22} = -e^2\mathbf{a}^{\delta,\chi} f_{\omega,e}$ is $O(e^2)$ in every L^p by the bounds in §2.1.4. Together with the fact that, $\|\partial_\lambda \mathbf{A}_{SC,e}\|_{L^p} = O(e)$ for $p > 3$, by (51) and the results of appendix A.1.2, this implies that $\langle \partial_\lambda \mathbf{A}_{SC,e}, \Phi_{22} \rangle_{L^2} = O(e^3)$, and so by (102) the first line is $O(e^4)$.

Estimation of the second line in Q_1. The second line is smaller than appears due to a cancellation which is a consequence of the modulation equations, (115) or (116). To see this, we refer to the decomposition of the force on the right hand side of (116) given in §3.2.2, and using the definitions of the Φ_{IJ} in (134),(135), we see that

$$\begin{aligned} \left\langle \widetilde{\partial_{\lambda_A}} \phi_{SC,e}, \Phi_{21} \right\rangle_{L^2} \quad - \quad \left\langle \widetilde{\partial_{\lambda_A}} \psi_{SC,e}, \Phi_{11} \right\rangle_{L^2} &= -\mathbf{F}^L_A - \mathbf{F}^n_A + Err_A \\ &= -\left(\mathbb{M}(e)_{AB} + \beth_{AB} \right) \left(\dot{\lambda} - V_0 \right)_B + \mathbf{F}^p_A + Err_A \end{aligned}$$

where

$$Err_A = \left\langle \widetilde{\partial_{\lambda_A}} \phi_{SC,e} - \widetilde{\partial_{\lambda_A}} \phi_{S,0}, \Phi_{21} \right\rangle_{L^2} - \left\langle \widetilde{\partial_{\lambda_A}} \psi_{SC,e} - \widetilde{\partial_{\lambda_A}} \psi_{S,0}, \Phi_{11} \right\rangle_{L^2}.$$

Using lemma 35, the bound (95) for $\mathcal{N}$, and the fact that from §2.1.4 $\Phi_{11} = \jmath_1^{II}$ and $\Phi_{21} - \mathcal{N} = \jmath_2^{II}$ are $O(e)$, we deduce that $|Err_A| \leq ce^2(e + \widetilde{W} + \widetilde{W}^{5/2} + e^2\widetilde{W}^{1/2})$. Next notice that lemma 35 implies that $\mathbb{M}(e)_{AB} - \mathbb{M}(0)_{AB} = O(e^2)$. Therefore since $\mathbb{M}(0)_{AB} = -\mathbb{M}(0)_{BA}$ the largest term drops out and the second line of Q_1 can be rewritten as

$$\left(\mathbb{M}(e)_{AB} - \mathbb{M}(0)_{AB} + \beth_{AB} \right) \left(\dot{\lambda} - V_0 \right)_A \left(\dot{\lambda} - V_0 \right)_B - \left(\mathbf{F}^p_A + Err_A \right) \left(\dot{\lambda} - V_0 \right)_A$$

which, by the above and (123),(127) is $O(e^4 + e^2\widetilde{W}^{1/2})$, for small e and $\widetilde{W}$.

Estimation of the third and fourth lines in Q_1. Using lemma 39,(95), the bounds in §2.1.4 and the properties of $f_{\omega,e}$ in appendix A.1.1, the third and fourth lines can be estimated immediately to be $O\left(e^3\widetilde{W}^{1/2} + e^2\widetilde{W}^{3/2} \right)$.

Estimation of the fifth and sixth line in Q_1. This requires care because h is unbounded as a function of x. This makes it essential to separate the nonlinear term $\mathcal{N}$ in Φ_{21} from the other terms (which are exponentially decreasing in x and can thus absorb the unboundedness of h). Therefore we estimate first of all the quantity

$$\langle ihv, \Phi_{21} - \mathcal{N}(f_{\omega,e}, f_\omega, v)\rangle_{L^2} - \langle ihw, \Phi_{11}\rangle_{L^2} = O\left(e^2 \widetilde{W}^{\frac{1}{2}}\right), \tag{144}$$

by (102) and the bounds for h recorded in §2.1.4. Next, write the first term on line five, together with the missing piece $\langle ihv, \mathcal{N}\rangle_{L^2}$ from the previous estimation, as the sum of two quantities:

$$\langle (\partial_t + ih + \mathbf{u}\cdot\nabla)\, v, \mathcal{N}\rangle_{L^2} + Rem,$$

where $Rem = \langle (\partial_t + \mathbf{u}\cdot\nabla)\, v, \Phi_{21} - \mathcal{N}\rangle_{L^2}$. It is shown in lemma 40 that the first of these quantities is $\partial_t(o(\widetilde{W})) + O(e\widetilde{W} + e^3\widetilde{W}^{\frac{1}{2}})$. To complete the proof of proposition 24 we need to estimate the sixth line and the quantity Rem defined above. This is done by means of the integration by parts identity (195), and taking advantage of the fact that

$$(\partial_t + \mathbf{u}\cdot\nabla) f_{\omega,e} = (\dot{\lambda} - V_0(\lambda)) \cdot \partial_\lambda f_{\omega,e}, \tag{145}$$

is $O(e)$ by (102). Together with (191), this implies that

$$\|(\partial_t + \mathbf{u}\cdot\nabla)\Phi_{IJ}\|_{L^p} = O\big(e(e+\delta)\big) \tag{146}$$

for all p and all IJ except for $IJ = 21$; but in that case (146) holds instead for $\Phi_{21} - \mathcal{N} = \jmath_2^{II}$, (which is what is actually needed to estimate Rem). Putting this information into (195), we infer that the sixth line and Rem are $\partial_t(O(e\widetilde{W}^{\frac{1}{2}})) + O(e(e+\delta)\widetilde{W}^{\frac{1}{2}})$, which is sufficient to complete the proof of the proposition. □

