# Partial Differential Equations Example sheet 1

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## Books

In addition to the sets of lecture notes written by previous lecturers ([1, 2]) which are still useful, the books [4, 3, 5] are very good for the PDE topics in the course, and go well beyond the course also. If you want to read more on distributions [6] is most relevant. Also [7, 8] are useful; the books [9] are more advanced, but the first volume may be helpful.

## References

[1] T.W. Körner, Cambridge Lecture notes on PDE, available at https://www.dpmms.cam.ac.uk/ twk

[2] M. Joshi and A. Wassermann, Cambridge Lecture notes on PDE,available at http://www.damtp.cam.ac.uk/user/dmas2

[3] G.B. Folland, Introduction to Partial Differential Equations, *Princeton 1995, QA 374 F6*

[4] L.C. Evans, Partial Differential Equations,*AMS Graduate Studies in Mathematics Vol 19, QA377.E93 1990*

[5] E DiBenedetto, Partial Differential Equations, *Birkhauser, Boston 2010 QA377 D534 2010*

[6] F.G. Friedlander, Introduction to the Theory of Distributions, *CUP 1982, QA324*

[7] F. John, Partial Differential Equations, *Springer-Verlag 1982, QA1.A647*

[8] Rafael Jos Iorio and Valria de Magalhes Iorio, Fourier analysis and partial differential equations *CUP 2001, QA403.5 .I57 2001*

[9] M.E. Taylor, Partial Differential Equations,Vols I-III*Springer 96, QA1.A647*

# 1 Introduction

## 1.1 Notation

We write partial derivatives as $\partial_t = \frac{\partial}{\partial t}, \partial_j = \frac{\partial}{\partial x_j}$ etc and also use suffix on a function to indicate partial differentiation: $u_t = \partial_t u$ etc. A general $k^{th}$ order linear partial differential operator (pdo) acting on functions $u = u(x_1, \dots x_n)$ is written:

$$P = \sum_{|\alpha \leq k} a_\alpha \partial^\alpha u \,. \tag{1.1}$$

Here $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{Z}_+^n$ is a multi-index of order $|\alpha| = \sum \alpha_j$ and

$$\partial^\alpha = \prod \partial_j^{\alpha_j} \,, \quad x^\alpha = \prod x_j^{\alpha_j} \,. \tag{1.2}$$

For a multi-index we define the factorial $\alpha! = \prod \alpha_j!$. For (real or complex) constants $a_\alpha$ the formula (1.1) defines a constant coefficient linear pdo of order $k$. (Of course assume always that at least one of the $a_\alpha$ with $|\alpha| = k$ is non-zero so that it is genuinely of order $k$.) If the coefficients depend on $x$ it is a variable coefficient linear pdo. The word linear means that

$$P(c_1 u_1 + c_2 u_2) = c_1 P u_1 + c_2 P u_2 \tag{1.3}$$

holds for $P$ applied to $C^k$ functions $u_1, u_2$ and arbitrary constants $c_1, c_2$.

## 1.2 Basic definitions

If the coefficients depend on the partial derivatives of a function of order strictly less than $k$ the operator

$$u \mapsto Pu = \sum_{|\alpha \leq k} a_\alpha(x, \{\partial^\beta u\}_{|\beta| < k})\, \partial^\alpha u \tag{1.4}$$

is called quasi-linear and (1.3) no longer holds. The corresponding equation $Pu = f$ for $f = f(x)$ is a quasi-linear partial differential equation (pde). In such equations the partial derivatives of highest order - which are often most important - occur linearly. If the coefficients of the partial derivatives of highest order in a quasi-linear operator $P$ depend only on $x$ (not on $u$ or its derivatives) the equation is called semi-linear. If the partial derivatives of highest order appear nonlinearly the equation is called fully nonlinear; such a general pde of order $k$ may be written

$$F(x, \{\partial^\alpha u\}_{|\alpha| \leq k}) = 0 \,. \tag{1.5}$$

**Definition 1.2.1** *A classical solution of the pde* (1.5) *on an open set* $\Omega \subset \mathbb{R}^n$ *is a function* $u \in C^k(\Omega)$ *which is such that* $F(x, \{\partial^\alpha u(x)\}_{|\alpha| \leq k}) = 0$ *for all* $x \in \Omega$ *.*

Classical solutions do not always exist and we will define generalized solutions later in the course. The most general existence theorem for classical solutions is the Cauchy-Kovalevskaya theorem, to state which we need the following definitions:

