Partial Differential Equations Example sheet 4

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3 Parabolic equations

3.1 The heat equation on an interval

Next consider the heat equation $x \in [0,1]$ with Dirichlet boundary conditions $u(0,t) = 0 = u(1,t)$. Introduce the Sturm-Liouville operator $Pf = -f''$, with these boundary conditions. Its eigenfunctions $\phi_m = \sqrt{2}\sin m\pi x$ constitute an orthonormal basis for $L^2([0,1])$ (with inner product $(f,g) = \int f(x)g(x)dx$, considering here real valued functions). The eigenvalue equation is $P\phi_m = \lambda_m \phi_m$ with $\lambda_m = (m\pi)^2$. In terms of P the equation is:

$$u_t + Pu = 0$$

and the solution with initial data

$$u(0,x) = u_0(x) = \sum (\phi_m, u_0)\phi_m\,,$$

is given by

$$u(x,t) = \sum e^{-t\lambda_m}(\phi_m, u_0)\phi_m\,. \tag{3.1}$$

(In all these expressions $\sum$ means $\sum_{m=1}^{\infty}$.) An appropriate Hilbert space is to solve for $u(\cdot,t) \in L^2([0,1])$ given $u_0 \in L^2$, but the presence of the factor $e^{-t\lambda_m} = e^{-tm^2\pi^2}$ means the solution is far more regular for $t > 0$ than for $t = 0$.

3.2 The heat kernel

The heat equation is $u_t = \Delta u$ where Δ is the Laplacian on the spatial domain. For the case of spatial domain $\mathbb{R}^n$ the distribution defined by the function

$$K(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi t^n}}\exp[-\frac{\|x\|^2}{4t}] & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases} \tag{3.2}$$

is the fundamental solution for the heat equation (in n space dimensions). This can be derived slightly indirectly: first using the Fourier transform (in the space variable x only) the following formula for the solution of the initial value problem

$$u_t = \Delta u\,, \qquad u(x,0) = f(x)\ f \in \mathcal{S}(\mathbb{R}^n)\,. \tag{3.3}$$

Let $K_t(x) = K(x,t)$ and let $*$ indicate convolution in the space variable only, then

$$u(x,t) = K_t * f(x) \tag{3.4}$$

defines for $t > 0$ a solution to the heat equation and by the approximation lemma (see question 2 sheet 3) $\lim_{t\to 0+} u(x,t) = f(x)$. Once this formula has been derived for $f \in \mathcal{S}(\mathbb{R}^n)$ using the fourier transform it is straightforward to verify directly that it defines a solution for a much larger class of initial data, e.g. $f \in L^\infty(\mathbb{R}^n)$.

Now the *Duhamel principle* gives the formula for the inhomogeneous equation

$$u_t = \Delta u + F\,, \qquad u(x,0) = 0 \tag{3.5}$$

as $u(x,t) = \int_0^t U(x,t,s)ds$ where $U(x,t,s)$ is obtained by solving the family of homogeneous initial value problems:

$$U_t = \Delta U\,, \qquad U(x,s,s) = F(x,s)\,. \tag{3.6}$$

This gives the formula

$$u(x,t) = \int_0^t K_{t-s} * F(\cdot,s)\,ds = \int_0^t K_{t-s}(x-y)F(y,s)\,ds = K \circledast F(x,t)\,,$$

for the solution of (3.5), where $\circledast$ means space time convolution.

3.3 Parabolic equations and semigroups

Lemma 3.3.1 (Semigoup property) *The solution operator for the heat equation given by* (3.1) *(respectively* (3.4)*):*

$$S(t) : u_0 \mapsto u(\cdot,t)$$

defines a strongly continuous one parameter semigroup *(of contractions) on the Hilbert space* $L^2([0,1])$ *(respectively* $L^2(\mathbb{R}^n)$*).*

Noting the following properties of the heat kernel:

- $K_t(x) > 0$ for all $t > 0, x \in \mathbb{R}^n$,
- $\int_{\mathbb{R}^n} K_t(x)dx = 1$ for all $t > 0$,
- $K_t(x)$ is smooth for $t > 0, x \in \mathbb{R}^n$, and for t fixed $K_t(\cdot) \in \mathcal{S}(\mathbb{R}^n)$,

the following result concerning the solution $u(\cdot,t) = S(t)u_0 = K_t * u_0$ follows from basic properties of integration (see appendix):

- for $u_0 \in L^1(\mathbb{R}^n)$ the function $u(x,t)$ is smooth for $t > 0, x \in \mathbb{R}^n$ and satisfies $u_t - \Delta u = 0$,
- $\|u(\cdot,t)\|_{L^p} \le \|u_0\|_{L^p}$ and $\lim_{t\to 0+} \|u(\cdot,t) - u_0\|_{L^p} = 0$ for $1 \le p < \infty$,
- if $a \le u_0 \le b$ then $a \le u(x,t) \le b$ for $t > 0, x \in \mathbb{R}^n$.

From these and the approximation lemma (see question 2 sheet 3) we can read off the theorem:

Theorem 3.3.2 *(i) The formula $u(\cdot, t) = S(t)u_0 = K_t * u_0$ defines for $u_0 \in L^1$ a smooth solution of the heat equation for $t > 0$ which takes on the initial data in the sense that $\lim_{t\to 0+} \|u(\cdot, t) - u_0\|_{L^1} = 0$.*

(ii) The family $\{S(t)\}_{t\geq 0}$ also defines a strongly continuous semigroup of contractions on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

(iii) If in addition u_0 is continuous then $u(x, t) \to u_0(x)$ as $t \to 0+$ and the convergence is uniform if u_0 is uniformly continuous.

The final property of the kernel above implies a maximum principle for the heat equation, as is now discussed in generality.