5.2.4 Estimation of Q_2

The terms in Q_2 arising from $\jmath_1^0, \jmath_2^0$ can be estimated in a straightforward way by the Hölder and Sobolev inequalities, because of the exponential decay of $f_{\omega,e}$, and using lemma 39 to bound $\tilde{A}_0$. For example,

$$\left\langle w - i\gamma\omega v + \mathbf{u}\cdot\nabla v, ie\tilde{A}_0\left(i\gamma\left(\omega - e\alpha_{\omega,e}\right) - \mathbf{u}.\nabla\right) f_{\omega,e}\right\rangle_{L^2} = O\left(e^2\widetilde{W}\right) \tag{147}$$

by Hölder's inequality, since $f_{\omega,e}$ and $\nabla f_{\omega,e}$ are bounded in every L^p norm and $\|\tilde{A}_0\|_{L^p} = O\left(e\widetilde{W}^{\frac{1}{2}}\right)$ for $3 < p < \infty$. For the terms involving $\jmath_3^0 = \nabla\tilde{A}_0$ we can estimate,

$$\left\langle \mathbf{u}\cdot\nabla\tilde{\mathbf{E}}, \nabla\tilde{A}_0\right\rangle_{L^2} = \left\langle \operatorname{div}\tilde{\mathbf{E}}, \mathbf{u}\cdot\nabla\tilde{A}_0\right\rangle_{L^2} = O\left(e^2\widetilde{W}\right), \tag{148}$$

since $\|\operatorname{div}\tilde{\mathbf{E}}\|_{L^{3/2}} = O\left(e\widetilde{W}^{\frac{1}{2}}\right)$ and $\|\nabla\tilde{A}_0\|_{L^3} = O\left(e\widetilde{W}^{\frac{1}{2}}\right)$. Consider next the terms $\left\langle \jmath_3^0, -D_{\tilde{A}}\tilde{H}\right\rangle_{L^2}$. Referring to the explicit expressions for $D_{\tilde{A}}\tilde{H}$ given in §2.1.3, starting with (88), we see that the resulting terms can all be estimated in a straightforward way (using the bounds for $\nabla\tilde{A}_0$ in appendix A.2.2) to be $O(e^2\widetilde{W})$, except for one, namely:

$$\langle \triangle\tilde{\mathbf{A}}, \nabla\tilde{A}_0\rangle_{L^2},$$

but this vanishes by the Coulomb condition, and so $Q_2 = O(e^2\widetilde{W})$.

5.2.5 Estimation of Q_3

The quantity $\tilde{Q}_3$ is smaller than it appears due to the constraints. To see this first recall that, as used above already, $\|\partial_\lambda\mathbf{A}_{SC,e}\|_{L^p} = O(e)$ for $p > 3$, and $\|\partial_\lambda\mathbf{E}_{SC,e}\|_{L^p} = O(e)$ for $p > 3/2$, by (51) and the results of appendices A.1.2 and A.1.3 Referring to the expressions for $D_{\tilde{A}}\tilde{H}$ in §2.1.3, this means that the electromagnetic contributions to $\tilde{Q}_3$ can be bounded as $O(e\widetilde{W}^{\frac{1}{2}})$. But also, the expressions for $D_v\tilde{H}$ in §2.1.3 imply that

$$\left\langle \widetilde{\partial_\lambda}\phi_{SC,e}, -D_v\tilde{H} + M_\lambda v \right\rangle_{L^2} = O\left(e\widetilde{W}^{\frac{1}{2}}\right).$$

Therefore, up to $O(e\widetilde{W}^{\frac{1}{2}})$, we deduce that $\tilde{Q}_3$ is equal to

$$\left\langle \mathbf{u}\cdot\nabla w - i\omega\gamma w - M_\lambda v, \widetilde{\partial_\lambda}\phi_{SC,e}, \right\rangle_{L^2} - \left\langle \mathbf{u}\cdot\nabla v - i\omega\gamma v + w, \widetilde{\partial_\lambda}\psi_{SC,e} \right\rangle_{L^2}.$$

Now the identities in appendix A.1.4 and the constraints (109) imply that this expression vanishes if $\phi_{SC,e}, \psi_{SC,e}$ are replaced by $\phi_{S,0}, \psi_{S,0}$. But by lemma 35 this can be done at the expense of an $O\left(e^2\widetilde{W}^{\frac{1}{2}}\right)$ error. Therefore, since $(\dot{\lambda} - V_0) = O(e)$ by (102), we deduce that $Q_3 = O\left(e^2\widetilde{W}^{\frac{1}{2}}\right)$.

5.2.6 Completion of proof of lemma 23

The previous subsections have provided the requisite information on the $Q's$, and so it now suffices to control the remaining quantities in (142) appearing after the $Q's$. The following two propositions treat the two quantities on the first line of (142).

Proposition 25 *Assume the hypotheses of lemma 23. It follows that,*

$$\begin{aligned}\int(\partial_t + \mathbf{u}\cdot\nabla)\tilde{h}(v,\tilde{\mathbf{A}})\,dx &= -\frac{1}{2}\int\sum_{n=2}^{4}\left\langle (v,\tilde{\mathbf{A}}), (\partial_t + \mathbf{u}\cdot\nabla)\mathbb{B}^{(n-1)}(v,\tilde{\mathbf{A}})\right\rangle\,dx \\ &= O\left(e\widetilde{W} + \frac{e^2}{\delta}\widetilde{W}\right).\end{aligned}$$

Proof Observe

- the fact that $\mathbf{a}^{\delta,\chi}$ is pointwise $O(\frac{1}{\delta})$, but its derivatives are $O(1)$, in particular $\|(\partial_t + \mathbf{u}\cdot\nabla)\mathbf{a}^{\delta,\chi}\|_{L^\infty} \leq 2(\|\dot{\mathbf{a}}\|_{L^\infty} + \|\nabla\mathbf{a}\|_{L^\infty})$.
- the identity
$$(\partial_t + \mathbf{u}\cdot\nabla)f_{\omega,e} = (\dot{\lambda} - V_0)\cdot\partial_\lambda f_{\omega,e},$$
which shows that the left hand side is $O(e)$ in every L^p, by (102) and the exponential decay properties in appendix A.1. Similarly, $\|(\partial_t + \mathbf{u}\cdot\nabla)\alpha_{\omega,e}\|_{W^{1,\infty}}$ is $O(e^2)$ by (102) and the bounds for $\alpha_{\omega,e}$ in appendix A.1.2.

