Partial Differential Equations Example sheet 3

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2.7 Sobolev spaces

We define the Sobolev spaces for $s = 0, 1, 2, \dots$ on various domains:
On $\mathbb{R}^n$ we have the following equivalent definitions:

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{u \in L^2(\mathbb{R}^n) : \|u\|^2_{H^s} = \sum_{\alpha:|\alpha|\leq s} \|\partial^\alpha u\|^2_{L^2} < \infty\} \\ &= \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^s |\hat{u}(\xi)|^2 \, d\xi < \infty\} \\ &= \overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_{H^s}} . \end{aligned}$$

In the first line the partial derivatives are taken in the distributional sense: the precise meaning is that all *distributional (=weak)* partial derivatives up to order s of the distribution T_u determined by u are distributions which are determined by square integrable funtions which are designated $\partial^\alpha u$ (i.e. $\partial^\alpha T_u = T_{\partial^\alpha u}$ with $\partial^\alpha u \in L^2$ in the notation introduced previously). The final line means that H^s is the closure of the space of smooth compactly supported functions $C_0^\infty(\mathbb{R}^n)$ in the Sobolev norm $\|\cdot\|_{H^s}$. The quantity $\tilde{\|}u\tilde{\|}^2_{H^s} = \int_{\mathbb{R}^n} (1+\|\xi\|^2)^s |\hat{u}(\xi)|^2 \, d\xi$ appearing in the middle definition defines a norm which is equivalent to the norm $\|u\|_{H^s}$ appearing in the first definition. (Recall that $\|\cdot\|$ and $\tilde{\|}\cdot\tilde{\|}$ are equivalent if there exist positive numbers C_1, C_2 such that $\|u\| \leq C_1 \tilde{\|}u\tilde{\|}$ and $\tilde{\|}u\tilde{\|} \leq C_2 \|u\|$ for all vectors u; equivalent norms give rise to identical notions of convergence (i.e. they define the same topologies).
On $(\mathbb{R}/(2\pi\mathbb{Z}))^n$: In the 2π-periodic case the following definitions are equivalent:

$$\begin{aligned} H^s_{per}([-\pi,\pi]^n) &= \{u \in L^2([-\pi,\pi]^n) : \|u\|^2_{H^s} = \sum_{\alpha:|\alpha|\leq s} \|\partial^\alpha u\|^2_{L^2} < \infty\} \\ &= \{\sum_{m\in\mathbb{Z}^n} \hat{u}(m) e^{im\cdot x} : \sum_{m\in\mathbb{Z}^n} (1 + \|m\|^2)^s |\hat{u}(m)|^2 < \infty\} \\ &= \overline{C^\infty_{per}([-\pi,\pi]^n)}^{\|\cdot\|_{H^s}} . \end{aligned}$$

Again the quantity apearing in the middle line defines an equivalent norm which can be used when it is more convenient. Since we are considering only the case $s = 0, 1, 2, \dots$ the Fourier series $\sum_{m\in\mathbb{Z}^n} \hat{u}(m) e^{im\cdot x}$ always defines a square integrable function, and as s increases the function so defined is more and more regular (exercise).

These definitions require some modifications for the case of general domains Ω, starting with the notion of the weak partial derivative (since we did not define distributions in Ω).

Definition 2.7.1 *A locally integrable function u defined on an open set Ω admits a weak partial derivative corresponding to the multi-index α if there exists a locally integrable function, designated $\partial^\alpha u$, with the property that*

$$\int_\Omega u\,\partial^\alpha \chi\,dx = (-1)^{|\alpha|}\int_\Omega \partial^\alpha u\,\chi\,dx\,,$$

for every $\chi \in C_0^\infty(\Omega)$.