**Definition 1.2.2** *Given an operator* (1.1) *we define*

- $P_{principal} = \sum_{|\alpha=k} a_\alpha \partial^\alpha u$ , *(principal part)*
- $p = \sum_{|\alpha \leq k} a_\alpha (i\xi)^\alpha$ , $\quad \xi \in \mathbb{R}^n$ , *(total symbol)*
- $\sigma = \sum_{|\alpha=k} a_\alpha (i\xi)^\alpha$ , $\quad \xi \in \mathbb{R}^n$ , *(principal symbol)*
- $Char_x(P) = \{\xi \in \mathbb{R}^n : \sigma(x,\xi) = 0\}$ , *(the set of characteristic vectors at x)*
- $Char(P) = \{(x,\xi) : \sigma(x,\xi) = 0\} = \cup_x Char_x(P)$ , *(characteristic variety)*.

*Clearly* $\sigma, p$ *depend on* $(x,\xi) \in \mathbb{R}^{2n}$ *for variable coefficient linear operators, but are independent of* $x$ *in the constant coefficient case. For quasi-linear operators we make these definitions by substituting in* $u(x)$ *into the coefficients, so that* $p, \sigma$ *and (also the definition of characteristic vector) depend on this* $u(x)$.

**Definition 1.2.3** *The operator* (1.1) *is elliptic at* $x$ *(resp. everywhere) if the principal symbol is non-zero for non-zero* $\xi$ *at* $x$ *(resp. everywhere). (Again the definition of ellipticity in the quasi-linear case depends upon the function* $u(x)$ *in the coefficients.)*

## 1.3 The Cauchy-Kovalevskaya theorem

The *Cauchy problem*is the problem of showing that for a given pde and given data on a hypersurface $\mathcal{S} \subset \mathbb{R}^n$ there is a unique solution of the pde which agrees with the data on $\mathcal{S}$. This is a generalization of the initial value problem for ordinary differential equations, and by analogy the appropriate data to be given on $\mathcal{S}$ consists of $u$ and its normal derivatives up to order $k-1$. A crucial condition is the following:

**Definition 1.3.1** *A hypersurface* $\mathcal{S}$ *is non-characteristic at a point* $x$ *if its normal vector* $n(x)$ *is non-characteristic, i.e.* $\sigma(x, n(x)) \neq 0$. *We say that* $\mathcal{S}$ *is non-characteristic if it is non-characteristic for all* $x \in \mathcal{S}$.*)*

Again for quasi-linear operators it is necessary to substitute $u(x)$ to make sense of this definition, so that whether or not a hypersurface is non-characteristic depends on $u(x)$, which amounts to saying it depends on the data which are given on $\mathcal{S}$.

**Theorem 1.3.2 (Cauchy-Kovalevskaya theorem)** *In the real analytic case there is a local solution to the Cauchy problem for a quasi-linear pde in a neighbourhood of a point as long as the hypersurface is non-characteristic at that point.*

This becomes clearer with a suitable choice of coordinates which emphasizes the analogy with ordinary differential equations: let the hypersurface be the level set $x_n = t = 0$ and let $x = (x_1, \dots x_{n-1})$ be the remaining $n-1$ coordinates. Then a quasi-linear $P$ takes the form

$$Pu = a_{0k}\partial_t^k + \sum_{|\alpha|+j \leq k, j<k} a_{j\alpha}\partial_t^j \partial^\alpha u \qquad (1.6)$$

with the coefficients depending on derivatives of order $< k$, as well as on $(x, t)$. Since the normal vector to $t = 0$ is $n = (0, 0, \ldots 0, 1) \in \mathbb{R}^n$ the non-characteristic condition is just $a_{0k} \neq 0$, and ensures that the quasi-linear equation $Pu = f$ can be solved for $\partial_t^k u$ in terms of $\{\partial_t^j \partial^\alpha u\}_{|\alpha|+j \leq k, j<k}$ to yield an equation of the form:

$$\partial_t^k u = G(x, t, \{\partial_t^j \partial^\alpha u\}_{|\alpha|+j \leq k, j<k}) \tag{1.7}$$

to be solved with data

$$u(x, 0) = \phi_0(x), \partial_t u(x, 0) = \phi_1(x) \ldots \partial_t^{k-1} u(x, 0) = \phi_{k-1}(x)\,. \tag{1.8}$$