To prove the proposition now, just use these observations to estimate with Hölder's inequality each of the terms arising from differentiation of the expressions for $\mathbb{B}^{(n)}$ in §2.1.3. □

Proposition 26 *Assume the hypotheses of lemma 23. It follows that*

$$\langle D_v\tilde{H}, ihv\rangle_{L^2} = O\left(e\widetilde{W} + \frac{e^2}{\delta}\widetilde{W}\right).$$

Proof Using the notation in (88) for the Frechet derivative $D_v\tilde{H}$, we have

$$|\langle D_v\tilde{H}, ihv\rangle_{L^2}| = |\langle\mathbb{B}(v,\tilde{\mathbf{A}}), (ihv, 0)\rangle_{L^2}| \tag{149}$$

and we can estimate term by term, but some care is needed since h is unbounded as a function of x, see (72). In addition to the first point in the proof of the previous proposition, we use the bounds for h recorded in §2.1.4. Those terms in (149) arising from $\mathbb{B}^{(3)}$ vanish identically, while of those arising from $\mathbb{B}^{(2)}$ the only non-zero ones are proportional to $e\langle hv, \nabla v\tilde{\mathbf{A}}\rangle_{L^2}$. By the Coulomb condition and the bound for ∇h

from §2.1.4, this term is $O(e^2\widetilde{W}^{3/2})$. It remains to bound those terms arising from $\mathbb{B}^{(1)}$. Of these, it is straightforward to bound those arising from $\mathbb{B}_{12}$ as $O(e\widetilde{W})$ by the second fact just mentioned, and the same goes for those arising from M_λ in $\mathbb{B}_{11} = -M_\lambda + e\mathbf{R} + \mathbf{S}$. However, there is a single non-zero term arising from $e\mathbf{R}v$ which is proportional to

$$\langle hv, \mathbf{a}^{\delta,\chi} \cdot \nabla v\rangle$$

which, with an integration by parts, can be bounded as $O(\frac{e^2}{\delta}\widetilde{W})$, but, again, only after taking into account the Coulomb condition $\nabla \cdot \mathbf{a}^{\delta,\chi} = 0$. Finally for the terms arising from $\mathbf{S}$ we see from (81) that

$$\langle ihv, \mathbf{S}v\rangle_{L^2} = e\gamma \int [2\alpha_{\omega,e}\langle v\mathbf{u} \cdot \nabla v\rangle + \mathbf{u} \cdot \nabla\alpha_{\omega,e}|v|^2]\, dx = 0,$$

so that $\langle ihv, \mathbf{S}v\rangle_{L^2} = 0$, and the proof of the proposition is completed. □

The remaining terms on the second line of formula (142) are easily estimated as $O(e\widetilde{W})$ by (102), and the proof of lemma 23 is completed.

A Appendices

A.1 Further properties of the solitons

A.1.1 Exponential decay properties of the solitons

The $e = 0$ solitons in the nonlinear Klein-Gordon equation (23) are exponentially localized: to be precise we have the following estimates for the profiles functions f_ω, g_ω:

$$\lim_{|x|\to\infty} \sup \sum_{|\alpha|\leq 3} \nabla^\alpha f_\omega Exp[|x|\left(\sqrt{m^2-\omega^2} - \varepsilon\right)] < \infty \quad \forall\varepsilon \in (0, \sqrt{m^2-\omega^2}), \tag{150}$$

together with

$$\lim_{|x|\to\infty} \sup \sum_{|\alpha|\leq 3} \nabla^\alpha g_\omega Exp[|x|\left(\sqrt{m^2-\omega^2} - \varepsilon\right)] < \infty \quad \forall\varepsilon \in (0, \sqrt{m^2-\omega^2}), \tag{151}$$

and

$$\lim_{|x|\to\infty} \frac{f'_\omega}{f_\omega} = -\sqrt{m^2-\omega^2}, \tag{152}$$

while $\forall\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that

$$f_\omega(|x|) > c(\varepsilon)Exp[-|x|\left(\sqrt{m^2-\omega^2} + \varepsilon\right)]. \tag{153}$$

(See theorem 1.4 in [24]). Exponential decay also holds for the solitons coupled to electromagnetism for small e:

Lemma 27 *Suppose that $|e| < e_1$, for some $e_1 > 0$. Under conditions (29-32) on U,*

$$|D^\alpha f_{\omega,e}(x)| \leq C Exp[-\kappa|x|] \tag{154}$$

for positive constants C and κ, and where α is any multi-index with $|\alpha| \leq 2$. Furthermore, the constants C and κ are independent of the coupling constant e.

Proof See [17]. □

A.1.2 Some estimates of the soliton electromagnetic potential α

Lemma 28 *For each $f \in H_r^2(\mathbb{R}^3)$, there exists a unique $\alpha \in \dot{H}_r^1(\mathbb{R}^3)$ such that*

$$-\triangle\alpha + e^2 f^2 \alpha = \omega e f^2. \tag{155}$$

Furthermore, the map $A : H^2(\mathbb{R}^3) \longrightarrow \dot{H}^1(\mathbb{R}^3)$ defined by $A(f) = \alpha$ is continuously Frechet-differentiable.