3.4 The maximum principle

Maximum principles for parabolic equations are similar to elliptic once the correct notion of boundary is understood. If $\Omega \subset \mathbb{R}^n$ is an open bounded subset with smooth boundary $\partial\Omega$ and for $T > 0$ we define $\Omega_T = \Omega \times (0, T]$ then the parabolic boundary of the space-time domain Ω_T is (by definition)

$$\partial_{par}\Omega_T = \overline{\Omega_T} - \Omega_T = \Omega \times \{t = 0\} \cup \partial\Omega \times [0, T] \,.$$

We consider variable coefficient parabolic operators of the form

$$Lu = \partial_t u + Pu$$

where

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u + cu \tag{3.7}$$

is an elliptic operator with continuous coefficients and throughout this section $a_{jk} = a_{kj}, b_j, c$ are continuous and

$$m\|\xi\|^2 \leq \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \leq M\|\xi\|^2 \tag{3.8}$$

for some positive constants m, M and all x, t and ξ.

Theorem 3.4.1 *Let $u \in C(\overline{\Omega_T})$ have derivatives up to second order in x and first order in t which are continuous in Ω_T, and assume $Lu = 0$. Then*

- *if $c = 0$ (everywhere) then $\max\limits_{\overline{\Omega_T}} u(x, t) = \max\limits_{\partial_{par}\Omega_T} u(x, t)$, and*
- *if $c \geq 0$ (everywhere) then $\max\limits_{\overline{\Omega_T}} u(x, t) \leq \max\limits_{\partial_{par}\Omega_T} u^+(x, t)$, and*
  $\max\limits_{\overline{\Omega_T}} |u(x, t)| = \max\limits_{\partial_{par}\Omega_T} |u(x, t)| \,.$

where $u^+ = \max\{u, 0\}$ is the positive part of the function u.

Proof We prove the first case (when $c = 0$ everywhere). To prove the maximum principle bound, consider $u^\epsilon(x,t) = u(x,t) - \epsilon t$ which verifies, for $\epsilon > 0$, the strict inequality $Lu^\epsilon < 0$. First prove the result for u^ϵ:

$$\max_{\overline{\Omega_T}} u^\epsilon(x,t) = \max_{\partial_{par}\Omega_T} u^\epsilon(x,t)$$

Since $\partial_{par}\Omega_T \subset \overline{\Omega_T}$ the left side is automatically $\geq$ the right side. If the left side were strictly greater there would be a point (x_*, t_*) with $0 < x_* < 1$ and $0 < t_* \leq T$ at which a positive value is attained:

$$u^\epsilon(x_*, t_*) = \max_{(x,t)\in\overline{\Omega_T}} u^\epsilon(x,t) > 0\,.$$

By calculus first and second order conditions: $\partial_j u^\epsilon = 0$, $u^\epsilon_t \geq 0$ and $\partial^2_{ij} u^\epsilon_x \leq 0$ (as a symmetric matrix - i.e. all eigenvalues are ≤ 0). These contradict $Lu^\epsilon < 0$ at the point (x_*, t_*). Therefore

$$\max_{\overline{\Omega_T}} u^\epsilon(x,t) = \max_{\partial_{par}\Omega_T} u^\epsilon(x,t)\,.$$

Now let $\epsilon \downarrow 0$ and the result follows.

3.5 Regularity for parabolic equations

Consider the Cauchy problem for the parabolic equation $Lu = \partial_t u + Pu = f$, where

$$Pu = -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n} b_j \partial_j u + cu \tag{3.9}$$

with initial data u_0. For simplicity assume that the coefficients are all smooth functions of $x, t \in \overline{\Omega_\infty}$. The weak formulation of $Lu = f$ is obtained by multiplying by a test function $v = v(x)$ and integrating by parts, leading to (where $(\,\cdot\,)$ means the L^2 inner product defined by integration over $x \in \Omega$):

$$(u_t\,,\,v\,) + B(u,v) = (f,v)\,, \tag{3.10}$$

$$B(u,v) = \int \big(\sum_{jk} a_{jk}\,\partial_j u \partial_k v + \sum b_j \partial_j u v\ + cuv\big)\,dx\,.$$

To give a completely precise formulation it is necessary to define in which sense the time derivative u_t exists. To do this in a natural and general way requires the introduction of Sobolev spaces H^s for negative s - see §5.9 and §7.1.1-§7.1.2 in the book of Evans. However stronger assumptions on the initial data and inhomogeneous term are made a simpler statement is possible:

Theorem 3.5.1 *For $u_0 \in H^1_0(\Omega)$ and $f \in L^2(\Omega_T)$ there exists*

$$u \in L^2([0,T]; H^2(\Omega) \cap L^\infty([0,T]; H^1_0(\Omega))$$

with time derivative $u_t \in L^2(\Omega_T)$ *which satisfies* (3.10) *for all* $v \in H^1_0(\Omega)$ *and almost every* $t \in [0,T]$ *and* $\lim_{t\to 0+} \|u(t) - u_0\|_{L^2} = 0$*. Furthermore it is unique and has the* parabolic regularity *property:*

$$\int_0^T \left(\|u(t)\|^2_{H^2(\Omega)} + \|u_t\|_{L^2(\Omega)}\right) dt + \text{ess} \sup_{0\le t\le T} \|u(t)\|^2_{H^1_0(\Omega)} \le C\left(\|f\|_{L^2(\Omega_T)} + \|u_0\|_{H^1_0(\Omega)}\right). \tag{3.11}$$

(The time derivative is here to be understood in a weak/distributional sense as discussed in the sections of Evans' book just referenced, and the proof of the regularity (3.11) is in §7.1.3 of the same book.)

4 Hyperbolic equations

A second order equation of the form

$$u_{tt} + \sum_j \alpha_j \partial_t \partial_j u + Pu = 0$$

with P as in (3.7) (with coefficients potentially depending upon t and x), is strictly hyperbolic if the principal symbol

$$\sigma(\tau, \xi; t, x) = -\tau^2 - (\alpha \cdot \xi)\tau + \sum_{jk} a_{jk}\xi_j\xi_k$$

considered as a polynomial in τ has two distinct real roots $\tau = \tau_\pm(\xi; t, x)$ for all nonzero ξ. We will mostly study the wave equation

$$u_{tt} - \Delta u = 0\,,$$

starting with some representations of the solution for the wave equation. In this section we write $u = u(t,x)$, rather than $u(x,t)$, for functions of space and time to fit in with the most common convention for the wave equation.