**Theorem 1.3.3** *Assume that $G, \phi_0, \ldots \phi_{k-1}$ are all real analytic functions in some neighbourhood of the origin. Then there exists a unique real analytic function which satisfies* (1.8)-(1.7) *in some neighbourhood of the origin.*

Notice that the non-characteristic condition ensures that the $k^{th}$ normal derivative $\partial_t^k u(x, 0)$ is determined by the data. Differentiation of (1.7) gives further relations which can be shown to determine all derivatives of the solution at $t = 0$, and the theorem can be proved by showing that the resulting Taylor series defines a real-analytic solution of the equation. Read section 1C of the book of Folland for the full proof.

In the case of first order equations with real coefficients the method of characteristics gives an alternative method of attack which does not require real analyticity. In this case we consider a pde of the form

$$\sum_{j=1}^{n} a_j(x, u)\partial_j u = b(x, u) \tag{1.9}$$

with data

$$u(x) = \phi(x)\,, \quad x \in \mathcal{S} \tag{1.10}$$

where $\mathcal{S} \subset \mathbb{R}^n$ is a hypersurface, given in paramteric form as $x_j = g_j(s)\,, s = (s_1, \ldots s_{n-1}) \in \mathbb{R}^{n-1}$. (Think of $\mathcal{S} = \{x_n = 0\}$ parametrized by $g(s_1, \ldots s_{n-1}) = (s_1, \ldots s_{n-1}, 0)$.)

**Theorem 1.3.4** *Let $\mathcal{S}$ be a $C^1$ hypersurface properly paramterized by a $C^1$ function $g$, and assume that the $a_j, b, \phi$ are all $C^1$ functions. Then there exists a unique $C^1$ solution of* (1.9)-(1.10) *defined on a neighbourhood of $\mathcal{S}$ as long as the non-characteristic condition $\Sigma_{j=1}^n a_j(x, \phi(x)) n_j(x) \neq 0$ holds.*

This is proved by considering the characteristic curves which are obtained by integrating the system of $n+1$ characteristic ordinary differential equations (ode):

$$\frac{dx_j}{dt} = a_j(x, z)\,, \quad \frac{dz}{dt} = b(x, z) \tag{1.11}$$

with data $x_j(s, 0) = g_j(s), z(s, 0) = \phi(g(s))$; let $(X(s, t), Z(s, t)) \in \mathbb{R}^n \times \mathbb{R}$ be this solution. The non-characteristic condition implies (via the inverse function theorem) that the "restricted flow map" which takes $(s, t) \mapsto X(s, t) = x$ is locally invertible, with inverse $s_j = \sigma_j(x), t = \tau(x)$ and this allows one to recover the solution as $u(x) = Z(\sigma(x), \tau(x))$.