Proof This follows from standard arguments. □

Lemma 29 *Suppose that $f \in H^1(\mathbb{R}^3)$. Suppose further that α solves*

$$-\triangle\alpha + e^2 f^2 \alpha = e\omega f^2. \tag{156}$$

It follows that $\nabla\alpha, \nabla^i\nabla^j\alpha \in L^2(\mathbb{R}^3)$ for any $i, j \in (1,2,3)$. Furthermore, $\left\|\nabla^i\nabla^j\alpha\right\|_{L^2}$, $\|\nabla\alpha\|_{L^2}$, $\|\alpha\|_{L^\infty} = O(e)$

Proof

$$\int |\nabla\alpha|^2 + e^2 f^2 \alpha^2 = e\omega \int f^2 \alpha \tag{157}$$

from which it easily follows via Sobolev's inequality that

$$\|\nabla\alpha\|_{L^2} \leq ce \|f\|_{L^2} \|f\|_{L^3}. \tag{158}$$

Next, since $-\triangle\alpha = e(\omega - e\alpha) f^2$, we have

$$\|\triangle\alpha\|_{L^2} \leq e\left(\omega \|f\|_{L^4}^2 + e\|\alpha_{\omega,e}\|_{L^6} \|f\|_{L^6}^2\right). \tag{159}$$

By the Calderon-Zygmund inequality, we have that for any $i, j \in (1,2,3)$,

$$\left\|\nabla^i\nabla^j\alpha\right\|_{L^2} = O(e). \tag{160}$$

By Sobolev's inequality, we have thus shown that $\alpha \in W^{1,6}$ and hence by Morrey's inequality, $\|\alpha\|_{L^\infty} = O(e)$. □

Corollary 30 *Suppose that $f_{\omega,e} \in H^2(\mathbb{R}^3)$ solves*

$$-\triangle f_{\omega,e} + m^2 f_{\omega,e} - (\omega - e\alpha_{\omega,e})^2 f_{\omega,e} = \beta(f_{\omega,e}) f_{\omega,e}, \tag{161}$$

where $\alpha_{\omega,e} \in \dot{H}_r^1(\mathbb{R}^3)$ is a non-local function of $f_{\omega,e}$ uniquely determined by

$$-\triangle\alpha_{\omega,e} + e^2 f_{\omega,e}^2 \alpha_{\omega,e} = \omega e f_{\omega,e}^2. \tag{162}$$

Then, $f_{\omega,e} \in H^4(\mathbb{R}^3)$.

Proof Differentiate the equation for $f_{\omega,e}$ and apply the Calderon-Zygmund inequality. □

This leads naturally to the following lemma.

Lemma 31 *Suppose that $f \in H^4(\mathbb{R}^3)$ and that α solves*

$$-\triangle\alpha + e^2 f^2 \alpha = e\omega f^2. \tag{163}$$

It follows that $\nabla\alpha \in W^{3,p}(\mathbb{R}^3)$ for any $p \in \left(\frac{3}{2}, \infty\right)$.

Proof Differentiate (163), and apply the Calderon-Zygmund inequality (using the Hölder and Sobolev inequalities if necessary) to get the result. □

Lemma 32 *Suppose that $f \in H^2(\mathbb{R}^3)$ and that α solves*

$$-\triangle\alpha + e^2 f^2 \alpha = e\omega f^2. \tag{164}$$

It follows that

$$0 \leq sgn\left(\frac{\omega}{e}\right)\alpha \leq \left|\frac{\omega}{e}\right|,$$

where $sgn(x) = x/|x|$ for $x \neq 0$ and $sgn(0) = 0$.

Proof Assume that f in $C_c^\infty(\mathbb{R}^3)$. Define $\alpha^+ = \max(\alpha, 0)$ and $\alpha^- = \max(-\alpha, 0)$. Suppose $\omega e > 0$, then by a weak maximum principle (theorem 8.1 in [8]), $\alpha > 0$. Now, $A_0 = \alpha - \frac{\omega}{e}$ solves $-\triangle A_0 + e^2 |f|^2 A_0 = 0$, therefore $A_0 \leq 0$ by the same weak maximum principle. Hence, $0 \leq \alpha \leq \frac{\omega}{e}$. Similarly, if $-\omega e > 0$, then $0 \geq \alpha \geq -\frac{\omega}{e}$ so that $\|\alpha\|_{L^\infty} \leq \left|\frac{\omega}{e}\right|$. The lemma follows by approximation. □

Lemma 33 *Suppose that $f_{\omega,e}$ and $\alpha_{\omega,e}$ are as given in theorem 6. Then,*

$$\left\|\nabla^i \nabla^j \frac{d\alpha_{\omega,e}}{d\lambda}\right\|_{L^p} = O(e) \tag{165}$$

for $p \in (1, \infty)$, and $i, j = 1, 2, 3$. In addition, $\left\|\nabla \frac{d\alpha_{\omega,e}}{d\lambda}\right\|_{W^{2,p}} = O(e)$ for any $p \in \left(\frac{3}{2}, \infty\right)$.

Proof From lemma 28 and theorem 6, $\frac{d\alpha_{\omega,e}}{d\lambda}$ is a well-defined object. We note that

$$\triangle \frac{d\alpha_{\omega,e}}{d\lambda_A} + e^2 f_{\omega,e}^2 \frac{d\alpha_{\omega,e}}{d\lambda_A} = e f_{\omega,e}^2 \delta_{-1\,A} + 2e f_{\omega,e}(\omega - e\alpha_{\omega,e}) \frac{df_{\omega,e}}{d\lambda_A} \tag{166}$$

from which $\left\|\triangle \frac{d\alpha_{\omega,e}}{d\lambda_A}\right\|_{L^p} = O(e)$ for $p \in (1, \infty)$ follows immediately. The lemma follows trivially from repeated differentiation, the Calderon-Zygmund inequality and the Hölder and Sobolev inequalities. □

Let $\zeta(x; \lambda)$ be the unique solution in $\dot{H}^1$ of (52), $-\triangle\zeta = -\gamma\mathbf{u} \cdot \nabla\alpha_{\omega,e}(\mathbf{Z})$, which takes the Lorentz transformed solitons into Coulomb gauge. Then

Lemma 34

$$\begin{aligned} \left\|\nabla^i \nabla^j \zeta\right\|_{L^p} &= O(e), \\ \left\|\nabla^i \nabla^j \partial_\lambda \zeta\right\|_{L^p} &= O(e), \end{aligned} \tag{167}$$

for $p \in (\frac{3}{2}, \infty)$ and $i, j = 1, 2, 3$.

