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Classical Dynamics

University of Cambridge Part II Mathematical Tripos

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Recommended Books and Resources

- L. Hand and J. Finch, *Analytical Mechanics*

This very readable book covers everything in the course at the right level. It is similar to Goldstein's book in its approach but with clearer explanations, albeit at the expense of less content.

There are also three classic texts on the subject

- H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*

In previous editions it was known simply as "Goldstein" and has been the canonical choice for generations of students. Although somewhat verbose, it is considered the standard reference on the subject. Goldstein died and the current, third, edition found two extra authors.

- L. Landau an E. Lifshitz, *Mechanics*

This is a gorgeous, concise and elegant summary of the course in 150 content packed pages. Landau is one of the most important physicists of the 20th century and this is the first volume in a series of ten, considered by him to be the "theoretical minimum" amount of knowledge required to embark on research in physics. In 30 years, only 43 people passed Landau's exam!

A little known fact: Landau originally co-authored this book with one of his students, Leonid Pyatigorsky. They subsequently had a falling out and the authorship was changed. There are rumours that Pyatigorsky got his own back by denouncing Landau to the Soviet authorities, resulting in his arrest.

- V. I. Arnold, *Mathematical Methods of Classical Mechanics*

Arnold presents a more modern mathematical approach to the topics of this course, making connections with the differential geometry of manifolds and forms. It kicks off with "The Universe is an Affine Space" and proceeds from there...

Contents

Acknowledgements

These notes rely heavily on the textbooks listed at the beginning and on notes from past courses given by others, in particular Anne Davis, Gary Gibbons, Robin Hudson, Michael Peskin and Neil Turok. My thanks also to Michael Efroimsky and Matt Headrick for useful comments. I am supported by the Royal Society.

1. Newton's Laws of Motion

"So few went to hear him, and fewer understood him, that oftimes he did, for want of hearers, read to the walls. He usually stayed about half an hour; when he had no auditors he commonly returned in a quarter of that time."

Appraisal of a Cambridge lecturer in classical mechanics, circa 1690

1.1 Introduction

The fundamental principles of classical mechanics were laid down by Galileo and Newton in the 16^{th} and 17^{th} centuries. In 1686, Newton wrote the *Principia* where he gave us three laws of motion, one law of gravity and pretended he didn't know calculus. Probably the single greatest scientific achievement in history, you might think this pretty much wraps it up for classical mechanics. And, in a sense, it does. Given a collection of particles, acted upon by a collection of forces, you have to draw a nice diagram, with the particles as points and the forces as arrows. The forces are then added up and Newton's famous "$F = ma$" is employed to figure out where the particle's velocities are heading next. All you need is enough patience and a big enough computer and you're done.

From a modern perspective this is a little unsatisfactory on several levels: it's messy and inelegant; it's hard to deal with problems that involve extended objects rather than point particles; it obscures certain features of dynamics so that concepts such as chaos theory took over 200 years to discover; and it's not at all clear what the relationship is between Newton's classical laws and quantum physics.

The purpose of this course is to resolve these issues by presenting new perspectives on Newton's ideas. We shall describe the advances that took place during the 150 years after Newton when the laws of motion were reformulated using more powerful techniques and ideas developed by some of the giants of mathematical physics: people such as Euler, Lagrange, Hamilton and Jacobi. This will give us an immediate practical advantage, allowing us to solve certain complicated problems with relative ease (the strange motion of spinning tops is a good example). But, perhaps more importantly, it will provide an elegant viewpoint from which we'll see the profound basic principles which underlie Newton's familiar laws of motion. We shall prise open "$F = ma$" to reveal the structures and symmetries that lie beneath.

Moreover, the formalisms that we'll develop here are the basis for *all* of fundamental modern physics. Every theory of Nature, from electromagnetism and general relativity, to the standard model of particle physics and more speculative pursuits such as string theory, is best described in the language we shall develop in this course. The new formalisms that we'll see here also provide the bridge between the classical world and the quantum world.

There are phenomena in Nature for which these formalisms are not particularly useful. Systems which are dissipative, for example, are not so well suited to these new techniques. But if you want to understand the dynamics of planets and stars and galaxies as they orbit and spin, or you want to understand what's happening at the LHC where protons are collided at unprecedented energies, or you want to know how electrons meld together in solids to form new states of matter, then the foundations that we'll lay in in this course are a must.

1.2 Newtonian Mechanics: A Single Particle

In the rest of this section, we'll take a flying tour through the basic ideas of classical mechanics handed down to us by Newton. More details can be found in the lectures on Dynamics and Relativity.

We'll start with a single particle.. A *particle* is defined to be an object of insignificant size. e.g. an electron, a tennis ball or a planet. Obviously the validity of this statement depends on the context: to first approximation, the earth can be treated as a particle when computing its orbit around the sun. But if you want to understand its spin, it must be treated as an extended object.

The motion of a particle of mass m at the position $\mathbf{r}$ is governed by *Newton's Second Law* $\mathbf{F} = m\mathbf{a}$ or, more precisely,

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}) = \dot{\mathbf{p}} \tag{1.1}$$

where $\mathbf{F}$ is the force which, in general, can depend on both the position $\mathbf{r}$ as well as the velocity $\dot{\mathbf{r}}$ (for example, friction forces depend on $\dot{\mathbf{r}}$) and $\mathbf{p} = m\dot{\mathbf{r}}$ is the momentum. Both $\mathbf{F}$ and $\mathbf{p}$ are 3-vectors which we denote by the bold font. Equation (1.1) reduces to $\mathbf{F} = m\mathbf{a}$ if $\dot{m} = 0$. But if $m = m(t)$ (e.g. in rocket science) then the form with $\dot{\mathbf{p}}$ is correct.

General theorems governing differential equations guarantee that if we are given $\mathbf{r}$ and $\dot{\mathbf{r}}$ at an initial time $t = t_0$, we can integrate equation (1.1) to determine $\mathbf{r}(t)$ for all t (as long as $\mathbf{F}$ remains finite). This is the goal of classical dynamics.

Equation (1.1) is not quite correct as stated: we must add the caveat that it holds only in an *inertial frame*. This is defined to be a frame in which a free particle with $\dot{m} = 0$ travels in a straight line,

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t \tag{1.2}$$

Newtons's first law is the statement that such frames exist.

An inertial frame is not unique. In fact, there are an infinite number of inertial frames. Let S be an inertial frame. Then there are 10 linearly independent transformations $S \to S'$ such that S' is also an inertial frame (i.e. if (1.2) holds in S, then it also holds in S'). These are

- 3 Rotations: $\mathbf{r}' = O\mathbf{r}$ where O is a 3×3 orthogonal matrix.
- 3 Translations: $\mathbf{r}' = \mathbf{r} + \mathbf{c}$ for a constant vector $\mathbf{c}$.
- 3 Boosts: $\mathbf{r}' = \mathbf{r} + \mathbf{u}t$ for a constant velocity $\mathbf{u}$.
- 1 Time Translation: $t' = t + c$ for a constant real number c

If motion is uniform in S, it will also be uniform in S'. These transformations make up the *Galilean Group* under which Newton's laws are invariant. They will be important in section 2.4 where we will see that these symmetries of space and time are the underlying reason for conservation laws. As a parenthetical remark, recall from special relativity that Einstein's laws of motion are invariant under Lorentz transformations which, together with translations, make up the Poincaré group. We can recover the Galilean group from the Poincaré group by taking the speed of light to infinity.

1.2.1 Angular Momentum

We define the *angular momentum* $\mathbf{L}$ of a particle and the *torque* $\boldsymbol{\tau}$ acting upon it as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad , \qquad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{1.3}$$

Note that, unlike linear momentum $\mathbf{p}$, both $\mathbf{L}$ and $\boldsymbol{\tau}$ depend on where we take the origin: we measure angular momentum with respect to a particular point. Let us cross both sides of equation (1.1) with $\mathbf{r}$. Using the fact that $\dot{\mathbf{r}}$ is parallel to $\mathbf{p}$, we can write $\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \dot{\mathbf{p}}$. Then we get a version of Newton's second law that holds for angular momentum:

$$\boldsymbol{\tau} = \dot{\mathbf{L}} \tag{1.4}$$

1.2.2 Conservation Laws

From (1.1) and (1.4), two important conservation laws follow immediately.

- If $\mathbf{F} = 0$ then $\mathbf{p}$ is constant throughout the motion
- If $\boldsymbol{\tau} = 0$ then $\mathbf{L}$ is constant throughout the motion

Notice that $\boldsymbol{\tau} = 0$ does not require $\mathbf{F} = 0$, but only $\mathbf{r} \times \mathbf{F} = 0$. This means that $\mathbf{F}$ must be parallel to $\mathbf{r}$. This is the definition of a *central force*. An example is given by the gravitational force between the earth and the sun: the earth's angular momentum about the sun is constant. As written above in terms of forces and torques, these conservation laws appear trivial. In section 2.4, we'll see how they arise as a property of the symmetry of space as encoded in the Galilean group.

1.2.3 Energy

Let's now recall the definitions of energy. We firstly define the *kinetic energy* T as

$$T = \tfrac{1}{2} m\, \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \tag{1.5}$$

Suppose from now on that the mass is constant. We can compute the change of kinetic energy with time: $\frac{dT}{dt} = \dot{\mathbf{p}} \cdot \dot{\mathbf{r}} = \mathbf{F} \cdot \dot{\mathbf{r}}$. If the particle travels from position $\mathbf{r}_1$ at time t_1 to position $\mathbf{r}_2$ at time t_2 then this change in kinetic energy is given by

$$T(t_2) - T(t_1) = \int_{t_1}^{t_2} \frac{dT}{dt}\, dt = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}}\, dt = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \tag{1.6}$$

where the final expression involving the integral of the force over the path is called the *work done* by the force. So we see that the work done is equal to the change in kinetic energy. From now on we will mostly focus on a very special type of force known as a *conservative* force. Such a force depends only on position $\mathbf{r}$ rather than velocity $\dot{\mathbf{r}}$ and is such that the work done is independent of the path taken. In particular, for a closed path, the work done vanishes.

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \quad \Leftrightarrow \quad \nabla \times \mathbf{F} = 0 \tag{1.7}$$

It is a deep property of flat space $\mathbf{R}^3$ that this property implies we may write the force as

$$\mathbf{F} = -\nabla V(\mathbf{r}) \tag{1.8}$$

for some *potential* $V(\mathbf{r})$. Systems which admit a potential of this form include gravitational, electrostatic and interatomic forces. When we have a conservative force, we necessarily have a conservation law for energy. To see this, return to equation (1.6) which now reads

$$T(t_2) - T(t_1) = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \nabla V \cdot d\mathbf{r} = -V(t_2) + V(t_1) \tag{1.9}$$

or, rearranging things,

$$T(t_1) + V(t_1) = T(t_2) + V(t_2) \equiv E \tag{1.10}$$

So $E = T + V$ is also a constant of motion. It is the energy. When the energy is considered to be a function of position $\mathbf{r}$ and momentum $\mathbf{p}$ it is referred to as the *Hamiltonian* H. In section 4 we will be seeing much more of the Hamiltonian.

1.2.4 Examples

- Example 1: The Simple Harmonic Oscillator

This is a one-dimensional system with a force proportional to the distance x to the origin: $F(x) = -kx$. This force arises from a potential $V = \frac{1}{2}kx^2$. Since $F \neq 0$, momentum is not conserved (the object oscillates backwards and forwards) and, since the system lives in only one dimension, angular momentum is not defined. But energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ is conserved.

- Example 2: The Damped Simple Harmonic Oscillator

We now include a friction term so that $F(x, \dot{x}) = -kx - \gamma\dot{x}$. Since F is not conservative, energy is not conserved. This system loses energy until it comes to rest.

- Example 3: Particle Moving Under Gravity

Consider a particle of mass m moving in 3 dimensions under the gravitational pull of a much larger particle of mass M. The force is $\mathbf{F} = -(GMm/r^2)\hat{\mathbf{r}}$ which arises from the potential $V = -GMm/r$. Again, the linear momentum $\mathbf{p}$ of the smaller particle is not conserved, but the force is both central and conservative, ensuring the particle's total energy E and the angular momentum $\mathbf{L}$ are conserved.

1.3 Newtonian Mechanics: Many Particles

It's easy to generalise the above discussion to many particles: we simply add an index to everything in sight! Let particle i have mass m_i and position $\mathbf{r}_i$ where $i = 1, \ldots, N$ is the number of particles. Newton's law now reads

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \tag{1.11}$$

where $\mathbf{F}_i$ is the force on the i^{th} particle. The subtlety is that forces can now be working between particles. In general, we can decompose the force in the following way:

$$\mathbf{F}_i = \sum_{j \neq i} \mathbf{F}_{ij} + \mathbf{F}_i^{\text{ext}} \tag{1.12}$$

where $\mathbf{F}_{ij}$ is the force acting on the i^{th} particle due to the j^{th} particle, while $\mathbf{F}_i^{\text{ext}}$ is the external force on the i^{th} particle. We now sum over all N particles

$$\begin{aligned} \sum_i \mathbf{F}_i &= \sum_{i,j \text{ with } j \neq i} \mathbf{F}_{ij} + \sum_i \mathbf{F}_i^{\text{ext}} \\ &= \sum_{i<j} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) + \sum_i \mathbf{F}_i^{\text{ext}} \end{aligned} \tag{1.13}$$

where, in the second line, we've re-written the sum to be over all pairs $i < j$. At this stage we make use of *Newton's third law of motion*: every action has an equal and opposite reaction. Or, in other words, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. We see that the first term vanishes and we are left simply with

$$\sum_i \mathbf{F}_i = \mathbf{F}^{\text{ext}} \tag{1.14}$$

where we've defined the total external force to be $\mathbf{F}^{\text{ext}} = \sum_i \mathbf{F}_i^{\text{ext}}$. We now define the total mass of the system $M = \sum_i m_i$ as well as the *centre of mass* $\mathbf{R}$

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{M} \tag{1.15}$$

Then using (1.11), and summing over all particles, we arrive at the simple formula,

$$\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}} \tag{1.16}$$

which is identical to that of a single particle. This is an important formula. It tells that the centre of mass of a system of particles acts just as if all the mass were concentrated there. In other words, it doesn't matter if you throw a tennis ball or a very lively cat: the center of mass of each traces the same path.

1.3.1 Momentum Revisited

The *total momentum* is defined to be $\mathbf{P} = \sum_i \mathbf{p}_i$ and, from the formulae above, it is simple to derive $\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}}$. So we find the conservation law of total linear momentum for a system of many particles: $\mathbf{P}$ is constant if $\mathbf{F}^{\text{ext}}$ vanishes.