Then employing this notion of partial derivative we define (for $s = 0, 1, 2, \dots$):

$$H^s(\Omega) = \{u \in L^2(\Omega) : \|u\|_{H^s}^2 = \sum_{\alpha : |\alpha| \le s} \|\partial^\alpha u\|_{L^2}^2 < \infty\}$$

(with all L^2 norms being defined by integration over Ω). This space is to be distinguished from the corresponding closure of the space of smooth functions supported in a compact subset of Ω:

$$H_0^s(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s}}\,.$$

Since these functions are limits of functions which vanish in a neighbourhood of Ω they are to be thought of as vanishing in some generalized sense on $\partial\Omega$ (at least in the case $s = 1, 2, \dots$ and if Ω has a smooth boundary $\partial\Omega$.) The case $s = 1$ gives the space $H_0^1(\Omega)$ which is the natural Hilbert space to use in order to give a weak formulation of the Dirichlet problem for the elliptic equation $Pu = f$ on Ω.

3 Elliptic equations

3.1 Notation

Let $B_R = \{w : |w| < R\}$ and $\overline{B_R} = \{w : |w| \le R\}$ be the open and closed balls of radius R and more generally let $B_R(x) = \{w : |w - x| < R\}$ and $\overline{B_R(x)} = \{w : |w - x| \le R\}$. We write $\partial B_R, \partial B_R(x)$ for the corresponding spheres, i.e. $\partial B_R(x) = \{w : |w - x| = R\}$ etc. In the following $\Omega \subset \mathbb{R}^n$ is always open unless otherwise stated, $\overline{\Omega}$ is its closure and $\partial\Omega$ is its boundary (always assumed smooth).

3.2 Harmonic functions

Definition 3.2.1 *A function $u \in C^2(\Omega)$ which satisfies $\Delta u(x) = 0$ (resp. $\Delta u(x) \ge 0$, resp. $\Delta u(x) \le 0$) for all $x \in \Omega$, for an open set $\Omega \subset \mathbb{R}^n$, is said to be harmonic (resp. subharmonic, resp. superharmonic) in Ω.*

Theorem 3.2.2 *Let u be harmonic in $\Omega \subset \mathbb{R}^n$ and assume $\overline{B_R(x)} \subset \Omega$. Then for $0 < r \le R$:*

$$u(x) = \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(y)\,dy\,, \qquad \textit{(mean value property)}\,. \tag{3.1}$$

Proof This is a consequence of the Green identity

$$\int_{\rho<|w-x|<r}(v\Delta u-u\Delta v)\,dx=\int_{|w-x|=r}(v\partial_\nu u-u\partial_\nu v)\,d\Sigma-\int_{|w-x|=\rho}(v\partial_\nu u-u\partial_\nu v)\,d\Sigma\,,$$

(where $\partial_\nu = n\cdot\nabla$ just means the normal derivative on the boundary) with the choice of $v(w)=N(w-x)$, where N is the fundamental solution for Δ:

$$\begin{aligned} N(x) &= \frac{|x|^{2-n}}{(2-n)\omega_n}\,, \qquad (n>2) \\ &= \frac{1}{2\pi}\ln|x|\,, \qquad (n=2)\,. \end{aligned}$$

Here $\omega_n = \int_{|x|=1} d\Sigma(x) = 2\pi^{\frac{n}{2}}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbb{R}^n$. Thus on $\partial B_r(x)$ we have $v=r^{2-n}/(2-n)\omega_n, n>2$ or $v=(\ln r)/(2\pi), n=2$. Substituting these and the corresponding ones for normal derivatives, $\partial_\nu v = r^{1-n}/\omega_n$ on $\partial B_r(x)$, and taking the limit $\rho\to 0$ gives the result.

Corollary 3.2.3 *If u is a C^2 harmonic function in an open set Ω then $u\in C^\infty(\Omega)$. In fact if u is any C^2 function in Ω for which the mean value property* (3.1) *holds whenever $\overline{B_r(x)}\subset\Omega$ then u is a smooth harmonic function.*

Corollary 3.2.4 *Let $\Omega\subset\mathbb{R}^n$ be a connected open set and $u\in C(\overline{\Omega})$ harmonic in Ω with $M=\sup_{x\in\overline{\Omega}}u(x)<\infty$. Then either $u(x)<M$ for all $x\in\Omega$ or $u(x)=M$ for all $x\in\Omega$. (In words, a harmonic function cannot have an interior maximum unless it is constant on connected components).*

Corollary 3.2.5 *Let $\Omega\subset\mathbb{R}^n$ be open with bounded closure $\overline{\Omega}$, and let $u_j\in C(\overline{\Omega})$, $j=1,2$ be two harmonic functions in Ω with boundary values $u_j|_{\partial\Omega}=f_j$. Then*

$$\sup_{x\in\Omega}|u_1(x)-u_2(x)|\le\sup_{x\in\partial\Omega}|f_1(x)-f_2(x)|\,,\qquad \textit{(stability or well-posedness).}$$

In particular if $f_1=f_2$ then $u_1=u_2$.