4.1 The one dimensional wave equation: general solution

The general C^2 solution of $u_{tt} - u_{xx} = 0$ is

$$u(t,x) = F(x-t) + G(x+t)$$

for arbitrary C^2 functions F, G. From this can be derived the solution at time $t > 0$ of the inhomogeneous initial value problem:

$$u_{tt} - u_{xx} = f \tag{4.12}$$

with initial data

$$u(0,x) = u_0(x)\,, \qquad u_t(0,x) = u_1(x)\,. \tag{4.13}$$

$$u(t,x) = \frac{1}{2}\big(u_0(x-t)+u_0(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,dy + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\,dyds\,. \tag{4.14}$$

Notice that there is again a "Duhamel principle" for the effect of the inhomogeneous term since

$$\frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\,dyds = \int_0^t U(t,s,x)ds$$

where $U(t,s,x)$ is the solution of the *homogeneous* problem with data $U(s,s,x) = 0$ and $\partial_t U(s,s,x) = f(s,x)$ specified at $t = s$.

Theorem 4.1.1 *Assuming that* $(u_0, u_1) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$ *and that* $f \in C^1(\mathbb{R}\times\mathbb{R})$ *the formula* (4.13) *defines a* $C^2(\mathbb{R}\times\mathbb{R})$ *solution of the wave equation.*

4.2 The one dimensional wave equation on an interval

Next consider the problem $x \in [0,1]$ with Dirichlet boundary conditions $u(t,0) = 0 = u(t,1)$. Introduce the Sturm-Liouville operator $Pf = -f''$, with these boundary conditions as in §3.1, its eigenfunctions being $\phi_m = \sqrt{2}\sin m\pi x$ with eigenvalues $\lambda_m = (m\pi)^2$. In terms of P the wave equation is:

$$u_{tt} + Pu = 0$$

and the solution with initial data

$$u(0,x) = u_0(x) = \sum \widehat{u_0}(m)\phi_m\,, \qquad u_t(0,x) = u_1(x) = \sum \widehat{u_1}(m)\phi_m\,,$$

is given by

$$u(t,x) = \sum_{m=1}^{\infty} \cos(t\sqrt{\lambda_m})\widehat{u_0}(m)\phi_m + \frac{\sin(t\sqrt{\lambda_m})}{\sqrt{\lambda_m}}\widehat{u_1}(m)\phi_m$$

with an analogous formula for u_t. Recall the definition of the Hilbert space $H_0^1((0,1))$ as the closure of the functions in $C_0^\infty((0,1))$[1] with respect to the norm given by $\|f\|_{H^1}^2 = \int_0^1 f^2 + f'^2\,dx$. In terms of the basis ϕ_m the definition is:

$$H_0^1((0,1)) = \{f = \sum \hat{f}_m\phi_m : \|f\|_{H^1}^2 = \sum_{m=1}^{\infty}(1+m^2\pi^2)|\hat{f}_m|^2 < \infty\}\,.$$

(In all these expressions $\sum$ means $\sum_{m=1}^{\infty}$.) As equivalent norm we can take $\sum \lambda_m|\hat{f}_m|^2$. An appropriate Hilbert space for the wave equation with these boundary conditions is to solve for $(u,u_t) \in X$ where $X = H_0^1 \oplus L^2$, and precisely we will take the following:

$$X = \{(f,g) = (\sum \hat{f}_m\phi_m, \sum \hat{g}_m\phi_m) : \|(f,g)\|_X^2 = \sum(\lambda_m|\hat{f}_m|^2 + |\hat{g}_m|^2) < \infty\}\,.$$

[1] i.e. smooth functions which are zero outside of a closed set $[a,b] \subset (0,1)$

Now the effect of the evolution on the coefficients $\widehat{u}(m,t)$ and $\widehat{u_t}(m,t)$ is the map

$$\begin{pmatrix}\widehat{u}(m,t)\\ \widehat{u_t}(m,t)\end{pmatrix} \mapsto \begin{pmatrix}\cos(t\sqrt{\lambda_m}) & \frac{\sin(t\sqrt{\lambda_m})}{\sqrt{\lambda_m}}\\ -\sqrt{\lambda_m}\sin(t\sqrt{\lambda_m}) & \cos(t\sqrt{\lambda_m})\end{pmatrix}\begin{pmatrix}\widehat{u}(m,0)\\ \widehat{u_t}(m,0)\end{pmatrix} \tag{4.15}$$

Lemma 4.2.1 *The solution operator for the wave equation*

$$S(t): \begin{pmatrix}u_0\\ u_1\end{pmatrix} \mapsto \begin{pmatrix}u(t,\cdot)\\ u_t(t,\cdot)\end{pmatrix}$$

defined by (4.15) *defines a strongly continuous* group *of unitary operators on the Hilbert space X, as in definition 5.3.1.*

4.3 The wave equation on $\mathbb{R}^n$

To solve the wave equation on $\mathbb{R}^n$ take the Fourier transform in the space variables to show that the solution is given by

$$u(t,x) = (2\pi)^{-n}\int \exp^{i\xi\cdot x}(\cos(t|\xi|)\widehat{u_0}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u_1}(\xi))d\xi$$

for initial values $u(0,x) = u_0(x), u_t(0,x) = u_1(x)$ in $\mathcal{S}(\mathbb{R}^n)$. The Kirchhoff formula arises from some further manipulations with the fourier transform in the case $n = 3$ and $u_0 = 0$ and gives the following formula

$$u(t,x) = \frac{1}{4\pi t}\int\limits_{y:\|y-x\|=t} u_1(y)\,d\Sigma(y) \tag{4.16}$$

for the solution at time $t > 0$ of $u_{tt} - \Delta u = 0$ with initial data $(u, u_t) = (0, u_1)$. The solution for the inhomogeneous initial value problem with general Schwartz initial data u_0, u_1 can then be derived from the Duhamel principle, which takes the same form as in one space dimension.

