Partial Differential Equations Example sheet 2

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2 Background analysis

2.1 Fourier series

Consider 2π-periodic functions on the real line:

$$C^r_{per}([-\pi,\pi]) = \{u \in C^r(\mathbb{R}) : u(x+2\pi) = u(x)\}\,,$$

for $r \in [0,\infty]$. The case $r = 0$ is the continuous 2π-periodic functions, while the case $r = \infty$ is ths smooth 2π-periodic functions. For functions $u = u(x_1, \dots x_n)$ we define the corresponding spaces $C^r_{per}([-\pi,\pi]^n)$ of C^r functions which are 2π-periodic in each coordinate. (All of these definitions generalize in obvious ways for classes of functions with periods other than 2π, e.g. $C^r_{per}(\prod_{j=1}^n [0, L_j])$ consists of C^r functions $u = u(x_1, \dots x_n)$ which are L_j-periodic in x_j.)

Given a function $u \in C^\infty_{per}([-\pi,\pi])$ the Fourier coefficients are the sequence of numbers $\hat{u}_m = \hat{u}(m)$ given by

$$\hat{u}(m) = \hat{u}_m = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-imx} u(x)\,dx\,, \qquad m \in \mathbb{Z}\,.$$

Integration by parts gives the formula $\widehat{\partial^\alpha u}(m) = (im)^\alpha \hat{u}(m)$ for positive integral α, which shows that the sequence of Fourier coefficients is a rapidly decreasing (bi-infinite) sequence: this means that $\hat{u} \in s(\mathbb{Z})$ where

$$s(\mathbb{Z}) = \{\hat{u} : \mathbb{Z} \to \mathbb{C} \text{ such that } |\hat{u}|_\alpha = \sup_{m\in\mathbb{Z}} |m^\alpha \hat{u}(m)| < \infty \;\forall \alpha \in \mathbb{Z}_+\}\,.$$

This in turn means that the series $\sum_{m\in\mathbb{Z}} \hat{u}(m) e^{imx}$ converges absolutely and uniformly to a smooth function. The central fact about Fourier series is that this series actually converges to u, so that each $u \in C^\infty_{per}([-\pi,\pi])$ can be represented as:

$$u(x) = \sum \hat{u}(m) e^{imx}\,, \quad \text{where} \quad \hat{u}(m) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-imx} u(x)\,dx\,.$$

The whole development works for periodic functions $u = u(x_1, \dots x_n)$ with the sequence space generalized to

$$s(\mathbb{Z}^n) = \{\hat{u} : \mathbb{Z}^n \to \mathbb{C} \text{ such that } |\hat{u}|_\alpha = \sup_{m\in\mathbb{Z}^n} |m^\alpha \hat{u}(m)| < \infty \;\forall \alpha \in \mathbb{Z}^n_+\}\,.$$

Here we use multi-index notation, in terms of which we have:

Theorem 2.1.1 *The mappping*

$$\begin{aligned} C^\infty_{per}([-\pi,\pi]^n) &\to s(\mathbb{Z}^n)\,, \\ u &\mapsto \hat{u} = \{\hat{u}(m)\}_{m\in\mathbb{Z}^n} \qquad \textit{where } \hat{u}(m) = \frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n} e^{-im\cdot x}u(x)\,dx \end{aligned}$$

is a linear bijection whose inverse is the map which takes $\hat{u}$ *to* $\sum_{m\in\mathbb{Z}^n}\hat{u}(m)e^{im\cdot x}$ *and the following hold:*

1. $u(x) = \sum_{m\in\mathbb{Z}^n}\hat{u}(m)e^{im\cdot x}$ *where* $\hat{u}(m) = \frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n} e^{-im\cdot x}u(x)\,dx$ *(Fourier inversion),*
2. $\widehat{\partial^\alpha u}(m) = (im)^\alpha\hat{u}(m)$ *for all* $m\in\mathbb{Z}^n, \alpha\in\mathbb{Z}^n_+$*,*
3. $\frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n}|u(x)|^2\,dx = \sum_{m\in\mathbb{Z}^n}|\hat{u}(m)|^2$ *(Parseval-Plancherel).*