Similarly, we define *total angular momentum* to be $\mathbf{L} = \sum_i \mathbf{L}_i$. Now let's see what happens when we compute the time derivative.

$$\begin{aligned}
\dot{\mathbf{L}} &= \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i \\
&= \sum_i \mathbf{r}_i \times \left(\sum_{j\neq i} \mathbf{F}_{ij} + \mathbf{F}_i^{\text{ext}} \right) && (1.17) \\
&= \sum_{i,j \text{ with } i\neq j} \mathbf{r}_i \times \mathbf{F}_{ji} + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} && (1.18)
\end{aligned}$$

The last term in this expression is the definition of *total external torque*: $\boldsymbol{\tau}^{\text{ext}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}$. But what are we going to do with the first term on the right hand side? Ideally we would like it to vanish! Let's look at the circumstances under which this will happen. We can again rewrite it as a sum over pairs $i < j$ to get

$$\sum_{i<j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} \qquad (1.19)$$

which will vanish if and only if the force $\mathbf{F}_{ij}$ is parallel to the line joining to two particles $(\mathbf{r}_i - \mathbf{r}_j)$. This is the strong form of Newton's third law. If this is true, then we have a statement about the conservation of total angular momentum, namely $\mathbf{L}$ is constant if $\boldsymbol{\tau}^{\text{ext}} = 0$.

Most forces do indeed obey both forms of Newton's third law: $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and $\mathbf{F}_{ij}$ is parallel to $(\mathbf{r}_i - \mathbf{r}_j)$. For example, gravitational and electrostatic forces have this property. And the total momentum and angular momentum are both conserved in these systems. But some forces don't have these properties! The most famous example is the Lorentz force on two moving particles with electric charge Q. This is given by,

$$\mathbf{F}_{ij} = Q\mathbf{v}_i \times \mathbf{B}_j \qquad (1.20)$$

where $\mathbf{v}_i$ is the velocity of the i^{th} particle and $\mathbf{B}_j$ is the magnetic field generated by the j^{th} particle. Consider two particles crossing each other in a "T" as shown in the diagram. The force on particle 1 from particle 2 vanishes. Meanwhile, the force on particle 2 from particle 1 is non-zero, and in the direction

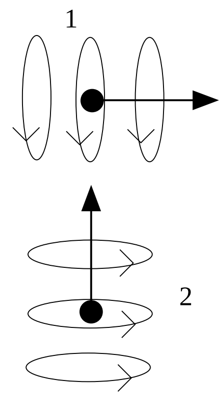

Figure 1: The magnetic field for two particles.

$$\mathbf{F}_{21} \sim \uparrow \times \otimes \sim \leftarrow \qquad (1.21)$$

Does this mean that conservation of total linear and angular momentum is violated? Thankfully, no! We need to realise that the electromagnetic field itself carries angular momentum which restores the conservation law. Once we realise this, it becomes a rather cheap counterexample to Newton's third law, little different from an underwater swimmer who can appear to violate Newton's third law if we don't take into account the momentum of the water.

1.3.2 Energy Revisited

The total kinetic energy of a system of many particles is $T = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i^2$. Let us decompose the position vector $\mathbf{r}_i$ as

$$\mathbf{r}_i = \mathbf{R} + \tilde{\mathbf{r}}_i \tag{1.22}$$

where $\tilde{\mathbf{r}}_i$ is the distance from the centre of mass to the particle i. Then we can write the total kinetic energy as

$$T = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\sum_i m_i \dot{\tilde{\mathbf{r}}}_i^2 \tag{1.23}$$

Which shows us that the kinetic energy splits up into the kinetic energy of the centre of mass, together with an *internal energy* describing how the system is moving around its centre of mass. As for a single particle, we may calculate the change in the total kinetic energy,

$$T(t_2) - T(t_1) = \sum_i \int \mathbf{F}_i^{\text{ext}} \cdot d\mathbf{r}_i + \sum_{i\neq j} \int \mathbf{F}_{ij} \cdot d\mathbf{r}_i \tag{1.24}$$

Like before, we need to consider conservative forces to get energy conservation. But now we need both

- Conservative external forces: $\mathbf{F}_i^{\text{ext}} = -\nabla_i V_i(\mathbf{r}_1, \ldots, \mathbf{r}_N)$
- Conservative internal forces: $\mathbf{F}_{ij} = -\nabla_i V_{ij}(\mathbf{r_1}, \ldots, \mathbf{r_N})$

where $\nabla_i \equiv \partial/\partial\mathbf{r}_i$. To get Newton's third law $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ together with the requirement that this is parallel to $(\mathbf{r}_i - \mathbf{r}_j)$, we should take the internal potentials to satisfy $V_{ij} = V_{ji}$ with

$$V_{ij}(\mathbf{r}_1, \ldots \mathbf{r}_, N) = V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \tag{1.25}$$

so that V_{ij} depends only on the distance between the i^{th} and j^{th} particles. We also insist on a restriction for the external forces, $V_i(\mathbf{r}_1, \ldots, \mathbf{r}_N) = V_i(\mathbf{r}_i)$, so that the force on particle i does not depend on the positions of the other particles. Then, following the steps we took in the single particle case, we can define the *total potential energy* $V = \sum_i V_i + \sum_{i<j} V_{ij}$ and we can show that $H = T + V$ is conserved.

1.3.3 An Example

Let us return to the case of gravitational attraction between two bodies but, unlike in Section 1.2.4, now including both particles. We have $T = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2$. The potential is $V = -Gm_1m_2/|\mathbf{r}_1 - \mathbf{r}_2|$. This system has total linear momentum and total angular mometum conserved, as well as the total energy $H = T + V$.

2. The Lagrangian Formalism

> When I was in high school, my physics teacher called me down one day after class and said, "You look bored, I want to tell you something interesting". Then he told me something I have always found fascinating. Every time the subject comes up I work on it.
>
> *Richard Feynman*

Feynman's teacher told him about the "Principle of Least Action", one of the most profound results in physics.

2.1 The Principle of Least Action

Firstly, let's get our notation right. Part of the power of the Lagrangian formulation over the Newtonian approach is that it does away with vectors in favour of more general coordinates. We start by doing this trivially. Let's rewrite the positions of N particles with coordinates $\mathbf{r}_i$ as x^A where $A = 1, \ldots 3N$. Then Newton's equations read

$$\dot{p}_A = -\frac{\partial V}{\partial x^A} \tag{2.1}$$

where $p_A = m_A \dot{x}^A$. The number of *degrees of freedom* of the system is said to be $3N$. These parameterise a $3N$-dimensional space known as the *configuration space* C. Each point in C specifies a configuration of the system (i.e. the positions of all N particles). Time evolution gives rise to a curve in C.

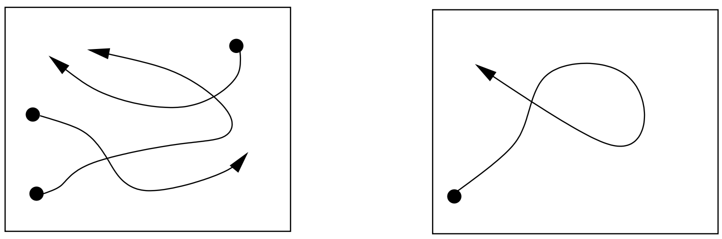

Figure 2: The path of particles in real space (on the left) and in configuration space (on the right).

The Lagrangian

Define the *Lagrangian* to be a function of the positions x^A and the velocities $\dot{x}^A$ of all the particles, given by

$$L(x^A, \dot{x}^A) = T(\dot{x}^A) - V(x^A) \tag{2.2}$$

where $T = \frac{1}{2}\sum_A m_A(\dot{x}^A)^2$ is the kinetic energy, and $V(x^A)$ is the potential energy. Note the minus sign between T and V! To describe the principle of least action, we consider all smooth paths $x^A(t)$ in C with fixed end points so that

$$x^A(t_i) = x^A_{\rm initial} \qquad \text{and} \qquad x^A(t_f) = x^A_{\rm final} \tag{2.3}$$

Of all these possible paths, only one is the true path taken by the system. Which one? To each path, let us assign a number called the *action* S defined as

$$S[x^A(t)] = \int_{t_i}^{t_f} L(x^A(t), \dot{x}^A(t))\; dt \tag{2.4}$$

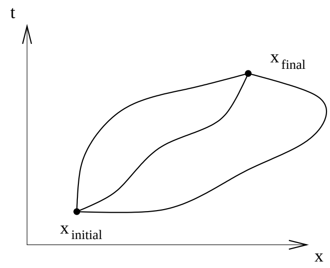

Figure 3:

The action is a functional (i.e. a function of the path which is itself a function). The principle of least action is the following result:

Theorem (Principle of Least Action): The actual path taken by the system is an extremum of S.

Proof: Consider varying a given path slightly, so

$$x^A(t) \to x^A(t) + \delta x^A(t) \tag{2.5}$$

where we fix the end points of the path by demanding $\delta x^A(t_i) = \delta x^A(t_f) = 0$. Then the change in the action is

$$\begin{aligned} \delta S &= \delta\left[\int_{t_i}^{t_f} L\, dt\right] \\ &= \int_{t_i}^{t_f} \delta L\, dt \\ &= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial x^A}\delta x^A + \frac{\partial L}{\partial \dot{x}^A}\delta \dot{x}^A\right)\, dt \end{aligned} \tag{2.6}$$

At this point we integrate the second term by parts to get

$$\delta S = \int_{t_i}^{t_f}\left(\frac{\partial L}{\partial x^A} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right)\right)\delta x^A\, dt + \left[\frac{\partial L}{\partial \dot{x}^A}\delta x^A\right]_{t_i}^{t_f} \tag{2.7}$$

But the final term vanishes since we have fixed the end points of the path so $\delta x^A(t_i) = \delta x^A(t_f) = 0$. The requirement that the action is an extremum says that $\delta S = 0$ for all changes in the path $\delta x^A(t)$. We see that this holds if and only if

$$\frac{\partial L}{\partial x^A} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) = 0 \qquad \text{for each } A = 1, \ldots 3N \tag{2.8}$$

These are known as *Lagrange's equations* (or sometimes as the Euler-Lagrange equations). To finish the proof, we need only show that Lagrange's equations are equivalent to Newton's. From the definition of the Lagrangian (2.2), we have $\partial L/\partial x^A = -\partial V/\partial x^A$, while $\partial L/\partial \dot{x}^A = p_A$. It's then easy to see that equations (2.8) are indeed equivalent to (2.1). $\square$

Some remarks on this important result:

- This is an example of a variational principle which you already met in the eponymous "variational principles" course.

- The principle of *least* action is a slight misnomer. The proof only requires that $\delta S = 0$, and does not specify whether it is a maxima or minima of S. Since $L = T - V$, we can always increase S by taking a very fast, wiggly path with $T \gg 0$, so the true path is never a maximum. However, it may be either a minimum or a saddle point. So "Principle of stationary action" would be a more accurate, but less catchy, name. It is sometimes called "Hamilton's principle".

- *All* the fundamental laws of physics can be written in terms of an action principle. This includes electromagnetism, general relativity, the standard model of particle physics, and attempts to go beyond the known laws of physics such as string theory. For example, (nearly) everything we know about the universe is captured in the Lagrangian
$$L = \sqrt{g}\left(R - \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\,\not{D}\psi\right) \tag{2.9}$$
where the terms carry the names of Einstein, Maxwell (or Yang and Mills) and Dirac respectively, and describe gravity, the forces of nature (electromagnetism and the nuclear forces) and the dynamics of particles like electrons and quarks. If you want to understand what the terms in this equation really mean, then you can find explanations in the lectures on General Relativity, Electromagnetism, and Quantum Field Theory.

- There is a beautiful generalisation of the action principle to quantum mechanics due to Feynman in which the particle takes *all paths* with some probability determined by S. We will describe this in Section 4.8.

- Back to classical mechanics, there are two very important reasons for working with Lagrange's equations rather than Newton's. The first is that Lagrange's equations hold in any coordinate system, while Newton's are restricted to an inertial frame. The second is the ease with which we can deal with constraints in the Lagrangian system. We'll look at these two aspects in the next two subsections.

2.2 Changing Coordinate Systems

We shall now show that Lagrange's equations hold in any coordinate system. In fact, this follows immediately from the action principle, which is a statement about paths and not about coordinates. But here we shall be a little more pedestrian in order to explain exactly what we mean by changing coordinates, and why it's useful. Let

$$q_a = q_a(x_1, \ldots, x_{3N}, t) \tag{2.10}$$

where we've included the possibility of using a coordinate system which changes with time t. Then, by the chain rule, we can write

$$\dot{q}_a = \frac{dq_a}{dt} = \frac{\partial q_a}{\partial x^A}\dot{x}^A + \frac{\partial q_a}{\partial t} \tag{2.11}$$

In this equation, and for the rest of this course, we're using the "summation convention" in which repeated indices are summed over. Note also that we won't be too careful about whether indices are up or down - it won't matter for the purposes of this course. To be a good coordinate system, we should be able to invert the relationship so that $x^A = x^A(q_a, t)$ which we can do as long as we have $\det(\partial x^A/\partial q_a) \neq 0$. Then we have,

$$\dot{x}^A = \frac{\partial x^A}{\partial q_a}\dot{q}_a + \frac{\partial x^A}{\partial t} \tag{2.12}$$

Now we can examine $L(x^A, \dot{x}^A)$ when we substitute in $x^A(q_a, t)$. Using (2.12) we have

$$\frac{\partial L}{\partial q_a} = \frac{\partial L}{\partial x^A}\frac{\partial x^A}{\partial q_a} + \frac{\partial L}{\partial \dot{x}^A}\left(\frac{\partial^2 x^A}{\partial q_a \partial q_b}\,\dot{q}_b + \frac{\partial^2 x^A}{\partial t \partial q_a}\right) \tag{2.13}$$

while

$$\frac{\partial L}{\partial \dot{q}_a} = \frac{\partial L}{\partial \dot{x}^A}\frac{\partial \dot{x}^A}{\partial \dot{q}_a} \tag{2.14}$$

We now use the fact that we can "cancel the dots" and $\partial\dot{x}^A/\partial\dot{q}_a = \partial x^A/\partial q_a$ which we can prove by substituting the expression for $\dot{x}^A$ into the LHS. Taking the time derivative of (2.14) gives us

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right)\frac{\partial x^A}{\partial q_a} + \frac{\partial L}{\partial \dot{x}^A}\left(\frac{\partial^2 x^A}{\partial q_a \partial q_b}\,\dot{q}_b + \frac{\partial^2 x^A}{\partial q_a \partial t}\right) \tag{2.15}$$

So combining (2.13) with (2.15) we find

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) - \frac{\partial L}{\partial x^A}\right]\frac{\partial x^A}{\partial q_a} \tag{2.16}$$

Equation (2.16) is our final result. We see that if Lagrange's equation is solved in the x^A coordinate system (so that $[\ldots]$ on the RHS vanishes) then it is also solved in the q_a coordinate system. (Conversely, if it is satisfied in the q_a coordinate system, so the LHS vanishes, then it is also satisfied in the x^A coordinate system as long as our choice of coordinates is invertible: i.e $\det(\partial x^A/\partial q_a) \neq 0$).