4.4 The energy identity and finite propagation speed

Lemma 4.4.1 (Energy identity) *If u is a C^2 solution of the wave equation show that*

$$\partial_t\big(\frac{u_t^2 + |\nabla u|^2}{2}\big) + \partial_i\big(-u_t\partial_i u\big) = 0$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

From this and the divergence theorem it follows that

Lemma 4.4.2 (Finite speed of propagation) *If $u \in C^2$ solves the wave equation and $u(0,x)$ and $u_t(0,x)$ both vanish for $|x| < R$ then $u(t,x)$ vanishes for $|x| < R - |t|$ if $|t| < R$.*

5 One-parameter semigroups and groups

If A is a bounded linear operator on a Banach space its norm is

$$\|A\| = \sup_{u\in X, u\neq 0} \frac{\|Au\|}{\|u\|}\,, \qquad \text{(operator or uniform norm).}$$

5.1 Definitions

Definition 5.1.1 *A one-parameter family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on a Banach space X forms a semigroup if*

1. *$S(0) = I$ (the identity operator) , and*
2. *$S(t+s) = S(t)S(s)$ for all $t, s \geq 0$ (semi-group property).*
3. *It is called a uniformly continuous semigroup if in addition to (1) and (2):*

$$\lim_{t\to 0+} \|S(t) - I\| = 0\,, \qquad \textit{(uniform continuity).}$$

4. *It is called a strongly continuous (or C_0) semigroup if in addition to (1) and (2):*

$$\lim_{t\to 0+} \|S(t)u - u\| = 0\,, \forall u \in X \qquad \textit{(strong pointwise continuity).}$$

5. *If $\|S(t)\| \leq 1$ for all $t \geq 0$ the semigroup $\{S(t)\}_{t\geq 0}$ is called a semigroup of contractions.*

Notice that in 3 the symbol $\|\cdot\|$ means the operator norm, while in 4 the same symbol means the norm on vectors in X. Also observe that uniform continuity is a stronger condition than strong continuity.

5.2 Semigroups and their generators

For ordinary differential equations $\dot{x} = Ax$, where A is an $n\times n$ matrix, the solution can be written $x(t) = e^{tA}x(0)$ and there is a $1-1$ corespondence between the matrix A and the semigroup $S(t) = e^{tA}$ on $\mathbb{R}^n$. In this subsection[2] we discuss how this generalizes.

Uniformly continuous semigroups have a simple structure which generalizes the finite dimensional case in an obvious way - they arise as solution operators for differential equations in the Banach space X:

$$\frac{du}{dt} + Au = 0\,, \qquad \text{for } u(0) \in X \text{ given}\,. \tag{5.17}$$

Theorem 5.2.1 *$\{S(t)\}_{t\geq 0}$ is a uniformly continuous semgroup on X if and only if there exists a unique bounded linear operator $A : X \to X$ such that $S(t) = e^{-tA} = \sum_{j=0}^{\infty}(-tA)^j/j!$. This semigroup gives the solution to* (5.17) *in the form $u(t) = S(t)u(0)$, which is continuously differentiable into X. The operator A is called the infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$.*

[2] This subsection is for background information only

This applies to ordinary differential equations when A is a matrix. It is not very useful for partial differential equations because partial differential operators are unbounded, whereas in the foregoing theorem the infinitesimal generator was necessarily bounded. For example for the heat equation we need to take $A = -\Delta$, the laplacian defined on some appropriate Banach space of functions. Thus it is necessary to consider more general semigroups, in particular the strongly continuous semigroups. An unbounded linear operator A is a linear map from a linear subspace $D(A) \subset X$ into X (or more generally into another Banach space Y). The subspace $D(A)$ is called the domain of A. An unbounded linear operator $A : D(A) \to Y$ is said to be

- *densely defined* if $\overline{D(A)} = X$, where the overline means closure in the norm of X, and
- *closed* if the graph $\Gamma_A = \{(u, Au)|_{u \in D(A)}\} \subset X \times Y$ is closed in $X \times Y$.

A class of unbounded linear operators suitable for understanding strongly continuous semigroups is the class of *maximal monotone* operators in a Hilbert space:

Definition 5.2.2 *1. A linear operator $A : D(A) \to X$ on a Hilbert space X is monotone if $(u, Au) \geq 0$ for all $u \in D(A)$.*

2. A monotone operator $A : D(A) \to X$ is maximal monotone if, in addition, the range of $I + A$ is all of X, i.e. if:

$$\forall f \in X \, \exists u \in D(A) : (I + A)u = f \, .$$

Maximal monotone operators are automatically densely defined and closed, and there is the following generalization of theorem 5.2.1:

Theorem 5.2.3 (Hille-Yosida) *If $A : D(A) \to X$ is maximal monotone then the equation*

$$\frac{du}{dt} + Au = 0 \, , \qquad \text{for } u(0) \in D(A) \subset X \text{ given} \, , \tag{5.18}$$

admits a unique solution $u \in C([0, \infty); D(A)) \cap C^1([0, \infty); X)$ with the property that $\|u(t)\| \leq \|u(0)\|$ for all $t \geq 0$ and $u(0) \in D(A)$. Since $D(A) \subset X$ is dense the map $D(A) \ni u(0) \to u(t) \in X$ extends to a linear map $S_A(t) : X \to X$ and by uniqueness this determines a strongly continuous semigroup of contractions $\{S_A(t)\}_{t \geq 0}$ on the Hilbert space X. Often $S_A(t)$ is written as $S_A(t) = e^{-tA}$.