Corollary 3.2.6 *A harmonic function $u\in C^2(\mathbb{R}^n)$ which is bounded is constant.*

Another consequence of the Green identity is the following. Let $N(x,y)=N(|x-y|)$ where N is the fundamental solution defined above.

Theorem 3.2.7 *Let u be harmonic in Ω with $\overline{\Omega}$ bounded and $u\in C^1(\overline{\Omega})$. Then*

$$u(x)=\int_{|y|=r}\Big[u(y)\partial_{\nu_y}N(x,y)-N(x,y)\partial_{\nu_y}u(y)\Big]\,d\Sigma(y)\,,$$

where $\partial_{\nu_y}=n\cdot\nabla_y$ just means the normal derivative in y, while ∂_ν is the normal in x. In fact the same formula holds with $N(x,y)$ replaced by any function $G(x,y)$ such that $G(x,y)-N(x,y)$ is harmonic in $y\in\Omega$ and C^1 for $y\in\overline{\Omega}$ for each $x\in\Omega$.

It is known from above that u is determined by its boundary values - to determine a harmonic function u from $u|_{\partial\Omega}$ is the *Dirichlet problem*. (The corresponding problem of determining u from its normal derivative $\partial_\nu u|_{\partial\Omega}$ is called the *Neumann problem*. To get a formula for (or understand) the solution of these problems it is sufficient to get a formula for (or understand) the correponding Green function:

Definition 3.2.8 *(i) A function $G_D = G_D(x, y)$ defined on $G_D : \Omega \times \overline{\Omega} - \{x = y\} \to \mathbb{R}$ such that (a) $G_D(x, y) - N(|x - y|)$ is harmonic in $y \in \Omega$ and continuous for $y \in \overline{\Omega}$ for each x, and (b) $G_D(x, y) = 0$ for $y \in \partial\Omega$, is a Dirichlet Green function.*

(ii) A function $G_N = G_N(x, y)$ defined on $G_N : \Omega \times \overline{\Omega} - \{x = y\} \to \mathbb{R}$ such that (a) $G_N(x, y) - N(|x - y|)$ is harmonic in $y \in \Omega$ and continuous for $y \in \overline{\Omega}$ for each x, and (b) $\partial_{\nu_y} G_N(x, y) = 0$ for $y \in \partial\Omega$, is a Neumann Green function.

Given such functions we obtain representation formulas:

$$\Delta u = 0\,, \quad u|_{\partial\Omega} = f \implies u(x) = \int_{|y|=r} f(y)\partial_{\nu_y} G_D(x, y)\, d\Sigma(y)\,,$$

and

$$\Delta u = 0\,, \quad \partial_\nu u|_{\partial\Omega} = g \implies u(x) = -\int_{|y|=r} g(y) G_N(x, y)\, d\Sigma(y)\,,$$

for $f, g \in C(\partial\Omega)$.

3.3 The maximum principle

We consider variable coefficient elliptic operators. Throughout this section $a_{jk}(x) = a_{kj}(x)$ is continuous and satisfies

$$m\|\xi\|^2 \le \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \le M\|\xi\|^2 \tag{3.2}$$

for some positive constants m, M and all $\xi \in \mathbb{R}^n$.

Theorem 3.3.1 *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Pu = 0$ where*

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u \tag{3.3}$$

is an elliptic operator with continuous coefficients and (3.2) *holds, then* $\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$.

Theorem 3.3.2 *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Pu = 0$ where*

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u + cu \tag{3.4}$$

is an elliptic operator with continuous coefficients and (3.2) *holds and $c \ge 0$ everywhere, then $\max_{x\in\overline{\Omega}} u(x) \le \max_{x\in\partial\Omega} u^+(x)$ where $u^+ = \max\{u, 0\}$ is the positive part of the function u.*

Corollary 3.3.3 *In the situation of theorem 3.3.2* $\max_{x \in \overline{\Omega}} |u(x)| = \max_{x \in \partial\Omega} |u(x)|$.

