Existence and Newtonian limit of nonlinear bound states in the Einstein-Dirac system

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Abstract

An analysis is given of particlelike nonlinear bound states in the Newtonian limit of the coupled Einstein-Dirac system introduced by Finster, Smoller and Yau. A proof is given of existence of these bound states in the almost Newtonianian regime, and it is proved that they may be approximated by the energy minimizing solution of the Newton-Schrödinger system obtained by Lieb.

1 Introduction

Since the gravitational interaction is attractive, there is the possibility that theories involving fields coupled to gravity will possess bound states held together by this force. At the level of Newtonian gravity an example is provided by the Newton-Schroödinger system, in which there are bound states which can be obtained by minimizing a nonlocal energy functional ([10]); see also more recent discussions in [8, 12, 5]. In general relativity, it was first established that this phenomenon occurs for the Yang-Mills-Einstein equations (see [2, 14] for the original work, and [4] for some more recent developments and further references) and then for various other systems involving Dirac fields ([6, 7]). In this article we discuss the bound states in the Einstein-Dirac system which were studied numerically in [6], and prove that in the almost Newtonian regime they may be rigorously approximated by the bound states in the Newton-Schrödinger system, and as a consequence give an existence proof.

The starting point is the action functional

$$S = \frac{c^3}{8\pi G}\int R\, d\mu_g + \sum_{A=1}^{2} \hbar \int \overline{\Psi}_A (\not{D} - \frac{mc}{\hbar})\Psi_A\, d\mu_g \tag{1.1}$$

describing the interaction of two Dirac spinor fields Ψ_1 and Ψ_2 with a gravitational metric g, whose scalar curvature is R and whose volume element is $d\mu_g$. We use $\not{D}$ for the Dirac operator derived from g, with associated γ matrices, as explained in [6, Section II]. The Euler-Lagrange equations are

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi G}{c^4}T_{ab}, \qquad (\not{D} - \frac{mc}{\hbar})\Psi_A = 0 \tag{1.2}$$

where R_{ab} is the Ricci curvature and $T_{ab} = \frac{\hbar c}{2}\sum \mathrm{Re}\left[\overline{\Psi}_A(i\gamma_a\partial_b + i\gamma_b\partial_a)\Psi_A\right]$ is the energy-momentum tensor; space-time indices a, b take values in $\{0, 1, 2, 3\}$. The system (1.2) is precisely that studied in [6], except that the dimensional constants $G, \hbar, c$ have been reinstated. It is always to be understood that $\hbar, m, G$ are fixed positive numbers, while the speed of light c is a positive number taking large values, i.e. we will study the Newtonian (or non-relativistic) limit in which $c \to +\infty$. The reason there are two Dirac spinor fields in the model is that this allows for spherical symmetry: indeed, in the article [6] a spherically symmetric ansatz was introduced, and the corresponding system of ordinary differential equations (ODEs) for static spherically symmetric solutions was derived. Numerical evidence was presented for the existence of solutions to this system of ODEs, which give particlelike, or solitonic, solutions of (1.2) corresponding to nonlinear bound states of two fermions; also the linear stability of these solutions was investigated. In this note we will show that in the Newtonian limit $c \to +\infty$ these solutions of the system (1.2) can be approximated by solutions of the Newton-Schrödinger system

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + u\psi, \qquad -\Delta u = -\kappa|\psi|^2 \tag{1.3}$$

for $\kappa = 8\pi G m$. In fact the rigorous analysis of the limit we provide yields a proof of existence for the bound state solutions of (1.2) introduced in [6], which does not seem to have appeared previously in the literature. The type of solution of (1.3) is one in which the time dependence is uniform phase rotation at frequency $\frac{\eta}{\hbar}$: explicitly $\psi = e^{-i\frac{\eta}{\hbar}t}\varphi(x)$, and φ solves

$$\eta\varphi = -\frac{\hbar^2}{2m}\Delta\varphi + mu\varphi, \qquad -\Delta u = -8\pi G m|\varphi|^2. \tag{1.4}$$

The existence of such nonlinear bound state solutions to the Newton-Schrödinger system was proved in [10] by variational methods. Pseudo-relativistic generalizations of Lieb's solutions have been given in [9], and we will make use of a result from this article on the non-degeneracy of the linearization of (1.4): see lemma 4 and the subsequent remark.

1.1 Spherical symmetry

Using the ansatz for bound states with spherical symmetry introduced in [6], we search for solutions with metric

$$g = c^2 e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 d\Omega^2 \tag{1.5}$$

and spinor fields of the form

$$\Psi_1 = e^{\nu/2} e^{-i\omega t} \begin{pmatrix} \Phi_1 e_1 \\ i\Phi_2 \sigma^r e_1 \end{pmatrix}, \qquad \Psi_2 = e^{\nu/2} e^{-i\omega t} \begin{pmatrix} \Phi_1 e_2 \\ i\Phi_2 \sigma^r e_2 \end{pmatrix} \tag{1.6}$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \sigma^r = \frac{1}{r}\sum_{i=1}^{3} x^i \sigma^i$$

where σ^i are the Pauli matrices. In (1.6) $\nu, \lambda, \Phi_1, \Phi_2$ all depend on r only, and

$$\omega\hbar = mc^2 + \eta$$

with η as in (1.4).

Remark 1 *The angular dependence in* (1.6) *is taken over directly from that displayed by the ground state Dirac wave functions for the relativistic hydrogen atom. Recall from [13, 16] that in problems involving the Dirac equation with spherical symmetry there are three commuting operators: J^2, J_3 and K (respectively the total angular momentum squared, the third component of the total angular momentum, and the spin-orbit coupling operator defined in [13, Equation 3.275]) which also commute with the Hamiltonian. The simultaneous eigenspaces of these three operators (corresponding to eigenvalues conventionally written $j(j+1)\hbar^2, j_3\hbar, \kappa\hbar$) are two dimensional and give the decomposition into irreducible subspaces. The possible values of j are in the set $\{\frac{1}{2}, \frac{3}{2}, \dots\}$, those of j_3 are in the set $\{-j, \dots, +j\}$ and $\kappa = \pm(j+\frac{1}{2})$. The ground state corresponds to the choice $j = \frac{1}{2}$, and the wave functions Ψ_1 (resp. Ψ_2) correspond to $j_3 = \frac{1}{2}$ (resp. $j_3 = -\frac{1}{2}$) and $\kappa = -1$. The value of κ is not important, but what is crucial is that the states Ψ_1, Ψ_2 have opposite values of j_3, which ensures that the energy momentum tensor T_{ab} is consistent with the spherically symmetric metric in* (1.5)*: see [6, Setion IV].*