Conversely, given a strongly continuous semgroup $\{S(t)\}_{t \geq 0}$ of contractions on X, there exists a unique maximal monotone operator $A : D(A) \to X$ such that $S_A(t) = S(t)$ for all $t \geq 0$. The operator A is the infinitesimal generator of $\{S(t)\}_{t \geq 0}$ in the sense that $\frac{d}{dt} S(t)u = Au$ for $u \in D(A)$ and $t \geq 0$ (interpreting the derivative as a right derivative at $t = 0$).

5.3 Unitary groups and their generators

Semigroups arise in equations which are not necessarily time reversible. For equations which are, e.g. the Schrödinger and wave equations, each time evolution operator has an inverse and the semigroup is in fact a group. In this subsection[3] we give the definitions and state the main result.

[3] In this subsection you only need to know definition 5.3.1. The remainder is for background information.

Definition 5.3.1 *A one-parameter family of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ on a Hilbert space X forms a group of unitary operators if*

1. *$U(0) = I$ (the identity operator) , and*
2. *$U(t+s) = U(t)U(s)$ for all $t, s \in \mathbb{R}$ (group property).*
3. *It is called a strongly continuous (or C_0) group of unitary operators if in addition to (1) and (2):*

$$\lim_{t\to 0} \|U(t)u - u\| = 0\,, \forall u \in X \qquad \textit{(strong pointwise continuity).}$$

A maximal monotone operator A which is symmetric (=hermitian), i.e. such that

$$(Au, v) = (u, Av) \qquad \text{for all } u, v \text{ in } D(A) \subset X \tag{5.19}$$

generates a one-parameter *group* of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$, often written $U(t) = e^{-itA}$, by solving the equation

$$\frac{du}{dt} + iAu = 0\,, \qquad \text{for } u(0) \in D(A) \subset X \text{ given}\,. \tag{5.20}$$

It is useful to introduce the adjoint operator $A*$ via the Riesz representation theorem: first of all let

$$D(A^*) = \{u \in X : \text{ the map } v \mapsto (u, Av) \text{ extends to a bounded linear functional } X \to \mathbb{C}\}$$

so that $D(A^*)$ is a linear space, and for $u \in D(A^*)$ there exists a vector w_u such that $(w_u, v) = (u, Av)$ (by Riesz representation). The map $u \to w_u$ is linear on $D(A^*)$ and so we can define an unbounded linear operator $A^* : D(A^*) \to X$ by $A^*u = w_u$, and since we started with a symmetric operator it is clear that $D(A) \subset D(A^*)$ and $A^*u = Au$ for $u \in D(A)$; the operator A^* is thus an extension of A.

Definition 5.3.2 *If $A : D(A) \to X$ is an unbounded linear operator which is symmetric and if $D(A^*) = D(A)$ then A is said to be self-adjoint and we write $A = A^*$.*

Theorem 5.3.3 *Maximal monotone symmetric operators are self-adjoint.*

Theorem 5.3.4 (Stone theorem) *If A is a self-adjoint operator the equation* (5.20) *has a unique solution for $u(0) \in D(A)$ which may be written $u(t) = U_A(t)u(0)$ with $\|u(t)\| = \|u(0\|$ for all $t \in \mathbb{R}$. It follows that the $U_A(t)$ extend uniquely to define unitary operators $X \to X$ and that $\{U_A(t)\}_{t\in\mathbb{R}}$ constitutes a strongly continuous group of unitary operators which are written $U_A(t) = e^{-itA}$.*

Conversely, given a strongly continuous group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ there exists a self-adjoint operator A such that $U(t) = U_A(t) = e^{-itA}$ for all $t \in \mathbb{R}$.

6 Appendix: integration

The aim of this appendix[4] is to give a brief review of facts from integration needed - completeness of the L^p spaces, dominated convergence and other basic theorems. We first consider the case of functions on the unit interval $[0,1]$. A main achievement of the Lebesgue integral is to construct *complete* vector spaces of functions where the completeness is with respect to a norm defined by an integral such as the L^2 norm $\|\cdot\|_{L^2}$ defined by

$$\|f\|_{L^2}^2 = \int_0^1 |f(x)|^2\,dx\,.$$

This is a perfectly good norm on the space of continuous functions $C([0,1])$, but the resulting normed vector space is not complete (and so not a Banach space) and is not so useful as a setting for analysis. The Lebesgue framework provides a larger class of functions which can be potentially integrated - the *measurable functions*. The complete Lebesgue space L^2 which this construction leads to then consists of (equivalence classes of) measurable functions f with $\|f\|_{L^2}^2 < \infty$; here it is necessary to consider equivalence classes of functions because functions which are non-zero only on sets which are very small (in a certain precise sense) are invisible to the integral, and so have to be factored out of the discussion. The “very small” sets in question are called null sets and are now defined.

6.1 Null sets and measurable functions on $[0,1]$

An interval in $[0,1]$ is a subset of the form (a,b) or $[a,b]$ or $(a,b]$ or $[a,b)$ (respectively open,closed, half open). In all cases the length of the interval is $|I| = b-a$. A collection of intervals $\{I_\alpha\}$ *covers* a subset A if $A \subset \cup_\alpha I_\alpha$.

Definition 6.1.1 (Null sets) *For a set $A \subset [0,1]$ we define the* outer measure *to be*

$$|A|_* = \inf_{\{I_n\}_{n=1}^\infty \in \mathcal{C}} \Big\{ \sum_n |I_n| \;:\; A \subset \cup I_n \Big\}\,,$$

where $\mathcal{C}$ consists of countable families of intervals in $[0,1]$. A set $N \subset [0,1]$ is null if $|I|_* = 0$, i.e. if for all $\epsilon > 0$ there exists $\{I_n\}_{n=1}^\infty \in \mathcal{C}$ which covers A with $\sum |I_n| < \epsilon$.