2.2 Fourier transform

Define the Schwartz space of test functions:

$$\mathcal{S}(\mathbb{R}^n) = \{u\in C^\infty(\mathbb{R}^n) : |u|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n}|x^\alpha\partial^\beta u(x)| < \infty\,,\ \forall\alpha\in\mathbb{Z}^n_+, \beta\in\mathbb{Z}^n_+\,.\}$$

This is a convenient space on which to define the Fourier transform because of the fact that Fourier integrals interchange rapidity of decrease with smoothness, so the space of functions which are smooth and rapidly decreasing is invariant under Fourier transform:

Theorem 2.2.1 *The Fourier transform, i.e. the mappping*

$$\begin{aligned} \mathcal{F}:\mathcal{S}(\mathbb{R}^n) &\to \mathcal{S}(\mathbb{R}^n)\,, \\ u &\mapsto \hat{u} \qquad \textit{where } \hat{u}(\xi) = \mathcal{F}_{x\to\xi}(u(x)) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}u(x)\,dx \end{aligned}$$

is a linear bijection whose inverse is the map $\mathcal{F}^{-1}$ *which takes* v *to the function* $\check{v} = \mathcal{F}^{-1}(v)$ *given by*

$$\check{v}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{+i\xi\cdot x}v(\xi)\,d\xi\,,$$

and the following hold:

1. $u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{u}(\xi)e^{i\xi\cdot x}d\xi$ *where* $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}u(x)\,dx$ *(Fourier inversion),*
2. $\widehat{\partial^\alpha u}(\xi) = \mathcal{F}_{x\to\xi}(\partial^\alpha u(x)) = (i\xi)^\alpha\hat{u}(\xi)$ *and* $(\partial^\alpha\hat{u})(\xi) = \mathcal{F}_{x\to\xi}((-ix)^\alpha u(x))$ *for all* $x,\xi\in\mathbb{R}^n, \alpha\in\mathbb{Z}^n_+$*,*
3. $\int_{\mathbb{R}^n}\hat{v}(\xi)u(\xi)\,d\xi = \int_{\mathbb{R}^n}v(x)\hat{u}(x)\,dx\,,$
4. $\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\overline{\hat{v}(\xi)}\hat{u}(\xi)\,d\xi = \int_{\mathbb{R}^n}\overline{v(x)}u(x)\,dx$ *(Parseval-Plancherel),*
5. $\widehat{u*v} = \hat{u}\hat{v}$ *where* $u*v = \int u(x-y)v(y)\,dy$ *(convolution).*

2.3 Banach spaces

A norm on a vector space X is a real function $x \mapsto \|x\|$ such that

1. $\|x\| \geq 0$ with equality iff $x = 0$,
2. $\|cx\| = |c|\|x\|$ for all $c \in \mathbb{C}$,
3. $\|x + y\| \leq \|x\| + \|y\|$.

(If the first condition is replaced by the weaker requirement $1'$ that $\|x\| \geq 0$ then the modified conditions $1', 2, 3$ define a semi-norm.) A normed vector space is a metric space with metric $d(x, y) = \|x - y\|$. Recall that a metric on X is a map $d : X \times X \to [0, \infty)$ such that

1. $d(x, y) \geq 0$ with equality iff $x = y$,
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z in X.

(This definition does not require that X be a vector space.) The metric space (X, d) is complete if every Cauchy sequence has a limit point: to be precise if $\{x_j\}_{j=1}^{\infty}$ has the property that $\forall \epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that $j, k \geq N(\epsilon) \implies d(x_j, x_k) < \epsilon$ then there exists $x \in X$ such that $\lim_{j\to\infty} d(x_j, x) = 0$.