Substitution of (1.5)-(1.6) into (1.2) leads, as in [6, Section IV], to the following system of equations:

$$(\frac{\omega e^{-\nu}}{c} - \frac{mc}{\hbar})\Phi_1 - e^{-\lambda}\frac{\partial \Phi_2}{\partial r} - \frac{(e^{-\lambda}+1)}{r}\Phi_2 = 0, \tag{1.7}$$

$$(\frac{\omega e^{-\nu}}{c} + \frac{mc}{\hbar})\Phi_2 + e^{-\lambda}\frac{\partial \Phi_1}{\partial r} + \frac{(e^{-\lambda}-1)}{r}\Phi_1 = 0 \tag{1.8}$$

to be solved coupled to

$$e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi G\frac{r^2}{c^2}\rho, \tag{1.9}$$

$$e^{-2\lambda}(2r\nu' + 1) - 1 = 8\pi G\frac{r^2}{c^4}p, \tag{1.10}$$

where $\rho = \rho(\nu, \Phi_1, \Phi_2, \eta, c)$ and $p = p(\nu, \Phi_1, \Phi_2, \eta, c)$ are given by

$$\rho = 2(m + c^{-2}\eta)e^{-2\nu}(\Phi_1^2 + \Phi_2^2) \tag{1.11}$$

$$p = -2(m + c^{-2}\eta)e^{-2\nu}(\Phi_1^2 + \Phi_2^2) \tag{1.12}$$

$$+ 4\frac{\hbar c}{r}e^{-\nu}\Phi_1\Phi_2 + 2mc^2 e^{-\nu}(\Phi_1^2 - \Phi_2^2) \tag{1.13}$$

$$= -2\eta e^{-2\nu}(\Phi_1^2 + \Phi_2^2) - 2mc^2 e^{-\nu}\Phi_1^2(e^{-\nu} - 1) \tag{1.14}$$

$$- 2mc^2 e^{-\nu}\Phi_2^2 + 4\frac{\hbar c}{r}\Phi_1\Phi_2. \tag{1.15}$$

1.2 Newtonian limit

We will solve the system (1.7)-(1.10) in the almost Newtonian regime. To be precise, we treat $\epsilon = c^{-1}$ as a small positive parameter, and show that the system (1.7)-(1.10) can be regarded as a perturbation of the Newton-Schrödinger system (1.3):

> *The spherically symmetric Einstein-Dirac system* (1.7)-(1.10) *admits nonlinear bound state solutions* $(\lambda^\epsilon, \nu^\epsilon, \Phi_1^\epsilon, \Phi_2^\epsilon)$ *for small positive* ϵ*, which can be approximated (in a strong weighted norm) by the bound state solution* φ *of the Newton-Schrödinger system which minimizes the energy* (2.25)*: in particular,* $(\Phi_1^\epsilon, \Phi_2^\epsilon)$ *converges uniformly to* $(\varphi, 0)$ *as* $\epsilon \to 0$.

To formulate more precisely and prove this we carry out a rescaling of the dependent variables, and also make an adjustment to facilitate the handling of the ADM mass. The precise statement appears in theorem 9, and asserts the validity of the Newtonian approximation in norms stronger than C^1 with exponential weights in the (massive) Dirac fields Φ_1, Φ_2, and polynomial weights for the (massless) metric components λ, ν: see (3.43). The bound state of the Newton-Schrödinger system determines space and time scales, and the requirement that ϵ be small (or, equivalently, that the speed of light c be large) should be regarded as being relative to the scale so determined.

1.3 Auxiliary conditions

We will work with boundary conditions corresponding to an asymptotically Minkowskian metric, i.e.

$$\lim_{r\to+\infty} e^{2\lambda(r)} = 1 = \lim_{r\to+\infty} e^{2\nu(r)},$$

and also require finite ADM mass, so that

$$\lim_{r\to+\infty} \frac{r}{2}(e^{2\lambda(r)} - 1) = \epsilon^2 l$$

exists and is finite (for each positive $\epsilon = c^{-1}$; the scaling factor ϵ^2 is introduced for later convenience in analyzing the limit $\epsilon \to 0$).

As $r \to 0$ the solutions will satisfy $e^{2\lambda} = 1 + O(r^2)$ and $\Phi_2 = O(r)$. The normalization condition $\|\Psi_A\|_{L^2(\{t=constant\})} = 1$, which ensures that the total probability equals one, can always be imposed on solutions by rescaling, see [6, Section VI]. We will therefore not impose it throughout this paper, since a simple rescaling ensures that the solutions we obtain in theorem 9 do satisfy it.

1.4 Rescaled variables

We now introduce new variables Q, N, ψ_2 (in place, respectively, of λ, ν, Φ_2) which take into account both the auxiliary conditions and the expected behaviour in the Newtonian limit:

$$\epsilon^{-2}(1 - e^{-2\lambda}) = 2lf_0(r) + Q \qquad \epsilon^{-2}\nu = N, \tag{1.16}$$

with $f_0(r) = r^2/(1+r)^3$, and

$$\Phi_2 = \epsilon\psi_2. \tag{1.17}$$

Remark 2 *The variable l determining the ADM mass is determined dynamically, as will become clear in the implicit function theorem set-up following* (3.36). *It turns out to be convenient for the analysis, however, to separate it off by introducing the function f_0 and requiring $Q = O(r^{-2})$ as $r \to +\infty$. Other choices of f_0 with the same asymptotic behaviour would be possible: the choice made here is just a simple function having the right asymptotics as $r \to \infty$, and vanishing to $O(r^2)$ as $r \to 0$ as is natural, see remark 7. Notice that l is only uniquely determined by the requirement $Q = O(r^{-2})$ as $r \to +\infty$, and this condition is built into the function spaces used in the implicit function theorem.*