So the form of Lagrange's equations holds in *any* coordinate system. This is in contrast to Newton's equations which are only valid in an inertial frame. Let's illustrate the power of this fact with a couple of simple examples

2.2.1 Example: Rotating Coordinate Systems

Consider a free particle with Lagrangian given by

$$L = \tfrac{1}{2}m\dot{\mathbf{r}}^2 \tag{2.17}$$

with $\mathbf{r} = (x, y, z)$. Now measure the motion of the particle with respect to a coordinate system which is rotating with angular velocity $\boldsymbol{\omega} = (0, 0, \omega)$ about the z axis. If $\mathbf{r}' = (x', y', z')$ are the coordinates in the rotating system, we have the relationship

$$\begin{aligned} x' &= x\cos\omega t + y\sin\omega t \\ y' &= y\cos\omega t - x\sin\omega t \\ z' &= z \end{aligned} \tag{2.18}$$

Then we can substitute these expressions into the Lagrangian to find L in terms of the rotating coordinates,

$$L = \tfrac{1}{2}m[(\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 + \dot{z}^2] = \tfrac{1}{2}m(\dot{\mathbf{r}}' + \boldsymbol{\omega} \times \mathbf{r}')^2 \tag{2.19}$$

In this rotating frame, we can use Lagrange's equations to derive the equations of motion. Taking derivatives, we have

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{r}'} &= m(\dot{\mathbf{r}}' \times \boldsymbol{\omega} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}'}\right) &= m(\ddot{\mathbf{r}}' + \boldsymbol{\omega} \times \dot{\mathbf{r}}') \end{aligned} \tag{2.20}$$

so Lagrange's equation reads

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}'}\right) - \frac{\partial L}{\partial \mathbf{r}'} = m(\ddot{\mathbf{r}}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2\boldsymbol{\omega} \times \dot{\mathbf{r}}') = 0 \tag{2.21}$$

The second and third terms in this expression are the centrifugal and coriolis forces respectively. These are examples of the "fictitious forces" that you were warned about in

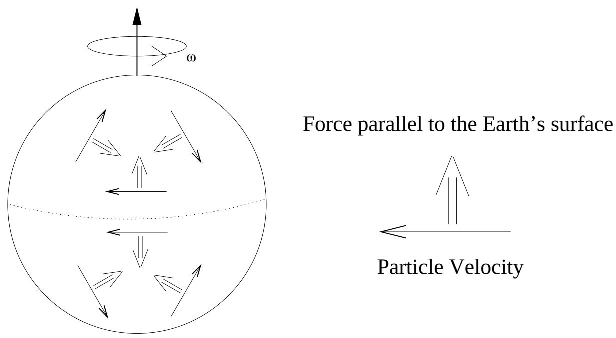

Figure 4: In the northern hemisphere, a particle is deflected in a clockwise direction; in the southern hemisphere in an anti-clockwise direction.

the first year. They're called fictitious because they're a consequence of the reference frame, rather than any interaction. But don't underestimate their importance just because they're "fictitious"! According to Einstein's theory of general relativity, the force of gravity is on the same footing as these fictitious forces.

The centrifugal force $\mathbf{F}_{\text{cent}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$ points outwards in the plane perpendicular to $\boldsymbol{\omega}$ with magnitude $m\omega^2|\mathbf{r}'_\perp| = m|\mathbf{v}_\perp|^2/\,|\mathbf{r}'_\perp|$ where $_\perp$ denotes the projection perpendicular to $\boldsymbol{\omega}$.

The coriolis force $\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \dot{\mathbf{r}}'$ is responsible for the large scale circulation of oceans and the atmosphere. For a particle travelling on the surface of the rotating earth, the direction of the coriolis force is drawn in figure 4. We see that a particle thrown in the northern hemisphere will be seen to rotate in a clockwise direction; a particle thrown in the southern hemisphere rotates in an anti-clockwise direction. For a particle moving along the equator, the coriolis force points directly upwards, so has no effect on the particle.

More details on the effect of the Coriolis force in various circumstances can be found in the Dynamics and Relativity lecture notes. Questions discussed include:

- The coriolis force is responsible for the formation of hurricanes. These rotate in different directions in the northern and southern hemisphere, and never form within 500 miles of the equator where the coriolis force is irrelevant. But hurricanes rotate *anti-clockwise* in the northern hemisphere. This is the opposite direction from what we deduced above for a projected particle! What did we miss?

- Estimate the magnitude of the coriolis force. Do you think that it *really* affects the motion of water going down a plughole? What about the direction in which a CD spins?

- Stand on top of a tower at the equator and drop a ball. As the ball falls, the earth turns underneath from west to east. Does the ball land

  1. At the base of the tower?
  2. To the east?
  3. To the west?

2.2.2 Example: Hyperbolic Coordinates

A particle moves in the (x, y) plane with a force directed towards the origin O with magnitude proportional to the distance from O. How does it move? In Cartesian coordinates, this problem is easy. We have the Lagrangian

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \tfrac{1}{2}k(x^2 + y^2) \tag{2.22}$$

Let's set $m = k = 1$ for simplicity. The equation of motion for this system is simply

$$\ddot{x} = -x \quad \text{and} \quad \ddot{y} = -y \tag{2.23}$$

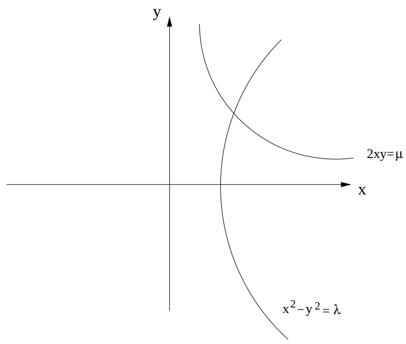

Figure 5: Hyperbolic coordinates.

Now suppose we want to know the motion of the system in *hyperbolic coordinates* defined as

$$2xy = \mu \quad , \quad x^2 - y^2 = \lambda \tag{2.24}$$

The coordinates μ and λ are curvilinear and orthogonal (i.e. two hyperbolics intersect at 90^o). We could try solving this problem by substituting the change of coordinates directly into the equations of motion. It's a mess. (Try if you don't believe me!). A much simpler way is to derive expressions for $x, y, \dot{x}$ and $\dot{y}$ in terms of the new coordinates and substitute into the Lagrangian to find,

$$L = \tfrac{1}{8}\frac{\dot{\lambda}^2 + \dot{\mu}^2}{\sqrt{\lambda^2 + \mu^2}} - \tfrac{1}{2}\sqrt{\lambda^2 + \mu^2} \tag{2.25}$$

From which we can easily derive the equation of motion for λ

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\lambda}\right)-\frac{\partial L}{\partial\lambda}&=\frac{d}{dt}\left(\frac{\dot\lambda}{4\sqrt{\lambda^2+\mu^2}}\right)+\tfrac{1}{8}(\dot\lambda^2+\dot\mu^2)\frac{\lambda}{(\lambda^2+\mu^2)^{3/2}}-\tfrac{1}{2}\frac{\lambda}{(\lambda^2+\mu^2)^{3/2}}\\&=0\end{aligned} \tag{2.26}$$

Which is also a mess! But it's a mess that was much simpler to derive. Moreover, we don't need to do any more work to get the second equation for μ: the symmetry of the Lagrangian means that it must be the same as (2.26) with $\lambda \leftrightarrow \mu$ interchanged.

2.3 Constraints and Generalised Coordinates

Now we turn to the second advantage of the Lagrangian formulation. In writing $\dot{\mathbf{p}}_i = -\nabla_i V$, we implicitly assume that each particle can happily roam anywhere in space $\mathbf{R}^3$. What if there are constraints? In Newtonian mechanics, we introduce "constraint forces". These are things like the tension of ropes, and normal forces applied by surfaces. In the Lagrangian formulation, we don't have to worry about such things. In this section, we'll show why.

An Example: The Pendulum

The simple pendulum has a single dynamical degree of freedom θ, the angle the pendulum makes with the vertical. The position of the mass m in the plane is described by two cartesian coordinates x and y subject to a constraint $x^2+y^2=l^2$. We can parameterise this as $x = l\sin\theta$ and $y = l\cos\theta$. Employing the Newtonian method to solve this system, we introduce the tension T as shown in the diagram and resolve the force vectors to find,

$$m\ddot{x} = -Tx/l \quad , \quad m\ddot{y} = mg - Ty/l \tag{2.27}$$

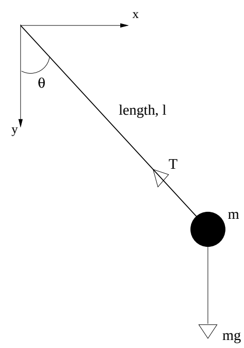

Figure 6:

To determine the motion of the system, we impose the constraints at the level of the equation of motion, and then easily find

$$\ddot\theta = -(g/l)\sin\theta \quad , \quad T = ml\dot\theta^2 + mg\cos\theta \tag{2.28}$$

While this example was pretty straightforward to solve using Newtonian methods, things get rapidly harder when we consider more complicated constraints (and we'll see plenty presently). Moreover, you may have noticed that half of the work of the calculation went into computing the tension T. On occasion we'll be interested in this. (For example, we might want to know how fast we can spin the pendulum before it breaks). But often we won't care about these constraint forces, but will only want to know the motion of the pendulum itself. In this case it seems like a waste of effort to go through the motions of computing T. We'll now see how we can avoid this extra work in the Lagrangian formulation. Firstly, let's define what we mean by constraints more rigorously.

2.3.1 Holonomic Constraints

Holonomic Constraints are relationships between the coordinates of the form

$$f_\alpha(x_A, t) = 0 \qquad \alpha = 1, \ldots, 3N - n \tag{2.29}$$

In general the constraints can be time dependent and our notation above allows for this. Holonomic constraints can be solved in terms of n *generalised coordinates* q_i, $i = 1, \ldots n$. So

$$x_A = x_A(q_1, \ldots, q_n) \tag{2.30}$$

The system is said to have n degrees of freedom. For the pendulum example above, the system has a single degree of freedom, $q = \theta$.

Now let's see how the Lagrangian formulation deals with constraints of this form. We introduce $3N - n$ new variables λ_α, called *Lagrange multipliers* and define a new Lagrangian

$$L' = L(x^A, \dot{x}^A) + \lambda_\alpha f_\alpha(x^A, t) \tag{2.31}$$

We treat λ_α like new coordinates. Since L' doesn't depend on $\dot{\lambda}_\alpha$, Lagrange's equations for λ_α are

$$\frac{\partial L'}{\partial \lambda_\alpha} = f_\alpha(x^A, t) = 0 \tag{2.32}$$

which gives us back the constraints. Meanwhile, the equations for x^A are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) - \frac{\partial L}{\partial x^A} = \lambda_\alpha \frac{\partial f_\alpha}{\partial x^A} \tag{2.33}$$

The LHS is the equation of motion for the unconstrained system. The RHS is the manifestation of the constraint forces in the system. We can now solve these equations as we did in the Newtonian formulation.

The Pendulum Example Again

The Lagrangian for the pendulum is given by that for a free particle moving in the plane, augmented by the Lagrange multiplier term for the constraints. It is

$$L' = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy + \tfrac{1}{2}\lambda(x^2 + y^2 - l^2) \tag{2.34}$$

From which we can calculate the two equations of motion for x and y,

$$m\ddot{x} = \lambda x \qquad \text{and} \qquad \ddot{y} = mg + \lambda y \tag{2.35}$$

while the equation of motion for λ reproduces the constraint $x^2+y^2-l^2 = 0$. Comparing with the Newtonian approach (2.27), we again see that the Lagrange multiplier λ is proportional to the tension: $\lambda = -T/l$.

So we see that we can easily incorporate constraint forces into the Lagrangian setup using Lagrange multipliers. But the big news is that we don't have to! Often we don't care about the tension T or other constraint forces, but only want to know what the generalised coordinates q_i are doing. In this case we have the following useful theorem

Theorem: For constrained systems, we may derive the equations of motion directly in generalised coordinates q_i

$$L[q_i, \dot{q}_i, t] = L[x^A(q_i, t), \dot{x}^A(q_i, \dot{q}_i, t)] \tag{2.36}$$

Proof: Let's work with $L' = L + \lambda_\alpha f_\alpha$ and change coordinates to

$$x_A \rightarrow \begin{cases} q_i & i = 1, \ldots, n \\ f_\alpha & \alpha = 1, \ldots 3N - n \end{cases} \tag{2.37}$$

We know that Lagrange's equations take the same form in these new coordinates. In particular, we may look at the equations for q_i,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} \tag{2.38}$$

But, by definition, $\partial f_\alpha / \partial q_i = 0$. So we are left with Lagrange's equations purely in terms of q_i, with no sign of the constraint forces. If we are only interested in the dynamics of the generalised coordinates q_i, we may ignore the Lagrange multipliers and work entirely with the unconstrained Lagrangian $L(q_i, \dot{q}_i, t)$ defined in (2.36) where we just substitute in $x_A = x_A(q_i, t)$. $\square$

The Pendulum Example for the Last Time

Let's see how this works in the simple example of the pendulum. We can parameterise the constraints in terms of the generalised coordinate θ so that $x = l\sin\theta$ and $y = l\cos\theta$. We now substitute this directly into the Lagrangian for a particle moving in the plane under the effect of gravity, to get

$$\begin{aligned} L &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy \\ &= \tfrac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \end{aligned} \tag{2.39}$$

From which we may derive Lagrange's equations using the coordinate θ directly

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = ml^2\ddot{\theta} + mgl\sin\theta = 0 \tag{2.40}$$

which indeed reproduces the equation of motion for the pendulum (2.28). Note that, as promised, we haven't calculated the tension T using this method. This has the advantage that we've needed to do less work. If we need to figure out the tension, we have to go back to the more laborious Lagrange multiplier method.

2.3.2 Non-Holonomic Constraints

For completeness, let's quickly review a couple of non-holonomic constraints. There's no general theory to solve systems of this type, although it turns out that both of the examples we describe here can be solved with relative ease using different methods. We won't discuss non-holonomic constraints for the rest of this course, and include a brief description here simply to inform you of the sort of stuff we won't see!