# 2 Example sheet 1

1. Write out the multinomial expansion for $(x_1 + \dots x_n)^N$ and the $n$-dimensional Taylor expansion using multi-index notation.
2. Consider the problem of solving the heat equation $u_t = \Delta u$ with data $u(x,0) = f(x)$. Is the non-characteristic condition satisfied? How about for the wave equation $u_{tt} = \Delta u$ with data $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$? For which of these problems, and for which data, does the Cauchy-Kovalevskaya theorem ensure the existence of a local solution? How about the Cauchy problem for the Schrödinger equation?
3. Find the characteristic vectors for the operator $P = \partial_1 \partial_2$ $(n = 2)$. Is it elliptic? Do the same for $P = \sum_{j=1}^m \partial_j^2 - \sum_{j=m+1}^n \partial_j^2$ $(1 < m < n)$.
4. Solve the linear PDE $x_1 u_{x_2} - x_2 u_{x_1} = u$ with boundary condition $u(x_1, 0) = f(x_1)$ for $f$ a $C^1$ function. Where is your solution valid? Classify the $f$ for which a global $C^1$ solution exists. (Global solution here means a solution which is $C^1$ on all of $\mathbb{R}^2$.Solve the linear PDE $x_1 u_{x_2} - x_2 u_{x_1} = u$ with boundary condition $u(x_1, 0) = f(x_1)$ for $f$ a $C^1$ function. Where is your solution valid? Classify the $f$ for which a global $C^1$ solution exists. (Global solution here means a solution which is $C^1$ on all of $\mathbb{R}^2$.
5. Solve Cauchy problem for the semi-linear PDE $u_{x_1} + u_{x_2} = u^4$, $u(x_1, 0) = f(x_1)$ for $f$ a $C^1$ function. Where is your solution $C^1$?
6. For the quasi-linear Cauchy problem $u_{x_2} = x_1 u u_{x_1}$, $u(x_1, 0) = x_1$
(a) Verify that the Cauchy-Kovalevskaya theorem implies existence of an analytic solution in a neighbourhood of all points of the initial hypersurface $x_2 = 0$ in $\mathbb{R}^2$,
(b) Solve the characteristic ode and discuss invertibility of the restricted flow map $X(s,t)$ (this may not be possible explicitly),
(c) give the solution to the Cauchy problem (implicitly).
7. For the quasi-linear Cauchy problem $u_{x_2} = A(u_{x_1} + 1)/(B - x_1 - u)$, $u(x_1, 0) = 0$:
(a) Find all points on the intial hypersurface where the Cauchy-Kovalevskaya theorem can be applied to obtain a local solution defined in a neighbourhood of the point.
(b) Solve the characteristic ode and invert (where possible) the restricted flow map, relating your answer to (a).
(c) Give the solution to the Cauchy problem, paying attention to any sign ambiguities that arise.
(In this problem take $A, B$ to be positive real numbers).
8. For the Cauchy problem
$$u_{x_1} + 4u_{x_2} = \alpha u \qquad u(x_1, 0) = f(x_1), \tag{2.12}$$
with $C^1$ initial data $f$, obtain the solution $u(x_1, x_2) = e^{\alpha x_2/4} f(x_1 - x_2/4)$ by the method of characteristics. For fixed $x_2$ write $u(x_2)$ for the function $x_1 \mapsto u(x_1, x_2)$ i.e. the solution restricted to "time" $x_2$. Derive the following *well-posedness* properties for solutions $u(x_1, x_2)$ and $v(x_1, x_2)$ corresponding to data $u(x_1, 0)$ and $v(x_1, 0)$ respectively:
(a) for $\alpha = 0$ there is *global well-posedness* in the supremum (or $L^\infty$) norm *uniformly in time* in the sense that if for fixed $x_2$ the distance between $u$ and

$v$ is taken to be

$$\|u(x_2) - v(x_2)\|_{L^\infty} \equiv \sup_{x_1} |u(x_1, x_2) - v(x_1, x_2)|$$

then

$$\|u(x_2) - v(x_2)\|_{L^\infty} \le |u(0) - v(0)|_{L^\infty} \qquad \text{for all } x_2.$$

Is the inequality ever strict?

(b) for all $\alpha$ there is *well-posedness* in supremum norm *on any finite time interval* in the sense that for any time interval $|x_2| \le T$ there exists a number $c = c(T)$ such that

$$\|u(x_2) - v(x_2)\|_{L^\infty} \le c(T)\|u(0) - v(0)\|_{L^\infty}.$$

and find $c(T)$. Also, for different $\alpha$, when can $c$ be assumed independent of time for positive (respectively negative) times $x_2$?

(c) Try to do the same for the $L^2$ *norm*, i.e. the norm defined by

$$\|u(x_2) - v(x_2)\|^2_{L^2(dx_1)} = \int |u(x_1, x_2) - v(x_1, x_2)|^2 dx_1.$$

9. For which real numbers $a$ can you solve the Cauchy problem

$$u_{x_1} + u_{x_2} = 0 \qquad u(x_1, ax_1) = f(x_1)$$

for any $C^1$ function $f$. Explain both in terms of the non-characteristic condition and by explicitly trying to invert the (restricted) flow map, interpreting your answer in relation to the line $x_2 = ax_1$ on which the initial data are given.

10. (a). Consider the equation

$$u_{x_1} + nu_{x_2} = f$$

where $n$ is an integer and $f$ is a smooth function which is $2\pi$- periodic in both variables:

$$f(x_1 + 2\pi, x_2) = f(x_1, x_2 + 2\pi) = f(x_1, x_2).$$

Apply the method of characteristics to find out for which $f$ there is a solution which is also $2\pi$- periodic in both variables:

$$u(x_1 + 2\pi, x_2) = u(x_1, x_2 + 2\pi) = u(x_1, x_2).$$

(b) Consider the problem in part (a) using fourier series representations of $f$ and $u$ (both $2\pi$- periodic in both variables) and compare your results.