Proof By (52), and its derivative: and

$$-\triangle \frac{d}{d\lambda_A}\zeta = -\gamma\mathbf{u} \cdot \nabla \frac{d}{d\lambda_A}\alpha_{\omega,e} - \left(\frac{d}{d\lambda_A}\gamma\mathbf{u}\right) \cdot \nabla\alpha_{\omega,e}.$$

the result follows by means of lemmas 29 and 33. □

A.1.3 Differentiability

Lemma 35 *Let $f_{\omega,e} \in H^2$ be given by theorem 6. Then it is a differentiable function of ω and satisfies, for small e:*

$$\|f_{\omega,e} - f_\omega\|_{H^2} + \|\partial_\omega f_{\omega,e} - \partial_\omega f_\omega\|_{H^2} = O(e^2). \tag{168}$$

Proof See [17]. □

Lemma 36 *Let* $\widetilde{h}_\omega = h_\omega - \omega q_\omega$, *where* $h_\omega = H(\Phi_{S,e}(0,\omega,0,0))$ *while* $q_\omega = Q(\Phi_{S,e}(0,\omega,0,0))$. *Then*

$$\frac{d}{d\omega}\widetilde{h}_\omega = -q_\omega. \tag{169}$$

Proof Following the argument given in [10], we note that

$$\frac{d}{d\omega}\widetilde{h}_\omega = -q_\omega + \left\langle H'(\Phi_{S,e}(\lambda_0)) - \omega Q'(\Phi_{S,e}(\lambda_0)), \frac{d}{d\omega}\Phi_{S,e}(\lambda_0)\right\rangle_{L^2}, \tag{170}$$

where $\lambda_0 = (\omega,0,0,0)$. The result follows from the fact that $H'(\Phi_{S,e}(\lambda_0)) - \omega Q'(\Phi_{S,e}(\lambda_0)) = 0$. □

A.1.4 Some identities involving $\left(\widetilde{\partial_\lambda}\phi_{S,0}, \widetilde{\partial_\lambda}\psi_{S,0}\right)$

The explicit calculation of the modulation equations can be carried out by making use of the following functions $(a_A(\mathbf{Z}(\mathbf{x},\lambda);\lambda), b_A(\mathbf{Z}(\mathbf{x},\lambda);\lambda)$ from [24]:

$$\begin{aligned}
b_{-1}(\mathbf{Z};\lambda) &= g_\omega - i\mathbf{u}.\mathbf{Z} f_\omega, &(171)\\
b_0(\mathbf{Z};\lambda) &= i f_\omega, &(172)\\
b_i(\mathbf{Z};\lambda) &= \nabla^i_{\mathbf{Z}} f_\omega(\mathbf{Z}), &(173)\\
b_{3+i}(\mathbf{Z};\lambda) &= \zeta_{ji}\nabla^j_{\mathbf{Z}} f_\omega(\mathbf{Z}) - i\omega\gamma((\gamma P_{\mathbf{u}} + Q_{\mathbf{u}})\,\mathbf{Z})_i f_\omega(\mathbf{Z}), &(174)
\end{aligned}$$

while

$$\begin{aligned}
a_{-1}(\mathbf{Z};\lambda) &= -\gamma^{-1} b_0 + (\gamma\mathbf{u}.\nabla_{\mathbf{Z}} - i\gamma\omega)\, b_{-1}, &(175)\\
a_0(\mathbf{Z};\lambda) &= (\gamma\mathbf{u}.\nabla_{\mathbf{Z}} - i\gamma\omega)\, b_0 &(176)\\
a_i(\mathbf{Z};\lambda) &= (\gamma\mathbf{u}.\nabla_{\mathbf{Z}} - i\gamma\omega)\, b_i, &(177)\\
a_{3+i}(\mathbf{Z};\lambda) &= (\gamma P_{\mathbf{u}} + Q_{\mathbf{u}})\,\mathbf{Z})_{ij} b_j + (\gamma\mathbf{u}.\nabla_{\mathbf{Z}} - i\gamma\omega)\, b_{3+i}, &(178)
\end{aligned}$$

where $i,j = 1,2,3$, $g_\omega = \frac{d}{d\omega} f_\omega$, and

$$\zeta_{ji} = \gamma^2(\mathbf{u}\cdot\mathbf{Z})(P_{\mathbf{u}})_{ji} + \frac{\gamma-1}{\gamma|\mathbf{u}|^2}(\mathbf{u}\cdot\mathbf{Z})(Q_{\mathbf{u}})_{ji} + \frac{\gamma-1}{|\mathbf{u}|^2}(Q_{\mathbf{u}}Z)_i\mathbf{u}_j.$$

These are convenient for computation of the modulation equations because the linear span of the $\widetilde{\partial_{\lambda_A}}(\phi_{S,0},\psi_{S,0})$ is the same as the linear span of the $(b_A, -a_A)$.(To be precise: except for $A = j \in \{1,2,3\}$, we have $\widetilde{\partial_{\lambda_A}}(\phi_{S,0},\psi_{S,0}) = (b_A,-a_A)$, and for $A = j$ we have $\widetilde{\partial_{\xi^j}}(\phi_{S,0},\psi_{S,0}) = -(\gamma P_{\mathbf{u}} + Q_{\mathbf{u}})_{jk}\,(b_k,-a_k) + \omega\gamma u^j(b_0,a_0)$.)