Inequalities

Consider a particle moving under gravity on the outside of a sphere of radius R. It is constrained to satisfy $x^2+y^2+z^2 \geq R^2$. This type of constraint, involving an inequality, is non-holonomic. When the particle lies close to the top of the sphere, we know that it will remain in contact with the surface and we can treat the constraint effectively as holonomic. But at some point the particle will fall off. To determine when this happens requires different methods from those above (although it is not particularly difficult).

Velocity Dependent Constraints

Constraints of the form $g(x^A, \dot{x}^A, t) = 0$ which cannot be integrated to give $f(x^A, t) = 0$ are non-holonomic. For example, consider a coin of radius R rolling down a slope as shown in figure 7. The coordinates (x, y) fix the coin's position on the slope. But the coin has other degrees of freedom as well: the angle θ it makes with the path of steepest

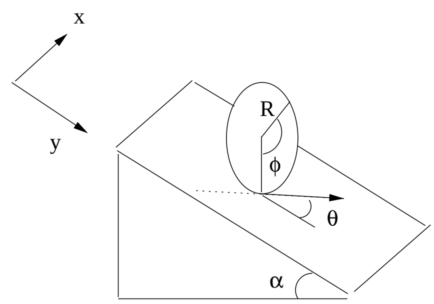

Figure 7: The coin rolling down a slope leads to velocity dependant, non-holonomic constraints.

descent, and the angle ϕ that a marked point on the rim of the coin makes with the vertical. If the coin rolls without slipping, then there are constraints on the evolution of these coordinates. We must have that the velocity of the rim is $v_{\rm rim} = R\dot{\phi}$. So, in terms of our four coordinates, we have the constraint

$$\dot{x} = R\dot{\phi}\sin\theta \qquad , \qquad \dot{y} = R\dot{\phi}\cos\theta \tag{2.41}$$

But these cannot be integrated to give constraints of the form $f(x, y, \theta, \phi) = 0$. They are non-holonomic.

2.3.3 Summary

Let's review what we've learnt so far. A system is described by n *generalised coordinates* q_i which define a point in an n-dimensional *configuration space* C. Time evolution is a curve in C governed by the *Lagrangian*

$$L(q_i, \dot{q}_i, t) \tag{2.42}$$

such that the q_i obey

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2.43}$$

These are n coupled $2^{\rm nd}$ order (usually) non-linear differential equations. Before we move on, let's take this opportunity to give an important definition. The quantity

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2.44}$$

is called the *generalised momentum* conjugate to q_i. (It only coincides with the real momentum in Cartesian coordinates). We can now rewrite Lagrange's equations (2.43) as $\dot{p}_i = \partial L/\partial q_i$. The generalised momenta will play an important role in Section 4.

Note: The Lagrangian L is not unique. We may make the transformation

$$\begin{aligned} & L' = \alpha L \qquad \text{for } \alpha \in \mathbf{R} \\ \text{or} \qquad & L' = L + \frac{df}{dt} \end{aligned} \tag{2.45}$$

for any function f and the equations of motion remain unchanged. To see that the last statement is true, we could either plug L' into Lagrange's equations or, alternatively, recall that Lagrange's equations can be derived from an action principle and the action (which is the time integral of the Lagrangian) changes only by a constant under the transformation. (As an aside: A system no longer remains invariant under these transformations in quantum mechanics. The number α is related to Planck's constant, while transformations of the second type lead to rather subtle and interesting effects related to the mathematics of topology).

2.3.4 Joseph-Louis Lagrange (1736-1813)

Lagrange[1] started off life studying law but changed his mind and turned to mathematics after reading a book on optics by Halley (of comet fame). Despite being mostly self-taught, by the age of 19 he was a professor in his home town of Turin.

He stayed in Italy, somewhat secluded, for the next 11 years although he communicated often with Euler and, in 1766, moved to Berlin to take up Euler's recently vacated position. It was there he did his famous work on mechanics and the calculus of variations that we've seen above. In 1787 he moved once again, now to Paris. He was just in time for the French revolution and only survived a law ordering the arrest of all foreigners after the intervention of the chemist Lavoisier who was a rather powerful political figure. (One year later, Lavoisier lost his power, followed quickly by his head.)

Lagrange published his collected works on mechanics in 1788 in a book called "Mechanique Analytique". He considered the work to be pure mathematics and boasts in the introduction that it contains no figures, thereby putting the anal in analytique.

Since I started with a quote about Newton's teaching, I'll include here a comment on Lagrange's lectures by one of his more famous students:

> "His voice is very feeble, at least in that he does not become heated; he has a very pronounced Italian accent and pronounces the s like z ... The students, of whom the majority are incapable of appreciating him, give him little welcome, but the professors make amends for it."
>
> *Fourier analysis of Lagrange*

[1] You can read all about the lives of mathematicians at http://www-gap.dcs.st-and.ac.uk/ history/BiogIndex.html

2.4 Noether's Theorem and Symmetries

In this subsection we shall discuss the appearance of conservation laws in the Lagrangian formulation and, in particular, a beautiful and important theorem due to Noether relating conserved quantities to symmetries.

Let's start with a definition. A function $F(q_i, \dot{q}_i, t)$ of the coordinates, their time derivatives and (possibly) time t is called a *constant of motion* (or a *conserved quantity*) if the total time derivative vanishes

$$\frac{dF}{dt} = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial q_j} \dot{q}_j + \frac{\partial F}{\partial \dot{q}_j} \ddot{q}_j \right) + \frac{\partial F}{\partial t} = 0 \tag{2.46}$$

whenever $q_i(t)$ satisfy Lagrange's equations. This means that F remains constant along the path followed by the system. Here's a couple of examples:

Claim: If L does not depend explicitly on time t (i.e. $\partial L/\partial t = 0$) then

$$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \tag{2.47}$$

is constant. When H is written as a function of q_i and p_i, it is known as the Hamiltonian. It is usually identified with the total energy of the system.

Proof

$$\frac{dH}{dt} = \sum_j \left(\ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \dot{q}_j - \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \right) \tag{2.48}$$

which vanishes whenever Lagrange's equations (2.43) hold. □

Claim: Suppose $\partial L/\partial q_j = 0$ for some q_j. Then q_j is said to be *ignorable* (or *cyclic*). We have the conserved quantity

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{2.49}$$

Proof:

$$\frac{dp_j}{dt} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = \frac{\partial L}{\partial q_j} = 0 \tag{2.50}$$

where we have used Lagrange's equations (2.43) in the second equality. □

2.4.1 Noether's Theorem

Consider a one-parameter family of maps

$$q_i(t) \to Q_i(s,t) \qquad\qquad s \in \mathbf{R} \tag{2.51}$$

such that $Q_i(0,t) = q_i(t)$. Then this transformation is said to be a continuous *symmetry* of the Lagrangian L if

$$\frac{\partial}{\partial s}\, L(Q_i(s,t), \dot{Q}_i(s,t), t) = 0 \tag{2.52}$$

Noether's theorem states that for each such symmetry there exists a conserved quantity.

Proof of Noether's Theorem:

$$\frac{\partial L}{\partial s} = \frac{\partial L}{\partial Q_i}\frac{\partial Q_i}{\partial s} + \frac{\partial L}{\partial \dot{Q}_i}\frac{\partial \dot{Q}_i}{\partial s} \tag{2.53}$$

so we have

$$\begin{aligned}
0 = \left.\frac{\partial L}{\partial s}\right|_{s=0} &= \left.\frac{\partial L}{\partial q_i}\frac{\partial Q_i}{\partial s}\right|_{s=0} + \left.\frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{Q}_i}{\partial s}\right|_{s=0} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\left.\frac{\partial Q_i}{\partial s}\right|_{s=0} + \left.\frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{Q}_i}{\partial s}\right|_{s=0} \qquad\qquad \text{(By Lagrange)} \\
&= \frac{d}{dt}\left(\left.\frac{\partial L}{\partial \dot{q}_i}\frac{\partial Q_i}{\partial s}\right|_{s=0}\right)
\end{aligned} \tag{2.54}$$

and the quantity $\sum_i (\partial L/\partial \dot{q}_i)(\partial Q_i/\partial s)$, evaluated at $s = 0$, is constant for all time. $\Box$

Example: Homogeneity of Space

Consider the closed system of N particles discussed in Section 1 with Lagrangian

$$L = \tfrac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i^2 - V(|\mathbf{r}_i - \mathbf{r}_j|) \tag{2.55}$$

This Lagrangian has the symmetry of translation: $\mathbf{r}_i \to \mathbf{r}_i + s\mathbf{n}$ for any vector $\mathbf{n}$ and for any real number s. This means that

$$L(\mathbf{r}_i, \dot{\mathbf{r}}_i, t) = L(\mathbf{r}_i + s\mathbf{n}, \dot{\mathbf{r}}_i, t) \tag{2.56}$$

This is the statement that space is homogeneous and a translation of the system by $s\mathbf{n}$ does nothing to the equations of motion. These translations are elements of the

Galilean group that we met in section 1.2. From Noether's theorem, we can compute the conserved quantity associated with translations. It is

$$\sum_i \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot \mathbf{n} = \sum_i \mathbf{p}_i \cdot \mathbf{n} \tag{2.57}$$

which we recognise as the the total linear momentum in the direction $\mathbf{n}$. Since this holds for all $\mathbf{n}$, we conclude that $\sum_i \mathbf{p}_i$ is conserved. But this is very familiar. It is simply the conservation of total linear momentum. To summarise

$$\begin{aligned} \text{Homogeneity of Space} &\Rightarrow \text{Translation Invariance of } L \\ &\Rightarrow \text{Conservation of Total Linear Momentum} \end{aligned}$$

This statement should be intuitively clear. One point in space is much the same as any other. So why would a system of particles speed up to get over there, when here is just as good? This manifests itself as conservation of linear momentum.

Example: Isotropy of Space

The isotropy of space is the statement that a closed system, described by the Lagrangian (2.55) is invariant under rotations around an axis $\hat{\mathbf{n}}$, so all $\mathbf{r}_i \to \mathbf{r}'_i$ are rotated by the same amount. To work out the corresponding conserved quantities it will suffice to work with the infinitesimal form of the rotations

$$\begin{aligned} \mathbf{r}_i &\to \mathbf{r}_i + \delta \mathbf{r}_i \\ &= \mathbf{r}_i + \alpha \hat{\mathbf{n}} \times \mathbf{r}_i \end{aligned} \tag{2.58}$$

where α is considered infinitesimal. To see that this is indeed a rotation, you could calculate the length of the vector and notice that it's preserved to linear order in α. Then we have

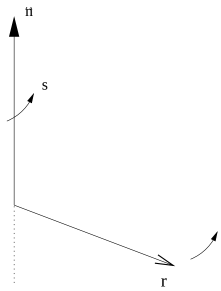

Figure 8:

$$L(\mathbf{r}_i, \dot{\mathbf{r}}_i) = L(\mathbf{r}_i + \alpha \hat{\mathbf{n}} \times \mathbf{r}_i, \dot{\mathbf{r}}_i + \alpha \hat{\mathbf{n}} \times \dot{\mathbf{r}}_i) \tag{2.59}$$

which gives rise to the conserved quantity

$$\sum_i \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot (\hat{\mathbf{n}} \times \mathbf{r}_i) = \sum_i \hat{\mathbf{n}} \cdot (\mathbf{r}_i \times \mathbf{p}_i) = \hat{\mathbf{n}} \cdot \mathbf{L} \tag{2.60}$$

This is the component of the total angular momentum in the direction $\hat{\mathbf{n}}$. Since the vector $\hat{\mathbf{n}}$ is arbitrary, we get the result

$$\begin{aligned} \text{Isotropy of Space} &\Rightarrow \text{Rotational Invariance of } L \\ &\Rightarrow \text{Conservation of Total Angular Momentum} \end{aligned}$$

Example: Homogeneity of Time

What about homogeneity of time? In mathematical language, this means L is invariant under $t \to t+s$ or, in other words, $\partial L/\partial t = 0$. But we already saw earlier in this section that this implies $H = \sum_i \dot{q}_i(\partial L/\partial \dot{q}_i) - L$ is conserved. In the systems we're considering, this is simply the total energy. We see that the existence of a conserved quantity which we call energy can be traced to the homogeneous passage of time. Or

Time is to Energy as Space is to Momentum

Recall from the lectures on Special Relativity that energy and 3-momentum fit together to form a 4-vector which rotates under spacetime transformations. Here we see that the link between energy-momentum and time-space exists even in the non-relativistic framework of Newtonian physics. You don't have to be Einstein to see it. You just have to be Emmy Noether.

Remarks: It turns out that *all* conservation laws in nature are related to symmetries through Noether's theorem. This includes the conservation of electric charge and the conservation of particles such as protons and neutrons (known as baryons).

There are also *discrete* symmetries in Nature which don't depend on a continuous parameter. For example, many theories are invariant under reflection (known as parity) in which $\mathbf{r}_i \to -\mathbf{r}_i$. These types of symmetries do not give rise to conservation laws in classical physics (although they do in quantum physics).

2.5 Applications

Having developed all of these tools, let's now apply them to a few examples.