Definition 2.3.1 *A Banach space is a normed vector space which is complete (using the metric $d(x, y) = \|x - y\|$).*

Examples are

- $\mathbb{C}^n$ with the Euclidean norm $\|z\| = (\sum_j |z_j|^2)^{\frac{1}{2}}$.
- $C([a, b])$ with $\|f\| = \sup_{[a,b]} |f(x)|$ (uniform norm).
- Spaces of p-summable (bi-infinite) sequences $\{u_m = u(m)\}_{m\in\mathbb{Z}}$

  $$l^p(\mathbb{Z}) = \{u : \mathbb{Z} \to \mathbb{C} \text{ such that } \|u\|_p = (\sum |u(m)|^p)^{\frac{1}{p}} < \infty\}$$

  and generalizations such as $l^p(\mathbb{Z}^n)$ and $l^p(\mathbb{N})$.
- Spaces of measurable L^p functions for $1 \leq p < \infty$

  $$L^p(\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{C} \text{ measurable with } \|u\|_p = (\int |u(x)\|^p \, dx\,)^{\frac{1}{p}} < \infty\}$$

  and generalizations such as $L^p([-\pi, \pi]^n)$ and $L^p([0, \infty)$ etc. For $p = \infty$ the space $L^\infty(\mathbb{R}^n)$ consists of measurable functions which are bounded on the complement of a null set, and the least such bound is called the essential supremum and gives the norm $\|u\|_{L^\infty}$. In this example we identify functions which agree on the complement of a null set (almost everywhere).

Completeness is important because methods for proving that an equation has a solution typically produce a sequence of "approximate solutions", e.g. Picard iterates for the case of ode. If this sequence can be shown to be Cauchy in some norm then completeness ensures the existence of a limit point which is the putative solution, and in good situations can be proved to be a solution. Review the proof of the existence theorem for ode via the contraction mapping theorem in the Banach space of continuous functions and the uniform norm.

Norms are used in the definition of well posed: if a pde can be solved for a solution u which is uniquely determined by some set of initial and/or boundary data $\{f_j\}$ then the problem is said to be well posed in a norm $\|\cdot\|$ if the solution changes a small amount in this norm as the data change. This would be satisfied if for example for any other solution v determined by data $\{g_j\}$ there holds:

$$\|u-v\| \leq C(\sum_j \|f_j - g_j\|_j)\,, \qquad \text{for some } C>0\,, \tag{2.1}$$

where $\|\cdot\|_j$ are some collection of norms which measure what kind of changes of data produce small changes of the solution. *Finding the appropriate norms such that* (2.1) *holds for a given problem is a crucial part of understanding the problem - they are generally not known in advance.* Once this is understood it is helpful with development of numerical methods for solving problems on computers, and tells you in an experimental situation how accurately you need to measure the data to make a good prediction.

To fix ideas consider the problem of solving an evolution equation of the form

$$\partial_t u = P(\partial_x)u$$

where P is a constant coefficient polynomial; e.g. the case $P(\partial_x) = i\partial_x^2$ corresponds to the Schrödinger equation $\partial_t u = i\partial_x^2 u$. If we are solving this with periodic boundary conditions $u(x,t) = u(x+2\pi,t)$ and with given initial data $u(x,0) = u_0(x)$ for $u_0 \in C^\infty_{per}([-\pi,\pi])$. *Formally* the solution can be given as

$$u(x,t) = \sum_{m\in\mathbb{Z}} e^{tP(im)+imx}\hat{u}_0(m) \tag{2.2}$$

and if the initial data $u_o = \sum \hat{u}_0(m)e^{imx}$ is a finite sum of exponentials then (2.2) is easily seen to define a solution since it reduces also to a finite sum. In the general case it is necessary to investigate convergence of the sum so that it does define a solution, then to prove uniqueness of this solution, and finally to find norms for which well-posedness holds. For this final step the Parseval identity is often very helpful, and for the case of the Schrödinger equation the series (2.2) does indeed define a solution for smooth periodic data u_0, v_0 and

$$\max_{t\in\mathbb{R}} \int |u(x,t) - v(x,t)|^2\,dx \leq \int |u_0(x) - v_0(x)|^2\,dx\,.$$

This inequality would be interpreted as saying that the Schrödinger equation is well posed in L^2 (globally in time since there is no restriction on t.) In question 7 of sheet II you are asked to prove that the solutions are unique.