Theorem 3.3.4 *Let* $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ *satisfy* $Pu = f$ *with Dirichlet data* $u|_{\partial\Omega} = 0$, *where*

$$Pu = -\sum_{j,k=1}^{n} a_{jk} \partial_j \partial_k u + \sum_{j=1}^{n} b_j \partial_j u + cu \tag{3.5}$$

is an elliptic operator with continuous coefficients and (3.2) *holds and*

$$c(x) \geq c_0 > 0 \tag{3.6}$$

everywhere, for some constant $c_0 > 0$, *and* $f \in C(\overline{\Omega})$, *then* $\max_{x \in \overline{\Omega}} u(x) \leq \max_{x \in \overline{\Omega}} f(x)/c_0$. *If* $Pu_j = f_j$ *are two such solutions then* $\max |u_1 - u_2| \leq \max |f_1 - f_2|/c_0$ *(stability or well-posedness in uniform norm).*

3.4 Stability in Sobolev spaces

In this section we consider results analogous to those in theorem 3.3.4 but using L^2 based norms. We consider P in a special form to facilitate integration by parts.

Theorem 3.4.1 *Let* $a_{jk} = a_{kj} \in C^\infty([-\pi, \pi]^n)$ *and* $c \in C^\infty([-\pi, \pi]^n)$ *be smooth periodic coefficents for the elliptic operator*

$$Pu = -\sum_{jk} \partial_j (a_{jk} \partial_k u) + cu$$

and assume (3.2) *and* (3.6) *hold for some poitive constants* m, M, c_0. *Assume* $Pu = f$ *with* $f \in L^2([-\pi, \pi]^n)$, *then there exists a number* L *such that then*

$$\|u\|_{H^1} \leq L \|f(x)\|_{L^2}.$$

If $Pu_j = f_j$ *are two such solutions then*

$$\|u_1 - u_2\| \leq L \|f_1 - f_2\|$$

(stability or well-posedness in H^1*).*

3.5 Existence of solutions

Definition 3.5.1 *A weak solution of* $Pu = f \in L^2([-\pi, \pi]^n)$, *with* P *as in theorem 3.4.1, is a function* $u \in H^1_{per}([-\pi, \pi]^n)$ *with the property that*

$$\int \sum_{jk} a_{jk} \, \partial_j u \partial_k v + cuv \, dx = \int f v \, dx$$

for all $v \in H^1_{per}([-\pi, \pi]^n)$.

Theorem 3.5.2 *Let P be as in theorem 3.4.1 and assume that* (3.2) *and* (3.6) *exist. Then given $f \in L^2([-\pi,\pi]^n)$ there exists a unique weak solution of $Pu = f$ in the sense of definition* (3.5.1).

This definition has various generalizations: to obtain the correct definition of weak solution for a given elliptic boundary value problem the general idea is to start with a classical solution and multiply by a test function and integrate by parts using the boundary conditions in their classical format. This will lead to a weak formulation of both the equation and the boundary conditions. For example the weak formulation of the Dirichlet problem

$$Pu = f\,, \qquad u|_{\partial\Omega} = 0\,,$$

where

$$Pu = -\sum_{j,k=1}^{n} \partial_j (a_{jk}\partial_k u) + \sum_{j=1}^{n} b_j \partial_j u + cu \tag{3.7}$$

for continuous functions $a_{jk} = a_{kj}, b_j$ and c, is to find a function $u \in H^1_0(\Omega)$ such that

$$B(u,v) = L(v)\,, \quad \forall v \in H^1_0(\Omega)\,,$$

where $L(v) = \int f v\,dx$ (a bounded linear map/functional), and B is the bilinear form:

$$B(u,v) = \int \big(\sum_{jk} a_{jk}\,\partial_j u \partial_k v + \sum b_j \partial_j u v \;+ cuv\big)\,dx\,.$$