Definition 6.1.2 *We say $f = g$ almost everywhere (a.e.) if $f(x) = g(x)$ for all $x \notin N$ for some null set N. We say a sequence of functions f_n converges to f a.e. if $f_n(x) \to f(x)$ for all $x \notin N$ for some null set N.*

Equality a.e. defines an equivalence relation, and two equivalent functions f, g are said to be Lebesgue or measure theoretically equivalent. One way to think about measurable functions is provided by the Lusin theorem, which says a measurable function is one which is “almost continuous” in the sense that it agrees with a continuous function on the complement of a set of arbitrarily small outer measure:

[4]This section gives a brief introduction to the results on Lebesgue integral which we make use of. You should be able to use the results listed here but will not be examined on the proofs or on any subtleties connected with the results.

Definition 6.1.3 (Measurable functions) *A function $f : [0,1] \to \mathbb{R}$ is measurable if for every $\epsilon > 0$ there exists a continuous function $f^\epsilon : [0,1] \to \mathbb{R}$ and a set F^ϵ such that $|F^\epsilon|_* < \epsilon$ and $f(x) = f^\epsilon(x)$ for all $x \notin F^\epsilon$. We write $L([0,1])$ for the space of all measurable functions so defined.*

Theorem 6.1.4 *$L([0,1])$ is a linear space closed under almost everywhere convergence: given a sequence $f_n \in L([0,1])$ of measurable functions which converges to a function f a.e. it follows that $f \in L([0,1])$.*

Definition 6.1.3 is not the usual definition of measurability - which involves the notion of a distinguished collection of sets, the σ-algebra of measurable sets - but is equivalent to it by what is called the *Lusin theorem* (see for example §2.4 and §7.2 in the book *Real Analysis* by Folland). The Lusin theorem gives a helpful way of thinking about measurability (the Littlewood 3 principles - see §3.3 in the book *Real Analysis* by Royden and Fitzpatrick). A companion to the Lusin theorem is the *Egoroff theorem* which states that given a sequence $f_n \in L([0,1])$ of measurable functions which converges to a function f a.e. then for every $\epsilon > 0$ it is possible to find a set $E \subset [0,1]$ with $|E|_* < \epsilon$ such that $f_n \to f$ uniformly on $E^c = [0,1] - E$. Thus two of Littlewood's principles say that " a measurable function is one which agrees with a continuous function except on a set which may be taken to have arbitrarily small size" and "a sequence of measurable functions which converges almost everywhere converges uniformly on the complement of a set which may be assumed to be arbitrarily small".

6.2 Definition of $L^p([0,1])$

Definition 6.2.1 *For $1 \le p < \infty$ define $L^p([0,1])$ to be the linear space of measurable functions on $[0,1]$ with the property that*

$$\|f\|_{L^p}^p = \int_0^1 |f(x)|^p \, dx < \infty \, .$$

For the case $p = \infty$: firstly, say that f is essentially bounded above with upper (essential) bound M if $f(x) \le M$ for $x \notin N$ for some null set N. Then let ess $\sup f$ be the infimum of all upper essential bounds. Then:

Definition 6.2.2 *$L^\infty([0,1])$ is the linear space of measurable functions on $[0,1]$ with the property that*

$$\|f\|_{L^\infty} = \text{ess} \sup |f| < \infty \, .$$

The crucial fact is that considering the spaces of equivalence classes of functions which agree almost everywhere we obtain Banach spaces , also written $L^p([0,1])$: these "Lebesgue spaces" are vector spaces of (equivalence classes of) functions which are complete with respect to the norm $\| \cdot \|_{L^p}$. (The fact that strictly speaking these the elements of these spaces are equivalence classes of functions which agree almost everywhere is often taken as understood and not repeatedly mentioned each time the spaces are made use of.)

The spaces $L^p([0,1])$ which arise in this way are special cases of $L^p(\mathcal{M})$ spaces which arise from abstract measure spaces $\mathcal{M}$ on which a measure μ (and a σ-algebra of measurable sets) is given. Other examples used in this course are

- $L^p([a,b])$ with norm $(\int_a^b |f(x)|^p \, dx)^{\frac{1}{p}}$, and
- $L^p(\mathbb{R}^n)$ with norm $(\int_{\mathbb{R}^n} |f(x)|^p \, dx)^{\frac{1}{p}}$.

6.3 Assorted theorems on integration

Theorem 6.3.1 (Holder inequality) *$\int fg dx \leq \|f\|_{L^p}\|g\|_{L^q}$ for any pair of functions $f \in L^p, g \in L^q$ (on any measure space) with $p^{-1} + q^{-1} = 1$ and $p, q \in [1, \infty]$.*

Corollary 6.3.2 (Young inequality) *If $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ then $f*g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^1}$ for $1 \leq p \leq \infty$.*

Theorem 6.3.3 (Dominated convergence theorem) *Let the sequence $f_n \in L^1$ converge to f almost everywhere (on any measure space) and assume that there exists a nonnegative measurable function $\Phi \geq 0$ such that $|f_n(x)| \leq \Phi(x)$ almost everywhere and $\int \Phi < \infty$. Then $\lim_{n\to\infty} \int f_n = \int f$ and $\lim_{n\to\infty} \|f_n - f\|_{L^1} = 0$.*

Corollary 6.3.4 (Differentiation through the integral) *Let $g \in C^1(\mathbb{R}^n \times \Omega)$ where $\Omega \subset \mathbb{R}^m$ is open, and consider $F(\lambda) = \int_{\mathbb{R}^n} g(x, \lambda)dx$. Assume there exists a measurable function $\Phi(x) \geq 0$ such that*

- $\int_{\mathbb{R}^n} \Phi(x)\, dx < \infty$,
- $\sup_\lambda(|g(x, \lambda) + |\partial_\lambda g(x, \lambda)|) \leq \Phi(x)$.

Then $F \in C^1(\Omega)$ and $\partial_\lambda F = \int_{\mathbb{R}^n} \partial_\lambda g(x, \lambda)\, dx$.