1.5 Formal consideration of the Newtonian limit

Equations (1.9)-(1.10) become

$$(rQ)' + 2l(rf_0)' - 8\pi G r^2 \rho_\epsilon = 0, \tag{1.18}$$

$$2rN' - \frac{2lf_0 + Q}{1 - \epsilon^2(2lf_0 + Q)} - \frac{8\pi G\epsilon^2 r^2 p_\epsilon}{1 - \epsilon^2(2lf_0 + Q)} = 0, \tag{1.19}$$

with $\rho_\epsilon(N, \Phi_1, \psi_2, \eta) = \rho(\epsilon^2 N, \Phi_1, \epsilon\psi_2, \eta, \epsilon^{-1})$ and $p_\epsilon(N, \Phi_1, \psi_2, \eta) = p(\epsilon^2 N, \Phi_1, \epsilon\psi_2, \eta, \epsilon^{-1})$, while (1.7)-(1.8) become

$$\left(\frac{\eta}{\hbar} - \frac{mN}{\hbar}\right)\Phi_1 - \frac{\partial\psi_2}{\partial r} - \frac{2}{r}\psi_2 - F_1 = 0, \tag{1.20}$$

$$\frac{2m}{\hbar}\psi_2 + \frac{\partial\Phi_1}{\partial r} - F_2 = 0, \tag{1.21}$$

where

$$\begin{aligned} F_1 = {} & \frac{m}{\hbar}\frac{(1 - \epsilon^2 N - e^{-\epsilon^2 N})}{\epsilon^2}\Phi_1 \\ & + \frac{\eta}{\hbar}(1 - e^{-\epsilon^2 N})\Phi_1 - \epsilon^2\big(Q + 2lf_0\big)\Big(\frac{\partial\psi_2}{\partial r} + \frac{1}{r}\psi_2\Big) \end{aligned} \tag{1.22}$$

and

$$F_2 = \epsilon^2(Q + 2lf_0)\Big(\frac{\partial\Phi_1}{\partial r} + \frac{1}{r}\Phi_1\Big) - \epsilon^2\frac{\eta}{\hbar}e^{-\epsilon^2 N}\psi_2 + \frac{m}{\hbar}(1 - e^{-\epsilon^2 N})\psi_2. \tag{1.23}$$

Note that in the limit $\epsilon \to 0$, the inhomogeneous terms F_1 and F_2 are *formally* $O(\epsilon^2)$, while

$$\begin{aligned}
\rho_\epsilon &= 2(m+\epsilon^2\eta)e^{-2\epsilon^2 N}(\Phi_1^2+\epsilon^2\psi_2^2) \\
&= 2m|\Phi_1|^2 + O(\epsilon^2) \\
p_\epsilon &= -2\eta e^{-2\epsilon^2 N}(\Phi_1^2+\epsilon^2\psi_2^2) - 2me^{-\epsilon^2 N}\Phi_1^2\frac{(e^{-\epsilon^2 N}-1)}{\epsilon^2} - 2me^{-\epsilon^2 N}\psi_2^2 + 4\frac{\hbar}{r}\Phi_1\psi_2 \\
&= O(1),
\end{aligned} \tag{1.24}$$

so that (1.18)-(1.19) imply that N becomes the Newtonian potential (i.e. the function u in (1.4)), and (eliminating ψ_2 between (1.20)-(1.21)) the function Φ_1 approaches φ, a solution of the Newton-Schrödinger system (1.4). So far this is completely formal, but it will be made into a rigorous statement in theorem 9 in section 3. In order to achieve this it is necessary to recall some facts about (1.4).

2 The Newton-Schrödinger system

We briefly summarize the approach to (1.4) adopted in the article [10] so as to provide sufficient details for our purposes.

Theorem 3 ([10]) *The nonlocal energy associated to* (1.4)

$$\frac{\hbar^2}{2m}\int |\nabla\varphi(x)|^2 dx - m^2 G \int\int \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|}dxdy \tag{2.25}$$

admits a finite lower bound subject to the constraint of having $\int|\varphi(x)|^2 dx = 1$ *fixed, and this lower bound is attained on a function which is unique up to translation. Further this minimizer is positive, spherically symmetric and a monotone non-increasing function of the radial coordinate satisfying* $|\varphi(r)| \le c_1 e^{-c_2 r}$ *for some positive numbers* c_1, c_2.

We summarize the points of the proof from [10] which will be needed below.

1. The existence of a spherically symmetric minimizer of the nonlocal energy (2.25) is proved by means of the Riesz rearrangement inequality, and a strict version of this inequality implies that any minimizer is spherically symmetric. The corresponding Euler-Lagrange equation is

$$-\frac{\hbar^2}{2m}\Delta\varphi(x) - 2m^2 G\int \frac{|\varphi(y)|^2}{|x-y|}dy\,\varphi(x) = \eta\varphi(x) \tag{2.26}$$

   where $\eta < 0$ is the Lagrange multiplier.

2. The relation between (1.4) and (2.26) follows quickly using the condition $\lim_{|x|\to+\infty}|u(x)| = 0$ and the formula for the solution of Poisson's equation $-\Delta u = f$ on $\mathbb{R}^3$, namely:

$$(-\Delta^{-1}f)(x) = \int \frac{f(y)}{4\pi|x-y|}dy. \tag{2.27}$$

3. For the case when $f(x) = \kappa\rho(r)$, where $\kappa > 0$ is a constant and ρ is a function of the radial coordinate $r = |x|$ only, a result of Newton ([11]) implies that (2.27) can be rewritten:

$$\begin{aligned}
(-\Delta^{-1}\kappa\rho)(r) &= -\kappa\int_0^r \Big(\frac{1}{s}-\frac{1}{r}\Big)\rho(s)s^2 ds + \kappa\int_0^\infty \rho(s)s ds \\
&= -\kappa\int_0^r \Big(\frac{1}{s}-\frac{1}{r}\Big)\rho(s)s^2 ds + u(0).
\end{aligned}$$

Define $K(r,s) \equiv 8\pi s^2(\frac{1}{s} - \frac{1}{r})$; this kernel is non-negative for $0 \le s \le r$. It is shown in [10] that any energy minimizing solution to (1.4) is radially symmetric, and so solves the equation

$$E\varphi = -\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\varphi + m^2 G\left(\int_0^r K(r,s)|\varphi(s)|^2 ds\right)\varphi \tag{2.28}$$

where $E = (\eta - mu(0))$.

4. It is proved in [10] that all positive solutions of (2.28) can be obtained by a scaling of the unique positive solution of

$$-\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\phi + \left(\int_0^r K(r,s)|\phi(s)|^2 ds\right)\phi = \phi. \tag{2.29}$$

(The proof of uniqueness of positive solutions of (2.29) is a crucial part of the article [10]). Conversely, as long as $E > 0$, there is a positive solution of (2.28), which can be obtained by scaling the unique positive solution of (2.29).