2.5.1 Bead on a Rotating Hoop

This is an example of a system with a time dependent holonomic constraint. The hoop is of radius a and rotates with frequency ω as shown in figure 9. The bead, of mass m, is threaded on the hoop and moves without friction. We want to determine its motion. There is a single degree of freedom ψ, the angle the bead makes with the vertical. In terms of Cartesian coordinates (x, y, z) the position of the bead is

$$x = a \sin\psi \cos\omega t \quad , \quad y = a \sin\psi \sin\omega t \quad , \quad z = a - a\cos\psi \tag{2.61}$$

To determine the Lagrangian in terms of the generalised coordinate ψ we must substitute these expressions into the Lagrangian for the free particle. For the kinetic energy T we have

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2}ma^2[\dot{\psi}^2 + \omega^2 \sin^2\psi] \tag{2.62}$$

while the potential energy V is given by (ignoring an overall constant)

$$V = mgz = -mga\cos\psi \tag{2.63}$$

So, replacing x, y and z by ψ, we have the Lagrangian

$$L = ma^2\left(\tfrac{1}{2}\dot{\psi}^2 - V_{\rm eff}\right) \tag{2.64}$$

where the effective potential is

$$V_{\rm eff} = \frac{1}{ma^2}\left(-mga\cos\psi - \tfrac{1}{2}ma^2\omega^2\sin^2\psi\right) \tag{2.65}$$

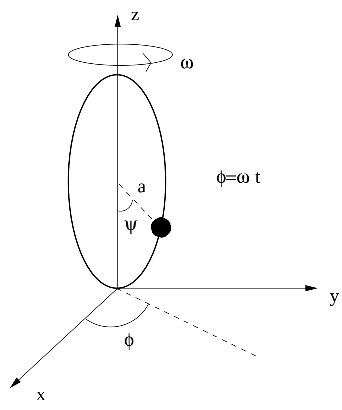

Figure 9:

We can now derive the equations of motion for the bead simply from Lagrange's equations which read

$$\ddot{\psi} = -\frac{\partial V_{\rm eff}}{\partial \psi} \tag{2.66}$$

Let's look for stationary solutions of these equations in which the bead doesn't move (i.e solutions of the form $\ddot{\psi} = \dot{\psi} = 0$). From the equation of motion, we must solve $\partial V_{\rm eff}/\partial\psi = 0$ to find that the bead can remain stationary at points satisfying

$$g\sin\psi = a\omega^2\sin\psi\cos\psi \tag{2.67}$$

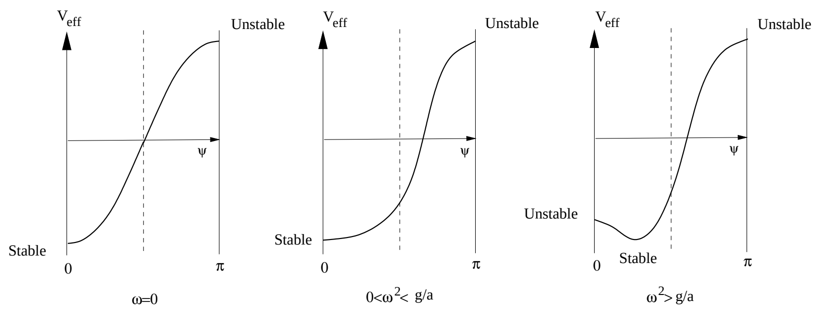

Figure 10: The effective potential for the bead depends on how fast the hoop is rotating

There are at most three such points: $\psi = 0$, $\psi = \pi$ or $\cos\psi = g/a\omega^2$. Note that the first two solutions always exist, while the third stationary point is only there if the hoop is spinning fast enough so that $\omega^2 \geq g/a$. Which of these stationary points is stable depends on whether $V_{\rm eff}(\psi)$ has a local minimum (stable) or maximum (unstable). This in turn depends on the value of ω. V_{eff} is drawn for several values of ω in figure 10. For $\omega^2 < g/a$, the point $\psi = 0$ at the bottom of the hoop is stable, while for $\omega^2 > g/a$, the position at the bottom becomes unstable and the new solution at $\cos\psi = g/a\omega^2$ is the stable point. For all values of ω the bead perched at the top of the hoop $\psi = \pi$ is unstable.

2.5.2 Double Pendulum

A double pendulum is drawn in figure 11, consisting of two particles of mass m_1 and m_2, connected by light rods of length l_1 and l_2. For the first particle, the kinetic energy T_1 and the potential energy V_1 are the same as for a simple pendulum

$$T_1 = \tfrac{1}{2}m_1 l_1^2\dot{\theta}_1^2 \quad \text{and} \quad V_1 = -m_1 g l_1 \cos\theta_1 \tag{2.68}$$

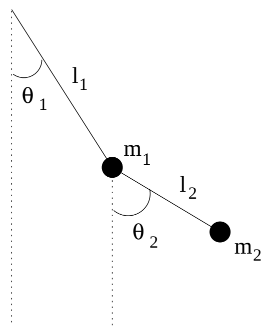

Figure 11:

For the second particle it's a little more involved. Consider the position of the second particle in the (x, y) plane in which the pendulum swings (where we take the origin to be the pivot of the first pendulum with y increasing downwards)

$$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2 \quad \text{and} \quad y_2 = l_1\cos\theta_1 + l_2\cos\theta_2 \tag{2.69}$$

Which we can substitute into the kinetic energy for the second particle

$$\begin{aligned} T_2 &= \tfrac{1}{2}m_2(\dot{x}^2 + \dot{y}^2) \\ &= \tfrac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right) \end{aligned} \tag{2.70}$$

while the potential energy is given by

$$V_2 = -m_2 g y_2 = -m_2 g\left(l_1\cos\theta_1 + l_2\cos\theta_2\right) \tag{2.71}$$

The Lagrangian is given by the sum of the kinetic energies, minus the sum of the potential energies

$$\begin{aligned} L = {}& \tfrac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \\ & + (m_1 + m_2)g l_1\cos\theta_1 + m_2 g l_2\cos\theta_2 \end{aligned} \tag{2.72}$$

The equations of motion follow by simple calculus using Lagrange's two equations (one for θ_1 and one for θ_2). The solutions to these equations are complicated. In fact, above a certain energy, the motion is chaotic.

2.5.3 Spherical Pendulum

The spherical pendulum is allowed to rotate in three dimensions. The system has two degrees of freedom drawn in figure 12 which cover the range

$$0 \le \theta < \pi \qquad \text{and} \qquad 0 \le \phi < 2\pi \tag{2.73}$$

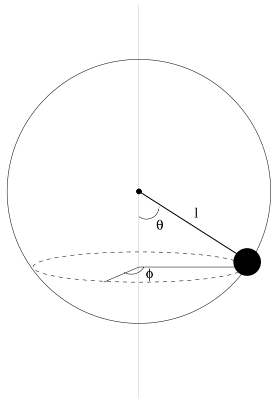

Figure 12:

In terms of cartesian coordinates, we have

$$x = l\cos\phi\sin\theta \quad , \quad y = l\sin\phi\sin\theta \quad , \quad z = -l\cos\theta$$

We substitute these constraints into the Lagrangian for a free particle to get

$$\begin{aligned} L &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\ &= \tfrac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl\cos\theta \end{aligned} \tag{2.74}$$

Notice that the coordinate ϕ is ignorable. From Noether's theorem, we know that the quantity

$$J = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta \tag{2.75}$$

is constant. This is the component of angular momentum in the ϕ direction. The equation of motion for θ follows from Lagrange's equations and is

$$ml^2\ddot{\theta} = ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta \tag{2.76}$$

We can substitute $\dot{\phi}$ for the constant J in this expression to get an equation entirely in terms of θ which we chose to write as

$$\ddot{\theta} = -\frac{\partial V_{\text{eff}}}{\partial \theta} \tag{2.77}$$

where the effective potential is defined to be

$$V_{\text{eff}}(\theta) = -\frac{g}{l}\cos\theta + \frac{J^2}{2m^2l^4}\frac{1}{\sin^2\theta} \tag{2.78}$$

An important point here: we must substitute for J into the equations of motion. If you substitute J for $\dot{\phi}$ directly into the Lagrangian, you will derive an equation that looks like the one above, but you'll get a minus sign wrong! This is because Lagrange's equations are derived under the assumption that θ and ϕ are independent.

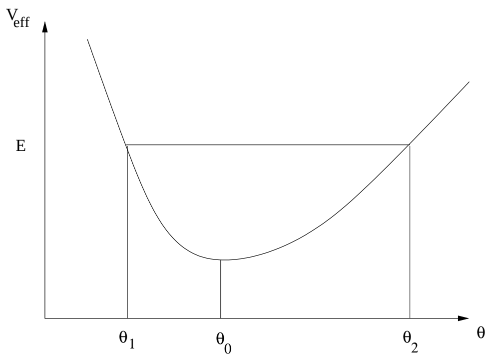

Figure 13: The effective potential for the spherical pendulum.

As well as the conservation of angular momentum J, we also have $\partial L/\partial t = 0$ so energy is conserved. This is given by

$$E = \tfrac{1}{2}\dot{\theta}^2 + V_{\rm eff}(\theta) \tag{2.79}$$

where E is a constant. In fact we can invert this equation for E to solve for θ in terms of an integral

$$t - t_0 = \frac{1}{\sqrt{2}} \int \frac{d\theta}{\sqrt{E - V_{\rm eff}(\theta)}} \tag{2.80}$$

If we succeed in writing the solution to a problem in terms of an integral like this then we say we've "reduced the problem to quadrature". It's kind of a cute way of saying we can't do the integral. But at least we have an expression for the solution that we can play with or, if all else fails, we can simply plot on a computer.

Once we have an expression for $\theta(t)$ we can solve for $\phi(t)$ using the expression for J,

$$\phi = \int \frac{J}{ml^2}\frac{1}{\sin^2\theta}dt = \frac{J}{\sqrt{2}ml^2}\int \frac{1}{\sqrt{E - V_{\rm eff}(\theta)}}\frac{1}{\sin^2\theta}d\theta$$

which gives us $\phi = \phi(\theta) = \phi(t)$. Let's get more of a handle on what these solutions look like. We plot the function $V_{\rm eff}$ in figure 13. For a given energy E, the particle is restricted to the region $V_{\rm eff} \leq E$ (which follows from (2.79)). So from the figure we see that the motion is pinned between two points θ_1 and θ_2. If we draw the motion of the pendulum in real space, it must therefore look something like figure 14, in which the bob oscillates between the two extremes: $\theta_1 \leq \theta \leq \theta_2$. Note that we could make more progress in understanding the motion of the spherical pendulum than for the double pendulum. The reason for this is the existence of two conservation laws for the spherical pendulum (energy and angular momentum) compared to just one (energy) for the double pendulum.

There is a stable orbit which lies between the two extremal points at $\theta = \theta_0$, corresponding to the minimum of $V_{\rm eff}$. This occurs if we balance the angular momentum J and the energy E just right. We can look at small oscillations around this point by expanding $\theta = \theta_0 + \delta\theta$. Substituting into the equation of motion (2.77), we have

$$\delta\ddot{\theta} = -\left(\left.\frac{\partial^2 V_{\rm eff}}{\partial\theta^2}\right|_{\theta=\theta_0}\right)\delta\theta + \mathcal{O}(\delta\theta^2) \qquad (2.81)$$

so small oscillations about $\theta = \theta_0$ have frequency $\omega^2 = (\partial^2 V_{\rm eff}/\partial\theta^2)$ evaluated at $\theta = \theta_0$.

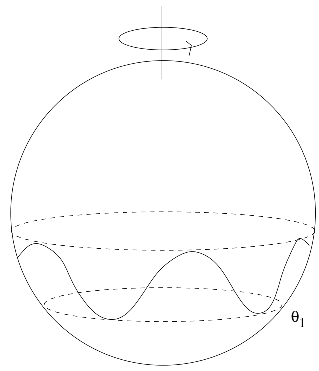

Figure 14:

2.5.4 Two Body Problem

We now turn to the study of two objects interacting through a central force. The most famous example of this type is the gravitational interaction between two bodies in the solar system which leads to the elliptic orbits of planets and the hyperbolic orbits of comets. Let's see how to frame this famous physics problem in the Lagrangian setting. We start by rewriting the Lagrangian in terms of the centre of mass $\mathbf{R}$ and the separation $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ and work with an arbitrary potential $V(|\mathbf{r}_{12}|)$

$$\begin{aligned} L &= \tfrac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \tfrac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(|\mathbf{r}_{12}|) \\ &= \tfrac{1}{2}(m_1+m_2)\dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu\dot{\mathbf{r}}_{12}^2 - V(|\mathbf{r}_{12}|) \end{aligned} \qquad (2.82)$$

where $\mu = m_1 m_2/(m_1+m_2)$ is the *reduced mass*. The Lagrangian splits into a piece describing the centre of mass $\mathbf{R}$ and a piece describing the separation. This is familiar from Section 1.3.2. From now on we neglect the centre of mass piece and focus on the separation. We know from Noether's theorem that $\mathbf{L} = \mathbf{r}_{12} \times \mathbf{p}_{12}$ is conserved, where $\mathbf{p}_{12}$ is the momentum conjugate to $\mathbf{r}_{12}$. Since $\mathbf{L}$ is perpendicular to $\mathbf{r}_{12}$, the motion of the orbit must lie in a plane perpendicular to $\mathbf{L}$. Using polar coordinates (r,ϕ) in that plane, the Lagrangian is

$$L = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) \qquad (2.83)$$

To make further progress, notice that ϕ is ignorable so, once again using Noether's theorem, we have the conserved quantity

$$J = \mu r^2\dot{\phi} \qquad (2.84)$$

This is also conservation of angular momentum: to reduce to the Lagrangian (2.83), we used the fact that the direction of $\mathbf{L}$ is fixed; the quantity J is related to the magnitude of $\mathbf{L}$. To figure out the motion we calculate Lagrange's equation for r from (2.83)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \mu\ddot{r} - \mu r\dot{\phi}^2 + \frac{\partial V}{\partial r} = 0 \tag{2.85}$$

We can eliminate $\dot{\phi}$ from this equation by writing it in terms of the constant J to get a differential equation for the orbit purely in terms of r,

$$\mu\ddot{r} = -\frac{\partial}{\partial r}V_{\rm eff}(r) \tag{2.86}$$

where the effective potential is given by

$$V_{\rm eff}(r) = V(r) + \frac{J^2}{2\mu r^2} \tag{2.87}$$

The last term is known as the "angular momentum barrier". Let me reiterate the warning of the spherical pendulum: do *not* substitute $J = \mu r^2\dot{\phi}$ directly into the Lagrangian – you will get a minus sign wrong! You must substitute it into the equations of motion.

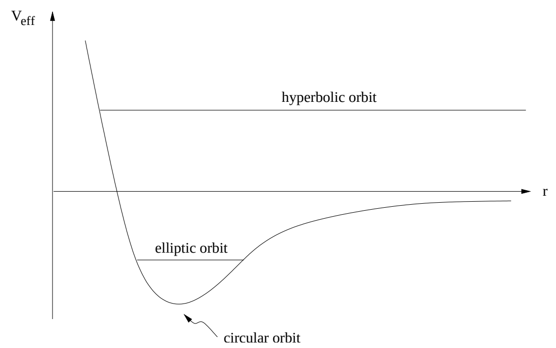

Figure 15: The effective potential for two bodies interacting gravitationally.

So far, you may recognise that the analysis has been rather similar to that of the spherical pendulum. Let's continue following that path. Since $\partial L/\partial t = 0$, Noether tells us that energy is conserved and

$$E = \tfrac{1}{2}\mu\dot{r}^2 + V_{\rm eff}(r) \tag{2.88}$$

is constant throughout the motion. We can use this fact to "reduce to quadrature",

$$t - t_0 = \sqrt{\frac{\mu}{2}} \int \frac{dr}{\sqrt{E - V_{\text{eff}}(r)}} \tag{2.89}$$

Up to this point the analysis is for an arbitrary potential $V(r)$. At this point let's specialise to the case of two bodies interacting gravitationally with

$$V(r) = -\frac{Gm_1 m_2}{r} \tag{2.90}$$

where G is Newton's constant. For this potential, the different solutions were studied in your Part I mechanics course where Kepler's laws were derived. The orbits fall into two categories: elliptic if $E < 0$ and hyperbolic if $E > 0$ as shown in figure 15.