In general an equation defines a well posed problem with respect to specific norms, which encode certain aspects of the behaviour of the solutions and have to be found as part of the investigation: the property of being well posed depends on the norm. This is related to the fact that norms on infinite dimensional vector spaces (like spaces of functions) can be inequivalent (i.e. can correspond to different notions of convergence), unlike in Euclidean space $\mathbb{R}^n$.

2.4 Hilbert spaces

A Hilbert space is a Banach space which is also an inner product space: the norm arises as $\|x\| = (x,x)^{\frac{1}{2}}$ where $(\cdot,\cdot) : X \times X \to \mathbb{C}$ satisfies:

1. $(x,x) \geq 0$ with equality iff $x = 0$,
2. $\overline{(x,y)} = (y,x)$,
3. $(ax+by,z) = \overline{a}\,(x,z) + \overline{b}\,(y,z)$ and $(x, ay+bz) = a\,(x,y) + b\,(x,z)$ for complex numbers a,b and vectors x,y,z.

(Funtions $X \times X \to \mathbb{C}$ like this which are linear in the second variable and anti-linear in the first are sometimes called sesqui-linear.) Crucial properties of the inner product in a Hilbert space are the Cauchy-Schwarz inequality $|\,(x,y)\,| \leq \|x\|\|y\|$ and the fact that the inner product can be recovered from the norm via

$$(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right)\,, \qquad \text{(polarization)}\,.$$

Examples include $l^2(\mathbb{Z}^n)$ with inner product $\sum_m \overline{u(m)}v(m)$ and $L^2(\mathbb{R}^n)$ with inner product $(u,v) = \int \overline{u(x)}v(x)\,dx$. Another example is the Sobolev spaces: firstly in the periodic case

$$H^1(\mathbb{R}/2\pi\mathbb{Z}) = \{u \in L^2([-\pi,\pi]) : \|u\|_1^2 = \sum_{m\in\mathbb{Z}} (1+|m|^2)|\hat{u}(m)|^2 < \infty\}\,, \qquad (2.3)$$

where $u = \sum \hat{u}(m)e^{imx}$ is the Fourier representation, and secondly

$$H^1(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \|u\|_1^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)|\hat{u}(\xi)|^2\,d\xi < \infty\}\,, \qquad (2.4)$$

where $\hat{u}$ is the Fourier transform.

The new structure in Hilbert (as compared to Banach) spaces is the notion of orthogonality coming from the the inner product. A set of vectors $\{e_n\}$ is called orthonormal if $(e_n, e_m) = \delta_{nm}$. We will consider only Hilbert spaces which have a countable orthonormal basis $\{e_n\}$ (*separable* Hilbert spaces). In such spaces it is possible to decompose arbitrary elements as $u = \sum u_n e_n$ where $u_n = (e_n, u)$. (The case of Fourier series with $e_m(x) = e^{imx}/\sqrt{2\pi}, m \in \mathbb{Z}$ is an example.) The Parseval identity in abstract form reads $\|u\|^2 = \sum |\,(e_n,u)\,|^2$ and:

Theorem 2.4.1 *Given an orthonormal set $\{e_n\}$ the following are equivalent:*

- $(e_n, u) = 0\ \forall n$ *implies* $u = 0$, *(completeness)*
- $\|u\|^2 = \sum |\,(e_n,u)\,|^2\ \forall u \in X$, *(Parseval),*
- $u = \sum (e_n,u)\,e_n\ \forall u \in X$ *(orthonormal basis).*

A closed subspace $X_1 \subset X$ of a Hilbert space is also a Hilbert space, and there is an orthogonal decomposition

$$X = X_1 \oplus X_1^\perp$$

where $X_1^\perp = \{y \in X : (x_1, y) = 0 \forall x_1 \in X_1\}$. This means that any $x \in X$ can be written uniquely as $x = x_1 + y$ with $x_1 \in X_1$ and $y \in X_1^\perp$, and there is a correponding projection $P_{X_1} x = x_1$.