The following identities are equivalent to lemma 2.2 in [24], and can be obtained by differentiating the Euler-Lagrange equation $F_0' = 0$, where F_0 is the augmented Hamiltonian (48):

$$\begin{aligned}
(i\gamma\omega - \mathbf{u}\cdot\nabla)\,\widetilde{\partial_{\lambda_0}}\phi_{S,0} - \widetilde{\partial_{\lambda_0}}\psi_{S,0} &= 0, &(179)\\
(i\gamma\omega - \mathbf{u}\cdot\nabla)\,\widetilde{\partial_{\lambda_0}}\psi_{S,0} - M_\lambda\widetilde{\partial_{\lambda_0}}\phi_{S,0} &= 0, &(180)\\
(i\gamma\omega - \mathbf{u}\cdot\nabla)\,\widetilde{\partial_{\lambda_j}}\phi_{S,0} - \widetilde{\partial_{\lambda_j}}\psi_{S,0} &= 0, &(181)\\
(i\gamma\omega - \mathbf{u}\cdot\nabla)\,\widetilde{\partial_{\lambda_j}}\psi_{S,0} - M_\lambda\widetilde{\partial_{\lambda_j}}\phi_{S,0} &= 0, &(182)\\
(i\gamma\omega - \mathbf{u}\cdot\nabla)\,\widetilde{\partial_{\lambda_{-1}}}\phi_{S,0} - \widetilde{\partial_{\lambda_{-1}}}\psi_{S,0} &= -\frac{1}{\gamma}\widetilde{\partial_{\lambda_0}}\phi_{S,0}, &(183)\\
(i\gamma\omega - \mathbf{u}\cdot\nabla)\,\widetilde{\partial_{\lambda_{-1}}}\psi_{S,0} - M_\lambda\widetilde{\partial_{\lambda_{-1}}}\phi_{S,0} &= -\frac{1}{\gamma}\widetilde{\partial_{\lambda_0}}\psi_{S,0}, &(184)\\
(i\gamma\omega - \mathbf{u}\cdot\nabla)\,\widetilde{\partial_{\lambda_{3+j}}}\phi_{S,0} - \widetilde{\partial_{\lambda_{3+j}}}\psi_{S,0} &= -\widetilde{\partial_{\lambda_j}}\phi_{S,0} - \gamma\omega u_j\widetilde{\partial_{\lambda_0}}\phi_{S,0}, &(185)\\
(i\gamma\omega - \mathbf{u}\cdot\nabla)\,\widetilde{\partial_{\lambda_{3+j}}}\psi_{S,0} - M_\lambda\widetilde{\partial_{\lambda_{3+j}}}\phi_{S,0} &= -\widetilde{\partial_{\lambda_j}}\psi_{S,0} - \gamma\omega u_j\widetilde{\partial_{\lambda_0}}\psi_{S,0} &(186)
\end{aligned}$$

where the index j runs from 1 to 3.

A.2 Some estimates

A.2.1 Estimates related to the external field

Lemma 37 *Let f be a measurable function with $(1+|x|)f \in L^1$. Then if $a^{\delta,\chi}$ is as in* (64)

$$\left\| e a_0^{\delta,\chi} f \right\|_{L^p} \leq ceL_1 \left\| (1+|x-\xi|) f \right\|_{L^p}, \tag{187}$$

and

$$\left\| e \mathbf{a}^{\delta,\chi} f \right\|_{L^p} \leq ceL_1 \left\| (1+|x-\xi|) f \right\|_{L^p} \tag{188}$$

for $p \in [1,\infty]$. If in addition f_{even} is an even function of $(\mathbf{x}-\boldsymbol{\xi})$ and $(1+|x|)^2 f_{even} \in L^1$ then

$$\left| \int a_\mu^{\delta,\chi} f_{even} d^3x \right| \leq cL_2 \delta \left\| (1+|x-\xi|)^2 f_{even} \right\|_{L^1} \tag{189}$$

with L_1, L_2 as in (7).

Proof Recall (64) and (65). Writing

$$a_0^\delta(t,x) - a_0^\delta(t,\xi) = (\mathbf{x}-\xi) \cdot \int \nabla a_0^\delta(t, \xi + s(x-\xi)) ds \tag{190}$$

etc, by the fundamental theorem of calculus, the result then follows, using the fact that the gradients of $a_0^\delta, \mathbf{a}^\delta$ are bounded independent of δ by assumption (see §1.2). For the proof of (189), it suffices to use the identity for ∇a_μ^δ corresponding to (190), and then substitute this back into (190) and use the fact that $\int (\mathbf{x}-\xi) f_{even} = 0$. □

Similarly, we have the following bounds:

Lemma 38

$$\begin{aligned} \left\| (1+|x-\xi|)^{-1} (\partial_t + \mathbf{u}\cdot\nabla) a_\mu^{\delta,\chi} \right\|_{L^\infty} &\leq C_1(|\delta| + |e|) \qquad (191) \\ \int_{\mathbb{R}^3} f(x) \left| \nabla_{t,x} a_\mu^\delta(t,\mathbf{x}) - (\nabla_{t,x} a_\mu^\delta)(t,\boldsymbol{\xi}) \right| d\mathbf{x} &\leq C_2 |\delta|, \qquad (192) \end{aligned}$$

where we use (102), (7), *and $C_1 = C_1(L_1, L_2)$ and $C_2 = C_2(L_2, \|(1+|x|)f\|_{L^1})$.*

A.2.2 Estimates for the time component of the electromagnetic potential

Lemma 39 *Given $(v,w) \in H^1 \times L^2$ and $\lambda \in \widetilde{O}$ there exists a unique $\tilde{A}_0 \in \dot{H}^1$ solving* (71) *such that*

$$\left\| \nabla \tilde{A}_0 \right\|_{L^p} = O\left(e \|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}} + e \|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}}^2 \right), \tag{193}$$

for $p \in (\frac{3}{2}, 3]$. Consequently $\left\| \tilde{A}_0 \right\|_{L^q}$ satisfies the same bound for $3 < q < \infty$ by Sobolev's inequality.