It's worth noting the methodology we used to solve this problem. We started with 6 degrees of freedom describing the positions of two particles. Eliminating the centre of mass reduced this to 3 degrees of freedom describing the separation. We then used conservation of the direction of $\mathbf{L}$ to reduce to 2 degrees of freedom (r and ϕ), and conservation of the magnitude of $\mathbf{L}$ to reduce to a single variable r. Finally conservation of E allowed us to solve the problem. You might now be getting an idea about how important conservation laws are to help us solve problems!

2.5.5 Restricted Three Body Problem

Consider three masses m_1, m_2 and m_3 interacting gravitationally. In general this problem does not have an analytic solution and we must resort to numerical methods (i.e. putting it on a computer). However, suppose that $m_3 \ll m_1$ and m_2. Then it is a good approximation to first solve for the motion of m_1 and m_2 interacting alone, and then solve for the motion of m_3 in the time dependent potential set up by m_1 and m_2. Let's see how this works.

For simplicity, let's assume m_1 and m_2 are in a circular orbit with $\phi = \omega t$. We saw in the previous section that the circular orbit occurs for $\partial V_{\text{eff}}/\partial r = 0$, from which we get an expression relating the angular velocity of the orbit to the distance

$$\omega^2 = \frac{G(m_1 + m_2)}{r^3} \tag{2.91}$$

which is a special case of Kepler's third law. Let's further assume that m_3 moves in the same plane as m_1 and m_2 (which is a pretty good assumption for the sun-earth-moon system). To solve for the motion of m_3 in this background, we use our ability to change coordinates. Let's go to a frame which rotates with m_1 and m_2 with the centre of mass at the origin. The particle m_1 is a distance $r\mu/m_1$ from the origin, while m_2 is a distance $r\mu/m_2$ from the origin.

Then, from the example of Section 2.2.1, the Lagrangian for m_3 in the rotating frame is

$$L = \tfrac{1}{2}m_3\left[(\dot{x}-\omega y)^2 + (\dot{y}+\omega x)^2\right] - V$$

where V is the gravitational potential for m_3 interacting with m_1 and m_2

$$V = -\frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} \tag{2.92}$$

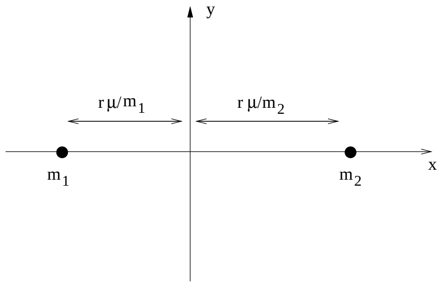

Figure 16:

The separations are given by

$$r_{13}^2 = (x + r\mu/m_1)^2 + y^2 \quad , \quad r_{23}^2 = (x - r\mu/m_2)^2 + y^2 \tag{2.93}$$

Be aware that x and y are the dynamical coordinates in this system, while r is the fixed separation between m_1 and m_2. The equations of motion arising from L are

$$\begin{aligned} m_3\ddot{x} &= 2m_3\omega\dot{y} + m_3\omega^2 x - \frac{\partial V}{\partial x} \\ m_3\ddot{y} &= -2m_3\omega\dot{x} + m_3\omega^2 y - \frac{\partial V}{\partial y} \end{aligned} \tag{2.94}$$

The full solutions to these equations are interesting and complicated. In fact, in 1889, Poincaré studied the restricted three-body system and discovered the concept of chaos in dynamical systems for the first time (and, in the process, won 2,500 krona and lost 3,500 krona). We'll be a little less ambitious here and try to find solutions of the form $\dot{x} = \dot{y} = 0$. This is where the third body sits stationary to the other two and the whole system rotates together. Physically, the centrifugal force of the third body exactly cancels its gravitational force. The equations we have to solve are

$$m_3\omega^2 x = \frac{\partial V}{\partial x} = Gm_1m_3\frac{x + r\mu/m_1}{r_{13}^3} + Gm_2m_3\frac{x - r\mu/m_2}{r_{23}^3} \tag{2.95}$$

$$m_3\omega^2 y = \frac{\partial V}{\partial y} = Gm_1m_3\frac{y}{r_{13}^3} + Gm_2m_3\frac{y}{r_{23}^3} \tag{2.96}$$

There are five solutions to these equations. Firstly suppose that $y = 0$ so that m_3 sits on the same line as m_1 and m_2. Then we have to solve the algebraic equation

$$\omega^2 x = Gm_1\frac{x + r\mu/m_1}{|x + r\mu/m_1|^3} + Gm_2\frac{x - r\mu/m_2}{|x - r\mu/m_2|^3} \tag{2.97}$$

In figure 17, we have plotted the LHS and RHS of this equation to demonstrate the three solutions, one in each of the regimes:

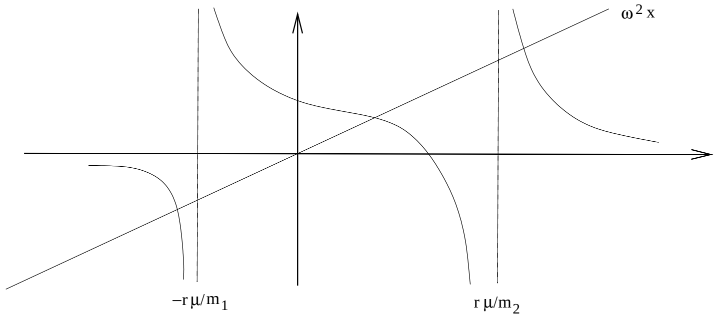

Figure 17: The three solutions sitting on $y = 0$.

$$x < -\frac{r\mu}{m_1} \quad , \quad -\frac{r\mu}{m_1} < x < \frac{r\mu}{m_2} \quad , \quad x > \frac{r\mu}{m_2} \tag{2.98}$$

Now let's look for solutions with $y \neq 0$. From (2.96) we have

$$\frac{Gm_2}{r_{23}^3} = \omega^2 - \frac{Gm_1}{r_{13}^3} \tag{2.99}$$

which we can substitute into (2.95) and, after a little algebra, we find the condition for solutions to be

$$\omega^2 = \frac{G(m_1+m_2)}{r_{13}^3} = \frac{G(m_1+m_2)}{r_{23}^3} \tag{2.100}$$

which means that we must have $r_{13} = r_{23} = r$. There are two such points.

In general there are five stationary points drawn in the figure. These are called Lagrange points. It turns out that L_1, L_2 and L_3 are unstable, while L_4 and L_5 are stable as long as m_2 is sufficiently less than m_1.

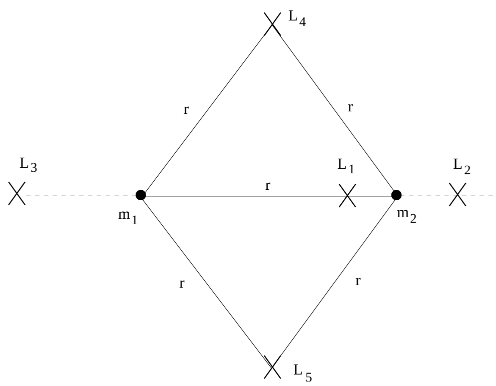

Figure 18: The five Lagrange points. X marks the spots.

For the earth-sun system, NASA and ESA make use of the Lagrange points L_2 and L_3 to place satellites. There are solar observatories at L_3; satellites such as WMAP and PLANCK which measure the cosmic microwave background radiation (the afterglow of the big bang) gather their data from L_2. Apparently, there is a large collection of cosmic dust which has accumulated at L_4 and L_5. Other planetary systems (e.g. the sun-jupiter and sun-mars systems) have large asteroids, known as trojans, trapped at their L_4 and L_5.

2.5.6 Purely Kinetic Lagrangians

Often in physics, one is interested in systems with only kinetic energy and no potential energy. For a system with n dynamical degrees of freedom q^a, $a = 1, \ldots, n$, the most general form of the Lagrangian with just a kinetic term is

$$L = \tfrac{1}{2}\, g_{ab}(q_c)\, \dot{q}^a \dot{q}^b \tag{2.101}$$

The functions $g_{ab} = g_{ba}$ depend on all the generalised coordinates. Assume that $\det(g_{ab}) \neq 0$ so that the inverse matrix g^{ab} exists ($g^{ab} g_{bc} = \delta^a{}_c$). It is a short exercise to show that Lagrange's equation for this system are given by

$$\ddot{q}^a + \Gamma^a_{bc} \dot{q}^b \dot{q}^c = 0 \tag{2.102}$$

where

$$\Gamma^a_{bc} = \tfrac{1}{2}\, g^{ad} \left(\frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d} \right) \tag{2.103}$$

The functions g_{ab} define a *metric* on the configuration space, and the equations (2.102) are known as the *geodesic* equations. They appear naturally in General Relativity where they describe a particle moving in curved spacetime. Lagrangians of the form (2.101) also appear in many other areas of physics, including the condensed matter physics, the theory of nuclear forces and string theory. In these contexts, the systems are referred to as *sigma models*.

2.5.7 Particles in Electromagnetic Fields

We saw from the beginning that the Lagrangian formulation works with conservative forces which can be written in terms of a potential. It is no good at dealing with friction forces which are often of the type $F = -k\dot{x}$. But there are other velocity dependent forces which arise in the fundamental laws of Nature. It's a crucial fact about Nature that all of these can be written in Lagrangian form. Let's illustrate this in an important example.

Recall that the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ can be written in terms of a vector potential $\mathbf{A}(\mathbf{r}, t)$ and a scalar potential $\phi(\mathbf{r}, t)$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad , \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{2.104}$$

Let's study the Lagrangian for a particle of electric charge e of the form,

$$L = \tfrac{1}{2} m\dot{\mathbf{r}}^2 - e\,(\phi - \dot{\mathbf{r}} \cdot \mathbf{A}) \tag{2.105}$$

The momentum conjugate to $\mathbf{r}$ is

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + e\mathbf{A} \tag{2.106}$$

Notice that the momentum is not simply $m\dot{\mathbf{r}}$; it's modified in the presence of electric and magnetic fields. Now we can calculate Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\right) - \frac{\partial L}{\partial \mathbf{r}} = \frac{d}{dt}\left(m\dot{\mathbf{r}} + e\mathbf{A}\right) + e\nabla\phi - e\nabla(\dot{\mathbf{r}}\cdot\mathbf{A}) = 0 \tag{2.107}$$

To disentangle this, let's work with indices $a, b = 1, 2, 3$ on the Cartesian coordinates, and rewrite the equation of motion as

$$m\ddot{r}^a = -e\left(\frac{\partial\phi}{\partial r^a} + \frac{\partial A_a}{\partial t}\right) + e\left(\frac{\partial A_b}{\partial r^a} - \frac{\partial A_a}{\partial r^b}\right)\dot{r}^b \tag{2.108}$$

Now we use our definitions of the $\mathbf{E}$ and $\mathbf{B}$ fields (2.104) which, in terms of indices, read

$$E_a = -\frac{\partial\phi}{\partial r^a} - \frac{\partial A_a}{\partial t} \quad , \quad B_c = \epsilon_{cab}\frac{\partial A_b}{\partial r^a} \tag{2.109}$$

The last of these can be equivalently written as $\partial A_b/\partial r^a - \partial A_a/\partial r^b = \epsilon_{abc}B_c$. The equation of motion then reads

$$m\ddot{r}^a = eE_a + e\epsilon_{abc}\dot{r}^b B_c$$

or, reverting to vector notation,

$$m\ddot{\mathbf{r}} = e\left(\mathbf{E} + \dot{\mathbf{r}}\times\mathbf{B}\right) \tag{2.110}$$

which is the Lorentz force law.

Gauge Invariance: The scalar and vector potentials are not unique. We may make a change of the form

$$\phi \to \phi - \frac{\partial\chi}{\partial t} \quad , \quad \mathbf{A} \to \mathbf{A} + \nabla\chi \tag{2.111}$$

These give the same $\mathbf{E}$ and $\mathbf{B}$ fields for any function χ. This is known as a *gauge transformation*. Under this change, we have

$$L \to L + e\frac{\partial\chi}{\partial t} + e\dot{\mathbf{r}}\cdot\nabla\chi = L + e\frac{d\chi}{dt} \tag{2.112}$$

but we know that the equations of motion remain invariant under the addition of a total derivative to the Lagrangian. This concept of gauge invariance underpins much of modern physics. You can learn more about this in the lectures on Electromagnetism.

2.6 Small Oscillations and Stability

> "Physics is that subset of human experience which can be reduced to coupled harmonic oscillators"
>
> *Michael Peskin*

Peskin doesn't say this to knock physics. He's just a fan of harmonic oscillators. And rightly so. By studying the simple harmonic oscillator and its relatives in ever more inventive ways we understand why the stars shine and why lasers shine and, thanks to Hawking, why even black holes shine.

In this section we'll see one reason why the simple harmonic oscillator is so important to us. We will study the motion of systems close to equilibrium and see that the dynamics is described by n *decoupled* simple harmonic oscillators, each ringing at a different frequency.

Let's start with a single degree of freedom x. We've already seen several examples where we get an equation of the form

$$\ddot{x} = f(x) \tag{2.113}$$

An *equilibrium point*, $x = x_0$, of this system satisfies $f(x_0) = 0$. This means that if we start with the initial conditions

$$x = x_0 \qquad \text{and} \qquad \dot{x} = 0 \tag{2.114}$$

then the system will stay there forever. But what if we start slightly away from $x = x_0$? To analyse this, we write

$$x(t) = x_0 + \eta(t) \tag{2.115}$$

where η is assumed to be small so that we can Taylor expand $f(x)$ to find

$$\ddot{\eta} = f'(x_0)\,\eta \qquad + \mathcal{O}(\eta^2) \tag{2.116}$$

and we neglect the terms quadratic in η and higher. There are two possible behaviours of this system

1. $f'(x_0) < 0$. In this case the restoring force sends us back to $\eta = 0$ and the solution is

$$\eta(t) = A\cos(\omega(t - t_0)) \tag{2.117}$$

   where A and t_0 are integration constants, while $\omega^2 = -f'(x_0)$. The system undergoes stable oscillations about $x = x_0$ at frequency ω.