Lemma 4 ([9]) *Let L be the linear operator obtained by linearizing* (2.26)*:*

$$L\chi(x) = \left(-\frac{\hbar^2}{2m}\Delta - \eta\right)\chi(x) - 2m^2G\int \frac{|\varphi(y)|^2}{|x-y|}dy\,\chi(x) - 4m^2G\int \frac{\langle\varphi(y),\chi(y)\rangle}{|x-y|}dy\,\varphi(x).$$

Then L is a linear homeomorphism from H^2_{rad} to L^2_{rad} (where these are the radial parts of the usual Sobolev and Lebesgue spaces).

Proof Recall that $\eta < 0$. It is proved in [9] that the operator L is self-adjoint on L^2_{rad} with domain H^2_{rad} and trivial kernel and a single negative eigenvalue with one dimensional eigenspace. (Going out of the radial sector, L has a three dimensional kernel associated with translation invariance, but this does not concern us in this article.) From this it follows that there exists $c > 0$ such that $\|L\chi\|_{L^2} \ge c\|\chi\|_{L^2}$, and that L maps H^2_{rad} continuously onto L^2_{rad}. Furthermore the standard elliptic estimate gives $\|L^{-1}f\|_{H^2} \le C\|f\|_{L^2}$ so that L is a linear homeomorphism $H^2_{rad} \to L^2_{rad}$. □

Remark 5 *Curiously in the present general relativistic problem it is possible to use diffeomorphism invariance to circumvent the need for this lemma. Briefly, by rescaling time, it is possible to linearize the system with the value of the gravitational potential fixed at the origin $r = 0$ rather than at infinity; for this linearization the corresponding nondegeneracy result is rather easy to prove. This was the approach taken in the first version of this article, and it has the advantage of not needing the more careful analysis of L given in [9]. On the other hand it has the disadvantage of giving solutions which are not asymptotically Minkowskian (without further co-ordinate changes), and this makes the physical picture less clear in various ways. Therefore it seems preferable to make use of the results of [9] and obtain solutions which are immediately asymptotically Minkowskian.*

3 Theorem on the Newtonian limit

3.1 The analytic set-up

We now introduce the spaces of functions with which we will be working. Let $BC^\delta = BC^\delta([0,\infty))$ be the space of bounded continuous functions on the half line $0 \le r < \infty$ with $\|f\|_{BC^\delta} = \sup(1+r)^\delta|f(r)| < \infty$; notice that elements of BC^δ satisfy $f = O(r^{-\delta})$ as $r \to \infty$. Let $BC^\delta_{\delta'}$ be the subspace of BC^δ consisting of functions f vanishing to order $\delta' > 0$ at the origin, in the sense that

$$\|f\|_{BC^\delta_{\delta'}} = \sup r^{-\delta'}|f(r)| + \sup(1+r)^\delta|f(r)| < \infty.$$

Further, let $BC^{1,\delta} = BC^{1,\delta}([0,\infty))$ be the subspace of BC^δ consisting of functions f which are also continuously differentiable with $f' \in BC^{1+\delta}$, with the norm

$$\|f\|_{BC^{1,\delta}} = \sup(1+r)^\delta|f(r)| + \sup(1+r)^{1+\delta}|f'(r)| < \infty,$$

and, for $\delta' \geq 1$ let $BC^{1,\delta}_{\delta'} \subset BC^{1,\delta}$ be the subset of those vanishing at the origin to order δ', with norm $\|f\|_{BC^{1,\delta}_{\delta'}} = \|f\|_{BC^{1,\delta}} + \|f\|_{BC^{\delta}_{\delta'}} + \|f'\|_{BC^{\delta+1}_{\delta'-1}} < \infty$. Let H^s_{rad} and L^2_{rad} be the radial parts of the usual Sobolev and Lebesgue spaces. We will use without comment the following inequalities: there exist positive constants C_1, C_2, C_3, C_4, R_* such that

$$\sup_{x\in\mathbb{R}^3} |f(x)| \leq C_1 \|f\|_{H^2(\mathbb{R}^3)} \tag{3.30}$$

$$\|f\|_{L^6(\mathbb{R}^3)} \leq C_2 \|f\|_{H^1(\mathbb{R}^3)} \tag{3.31}$$

$$\|f/|x|\|_{L^2(\mathbb{R}^3)} \leq C_3 \|f\|_{H^1(\mathbb{R}^3)} \tag{3.32}$$

$$\sup_{x\in\mathbb{R}^3:|x|\geq R_*} |xf(x)| \leq C_4 \|f\|_{H^1(\mathbb{R}^3)} \qquad \text{for } \textit{radial} \text{ functions } f \tag{3.33}$$

(see [15], or [3]), and also the fact that $H^2_{rad}(\mathbb{R}^3)$ functions are continuous and $O(r^{-1})$ as $r \to +\infty$. The inequality (3.32) is Hardy's inequality. The $_{rad}$ suffix will be ommitted when writing a norm.

We also need the exponentially weighted Sobolev space $H^{\{s,\delta\}}$ defined, for $s \in \{0, 1, 2, \dots\}$ and $\delta > 0$ as the subspace of $L^2(\mathbb{R}^3)$ with the norm

$$\|f\|_{H^{\{s,\delta\}}} = \sum_{m:|m|=0}^{s} \|e^{\delta|x|}\partial^m f\|_{L^2}$$

finite; we write $H^{\{s,\delta\}}_{rad}$ for the radial part of $H^{\{s,\delta\}}$. Since $f \in H^{\{1,\delta\}}_{rad}$ immediately implies $e^{\delta|x|}f \in H^1_{rad}$, with a bound for the norm, we have by (3.33):

$$\sup_{x\in\mathbb{R}^3:|x|\geq R_*} |xe^{\delta|x|}f(x)| \leq C_5 \|f\|_{H^{\{1,\delta\}}} \qquad \text{for } \textit{radial} \text{ functions } f. \tag{3.34}$$