Corollary 6.3.5 *If f is a $C^k(\mathbb{R}^n)$ function with all partial derivatives $\partial^\alpha f$ of order $|\alpha| \leq k$ bounded, and $g \in L^1(\mathbb{R}^n)$ then $f * g \in C^k(\mathbb{R}^n)$ and $\partial^\alpha(f * g) = (\partial^\alpha f) * g$ for $|\alpha| \leq k$.*

Theorem 6.3.6 (Tonelli) *If $f \geq 0$ is a nonnegative measurable function $f : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ then*

$$\iint\limits_{\mathbb{R}^l\times\mathbb{R}^m} f(x, y)\, dxdy = \int\limits_{\mathbb{R}^l} \left(\int\limits_{\mathbb{R}^m} f(x, y)\, dy \right) dx = \int\limits_{\mathbb{R}^m} \left(\int\limits_{\mathbb{R}^l} f(x, y)\, dx \right) dy \,.$$

Theorem 6.3.7 (Fubini) *If f is a measurable function $f : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ such that*

$$\iint\limits_{\mathbb{R}^l\times\mathbb{R}^m} |f(x, y)|\, dxdy \ < \infty$$

then

$$\iint\limits_{\mathbb{R}^l\times\mathbb{R}^m} f(x, y)\, dxdy = \int\limits_{\mathbb{R}^l} \left(\int\limits_{\mathbb{R}^m} f(x, y)\, dy \right) dx = \int\limits_{\mathbb{R}^m} \left(\int\limits_{\mathbb{R}^l} f(x, y)\, dx \right) dy \,.$$

Remark 6.3.8 *In these two results it is to be understood that when we write down repeated integrals that an implicit assertion is that the functions $y \mapsto \int f(x, y)dx$ and $x \mapsto \int f(x, y)dy$ are measurable and integrable.*

Theorem 6.3.9 (Minkowski inequality) *If f is a measurable function $f : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ is measurable, then*

$$\| \int_{\mathbb{R}^m} f(x,y)g(y)\,dy \|_{L^p(dx)} \leq \int_{\mathbb{R}^m} \|f(x,y)\|_{L^p(dx)} |g(y)|\,dy\,. \tag{6.21}$$

where

$$\|f(x,y)\|^p_{L^p(dx)} = \int_{\mathbb{R}^l} |f(x,y)|^p\,dx\,,$$

with the understanding as above that this means that if the right hand side of (6.21) *is finite then the function $f(x,y)g(y)$ is integrable in y for almost every x and the resulting function $x \mapsto \int f(x,y)g(y)\,dy$ is measurable and* (6.21) *holds.*

7 Example sheet 4

1. (a) Use the change of variables $v(t,x) = e^t u(t,x)$ to obtain an "x-space" formula for the solution to the initial value problem:

   $$u_t + u = \Delta u \qquad u(0,\,\cdot\,) = u_0(\cdot) \in \mathcal{S}(\mathbb{R}^n).$$

   Hence show that $|u(t,x)| \le \sup_x |u_0(x)|$ and use this to deduce well-posedness in the supremum norm (for $t > 0$ and all x).

   If $a \le u_0(x) \le b$ for all x what can you say about the possible values of $u(t,x)$ for $t > 0$.

   (b) Use the Fourier transform in x to obtain a (Fourier space) formula for the solution of:

   $$u_{tt} - 2u_t + u = \Delta u \qquad u(0,\,\cdot\,) = u_0(\cdot) \in \mathcal{S}(\mathbb{R}^n),\ u_t(0,\cdot) = u_1(\cdot) \in \mathcal{S}(\mathbb{R}^n).$$

2. Show that if $u \in C([0,\infty) \times \mathbb{R}^n) \cap C^2((0,\infty) \times \mathbb{R}^n)$ satisfies (i) the heat equation, (ii) $u(0,x) = 0$ and (iii) $|u(t,x)| \le M$ and $|\nabla u(t,x)| \le N$ for some M, N then $u \equiv 0$. (Hint: multiply heat equation by $K_{t_0 - t}(x - x_0)$ and integrate over $|x| < R, a < t < b$. Apply the divergence theorem, carefully let $R \to \infty$ and then $b \to t_0$ and $a \to 0$ to deduce $u(t_0, x_0) = 0$.)

3. Show that if $S(t)$ is a strongly continuous semigroup on a Banach space X with norm $\|\cdot\|$ then

   $$\lim_{t \to 0+} \|S(t_0 + t)u - S(t_0)u\| = 0\,, \qquad \forall u \in X \text{ and } \forall t_0 > 0\,.$$

4. Let $Pu = -(pu')' + qu$, with p and q smooth, be a Sturm-Liouville operator on the unit interval $[0,1]$ and assume there exist constants m, c_0 such that $p \ge m > 0$ and $q \ge c_0 > 0$ everywhere, and consider Dirichlet boundary conditions $u(0) = 0 = u(1)$. Assume $\{\phi_n\}_{n=1}^{\infty}$ are smooth functions which constitute an orthonormal basis for $L^2([0,1])$ of eigenfunctions: $P\phi_n = \lambda_n \phi_n$. Show that there exists a number $\gamma > 0$ such that $\lambda_n \ge \gamma$ for all n. Write down the solution to the equation $\partial_t u + Pu = 0$ with initial data $u_0 \in L^2([0,1])$ and show that it defines a strongly continuous semigroup of contractions on $L^2([0,1])$, and describe the large time behaviour.