Associated to a Hilbert space X is its dual space X' which is defined to be the space a bounded linear maps:

$$X' = \{L : X \to \mathbb{C}\,,\ \text{with } L \text{ linear and } \|L\| = \sup_{x \in X, \|x\|=1} |Lx| < \infty\}.$$

The definition of the norm on X' ensures that $|L(x)| \leq \|L\|\|x\|$.

Theorem 2.4.2 (Riesz representation) *Given a bounded linear map L on a Hilbert space X there exists a unique vector $y \in X$ such that $Lx = (y, x)$; also $\|L\| = \|y\|$. The correspondence between L and y gives an identification of the dual space X' with the original Hilbert space X.*

A generalization of this (for non-symmetric situations) is:

Theorem 2.4.3 (Lax-Milgram lemma) *Given a bounded linear map $L : X \to \mathbb{R}$ on a Hilbert space X, and a bilinear map $B : X \times X \to \mathbb{R}$ which satisfies (for some positive numbers $\|B\|, \gamma$):*

- $|B(x, y)| \leq \|B\|\|x\|\|y\| \quad \forall x, y \in X$ *(continuity),*
- $B(x, x) \geq \gamma \|x\|^2 \quad \forall x \in X$ *(coercivity),*

there exists a unique vector $y \in X$ such that $Lx = B(y, x) \forall x \in X$.

A bounded linear operator $B : X \to X$ means a linear map $X \to X$ with the property that there exists a number $\|B\| \geq 0$ such that $\|Bu\| \leq \|B\|\|u\| \forall u \in X$. As in Sturm-Liouville theory we say a bounded linear operator is diagonalizable if there is an orthonormal basis $\{e_n\}$ such that $Be_n = \lambda_n e_n$ for some collection of complex numbers λ_n which are the eigenvalues.

2.5 Distributions

Definition 2.5.1 *A periodic distribution $T \in C^\infty_{per}([-\pi, \pi]^n)'$ is a continuous linear map $T : C^\infty_{per}([-\pi, \pi]^n) \to \mathbb{C}$, where continuous means that if f_n and all its partial derivatives $\partial^\alpha f_n$ converge uniformly to f then $T(f_n) \to T(f)$. Here we call $C^\infty_{per}([-\pi, \pi]^n)$ the space of test functions.*

A tempered distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ is a continuous linear map $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$, where continuous means that if $\|f_n - f\|_{\alpha,\beta} \to 0$ for every Schwartz semi-norm then $T(f_n) \to T(f)$. Here we call $\mathcal{S}(\mathbb{R}^n)$ the space of test functions.

In both cases for $x_0 \in \mathbb{R}^n$ any fixed point (which may be taken to lie in $[-\pi, \pi]^n$ in the periodic case) the Dirac distribution defined by $\delta_{x_0}(f) = f(x_0)$ gives an example.