Introduce the Banach spaces

$$X = \mathbb{R} \times BC^{1,2}_2 \times BC^{1,1} \times \big(H^{\{2,\delta\}}_{rad} \cap BC^{1,0}\big) \times \big(H^{\{1,\delta\}}_{rad} \cap BC^{1,0}_1\big) \tag{3.35}$$

$$Y = BC^2_2 \times BC^1 \times \big(H^{\{1,\delta\}}_{rad} \cap BC\big) \times \big(H^{\{0,\delta\}}_{rad} \cap BC\big), \tag{3.36}$$

where for an intersection of two normed spaces we use the norm $\|f\|_{A\cap B} = \|f\|_A + \|f\|_B$. Now notice that (1.18)-(1.21) define a system of equations $\mathcal{F}(l, Q, N, \Phi_1, \psi_2; \epsilon) = 0$, where $\mathcal{F} : X \times \mathbb{R} \to Y$ maps $(l, Q, N, \Phi_1, \psi_2) \in X$ and $\epsilon \in \mathbb{R}$ (assumed small) to

$$\Bigg((rQ)' + 2l(rf_0)' - 8\pi G r^2 \rho_\epsilon,\ 2rN' - \frac{2lf_0 + Q}{1 - \epsilon^2(2lf_0 + Q)} - \frac{8\pi G\epsilon^2 r^2 p_\epsilon}{1 - \epsilon^2(2lf_0 + Q)},$$
$$\frac{2m}{\hbar}\psi_2 + \frac{\partial \Phi_1}{\partial r} - F_2,\ \Big(\frac{\eta}{\hbar} - \frac{mN}{\hbar}\Big)\Phi_1 - \frac{\partial \psi_2}{\partial r} - \frac{2}{r}\psi_2 - F_1 \Bigg)$$

in Y.

Remark 6 *As promised in remark 2 the field Q is $O(r^{-2})$ at infinity (since we are solving for $Q \in BC^{1,2}_2 \subset BC^2$), so that the parameter l encoding the ADM mass is well defined, by integration of the first equation:*

$$2l = \int_0^\infty 8\pi G r^2 \rho_\epsilon dr.$$

Remark 7 *Notice that $Q \in BC^{1,2}_2$, so that $Q = O(r^2)$ as $r \to 0$. To see that this is natural, integrate the first equation up to r, to deduce $rQ + 2lrf_0(r) = \int_0^r 8\pi G s^2 \rho_\epsilon ds = O(r^3)$, and recall that we chose f_0 to be $O(r^2)$ as $r \to 0$.*

Remark 8 *The fact that $\mathcal{F}(\,\cdot\,;\epsilon)$ maps X into Y (for small $\epsilon > 0$) can be read off from its definition and the formulae* (1.22)-(1.24)*, by means of the inequalities* (3.30)-(3.34) *above. For example consider the third component of $\mathcal{F}$: ψ_2 and $\frac{\partial \Phi_1}{\partial r}$ are clearly in $H^{\{1,\delta\}}_{rad} \cap BC$, so it remains to consider F_2. Referring to* (1.23)*, it follows from the fact that both Q and $f_0(r)$ are $O(r^2)$, that the first term is continuous at $r = 0$, while the same is obviously true for the second and third terms, since $\psi_2 \in BC^{1,0}_1$ and $N \in BC^{1,1}$. Furthermore, functions in $H^{\{1,\delta\}}$ are continuous away from the origin and exponentially decreasing by* (3.34)*, so that altogether $F_2 \in BC$. Similarly $F_2 \in H^{\{1,\delta\}}_{rad}$: the only term for which this is not immediately evident is $\frac{1}{r}\Phi_1$, but this is premultiplied by $Q + 2lf_0 = O(r^2)$ as $r \to 0$. These observations show that $\mathcal{F}$ is continuous from $X \times \mathbb{R} \to Y$ (locally, near $\epsilon = 0$); this will be strengthened to locally C^1 in lemma 11.*

3.2 The Newtonian limit and statement of the main theorem

When $\epsilon = 0$ the formal Newtonian limit solution mentioned previously corresponds to the observation that

$$\mathcal{F}\big(2mG, 2ru' - 4mGf_0, u, \varphi, \frac{-\hbar}{2m}\varphi'; 0\big) = 0 \tag{3.37}$$

where φ and u solve (1.4). To see this, observe that by (1.24) and the preceding remarks, the equation $\mathcal{F}(l, Q, N, \Phi_1, \psi_2; 0) = 0$ amounts to

$$(rQ)' + 2l(rf_0)' - 16m\pi G r^2|\Phi_1|^2 = 0, \tag{3.38}$$

$$2rN' - (2lf_0 + Q) = 0, \tag{3.39}$$

$$\big(\frac{\eta}{\hbar} - \frac{mN}{\hbar}\big)\Phi_1 - \frac{\partial \psi_2}{\partial r} - \frac{2}{r}\psi_2 = 0, \tag{3.40}$$

$$\frac{2m}{\hbar}\psi_2 + \frac{\partial \Phi_1}{\partial r} = 0. \tag{3.41}$$

The first two equations imply $-\Delta N = -8\pi mG|\Phi_1|^2$, and substituting this into (3.40) and then substituting for ψ_2 from (3.41), implies that (Φ_1, N) solves (1.4). We choose, for $\epsilon = 0$, $(\Phi_1, N)$to be the energy minimizing solution (φ, u) of (1.4) described in section 2. The multipole expansion for $N = u$ and the requirement that $Q \in BC^{1,2}$ imply that $l = 2mG$, since $\int |\varphi(x)|^2 dx = 1$. Collecting this together we write $\Xi_N = \big(2mG, 2ru' - 4mGf_0, u, \varphi, -\frac{\hbar}{2m}\varphi'\big)$ and call this the Newtonian limit point in X. We can now state the main theorem:

Theorem 9 *There exists an interval $(-\epsilon_1, +\epsilon_1)$ on which is defined a C^1 curve $\epsilon \to \Xi^\epsilon = (l^\epsilon, Q^\epsilon, N^\epsilon, \Phi_1^\epsilon, \psi_2^\epsilon) \in X$ of solutions to the Einstein-Dirac system* (1.18)-(1.21) *such that $\|\Xi^\epsilon - \Xi_N\|_X = O(\epsilon)$. More explicitly, $l^\epsilon \to 2mG$ and*

$$\|Q^\epsilon + 4mGf_0 - 2ru'\|_{BC^{1,2}_2} + \|N^\epsilon - u\|_{BC^{1,1}} + \|\Phi_1^\epsilon - \varphi\|_{H^{\{2,\delta\}}_{rad} \cap BC^{1,0}} + \|\psi_2^\epsilon + \frac{\hbar}{2m}\varphi'\|_{H^{\{1,\delta\}}_{rad} \cap BC^{1,0}_1} = O(\epsilon). \tag{3.42}$$