5. (i) Let $\partial_t u_j + P u_j = 0\,,\ j = 1,2$ where P is as in (3.7) and the functions u_j have the regularity assumed in theorem 3.4.1 and satisfy Dirichlet boundary conditions: $u_j(x,t) = 0\, \forall x \in \partial\Omega, t \ge 0$. Assuming, in addition to (3.8), that

   $$c \ge c_0 > 0 \tag{7.22}$$

   for some positive constant c_0 prove that for all $0 \le t \le T$:

   $$\sup_{x \in \Omega} |u_1(x,t) - u_2(x,t)| \le e^{-tc_0} \sup_{x \in \Omega} |u_1(x,0) - u_2(x,0)|.$$

(ii) In the situation of part (i) with

$$Pu = -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n} b_j\partial_j u + cu\,, \tag{7.23}$$

assuming in addition to (3.8) and (7.22) also that a_{jk}, b_j are C^1 and that

$$\sum_{j=1}^{n} \partial_j b_j = 0\,, \quad \text{in } \overline{\Omega_T}\,,$$

prove that for all $0 \le t \le T$:

$$\int_\Omega |u_1(x,t) - u_2(x,t)|^2\,dx \le e^{-tc_0}\int_\Omega |u_1(x,0) - u_2(x,0)|^2\,dx.$$

6. (i) Let K_t be the heat kernel on $\mathbb{R}^n$ at time t and prove directly by integration that

$$K_t * K_s = K_{t+s}$$

for $t, s > 0$ (semi-group property). Use the Fourier transform and convolution theorem to give a second simpler proof.
(ii) Deduce that the solution operators $S(t) = K_t *$ define a strongly continuous semigroup of contractions on $L^p(\mathbb{R}^n)\,\forall p < \infty$.
(iii) Show that the solution operator $S(t) : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ for the heat initial value problem satisfies $\|S(t)\|_{L^1\to L^\infty} \le ct^{-\frac{n}{2}}$ for positive t, or more explicitly, that the solution $u(t) = S_t u(0)$ satisfies $\|u(t)\|_{L^\infty} \le ct^{-\frac{n}{2}}\|u(0)\|_{L^1}$, or:

$$\sup_x |u(x,t)| \le ct^{-n/2}\int |u(x,0)|dx$$

for some positive number c, which should be found.
(iii) Now let $n = 4$. Deduce, by considering $v = u_t$, that if the inhomogeneous term $F \in \mathcal{S}(\mathbb{R}^4)$ is a function of x only, the solution of $u_t - \Delta u = F$ with zero initial data converges to some limit as $t \to \infty$. Try to identify the limit.

7. (i) Let $u(t,x)$ be a twice continuously differentiable solution of the wave equation on $\mathbb{R}\times\mathbb{R}^n$ for $n = 3$ which is radial, i.e. a function of $r = \|x\|$ and t. By letting $w = ru$ deduce that u is of the form

$$u(t,x) = \frac{f(r-t)}{r} + \frac{g(r+t)}{r}.$$

(ii) Show that the solution with initial data $u(0,\cdot) = 0$ and $u_t(0,\cdot) = G$, where G is radial and even function, is given by

$$u(t,r) = \frac{1}{2r}\int_{r-t}^{r+t} \rho G(\rho)d\rho.$$

(iii) Hence show that for initial data $u(0,\cdot) \in C^3(\mathbb{R}^n)$ and $u_t(0,\cdot) \in C^2(\mathbb{R}^n)$ the solution $u = u(t,x)$ need only be in $C^2(\mathbb{R}\times\mathbb{R}^n)$. Contrast this with the case of one space dimension.

8. Write down the solution of the Schrodinger equation $u_t = iu_{xx}$ with 2π-periodic boundary conditions and initial data $u(x, 0) = u_0(x)$ smooth and 2π-periodic in x, and show that the solution determines a strongly continuous group of unitary operators on $L^2([-\pi, \pi])$. Do the same for Dirichlet boundary conditions i.e. $u(-\pi, t) = 0 = u(\pi, t)$ for all $t \in \mathbb{R}$.

9. (i) Write the one dimensional wave equation $u_{tt} - u_{xx} = 0$ as a first order in time evolution equation for $U = (u, u_t)$.

   (ii) Use Fourier series to write down the solution with initial data $u(0, \cdot) = u_0$ and $u_t(0, \cdot) = u_1$ which are smooth 2π-periodic and have zero mean: $\hat{u}_j(0) = 0$.

   (iii) Show that $\|u\|_{\dot{H}^1_{per}} = \sum_{m\neq 0} |m|^2 |\hat{u}(m)|^2$ defines a norm on the space of smooth 2π-periodic functions with zero mean. The corresponding complete Sobolev space is the case $s = 1$ of

$$\dot{H}^s_{per} = \{\sum_{m\neq 0} \hat{u}(m)e^{im\cdot x} : \|u\|_{\dot{H}^s_{per}} = \sum_{m\neq 0} |m|^{2s}|\hat{u}(m)|^2 < \infty\} ,$$

   the Hilbert space of zero mean 2π-periodic H^s functions.

   (iv) Show that the solution defines a group of unitary operators in the Hilbert space

$$X = \{U = (u, v) \ : \ u \in \dot{H}^1_{per} \text{ and } v \in L^2([-\pi, \pi])\} .$$

   (v) Explain the "unitary" part of your answer to (iv) in terms of the energy

$$E(t) = \int_{-\pi}^{\pi} (u_t^2 + u_x^2)\, dx .$$

   (vi) Show that $\|U(t)\|_{\dot{H}^{s+1}_{per}\oplus\dot{H}^s_{per}} = \|(u_0, u_1)\|_{\dot{H}^{s+1}_{per}\oplus\dot{H}^s_{per}}$ (preservation of regularity).

10. (a) Deduce from the finite speed of propagation result for the wave equation (lemma 4.4.2) that a classical solution of the initial value problem, $\Box u = 0$, $u(0, t) = f$, $u_t(0, x) = g$, with $f, g \in \mathcal{D}(\mathbb{R}^n)$ given is unique.

    (b) The Kirchhoff formula for solutions of the wave equation $n = 3$ for initial data $u(0, \cdot) = 0, u_t(0, \cdot) = g$ is derived using the Fourier transform when $g \in \mathcal{S}(\mathbb{R}^n)$. Show that the validity of the formula can be extended to any smooth function $g \in C^\infty(\mathbb{R}^n)$.(Hint: finite speed of propagation).