By the Lax-Milgram lemma we have

Theorem 3.5.3 *In the situation just described assume* (3.2) *and* (3.6) *hold. Then if $\|b_j\|_{L^\infty} = \sup_x |b_j(x)|$ is sufficiently small (for all j) there exists a unique weak solution.*

This solution has various regularity properties, the simplest of which is that if in addition $a_{jk} \in C^1(\Omega)$ then in any ball such that $\overline{B_r(y)} \subset \Omega$ there holds for some constant $C > 0$:

$$\|u\|_{H^2(B_r(y))} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})\,, \qquad \text{(interior } H^2 \text{ regularity)}\,,$$

and if in addition all the coefficients are smooth then we have, for arbitrary $s \in \mathbb{N}$ and some $C_s > 0$:

$$\|u\|_{H^{s+2}(B_r(y))} \le C_s(\|f\|_{H^s(\Omega)} + \|u\|_{L^2(\Omega)})\,, \qquad \text{(higher interior regularity)}\,.$$

To get regularity up to the boundary it is necessary to assume that the boundary itself is smooth: in this case the interior regularity estimate can be improved to

$$\|u\|_{H^2(\Omega)} \le C'(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})\,, \qquad \text{(boundary } H^2 \text{ regularity)}\,,$$

4 Example sheet 3

1. Recall that if $u \in C^2(\mathbb{R}^3)$ and $\Delta u \geq 0$ then u is called subharmonic. State and prove a mean value property for subharmonic functions. Also state the analogous result for superharmonic functions, i.e. those C^2 functions which satisfy $\Delta u \leq 0$.

2. Let $\phi \in C(\mathbb{R}^n)$ be absolutely integrable with $\int \phi(x)dx = 1$. Assume $f \in C(\mathbb{R}^n)$ is bounded with $\sup|f(x)| \leq M < \infty$. Define $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ and show
$$\phi_\epsilon * f(x) - f(x) = \int \big(f(x - \epsilon w) - f(x)\big)\phi(w)dw$$
(where the integrals are over $\mathbb{R}^n$). Now deduce the *approximation lemma*:
$$\phi_\epsilon * f(x) \to f(x) \quad \text{as } \epsilon \to 0$$
and uniformly if f is uniformly continuous. (Hint: split up the w integral into an integral over the ball $B_R = \{w : |w| < R\}$ and its complement B_R^c for large R).*Prove that if $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$ then $\lim_{\epsilon \to 0} \|\phi_\epsilon * f(x) - f(x)\|_{L^p} = 0$.

3. Starting with the mean value property for harmonic $u \in C^2(\mathbb{R}^3)$ deduce that if $\phi \in C_0^\infty(\mathbb{R}^3)$ has total integral $\int \phi(x)dx = 1$ and is radial $\phi(x) = \psi(|x|)\,, \psi \in C_0^\infty(\mathbb{R})$ then $u = \phi_\epsilon * u$ where $\phi_\epsilon(x) = \epsilon^{-3}\phi(x/\epsilon)$. Deduce that harmonic functions $u \in C^2(\mathbb{R}^3)$ are in fact C^∞. Also for $u \in C^2(\Omega)$ harmonic in an open set $\Omega \in \mathbb{R}^3$ deduce that u is smooth in the interior of Ω (interior regularity).

4. If u_i, u_2 are two C^2 harmonic functions in $B_R = \{w : |w| < R\}$ which agree on the boundary $\partial B_R = \{w : |w| = R\}$ show that $u_i = u_2$ thoroughout out B_R.

5. (i) Using the Green identities show that if f_1, f_2 both lie in $\mathcal{S}(\mathbb{R}^n)$ then the corresponding Schwartzian solutions u_1, u_2 of the equation $-\Delta u + u = f$, i.e.
$$(-\Delta + 1)u_1 = f_1 \qquad (-\Delta + 1)u_2 = f_2$$
satisfy
$$(*) \qquad \int |\nabla(u_1 - u_2)|^2 + |u_1 - u_2|^2 \leq c \int |f_1 - f_2|^2$$
where the integrals are over $\mathbb{R}^n$. (This is interpreted as implying the equation $-\Delta u + u = f$ is well-posed in the H^1 norm (or "energy" norm) defined by the left hand side of (*).) Now try to improve the result so that the H^2 norm:
$$\|u\|_{H^2}^2 \equiv \sum_{|\alpha| \leq 2} \int |\partial^\alpha u|^2 dx,$$
appears on the left. (The sum is over all multi-indices of order less than or equal to 2).