2. $f'(x_0) > 0$. In this case, the force pushes us away from equilibrium and the solution is

$$\eta(t) = Ae^{\lambda t} + Be^{-\lambda t} \tag{2.118}$$

where A and B are integration constants, while $\lambda^2 = f'(x_0)$. In this case, there is a very special initial condition $A = 0$ such that $x \to x_0$ at late times. But for generic initial conditions, η gets rapidly large and the approximation that η is small breaks down. We say the system has a *linear instability.*

Now let's generalise this discussion to n degrees of freedom with equations of motion of the form,

$$\ddot{q}_i = f_i(q_1, \ldots, q_n) \qquad i = 1, \ldots, n \tag{2.119}$$

An equilibrium point q_i^0 must satisfy $f_i(q_1^0, \ldots, q_n^0) = 0$ for all $i = 1 \ldots, n$. Consider small perturbations away from the equilibrium point

$$q_i(t) = q_i^0 + \eta_i(t) \tag{2.120}$$

where, again, we take the η_i to be small so that we can Taylor expand the f_i, and neglect the quadratic terms and higher. We have

$$\ddot{\eta}_i \approx \left.\frac{\partial f_i}{\partial q_j}\right|_{q_k = q_k^0} \eta_j \tag{2.121}$$

where the sum over $j = 1, \ldots, n$ is implicit. It's useful to write this in matrix form. We define the vector $\boldsymbol{\eta}$ and the $n \times n$ matrix F as

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \quad , \quad F = \begin{pmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial q_1} & \cdots & \frac{\partial f_n}{\partial q_n} \end{pmatrix} \tag{2.122}$$

where each partial derivative in the matrix F is evaluated at $q_i = q_i^0$. The equation now becomes simply

$$\ddot{\boldsymbol{\eta}} = F\boldsymbol{\eta} \tag{2.123}$$

Our strategy is simple: we search for eigenvectors of F. If F were a symmetric matrix, it would have a complete set of orthogonal eigenvectors with real eigenvalues. Unfortunately, we can't assume that F is symmetric. Nonetheless, it is true that for equations of the form (2.123) arising from physical Lagrangian systems, the eigenvalues will be real. We shall postpone a proof of this fact for a couple of paragraphs and continue under the assumption that F has real eigenvalues. In general, F will have different left and right eigenvectors,

$$F\boldsymbol{\mu}_a = \lambda_a^2\boldsymbol{\mu}_a \quad , \quad \boldsymbol{\zeta}_a^T F = \lambda_a^2\,\boldsymbol{\zeta}_a^T \qquad a = 1,\ldots,n \tag{2.124}$$

where there's no sum over a in these equations. The left and right eigenvectors satisfy $\boldsymbol{\zeta}_a \cdot \boldsymbol{\mu}_b = \delta_{ab}$. Note that although the eigenvectors differ, the eigenvalues λ_a^2 for $a = 1,\ldots,n$ are the same. Although λ_a^2 are real for the physical systems of interest (to be proved shortly) they are not always positive. The most general solution to $\ddot{\boldsymbol{\eta}} = F\boldsymbol{\eta}$ is

$$\boldsymbol{\eta}(t) = \sum_a \boldsymbol{\mu}_a \left[A_a e^{\lambda_a t} + B_a e^{-\lambda_a t}\right] \tag{2.125}$$

where A_a and B_a are $2n$ integration constants. Again, we have two possibilities for each eigenvalue

1. $\lambda_a^2 < 0$ In this case $\pm\lambda_a = i\omega_a$ for some real number ω_a. The system will be stable in the corresponding direction $\boldsymbol{\eta} = \boldsymbol{\mu}_a$.

2. $\lambda_a^2 > 0$. Now $\pm\lambda_a$ are real and the system exhibits a linear instability in the direction $\boldsymbol{\eta} = \boldsymbol{\mu}_a$

The eigenvectors $\boldsymbol{\mu}_a$ are called *normal modes*. The equilibrium point is only stable if $\lambda_a^2 < 0$ for every $a = 1,\ldots,n$. If this is the case the system will typically oscillate around the equilibrium point as a linear superposition of all the normal modes, each at a different frequency.

To keep things real, we can write the most general solution as

$$\boldsymbol{\eta}(t) = \sum_{a,\lambda_a^2>0} \boldsymbol{\mu}_a \left[A_a e^{\lambda_a t} + B_a e^{-\lambda_a t}\right] + \sum_{a,\lambda_a^2<0} \boldsymbol{\mu}_a A_a \cos(\omega_a(t - t_a)) \tag{2.126}$$

where now A_a, B_a and t_a are the $2n$ integration constants.

The Reality of the Eigenvalues

Finally, let's show what we put off above: that the eigenvalues λ_a^2 are real for matrices F derived from a physical Lagrangian system. Consider a general Lagrangian of the form,

$$L = \tfrac{1}{2}T_{ij}(q)\dot{q}_i\dot{q}_j - V(q) \tag{2.127}$$

We will require that $T_{ij}(q)$ is invertible and positive definite for all q. Expanding about an equilibrium point as in (2.120), to linear order in η_i the equations read

$$T_{ij}\ddot{\eta}_j = -V_{ij}\eta_j \tag{2.128}$$

where $T_{ij} = T_{ij}(q^0)$ and $V_{ij} = \partial^2 V/\partial q_i \partial q_j$, evaluated at $q_i = q_i^0$. Then in the matrix notation of (2.123), we have $F = -T^{-1}V$. Both T_{ij} and V_{ij} are symmetric, but not necessarily simultaneously diagonalisable. This means that F_{ij} is not necessarily symmetric. Nevertheless, F does have real eigenvalues. To see this, look at

$$F\boldsymbol{\mu} = \lambda^2\boldsymbol{\mu} \quad \Rightarrow \qquad V\boldsymbol{\mu} = -\lambda^2\, T\boldsymbol{\mu} \tag{2.129}$$

So far, both $\boldsymbol{\mu}$ and λ^2 could be complex. We will now show that they're not. Take the inner product of this equation with the complex conjugate eigenvector $\bar{\boldsymbol{\mu}}$. We have $\bar{\boldsymbol{\mu}}\cdot V\boldsymbol{\mu} = \lambda^2\,\bar{\boldsymbol{\mu}}\cdot T\boldsymbol{\mu}$. But for any symmetric matrix S, the quantity $\bar{\boldsymbol{\mu}}\cdot S\boldsymbol{\mu}$ is real. (This follows from expanding $\boldsymbol{\mu}$ in the complete set of real, orthogonal eigenvectors of S, each of which has a real eigenvalue). Therefore both $\bar{\boldsymbol{\mu}}\cdot V\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}\cdot T\boldsymbol{\mu}$ are both real. Since we have assumed that T is invertible and positive definite, we know that $\bar{\boldsymbol{\mu}}\cdot T\boldsymbol{\mu} \neq 0$ so, from (2.129), we conclude that the eigenvalue λ^2 is indeed real.

2.6.1 Example: The Double Pendulum

In section 2.5.2, we derived the Lagrangian for the double pendulum. Restricting to the case where the two masses are the same $m_1 = m_2 = m$ and the two lengths are the same $l_1 = l_2 = l$, we derived the Lagrangian (2.72) for arbitrary oscillations

$$L = ml^2\dot{\theta}_1^2 + \tfrac{1}{2}ml^2\dot{\theta}_2^2 + ml^2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + 2mgl\cos\theta_1 + mgl\cos\theta_2$$

The stable equilibrium point is clearly $\theta_1 = \theta_2 = 0$. (You could check mathematically if you're dubious). Let's expand for small θ_1 and θ_2. If we want to linearise the equations of motion for θ, then we must expand the Lagrangian to second order (so that after we take derivatives, there's still a θ left standing). We have

$$L \approx ml^2\dot{\theta}_1^2 + \tfrac{1}{2}ml^2\dot{\theta}_2^2 + ml^2\dot{\theta}_1\dot{\theta}_2 - mgl\theta_1^2 - \tfrac{1}{2}mgl\theta_2^2 \tag{2.130}$$

where we've thrown away an irrelevant constant. From this we can use Lagrange's equations to derive the two linearised equations of motion

$$\begin{aligned} 2ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2 &= -2mgl\theta_1 \\ ml^2\ddot{\theta}_2 + ml^2\ddot{\theta}_1 &= -mgl\theta_2 \end{aligned} \tag{2.131}$$

Or, writing $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$, this becomes

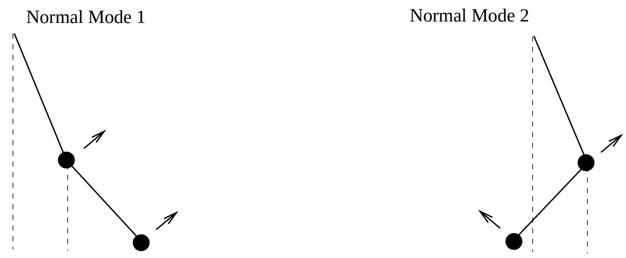

Figure 19: The two normal modes of the double pendulum

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \ddot{\boldsymbol{\theta}} = -\frac{g}{l}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\boldsymbol{\theta} \qquad \Rightarrow \qquad \ddot{\boldsymbol{\theta}} = -\frac{g}{l}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}\boldsymbol{\theta} \tag{2.132}$$

We have two eigenvectors. They are

1. $\boldsymbol{\mu}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ which has eigenvalue $\lambda_1^2 = -(g/l)(2-\sqrt{2})$. This corresponds to the motion shown in figure 19 for the first normal mode.
2. $\boldsymbol{\mu}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$ which has eigenvalue $\lambda_2^2 = -(g/l)(2+\sqrt{2})$. This corresponds to the motion shown in figure 19 for the second normal mode.

We see that the frequency of the mode in which the two rods oscillate in different directions should be higher than that in which they oscillate together.

2.6.2 Example: The Linear Triatomic Molecule

Consider the molecule drawn in the figure. It's a rough approximation of CO_2. We'll only consider motion in the direction parallel to the molecule for each atom, in which case the Lagrangian for this object takes the form,

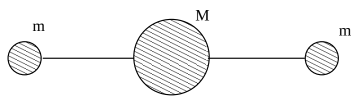

Figure 20:

$$L = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}M\dot{x}_2^2 + \tfrac{1}{2}m\dot{x}_3^2 - V(x_1 - x_2) - V(x_2 - x_3) \tag{2.133}$$

The function V is some rather complicated interatomic potential. But, the point of this section is that if we're interested in oscillations around equilibrium, this doesn't matter. Assume that $x_i = x_i^0$ in equilibrium. By symmetry, we have $|x_1^0 - x_2^0| = |x_2^0 - x_3^0| = r_0$. We write deviations from equilibrium as

$$x_i(t) = x_i^0 + \eta_i(t) \tag{2.134}$$

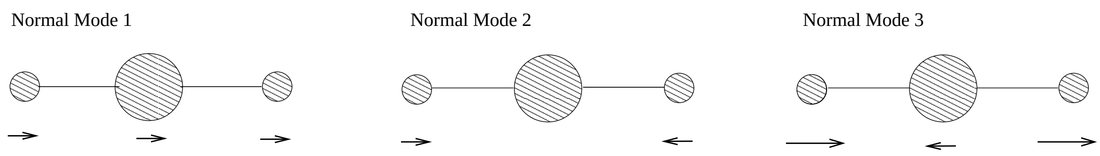

Figure 21: The three normal modes of the triatomic molecule

Taylor expanding the potential about the equilibrium point,

$$V(r) = V(r_0) + \left.\frac{\partial V}{\partial r}\right|_{r=r_0} (r - r_0) + \tfrac{1}{2}\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0} (r - r_0)^2 + \ldots \tag{2.135}$$

Here the first term $V(r_0)$ is a constant and can be ignored, while the second term $\partial V/\partial r$ vanishes since we are in equilibrium. Substituting into the Lagrangian, we have

$$L \approx \tfrac{1}{2}m\dot{\eta}_1^2 + \tfrac{1}{2}M\dot{\eta}_2^2 + \tfrac{1}{2}m\dot{\eta}_3^2 - \frac{k}{2}\left[(\eta_1 - \eta_2)^2 + (\eta_2 - \eta_3)^2\right] \tag{2.136}$$

where $k = \partial^2 V/\partial r^2$ evaluated at $r = r_0$. Then the equations of motion are

$$\begin{pmatrix} m\ddot{\eta}_1 \\ M\ddot{\eta}_2 \\ m\ddot{\eta}_3 \end{pmatrix} = -k \begin{pmatrix} \eta_1 - \eta_2 \\ (\eta_2 - \eta_1) + (\eta_2 - \eta_3) \\ \eta_3 - \eta_2 \end{pmatrix} \tag{2.137}$$

or, putting it in the form $\ddot{\boldsymbol{\eta}} = F\boldsymbol{\eta}$, we have

$$F = \begin{pmatrix} -k/m & k/m & 0 \\ k/M & -2k/M & k/M \\ 0 & k/m & -k/m \end{pmatrix} \tag{2.138}$$

Again, we must look for eigenvectors of F. There are three:

1. $\boldsymbol{\mu} = (1,1,1)^T$ which has eigenvalue $\lambda_1^2 = 0$. But this is just an overall translation of the molecule. It's not an oscillation.

2. $\boldsymbol{\mu}_2 = (1,0,-1)^T$ which has eigenvalue $\lambda_2^2 = -k/m$. In this motion, the outer two atoms oscillate out of phase, while the middle atom remains stationary. The oscillation has frequency $\omega_2 = \sqrt{k/m}$.

3. $\boldsymbol{\mu}_3 = (1, -2m/M, 1)^T$ which has eigenvalue $\lambda_3^2 = -(k/m)(1 + 2m/M)$. This oscillation is a little less obvious. The two outer atoms move in the same direction, while the middle atom moves in the opposite direction. The frequency of this vibration $\omega_3 = \sqrt{-\lambda_3^2}$ is greater than that of the second normal mode.

All motions are drawn in figure 21. For small deviations from equilibrium, the most general motion is a superposition of all of these modes.

$$\boldsymbol{\eta}(t) = \boldsymbol{\mu}_1(A + Bt) + \boldsymbol{\mu}_2 C \cos(\omega_2(t - t_2)) + \boldsymbol{\mu}_3 D \cos(\omega_3(t - t_3)) \tag{2.139}$$

3. The Motion of Rigid Bodies

Figure 22: Wolfgang Pauli and Niels Bohr stare in wonder at a spinning top.

Having now mastered the technique of Lagrangians, this section will be one big application of the methods. The systems we will consider are the spinning motions of extended objects. As we shall see, these can often be counterintuitive. Certainly Pauli and Bohr found themselves amazed!

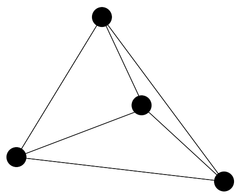

Figure 23:

We shall consider extended objects that don't have any internal degrees of freedom. These are called "rigid bodies", defined to be a collection of N points constrained so that the distance between the points is fixed. i.e.

$$|\mathbf{r}_i - \mathbf{r}_j| = \text{constant} \tag{3.1}$$

for all $i, j = 1, \ldots, N$. A simple example is a dumbbell (two masses connected by a light rod), or the pyramid drawn in the figure. In both cases, the distances between the masses is fixed.