In terms of the original variables of the problem, we define a metric

$$g^\epsilon = \epsilon^{-2}e^{2\nu^\epsilon}dt^2 - e^{2\lambda^\epsilon}dr^2 - r^2 d\Omega^2$$

where $\nu^\epsilon = \epsilon^2 N^\epsilon$ and $(1 - e^{-2\lambda^\epsilon}) = \epsilon^2(2l^\epsilon f_0(r) + Q^\epsilon)$. Define also $\Phi_2^\epsilon = \epsilon\psi_2^\epsilon$, then for $\epsilon = c^{-1}$ small we have a solution $(\lambda^\epsilon, \nu^\epsilon, \Phi_1^\epsilon, \Phi_2^\epsilon)$ to (1.7)-(1.10), and

$$\epsilon^{-2}\|(1 - e^{-2\lambda^\epsilon}) - 2\epsilon^2 ru'\|_{BC^{1,2}_2} + \epsilon^{-2}\|\nu^\epsilon - \epsilon^2 u\|_{BC^{1,1}} + \|\Phi_1^\epsilon - \varphi\|_{H^{\{2,\delta\}}_{rad} \cap BC^{1,0}} + \epsilon^{-1}\|\Phi_2^\epsilon + \epsilon\frac{\hbar}{2m}\varphi'\|_{H^{\{1,\delta\}}_{rad} \cap BC^{1,0}_1} = O(\epsilon). \tag{3.43}$$

Remark 10 *We briefly discuss regularity of the solutions just obtained, omitting the ϵ index for clarity. The definition of the last two factors in X ensures that Φ_1, ψ_2 are C^1 for $r > 0$, and $\psi_2 = O(r)$ near $r = 0$; notice that* (1.21) *implies that $\frac{\partial \Phi_1}{\partial r} = O(r)$ as $r \to 0$ also. It follows that the corresponding Dirac spinor given by* (1.6) *is also C^1 on $\mathbb{R}^3$. To see this, first oberve that the partial derivatives at $r = 0$ of the function $x^i\psi_2/r$ exist and given by $\partial_j(x^i\psi_2)(0) = \delta_{ij}\psi_2'(0)$. But also calculate*

$$\partial_j\Big(\frac{x^i\psi_2}{r}\Big) = \frac{\delta_{ij}}{r}\psi_2 + \frac{x^i x^j}{r^2}\Big(\psi_2' - \frac{\psi_2}{r}\Big),$$

and notice that this is continuous everywhere, including at $r = 0$ since

$$\psi_2'(0) = \lim_{r\to 0}\frac{\psi_2(r)}{r} = \lim_{r\to 0}\psi_2'(r).$$

Also the functions $e^{2\lambda}$ and N are C^1 functions for $r > 0$, whose derivatives have limit zero as $r \to 0$, and hence define C^1 functions on $\mathbb{R}^3$. Thus overall we have a classical C^1 solution of the equations of motion.

3.3 Linearization and proof of main theorem

Our aim is to use the implicit function theorem to obtain solutions of $\mathcal{F} = 0$ for small ϵ, thus proving theorem 9. To achieve this it is sufficient to check that $\mathcal{F}$ is locally C^1 and its partial derivative in the X direction, $D_1\mathcal{F}$, evaluated at the Newtonian limit is a bounded linear bijection (and hence a linear homeomorphism by the open mapping theorem). Let $B_1(\Xi)$ be the ball of unit radius centered at Ξ in the Banach space X.

Lemma 11 *For ϵ_* sufficiently small the map $\mathcal{F}$ is C^1 in the neighbourhood $B_1(\Xi_N) \times (-\epsilon_*, \epsilon_*)$ of the Newtonian limit point $(\Xi_N; 0)$. The partial derivative $D_1\mathcal{F}(\Xi_N; 0)$ is the linear map $X \to Y$ which takes $\xi = (\delta l, q, n, \chi_1, \chi_2) \in X$ to*

$$\Bigg((rq)' + 2\delta l(rf_0)' - 32\pi mGr^2\langle\varphi,\chi_1\rangle,\ 2rn' - 2\delta l f_0 - q,$$

$$\frac{2m}{\hbar}\chi_2 + \frac{\partial\chi_1}{\partial r},\ \ (\frac{\eta}{\hbar} - \frac{mu}{\hbar})\chi_1 - \frac{\partial\chi_2}{\partial r} - \frac{2}{r}\chi_2 - \frac{mn}{\hbar}\varphi\Bigg)$$

in Y.

Proof This can mostly be read off by inspection. Consider for example the first component: $(rQ)' + 2l(rf_0)' - 8\pi Gr^2\rho_\epsilon$. This will define a smooth map from $X \times \mathbb{R}$ to BC^2 if each of the three terms does so separately. Clearly $Q \mapsto (rQ)'$ is a continuous linear map $BC_2^{1,2} \to BC^2$, while the function $(rf_0(r))'$ lies in BC^2, so that $l \mapsto l(rf_0)'$ is continuous and linear from $\mathbb{R}$ to BC^2. Next, from the formula (1.24) it is clear (using the inequalities (3.30)-(3.34)) that $\rho_\epsilon \in BC^2$, and that the corresponding map is C^1 (in fact smooth) from $X \times \mathbb{R}$ to BC^2. The second component of $\mathcal{F}$ is handled similarly, after choosing ϵ_* to be sufficiently small that $|1 - \epsilon^2(2lf_0 + Q)| > \frac{1}{2}$ on $B_1(\Xi_N) \times (-\epsilon_*, \epsilon_*)$. The third and fourth components can also be handled in a straightforward way, noting that the coefficients in the expressions (1.22)-(1.23) (i.e. the quantities multiplying Φ_1, ψ_2 and their derivatives) are smooth functions on $B_1(\Xi_N) \times (-\epsilon_*, \epsilon_*)$ taking their values in $BC^{1,1}$; see also remark 8. □

Lemma 12 *Let φ be an energy minimizing solution of* (1.4) *as described in theorem 3, and let Ξ_N be the corresponding Newtonian limit point satisfying $\mathcal{F}(\Xi_N; 0) = 0$ as just described. Then the partial derivative in the X direction, $D_1\mathcal{F}(\Xi_N; 0)$, is a linear homeomorphism $X \to Y$.*

Proof To establish this we need to prove the unique solvability (with bounds for $\xi \in X$) of the equation $D_1\mathcal{F}(\Xi_N;0)(\xi) = y$, for $y = (\alpha, \beta, j_1, j_2) \in Y$. Thus we consider the system

$$(rq)' + 2\delta l(rf_0)' - 32\pi mGr^2\langle\varphi, \chi_1\rangle = \alpha, \tag{3.44}$$

$$2rn' - 2\delta lf_0 - q = \beta, \tag{3.45}$$

$$\frac{2m}{\hbar}\chi_2 + \frac{\partial\chi_1}{\partial r} = j_1, \tag{3.46}$$

$$(\frac{\eta}{\hbar} - \frac{mu}{\hbar})\chi_1 - \frac{\partial\chi_2}{\partial r} - \frac{2}{r}\chi_2 - \frac{mn}{\hbar}\varphi = j_2. \tag{3.47}$$