   (ii) Prove a maximum principle bound for u in terms of f and deduce that $\sup_{\mathbb{R}^n} |u_1 - u_2| \leq \sup_{\mathbb{R}^n} |f_1 - f_2|$.

6. Prove a maximum principle for solutions of $-\Delta u + V(x)u = 0$ (on a bounded domain Ω with smooth boundary $\partial\Omega$) with $V > 0$: if $u|\partial\Omega = 0$ then $u \leq 0$ in Ω. (Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Hint: exclude the possibility of u having a strictly positive interior maximum).

   What does the maximum principle reduce to for one dimensional harmonic functions i.e. C^2 functions such that $u_{xx} = 0$?

7. Write down the definition of a weak H^1 solution for the equation $-\Delta u + u + V(x)u = f \in L^2(\mathbb{R}^3)$ on the domain $\mathbb{R}^3$. Assuming that V is real valued, continuous, bounded and $V(x) \geq 0$ for all x prove the existence and uniqueness of a weak solution. Formulate and prove well posedness (stability) in H^1 for this solution.
   How about the case that V is pure imaginary valued?

8. The Dirichlet problem in half-space:
   Let $H = \{(x, y) : x \in \mathbb{R}^n, y > 0\}$ be the half-space in $\mathbb{R}^{n+1}$. Consider the problem $\Delta_x u + \partial_y^2 u = 0$, where Δ_x is the Laplacian in the x variables only) and $u(x, 0) = f(x)$ with f a bounded and uniformly continuous function on $\mathbb{R}^n$. Define
$$u(x, y) = P_y * f(x) = \int_{\mathbb{R}^n} P_y(x - z) f(z) dz$$
   where $P_y(x) = \frac{2y}{\omega_n(|x|^2+y^2)^{\frac{n+1}{2}}}$ for $x \in \mathbb{R}^n$ and $y > 0$. (This is the Poisson kernel for half-space.) Show that for an appropriate choice of ω_n u is harmonic on the half-space H and is equal to f for $y = 0$.
   (Hint: first differentiate carefully under the integral sign; then note that $P_y(x) = y^{-n} P_1(\frac{x}{y})$ where $P_1(x) = \frac{2}{\omega_n(1+|x|^2)^{\frac{n+1}{2}}}$, i.e. an approximation to the identity) and use the approximation lemma to obtain the boundary data).

   (ii) Assume instead that $f \in \mathcal{S}(\mathbb{R}^n)$. Take the Fourier transform in the x variables to prove the same result.

9. Formulate and prove a maximum principle for a 2nd order elliptic equation $Pu = f$ in the case of periodic boundary conditions. Take $Pu = -\sum_{jk=1}^n a_{jk}\partial_{jk}^2 u + \sum_{j=1}^n b_j \partial_j u + cu$ with $a_{jk} = a_{kj}, b_j, c$ and f all continuous and 2π - periodic in each variable and assume u is a C^2 function with same periodicity. Assume uniform ellipticity (3.2) and $c(x) \geq c_0 > 0$ for all x. Formulate and prove well-posedness for $Pu = f$ in the uniform norm.

10. Formulate a notion of weak H^1 solution for the Sturm-Liouville problem $Pu = f$ on the unit interval $[0, 1]$ with inhomogeneous Neumann data: assume $Pu = -(pu')' + qu$ with $p \in C^1([0, 1])$ and $q \in C^([0, 1])$ and assume there exist constants m, c_0 such that $p \geq m > 0$ and $q \geq c_0 > 0$ everywhere, and consider boundary conditions $u'(0) = \alpha$ and $u'(1) = \beta$. (Hint : start with a classical solution, multiply by a test function $v \in C^1([0, 1])$ and integrate by parts). Prove the existence and uniqueness of a weak H^1 solution for given $f \in L^2$.