Remark 2.5.2 *The notion of convergence on $C^\infty_{per}([-\pi,\pi]^n$ and $\mathcal{S}(\mathbb{R}^n)$ used in this definition makes these spaces into topological spaces in which the convergence must be with respect to a countable family of semi-norms. These are examples of Frechet spaces, a class of topological vector spaces which generalize the notion of Banach space by using a countable family of semi-norms rather than a single norm to define a notion of convergence. Using this notion of convergence one can check that the Fourier transform $\mathcal{F}$ is continuous as is its inverse, and the Fourier inversion theorem can be summarized by the assertion that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear homeomorphism with inverse $\mathcal{F}^{-1}$.*

Remark 2.5.3 *Notice that integrable functions define distributions in a natural way: in the simplest case if g is continuous 2π-periodic function then the formula $T_g(f) = \int_{[-\pi,\pi]} g(x)f(x)\,dx$ defines a periodic distribution and clearly the mapping $g \mapsto T_g$ is an injection of $C_{per}([-\pi,\pi])$ into $(C^\infty_{per}([-\pi,\pi]))'$. Similarly if g is absolutely integrable on $\mathbb{R}^n$ then the formula $T_g(f) = \int_{\mathbb{R}^n} g(x)f(x)\,dx$ defines a tempered distribution. The mapping $g \mapsto T_g$ is, properly interpreted, injective: if $g \in L^1(\mathbb{R}^n)$ then $T_g(f) = 0$ for all $f \in \mathcal{S}(\mathbb{R}^n)$ implies that $g = 0$ almost everywhere. On account of this remark distributions are often called "generalized functions". The Dirac example indicates that there are distrubutions which do not arise as T_g.*

Remark 2.5.4 *In these definitions distributions are elements of the dual space of a space of test functions with a specified notion of convergence (a topology). Another frequently used class of distributions is the dual space of $C^\infty_0(\mathbb{R}^n)$ the space of compactly supported smooth functions, topologized as follows: $f_n \to f$ in C^∞_0 if there is a fixed compact set K such that all f_n, f are supported in K and if all partial derivatives of $\partial^\alpha f_n$ converge (uniformly) to $\partial^\alpha f$. This class of distributions is more convenient for some purposes, but not for using the Fourier transform, for which purpose the tempered distributions are most convenient because of remark 2.5.2, which allows the fourier transform to be defined on tempered distributions "by duality" as we now discuss.*

Operations are defined on distributions by using duality to transfer them to the test functions, e.g.:

- Given $T \in \mathcal{S}'(\mathbb{R}^n)$ an arbitrary partial derivative $\partial^\alpha T$ is defined by $\partial^\alpha T(f) = (-1)^{|\alpha|}T(\partial^\alpha f)$.
- Given $T \in \mathcal{S}'(\mathbb{R}^n)$ its fourier transform $\hat{T}$ is defined by $\hat{T}(f) = T(\hat{f})$.
- Given $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\chi \in \mathcal{S}(\mathbb{R}^n)$ the distribution χT is defined by $\chi T(f) = T(\chi f)$.

It is useful to check, with reference to the fact in remark 2.5.3 that distributions are generalized functions, that all such defintions of operations on distributions are designed to extend the corresponding definitions on functions: e.g. for a Schwartz function g we have

$$\partial^\alpha T_g = T_{\partial^\alpha g}\,,$$

where on the left ∂^α means distributional derivative while on the right it is the usual derivative from calculus applied to the test function g. The same principle is behind the other definitions.

There are various alternate notations used for distributions:

$$T(f) \;=\; \langle T\,,\,f\rangle \;=\; \int T(x)f(x)\,dx$$

where in the right hand version it should be remembered that the expression is purely formal in general: the putative function $T(x)$ has not been defined, and the integral notation is not an integral - just shorthand for the duality pairing of the definition. It is nevertheless helpful to use it to remember some formulae: for example the formula for the distributional derivative takes the form

$$\partial^\alpha T(f) \;=\; \int \partial^\alpha T(x) f(x)\, dx \;=\; (-1)^{|\alpha|} \int T(x) \partial^\alpha f(x)\, dx = (-1)^{|\alpha|} T(\partial^\alpha f)\,,$$

which is "familiar" from integration by parts. The formula $\int \delta(x-x_0) f(x)\, dx = f(x_0)$ and related ones are to be understood as formal expressions for the proper definition of the delta distribution above.