To start with, multiply the second equation by r, differentiate, add the first equation and then divide by $2r^2$, leading to

$$-\Delta n + 16\pi mG\langle\varphi, \chi_1\rangle + \frac{\alpha + (r\beta)'}{2r^2} = 0. \tag{3.48}$$

Now use the representation (2.27) for n, and substitute the resulting formula for n into the equation obtained by eliminating χ_2 between the third and fourth equations:

$$-\frac{\hbar^2}{2m}\Delta\chi_1 - \eta\chi_1 + mu\chi_1 + mn\varphi = -\hbar j_2 - \frac{\hbar^2}{2m}\Big(\frac{\partial}{\partial r} + \frac{2}{r}\Big)j_1.$$

This yields

$$L\chi_1 = J$$

where L is the operator in lemma 4 and J is the function

$$J(r) = -\hbar j_2 - \frac{\hbar^2}{2m}\Big(\frac{\partial j_1}{\partial r} + \frac{2j_1}{r}\Big) + m\varphi\Big[(-\Delta)^{-1}(\frac{\alpha + (r\beta)'}{2r^2})\Big].$$

In order to apply lemma 4 we need to check that $J \in L^2_{rad}$: the first two terms clearly are, so we just need to show the same is true of the third term. Let $\dot{H}^1_{rad}$ be the homogeneous Sobolev space with norm $\|f\|^2_{\dot{H}^1} = \int |\nabla f|^2 d^3x$, and let $\dot{H}^{-1}_{rad}$ be the dual space. Recall that $\dot{H}^1_{rad} \subset L^6_{rad}$ with a continuous embedding, by Sobolev's inequality. Now $\alpha/r \in L^2_{rad}$, so $r^{-2}\alpha \in \dot{H}^{-1}_{rad}$ since, by Hardy's inequality,

$$|\int f\frac{\alpha}{r^2}r^2dr| \leq \|\alpha/r\|_{L^2}\|f/r\|_{L^2} \leq C\|\alpha\|_{BC^1}\|f\|_{\dot{H}^1}.$$

(Here the L^2 norm is understood to be with respect to the measure $4\pi r^2dr$ since all functions are radial.) Also $r^{-2}(r\beta)' \in \dot{H}^{-1}_{rad}$ since

$$|\int f\frac{(r\beta)'}{r^2}r^2dr| = |\int r\beta f'dr| \leq \|\beta/r\|_{L^2}\|f'\|_{L^2} \leq C\|\beta\|_{BC^1}\|f\|_{\dot{H}^1}.$$

Given this it follows from the Riesz representation theorem that $(-\Delta)^{-1}\big(\frac{\alpha+(r\beta)'}{2r^2}\big) \in \dot{H}^1_{rad} \subset L^6_{rad}$ is well-defined, and the bound

$$\big\|(-\Delta)^{-1}(\frac{\alpha + (r\beta)'}{2r^2})\big\|_{\dot{H}^1\cap L^6} \leq C\|(\alpha, \beta)\|_{BC^1\times BC^1}$$

holds. Therefore $\varphi(-\Delta)^{-1}\big(\frac{\alpha+(r\beta)'}{2r^2}\big) \in L^2_{rad}$ (since φ is smooth and decays exponentially and so is in L^3_{rad}, and $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$). As a consequence there exists a unique $\chi_1 \in H^2_{rad}$ such that $L\chi_1 = J$ satisfying the bound

$$\|\chi_1\|_{H^2} \leq C\Big(\|(\alpha, \beta)\|_{BC^1\times BC^1} + \|j_2\|_{L^2} + \|j_1\|_{H^1}\Big).$$

To derive the exponentially weighted bound, we apply lemma 13 below to the equation:

$$(-\Delta + M + V(r))\chi_1 = f = \frac{2m}{\hbar^2}\Big[-\hbar j_2 - \frac{\hbar^2}{2m}(j_1' + \frac{2}{r}j_1) - mn\varphi\Big]$$

where $M = -2m\eta/\hbar^2 > 0$ and $V = 2m^2 u/\hbar^2$. This implies

$$\|\chi_1\|_{H^{\{2,\delta\}}} \leq c\Big(\|\chi_1\|_{H^2} + \|f\|_{H^{\{0,\delta\}}}\Big)$$
$$\leq c\Big(\|(\alpha,\beta)\|_{BC^1\times BC^1} + \|j_2\|_{H^{\{0,\delta\}}} + \|j_1\|_{H^{\{1,\delta\}}} + \|n\|_{L^6}\|e^{\delta r}\varphi\|_{L^3}\Big)$$

for δ small ($\delta < \sqrt{M}$ and less than the decay rate of φ in theorem 3). The third equation above then determines $\chi_2 \in H^{\{1,\delta\}}_{rad}$ uniquely: $\chi_2 = \frac{\hbar}{2m}(j_1 - \frac{\partial\chi_1}{\partial r})$, with the same bound for $\|\chi_2\|_{H^{\{1,\delta\}}}$ as for $\|\chi_1\|_{H^{\{2,\delta\}}}$.

We show that also $\chi_2 \in BC^{1,0}_1$ below. First we must discuss q and n. Since we are attempting to solve for $q = O(r^{-2})$ as $r \to +\infty$, integration of the first equation gives

$$2\delta l = \int_0^\infty \big(32\pi mGr^2\langle\varphi,\chi_1\rangle + \alpha\big)dr$$

and

$$q(r) = 2\delta l\big(\frac{1}{r} - f_0(r)\big) - \frac{1}{r}\int_r^\infty \big(32\pi mGs^2\langle\varphi(s),\chi_1(s)\rangle + \alpha(s)\big)ds$$
$$= -\frac{1}{r}\int_r^\infty \Big[-2\delta l\big(sf_0(s)\big)' + \big(32\pi mGs^2\langle\varphi(s),\chi_1(s)\rangle + \alpha(s)\big)\Big]ds$$
$$= +\frac{1}{r}\int_0^r \Big[-2\delta l\big(sf_0(s)\big)' + \big(32\pi mGs^2\langle\varphi(s),\chi_1(s)\rangle + \alpha(s)\big)\Big]ds$$