Proof From Gauss's law (71), we have explicitly

$$-\triangle \tilde{A}_0 = e \langle i f_{\omega,e}, w \rangle + e \langle iv, (i\gamma(\omega - e\alpha_{\omega,e}) - \mathbf{u}.\nabla) f_{\omega,e} + w \rangle. \tag{194}$$

By Sobolev's and Hölder's respective inequalities,

$$\left\| \triangle \tilde{A}_0 \right\|_{L^q} = O\left(e \|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}} + e \|(v,w,\tilde{\mathbf{A}},\tilde{\mathbf{E}})\|_{\mathcal{H}}^2 \right).$$

for $q \in [1, \frac{3}{2}]$. The lemma follows from the Sobolev inequality and from the Calderon-Zygmund inequality, [8, section 9.4]. □

A.2.3 Integration by parts and simple averaging

First we recall the phenomenon of averaging in the context of ordinary differential equations, in the simplest possible case of the perturbed harmonic oscillator. Let g be a C^1 function of $t \in \mathbb{R}$, with $|g| \leq M$ and $|\dot{g}| \leq N$. For $0 < \epsilon << 1$ let y^ϵ be the solution of $\ddot{y} + y = \epsilon g(\epsilon t)$ with initial data $y^\epsilon(0) = y_0, \dot{y}^\epsilon(0) = y_1$ (fixed independent of ϵ). Then $y^\epsilon - y^0$ is $O(\epsilon)$ in $C^1([-T, T])$ norm for times of $T = O(\frac{1}{\epsilon})$. One way to prove this is to define $f = \epsilon^{-1}(y^\epsilon - y^0)$, which solves $\ddot{f} + f = g(\epsilon t)$ with zero initial data. Let $E(t) = (f^2 + \dot{f}^2)/2$ be the energy; it satisfies $E(0) = 0$ and $\dot{E}(t) = \dot{f}(t) g(\epsilon t)$. Now an integration by parts gives

$$\begin{aligned} \left| \int_0^T \dot{f}(t) g(\epsilon t) dt \right| &\leq M|f(T)| + \epsilon N \int_0^T |f(t)| dt \\ &\leq M|f(T)| + \frac{\epsilon T N^2}{2} + \epsilon \int_0^T \frac{|f(t)|^2}{2} dt, \end{aligned}$$

which, by Gronwall's inequality, implies $E(t) = O(1)$ for $t = O(\frac{1}{\epsilon})$ as claimed. To conclude, this simple fact - that a small slowly varying inhomogeneous $\epsilon g(\epsilon t)$ term only influences a simple harmonic oscillator to $O(\epsilon)$ on time scales of $O(\frac{1}{\epsilon})$ - expresses a weak averaging effect, and can be proved by integration by parts. Of course, this argument can be modified to give information about perturbed oscillators on longer times scales of $O(\frac{1}{\epsilon^a})$, $a < 2$, and many different generalizations are possible.

A simple generalization, which is usful for the study of slow motion of solitons, can be obtained by integrating the identity

$$\langle (\partial_t + \mathbf{u} \cdot \nabla) F, G \rangle_{L^2} = \partial_t \langle F, G \rangle_{L^2} - \langle F, (\partial_t + \mathbf{u} \cdot \nabla) G \rangle_{L^2} \tag{195}$$

where F, G are sufficiently regular functions of t, x but $\mathbf{u} = \mathbf{u}(t)$ depends on t only and the inner product is $L^2(dx)$. This is often useful because in perturbation theory for solitons functions often arise with $(\partial_t + \mathbf{u} \cdot \nabla) G$ small - see (146).

The following result, used in the proof of proposition 24, is a more complicated version of this idea:

Proposition 40 *In the situation of lemma 23,*

$$\langle (\partial_t + ih + \mathbf{u} \cdot \nabla) v, \mathcal{N}(f_{\omega,e}, f_\omega, v) \rangle_{L^2} = \partial_t \left(o\left(\widetilde{W} \right) \right) + O\left(e\widetilde{W} + e^3 \widetilde{W}^{\frac{1}{2}} \right),$$

where a function f satisfies $f = d/dt \left(o\left(\widetilde{W} \right) \right)$ if there exists a C^1 function $g = o\left(\widetilde{W} \right)$ and $f = \frac{d}{dt} g$.

Proof We work mostly with the potential $\mathcal{V}_1(\phi) = -U(|\phi|)$ which determines $\mathcal{N}$: recall that $\mathcal{V}_1'(\phi) = -\beta(|\phi|)\phi$, and (being slightly cavalier with notation) (76) can be rewritten

$$\mathcal{N}(f_{\omega,e}, f_\omega, v) = -\mathcal{V}_1^{'}(f_{\omega,e} + v) + \mathcal{V}_1^{'}(f_{\omega,e}) + \mathcal{V}_1^{''}(f_\omega)(v)$$

Define

$$\overline{\Theta} = \int_0^t h ds, \tag{196}$$

$$f_{\omega,e}^* = Exp[i\overline{\Theta}] f_\omega \tag{197}$$

and

$$v^* = Exp[i\overline{\Theta}] v. \tag{198}$$

Then, as with (93), and using the fact that $\|\partial_t v^*\|_{L^2} = O(e + \widetilde{W}^{\frac{1}{2}})$ by (66), we have

$$\begin{aligned} &\langle \partial_t v + ihv, \mathcal{N}(f_{\omega,e}, f_\omega, v) \rangle_{L^2} = \\ &\qquad - \left\langle \partial_t v^*, \mathcal{V}_1'(f_{\omega,e}^* + v^*) - \mathcal{V}_1'(f_{\omega,e}^*) - \mathcal{V}_1''(f_{\omega,e}^*)[v^*] \right\rangle_{L^2} + O\left(e^3 \widetilde{W}^{\frac{1}{2}} + e^2 \widetilde{W} \right). \end{aligned} \tag{199}$$