2.6 Positive distributions and Measures

In this section[1] we restrict to 2π-periodic distributions on the real line for simplicity. The delta distribution δ_{x_0} has the property that if $f \geq 0$ then $\delta_{x_0}(f) \geq 0$; such distributions are called positive. Positive distributions have an important continuity property as a result: if T is any positive periodic distribution, then since

$$-\|f\|_{L^\infty} \leq f(x) \leq \|f\|_{L^\infty}\,, \qquad \|f\| = \sup |f(x)|$$

for each $f \in C^\infty_{per}([-\pi,\pi])$ it follows from positivity that $T(\|f\|_{L^\infty} \pm f) \geq 0$ and hence by linearity that

$$-c\|f\|_{L^\infty} \leq T(f) \leq c\|f\|_{L^\infty}$$

where $c = T(1)$ is a positive number. This inequality, applied with f replaced by $f - f_n$, means that if $f_n \to f$ uniformly then $T(f_n) \to T(f)$, i.e. *positive distributions* are automatically continuous with respect to *uniform convergence*, in strong contrast to the continuity property required in the original definition. In fact this new continuity property ensures that a positive distribution can be extended uniquely as a map

$$L : C_{per}([-\pi,\pi]) \to [0,\infty)$$

i.e. as a continuous linear functional on the space of continuous functions. This extension is an immediate consquence of the density of smooth functions in the continuous functions in the uniform norm (which can be deduced from the Weierstrass approximation theorem). A much more lengthy argument allows such a functional to be extended as an integral $L(f) = \int f\, d\mu$ which is defined for a class of measurable functions f which contains and is bigger than the class of continuous functions. To conclude: positive distributions automatically extend to define continuous linear functional on the space of continuous functions, and hence can be identified with a class of *measures* (Radon measures) which can be used to integrate much larger classes of functions (extending further the domain of the original distribution).

[1] This is an optional section, for background only

3 Example sheet 2

1. Let $\Delta = \sum_{j=1}^{n-1} \partial_j^2$ be the laplacian. For which vectors $a \in \mathbb{R}^{n-1}$ is the operator $P = \partial_t^2 u + \partial_t \sum_{j=1}^{n-1} a_j \partial_j u - \Delta u$ hyperbolic.

2. Consider 2nd order constant coeffcient operator s $P = \sum_{i,j=1,\dots n} a_{ij} \partial_i \partial_j$, determined by a non-degenerate (i.e. invertible) $n \times n$ matrix $a_{ij} = a_{ji}$. For $n = 4$ find a matrix a_{ij} such that P cannot be written as either elliptic or hyperbolic even after making a linear transformation of the coordinates. How about for $n = 1, 2, 3$?

3. Obtain and solve the ode satisfied by characteristic curves $y = y(x)$ for the equation $(x^2 + 2)^2 u_{xx} - (x^2 + 1)^2 u_{yy} = 0$. Show that there are two families of such curves which can be written in the form $y - x + 2^{-\frac{1}{2}} \arctan 2^{-\frac{1}{2}} x = \xi$ and $y + x - 2^{-\frac{1}{2}} \arctan 2^{-\frac{1}{2}} x = \eta$, for arbitrary real numbers ξ, η. Now considering the change of coordinates $(x, y) \to (\xi, \eta)$ so determined find the form of the equation in the coordinate system (ξ, η).

4. Which of the following functions of x lie in Schwartz space $\mathcal{S}(\mathbb{R})$: (a) $(1 + x^2)^{-1}$,(b) e^{-x}, (c)$e^{-x^4}/(1 + x^2)$? Show that if $f \in \mathcal{S}(\mathbb{R})$ then so is $f(x)/P(x)$ where P is any strictly positive polynomial (i.e. $P(x) \geq \theta > 0$ for some real θ.