Since χ_1 is bounded and continuous, the first line makes it clear that $q = O(r^{-2})$ as $r \to +\infty$, while the third implies that $\lim_{r\to 0} r^{-2}q(r)$ exists (since $\alpha \in BC^2_2$), and hence that $q \in BC^{1,2}_2$. Recalling that φ is exponentially decaying, and putting together the obvious estimates gives the bounds $|\delta l| \leq c(\|\alpha\|_{BC^2} + \|\chi_1\|_{H^2})$ and $\|q\|_{BC^2_2} \leq c(|\delta l| + \|\alpha\|_{BC^2_2} + \|\chi_1\|_{H^2})$. The similar bound for $\|q'\|_{BC^3_1}$ then follows from (3.44), yielding the bound for $\|q\|_{1,2;2}$ the norm on $BC^{1,2}_2$. Next for n which was defined to solve (3.48); we must show it satisfies (3.45). So let $H = 2rn' - 2\delta lf_0 - q - \beta$, then (3.48) and (3.44) imply that $(rH)' = 0$ so that $H = c/r$. But $r^{-1}H \in L^2_{rad}$ (since $n' \in L^2_{rad}$) so $c = 0$, and hence $H = 0$ i.e. (3.45) holds. This implies immediately that $n' \in BC^2$ and $n \in BC^1$ with a bound $\|n\|_{BC^{1,1}} \leq c(|\delta l| + \|\beta\|_{BC^1} + \|q\|_{BC^1})$.

Next define $\chi_2 \in H^{\{1,\delta\}}_{rad}$ by (3.46); it follows that (3.47) holds, and so χ_2 is C^1 for $r > 0$. To establish $\chi_2 \in BC^{1,0}_1$ it is necessary to analyze the behaviour at the origin. To achieve this, integrate up the fourth equation:

$$\chi_2(r) = \frac{1}{r^2}\int_0^r \Big[\big(\frac{\eta}{\hbar} - \frac{mu}{\hbar}\big)\chi_1 - j_2 - \frac{mn}{\hbar}\varphi\Big]s^2 ds.$$

The quantity in square brackets is continuous by assumption, so that $\lim_{r\to 0} r^{-1}\chi_2(r)$ exists and is finite; it follows from (3.47) that $\lim_{r\to 0}\chi_2'(r)$ exists and is finite also, so that $\chi_2 \in BC^{1,0}_1$ as required. Finally (3.46) then implies that $\chi_1 \in BC^{1,0}$. □

Lemma 13 (Exponentially weighted bounds) *Assume $u \in H^2_{rad}(\mathbb{R}^3)$ solves*

$$\big(-\Delta + M + V(r)\big)u = f$$

where $f \in H^{\{0,\delta\}}_{rad}(\mathbb{R}^3)$, for some $\delta < \sqrt{M}$. Assume further that V is continuous and $\lim_{r\to+\infty} V(r) = 0$. Then u also satisfies the bound $\|u\|_{H^{\{2,\delta\}}} \leq c\big(\|u\|_{H^2} + \|f\|_{H^{\{0,\delta\}}}\big)$ for some $c = c(\delta) > 0$.

Proof $v = ru$ solves

$$-v'' + Mv + Vv = rf \tag{3.49}$$

For any $S > R+1 > R$ let $b(r)$ be a smooth function with $b(r) = 0$ if $r \leq R$ or $r \geq S+1$, and $b(r) = 1$ if $S > r \geq R+1$. Multiply by $e^{2\delta r} b(r)$ and integrate, estimate $\left|\int 2\delta e^{2\delta r} bvv' dr\right| \leq \int \big(e^{2\delta r}\delta^2 bv^2/(1-\epsilon) + (1-\epsilon)e^{2\delta r} bv'^2\big) dr$ and integrate by parts all other terms involving vv'. This leads to

$$\int_0^\infty \Big[\epsilon e^{2\delta r} bv'^2 + (M - \frac{\delta^2}{(1-\epsilon)} + V) be^{2\delta r} v^2\Big] dr \leq c \int_0^\infty \Big[\delta e^{2\delta r} |b'| v^2 + e^{2\delta r} |b''| v^2 + re^{2\delta r} |bfv|\Big] dr.$$

For any $\epsilon > 0$ let R be such that $\sup_{r \geq R} |V(r)| < \epsilon$, and let $\delta^2 < (M - 2\epsilon)(1-\epsilon)$, so that $|M - \frac{\delta^2}{(1-\epsilon)} + V| > \epsilon$ for $r \geq R$. Now let $S \to +\infty$, to deduce

$$\int_{R+1}^\infty \Big[e^{2\delta r} v'^2 + e^{2\delta r} v^2\Big] dr \leq c(\epsilon)\big(\|u\|_{L^2}^2 + \int_R^\infty re^{2\delta r} |fv| dr\big). \tag{3.50}$$

Using the fact that $\|u\|^2_{H^{\{1,\delta\}}} \leq c\big(\|u\|^2_{H^1} + \int_{R+1}^\infty e^{2\delta r}(v^2 + v'^2) dr\big)$, (3.50) implies $\|u\|_{H^{\{1,\delta\}}} \leq c\big(\|u\|_{H^1} + \|f\|_{H^{\{0,\delta\}}}\big)$. To improve this to $H^{\{2,\delta\}}$ it is only necessary to multiply the equation (3.49) by $e^{\delta r} b^{\frac{1}{2}}$, square and integrate, to obtain

$$\int_{R+1}^\infty e^{2\delta r} v''^2 dr \leq c\Big(\|u\|^2_{H^{\{0,\delta\}}} + \|f\|^2_{H^{\{0,\delta\}}}\Big)$$

since $\|f\|^2_{H^{\{0,\delta\}}} = 4\pi \int r^2 e^{2\delta r} f^2 dr$. But $\|u\|^2_{H^{\{2,\delta\}}} \leq c\big(\|u\|^2_{H^{\{1,\delta\}}} + \int_{R+1}^\infty e^{2\delta r} v''^2 dr\big)$ since for an arbitrary second order derivative $r^2 |\nabla_i \nabla_j u|^2 \leq cr^2(|u''|^2 + r^{-2}|u'|^2)$ so that in the region $r > R+1 > 1$ there holds $r^2 |\nabla_i \nabla_j u|^2 \leq c(|v''|^2 + r^2|u|^2 + r^2|u'|^2)$. This completes the proof. □

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