But,

$$\left\langle \partial_t v^*, \mathcal{V}_1'(f_{\omega,e}^*+v^*)-\mathcal{V}_1'(f_{\omega,e}^*)-\mathcal{V}_1''(f_{\omega,e}^*)[v^*]\right\rangle_{L^2} = \\ \partial_t \int \left(\mathcal{V}_1(f_{\omega,e}^*+v^*)-\mathcal{V}_1(f_{\omega,e}^*)-\mathcal{V}_1'(f_{\omega,e}^*)[v^*]-\frac{1}{2}\mathcal{V}_1''(f_{\omega,e}^*)[v^*]^2\right)dx \\ -\left\langle \partial_t f_{\omega,e}^*, \mathcal{V}_1'(f_{\omega,e}^*+v^*)-\mathcal{V}_1'(f_{\omega,e}^*)-\mathcal{V}_1''(f_{\omega,e}^*)[v^*]-\frac{1}{2}\mathcal{V}_1^{(3)}(f_{\omega,e}^*)[v^*]^2\right\rangle_{L^2}. \quad (200)$$

Hence,

$$\left\langle \partial_t v^*, \mathcal{V}_1'(f_{\omega,e}^*+v^*)-\mathcal{V}_1'(f_{\omega,e}^*)-\mathcal{V}_1''(f_{\omega,e}^*)[v^*]\right\rangle_{L^2} = \\ \partial_t \int \left(\mathcal{V}_1(f_{\omega,e}+v)-\mathcal{V}_1(f_{\omega,e})-\mathcal{V}_1'(f_{\omega,e})[v]-\frac{1}{2}\mathcal{V}_1''(f_{\omega,e})[v]^2\right)dx \\ -\left\langle (\partial_t+ih)\, f_{\omega,e}, \mathcal{V}_1'(f_{\omega,e}+v)-\mathcal{V}_1'(f_{\omega,e})-\mathcal{V}_1''(f_{\omega,e})[v]\right\rangle_{L^2} - \\ \left\langle (\partial_t+ih)\, f_{\omega,e}, \frac{1}{2}\mathcal{V}_1^{(3)}(f_{\omega,e})[v]^2\right\rangle_{L^2}. \quad (201)$$

Now,

$$\left\langle ihf_{\omega,e}, \mathcal{V}_1^{(3)}(f_{\omega,e})[v]^2\right\rangle_{L^2} \le c\int |f_{\omega,e}h|\left(1+|f_{\omega,e}|^3\right)|v|^2 dx, \quad (202)$$

by condition (13). Additionally,

$$\begin{aligned} &\left\langle ihf_{\omega,e}, \mathcal{V}_1'(f_{\omega,e}+v)-\mathcal{V}_1'(f_{\omega,e})-\mathcal{V}_1''(f_{\omega,e})[v]\right\rangle_{L^2} \quad (203)\\ &\quad = \int_0^1 (1-s)\left\langle ihf_{\omega,e}, \left(\mathcal{V}_1''(f_{\omega,e}+sv)-\mathcal{V}_1''(f_{\omega,e})\right)[v]\right\rangle_{L^2}, \\ &\quad \le c\int |f_{\omega,e}h|\left(1+|f_{\omega,e}|^3\right)\left(|v|^2+|v|^5\right)dx, \end{aligned}$$

by condition (13). Therefore, by the exponential decay of $f_{\omega,e}$ and the fact that $|f_{\omega,e}h|_{L^p} = O(e)$ by the bounds of §2.1.4,

$$\left\langle ihf_{\omega,e}, \mathcal{V}_1'(f_{\omega,e}+v)-\mathcal{V}_1'(f_{\omega,e})-\mathcal{V}_1''(f_{\omega,e})[v]-\frac{1}{2}\mathcal{V}_1^{(3)}(f_{\omega,e})[v]^2\right\rangle_{L^2} = O(e\widetilde{W}).$$

Integration by parts and lemma 35 imply that

$$\langle(\mathbf{u}\cdot\nabla v, \mathcal{N}(f_{\omega,e}, f_{\omega,0}, v)\rangle = O(e^2\widetilde{W}) + \\ \left\langle \mathbf{u}\cdot\nabla f_{\omega,e}, \mathcal{V}_1'(f_{\omega,e}+v)-\mathcal{V}_1'(f_{\omega,e})-\mathcal{V}_1''(f_{\omega,e})[v]-\frac{1}{2}\mathcal{V}_1^{(3)}(f_{\omega,e})[v]^2\right\rangle.$$

Next notice that the quantity

$$\left\langle (\partial_t+\mathbf{u}\cdot\nabla)f_{\omega,e}, \mathcal{V}_1'(f_{\omega,e}+v)-\mathcal{V}_1'(f_{\omega,e})-\mathcal{V}_1''(f_{\omega,e})[v]-\frac{1}{2}\mathcal{V}_1^{(3)}(f_{\omega,e})[v]^2\right\rangle$$

can be estimated to be $O(e\widetilde{W})$ in the same way as the bounds (202),(203) once we note that, for every $p\in[1,\infty]$,

$$\|\partial_t f_{\omega,e}+\mathbf{u}\cdot\nabla f_{\omega,e}\|_{L^p} = \left\|(\dot{\lambda}-V_0(\lambda))\cdot\partial_\lambda f_{\omega,e}\right\|_{L^p} = O(e), \quad (204)$$

by (102). The proof is now completed by noticing that Taylor's theorem and (13) imply that the quantity

$$\int\left(\mathcal{V}_1(f_{\omega,e}^*+v^*)-\mathcal{V}_1(f_{\omega,e}^*)-\mathcal{V}_1'(f_{\omega,e}^*)[v^*]-\frac{1}{2}\mathcal{V}_1''(f_{\omega,e}^*)[v^*]^2\right)dx$$

is $O(\widetilde{W}^{3/2})+O(\widetilde{W}^3) = o(\widetilde{W})$, since $\widetilde{W}$ is small by assumption. □

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