5. Solve the following initial value problem

$$\partial_t u = \partial_x^3 u \qquad u(0, x) = f(x)$$

   for $x \in [-\pi, \pi]$ with periodic boundary conditions $u(t, -\pi) = u(t, \pi)$ and f smooth and 2π-periodic. Discuss well-posedness properties of your solutions for the L^2 norm, i.e. $\|u(t)\|_{L^2} = (\int_{-\pi}^{+\pi} |u(t, x)|^2 dx)^{\frac{1}{2}}$, using the Parseval-Plancherel theorem.

6. Show that the heat equation $\partial_t u = \partial_x^2 u$, with boundary conditions as in 1 is well-posed forwards in time in L^2 norm, but not backwards in time (even locally). (Hint compute the L^2 norm of solutions u_n for negative t corresponding to initial values $u_n(0, x) = n^{-1} e^{inx}$.)

7. (i) Use fourier series to solve the Schrodinger equation

$$\partial_t u = i \partial_x^2 u \qquad u(0, x) = f(x)$$

   for initial value f smooth and periodic. Prove there is only one smooth solution.

   (ii) Use the fourier transform to solve the Schrodinger equation for $x \in \mathbb{R}$ and initial value $f \in \mathcal{S}(\mathbb{R})$. Find the solution for the case $f = e^{-x^2}$.

8. (i) Verify that the tempered distribution u on the real line defined by the function $(2m)^{-1} e^{-m|x|}$, (for positive m), solves

$$\left(\frac{-d^2}{dx^2} + m^2\right) u = \delta_0$$

in $\mathcal{S}'(\mathbb{R})$.

(ii) Verify that the function on the real line $g(x) = 1$ for $x \leq 0$ and $g(x) = e^{-x}$ for $x > 0$ defines a tempered distribution T_g which solves in $\mathcal{S}'(\mathbb{R})$

$$T'' + T' = -\delta_0.$$

9. Define, for positive integral s, the norm $\|\cdot\|_s$ on the space of smooth 2π-periodic function of x by

$$\|f\|_s^2 \equiv \sum_{m\in\mathbb{Z}} (1+m^2)^s |\hat{f}(m)|^2$$

where $\hat{f}(n)$ are the fourier coefficients of f. (This is called the Sobolev H^s norm).

(i) What are these norms if $s = 0$? Write down a formula for these norms for $s = 0, 1, 2 \ldots$ in terms of $f(x)$ and its derivatives directly. (Hint Parseval).

(ii) If $u(t, x)$ is the solution you obtained for the heat equation in 2 then for $t > 1$ and $s = 0, 1, 2, , ,$ find a number $C_s > 0$ such that

$$\|u(t, \cdot)\|_s \leq C_s \|u(0, \cdot)\|_0 \ .$$

(iii) Show that there exists a number $C > 0$ which does not depend on f so that $\max |f(x)| \leq C \|f\|_1$ for all smooth 2π-periodic f.

(iv)* Try to generalize (i)-(iii) to periodic functions $f = f(x_1, \ldots x_n)$ of n variables.

10. Write down the precise distributional meaning of the equation

$$-\Delta(|x|^{-1}) = 4\pi\delta_0 \qquad \text{in } \mathcal{S}'(\mathbb{R}^3)$$

in terms of test functions, and then use the divergence theorem to verify that it holds. (Hint: apply the divergence theorem on the region $\{0 < |x| < R\} - \{0 < |x| < \epsilon\}$ for R sufficiently large and take the limit $\epsilon \to 0$ carefully).

11. (i)* For each of the following equations find the most general tempered distribution T which satisfies it (see Friedlander §2.7)

$$xT = 0, \quad xdT/dx = 0, \quad x^2T = \delta_0, \quad xdT/dx = \delta_0$$

$$dT/dx = \delta_0, \quad dT/dx + T = \delta_0 \quad T - (d/dx)^2 T = \delta_0.$$

(ii)* Solve the equation $x^m T = 0$ in $\mathcal{S}'(\mathbb{R})$.