Often we will work with continuous, rather than discrete, bodies simply by replacing $\sum_i m_i \to \int d\mathbf{r}\, \rho(\mathbf{r})$ where $\rho(\mathbf{r})$ is the density of the object. A rigid body has six degrees of freedom

$$3 \text{ Translation} + 3 \text{ Rotation}$$

The most general motion of a free rigid body is a translation plus a rotation about some point P. In this section we shall develop the techniques required to describe this motion.

3.1 Kinematics

Consider a body fixed at a point P. The most general allowed motion is a rotation about P. To describe this, we specify positions in a *fixed space frame* $\{\tilde{\mathbf{e}}_a\}$ by embedding a *moving body frame* $\{\mathbf{e}_a\}$ in the body so that $\{\mathbf{e}_a\}$ moves with the body.

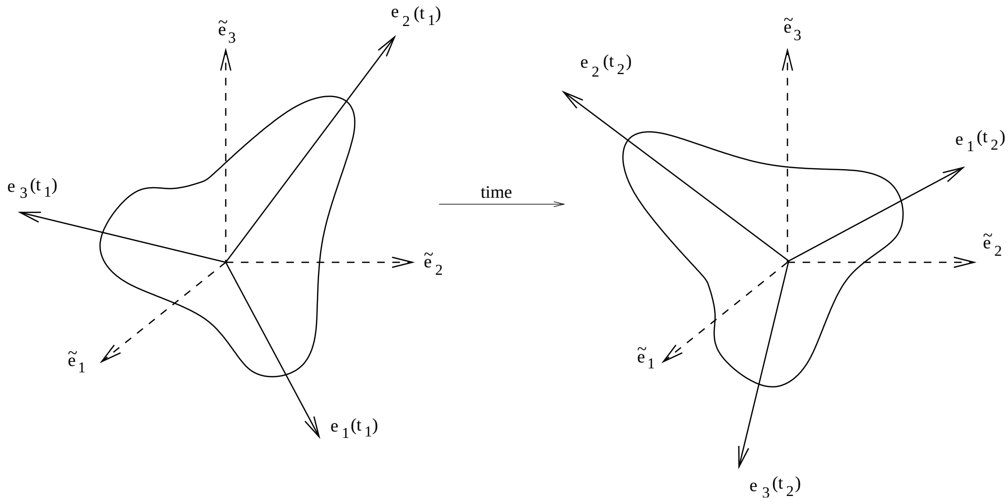

Figure 24: The fixed space frame and the moving body frame.

Both axes are orthogonal, so we have

$$\tilde{\mathbf{e}}_a \cdot \tilde{\mathbf{e}}_b = \delta_{ab} \quad , \quad \mathbf{e}_a(t) \cdot \mathbf{e}_b(t) = \delta_{ab} \tag{3.2}$$

We will soon see that there is a natural choice of the basis $\{\mathbf{e}_a\}$ in the body.

Claim: For all t, there exists a unique orthogonal matrix $R(t)$ with components $R_{ab}(t)$ such that $\mathbf{e}_a(t) = R_{ab}(t)\tilde{\mathbf{e}}_b$

Proof: $\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab} \Rightarrow R_{ac}R_{bd}\tilde{\mathbf{e}}_c \cdot \tilde{\mathbf{e}}_d = \delta_{ab} \Rightarrow R_{ac}R_{bc} = \delta_{ab}$ or, in other words, $(R^T R)_{ab} = \delta_{ab}$ which is the statement that R is orthogonal. The uniqueness of R follows by construction: $R_{ab} = \mathbf{e}_a \cdot \tilde{\mathbf{e}}_b$. $\square$.

So as the rigid body rotates it is described by a time dependent orthogonal 3×3 matrix $R(t)$. This matrix also has the property that its determinant is 1. (The other possibility is that its determinant is -1 which corresponds to a rotation and a reflection $\mathbf{e}_a \to -\mathbf{e}_a$). Conversely, every one-parameter family $R(t)$ describes a possible motion of the body. We have

$$C = \text{Configuration Space} = \text{Space of } 3 \times 3 \text{ Special Orthogonal Matrices} \equiv SO(3)$$

A 3×3 matrix has 9 components but the condition of orthogonality $R^T R = 1$ imposes 6 relations, so the configuration space C is 3 dimensional and we need 3 generalised coordinates to parameterise C. We shall describe a useful choice of coordinates, known as Euler angles, in section 3.5.

3.1.1 Angular Velocity

Any point $\mathbf{r}$ in the body can be expanded in either the space frame or the body frame:

$$\begin{aligned} \mathbf{r}(t) &= \tilde{r}_a(t)\,\tilde{\mathbf{e}}_a \qquad \text{in the space frame} \\ &= r_a\,\mathbf{e}_a(t) \qquad \text{in the body frame} \end{aligned} \tag{3.3}$$

where $\tilde{r}_b(t) = r_a R_{ab}(t)$. Taking the time derivative, we have

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \frac{d\tilde{r}_a}{dt}\,\tilde{\mathbf{e}}_a \qquad \text{in the space frame} \\ &= r_a\,\frac{d\mathbf{e}_a(t)}{dt} \qquad \text{in the body frame} \\ &= r_a\,\frac{dR_{ab}}{dt}\,\tilde{\mathbf{e}}_b \end{aligned} \tag{3.4}$$

Alternatively, we can ask how the body frame basis itself changes with time,

$$\frac{d\mathbf{e}_a}{dt} = \frac{dR_{ab}}{dt}\,\tilde{\mathbf{e}}_b = \frac{dR_{ab}}{dt}\,R^{-1}_{bc}\,\mathbf{e}_c \equiv \omega_{ac}\mathbf{e}_c \tag{3.5}$$

where, in the last equality, we have defined $\omega_{ac} = \dot{R}_{ab}(R^{-1})_{bc} = \dot{R}_{ab}R_{cb}$ using the fact that $R^T R = 1$.

Claim: $\omega_{ac} = -\omega_{ca}$ i.e. ω is antisymmetric.

Proof: $R_{ab}R_{cb} = \delta_{ac} \Rightarrow \dot{R}_{ab}R_{cb} + R_{ab}\dot{R}_{cb} = 0 \Rightarrow \omega_{ac} + \omega_{ca} = 0$ □

Since ω_{ac} is antisymmetric, we can use it to define an object with a single index (which we will also call ω) using the formula

$$\omega_a = \tfrac{1}{2}\epsilon_{abc}\omega_{bc} \tag{3.6}$$

so that $\omega_3 = \omega_{12}$ and so on. We treat these ω_a as the components of a vector *in the body frame*, so $\boldsymbol{\omega} = \omega_a \mathbf{e}_a$. Then finally we have our result for the change of the body frame basis with time

$$\frac{d\mathbf{e}_a}{dt} = -\epsilon_{abc}\omega_b \mathbf{e}_c = \boldsymbol{\omega} \times \mathbf{e}_a \tag{3.7}$$

where, in the second equality, we have used the fact that our body frame axis has a "right-handed" orientation, meaning $\mathbf{e}_a \times \mathbf{e}_b = \epsilon_{abc}\mathbf{e}_c$. The vector $\boldsymbol{\omega}$ is called the *instantaneous angular velocity* and its components ω_a are measured with respect to the body frame.

Since the above discussion was a little formal, let's draw a picture to uncover the physical meaning of $\boldsymbol{\omega}$. Consider a displacement of a given point $\mathbf{r}$ in the body by rotating an infinitesimal amount $d\phi$ about an axis $\hat{\mathbf{n}}$. From the figure, we see that $|d\mathbf{r}| = |\mathbf{r}|\, d\phi \sin\theta$. Moreover, this displacement is perpendicular to $\mathbf{r}$ since the distance to P is fixed by the definition of a rigid body. So we have

$$d\mathbf{r} = d\boldsymbol{\phi} \times \mathbf{r} \tag{3.8}$$

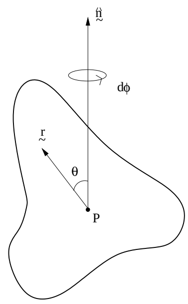

Figure 25:

with $d\boldsymbol{\phi} = \hat{\mathbf{n}} d\phi$. "Dividing" this equation by dt, we have the result

$$\dot{r} = \boldsymbol{\omega} \times \mathbf{r} \tag{3.9}$$

where $\boldsymbol{\omega} = d\boldsymbol{\phi}/dt$ is the instantaneous angular velocity. In general, both the axis of rotation $\hat{\mathbf{n}}$ and the rate of rotation $d\phi/dt$ will change over time.

Aside: One could define a slightly different type of angular velocity by looking at how the space frame coordinates $\tilde{r}_a(t)$ change with time, rather than the body frame axes $\mathbf{e}_a$. Since we have $\tilde{r}_b(t) = r_a R_{ab}(t)$, performing the same steps as above, we have

$$\dot{\tilde{r}}_b = r_a \dot{R}_{ab} = \tilde{r}_a (R^{-1}\dot{R})_{ab} \tag{3.10}$$

which tempts us to define a different type of angular velocity, sometimes referred to as "convective angular velocity" by $\Omega_{ab} = R^{-1}_{ac}\dot{R}_{cb}$ which has the R^{-1} and $\dot{R}$ in a different order. Throughout our discussion of rigid body motion, we will only deal with the original $\omega = \dot{R}R^{-1}$.

3.1.2 Path Ordered Exponentials

In the remainder of this chapter, we will devote much effort to determine the the angular velocity vector $\boldsymbol{\omega}(t)$ of various objects as they spin and turn. But how do we go from this to the rotation $R(t)$? As described above, we first turn the vector $\boldsymbol{\omega} = w_a \mathbf{e}_a$ into a 3×3 antisymmetric matrix $\omega_{ab} = \epsilon_{abc}\omega_c$. Then, from this, we get the rotation matrix R by solving the differential equation

$$\omega = \frac{dR}{dt} R^{-1} \tag{3.11}$$

If ω and R were scalar functions of time, then we could simply integrate this equation to get the solution

$$R(t) = \exp\left(\int_0^t \omega(t')\, dt'\right) \tag{3.12}$$

which satisfies the initial condition $R(0) = 1$. But things are more complicated because both ω and R are matrices. Let's first describe how we take the exponential of a matrix. This is defined by the Taylor expansion. For any matrix M, we have

$$\exp(M) \equiv 1 + M + \tfrac{1}{2}M^2 + \ldots \tag{3.13}$$

As our first guess for the solution to the matrix equation (3.11), we could try the scalar solution (3.12) and look at what goes wrong. If we take the time derivative of the various terms in the Taylor expansion of this putative solution, then problems first arise when we hit the $\frac{1}{2}M^2$ type term. The time derivative of this reads

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^t \omega(t')\, dt'\right)^2 = \frac{1}{2}\omega(t)\left(\int_0^t \omega(t')\, dt'\right) + \frac{1}{2}\left(\int_0^t \omega(t')\, dt'\right)\omega(t) \tag{3.14}$$

We would like to show that $\dot{R} = \omega R$. The first term on the right-hand side looks good since it appears in the Taylor expansion of ωR. But the second term isn't right. The problem is that we cannot commute $\omega(t)$ past $\omega(t')$ when $t' \neq t$. For this reason, equation (3.12) is *not* the solution to (3.11) when ω and R are matrices. But it does give us a hint about how we should proceed. Since the problem is in the ordering of the matrices, the correct solution to (3.11) takes a similar form as (3.12), but with a different ordering. It is the *path ordered exponential*,

$$R(t) = P\,\exp\left(\int_0^t \omega(t')\, dt'\right) \tag{3.15}$$

where the P in front means that when we Taylor expand the exponential, all matrices are ordered so that later times appear on the left. In other words

$$R(t) = 1 + \int_0^t \omega(t')\, dt' + \int_0^{t''} \int_{t'}^{t} \omega(t'')\, \omega(t')\; dt' dt'' + \ldots \tag{3.16}$$

The double integral is taken over the range $0 < t' < t'' < t$. If we now differentiate this double integral with respect to t, we get just the one term $\omega(t)\left(\int_0^t \omega(t')\, dt'\right)$, instead of the two that appear in (3.14). It can be checked that the higher terms in the Taylor expansion also have the correct property if they are ordered so that matrices evaluated at later times appear to the left in the integrals. This type of path ordered integral comes up frequently in theories involving non-commuting matrices, including the standard model of particle physics.

As an aside, the rotation matrix R is a member of the Lie group $SO(3)$, the space of 3×3 orthogonal matrices with unit determinant. The antisymmetric angular velocity matrix ω, corresponding to an instantaneous, infinitesimal rotation, lives in the Lie algebra $so(3)$.

3.2 The Inertia Tensor

Let's look at the kinetic energy for a rotating body. We can write

$$\begin{aligned} T &= \tfrac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2 \\ &= \tfrac{1}{2} \sum_i m_i \, (\boldsymbol{\omega} \times \mathbf{r}_i) \cdot (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &= \tfrac{1}{2} \sum_i m_i \left((\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_i \cdot \mathbf{r}_i) - (r_i \cdot \boldsymbol{\omega})^2 \right) \end{aligned} \tag{3.17}$$

Or, in other words, we can write the kinetic energy of a rotating body as

$$T = \tfrac{1}{2} \omega_a I_{ab} \omega_b \tag{3.18}$$

where I_{ab}, $a, b = 1, 2, 3$ are the components of the *inertia tensor* measured in the body frame, defined by

$$I_{ab} = \sum_i m_i \left((\mathbf{r}_i \cdot \mathbf{r}_i)\delta_{ab} - (\mathbf{r}_i)_a (\mathbf{r}_i)_b \right) \tag{3.19}$$

Note that $I_{ab} = I_{ba}$ so the inertia tensor is symmetric. Moreover, the components are independent of time since they are measured with respect to the body frame. For continuous bodies, we have the analogous expression

$$I = \int d^3\mathbf{r}\,\rho(\mathbf{r}) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \tag{3.20}$$

Since I_{ab} is a symmetric real matrix, we can diagonalise it. This means that there exists an orthogonal matrix O such that $OIO^T = I'$ where I' is diagonal. Equivalently, we can rotate the body frame axis $\{\mathbf{e}_a\}$ to coincide with the eigenvectors of I (which are $\{O\mathbf{e}_a\}$) so that, in this frame, the inertia tensor is diagonal. These preferred body axes, in which I is diagonal, are called the *principal axes*. In this basis,

$$I = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix} \tag{3.21}$$

The eigenvalues I_a are called the *principal moments of inertia*. The kinematical properties of a rigid body are fully determined by its mass, principal axes, and moments of inertia. Often the principal axes are obvious by symmetry.

Claim: The I_a are real and positive.

Proof: If $\mathbf{c}$ is an arbitrary vector, then

$$I_{ab}c^a c^b = \sum_i m_i(r_i^2 c^2 - (\mathbf{r}_i \cdot \mathbf{c})^2) \geq 0 \tag{3.22}$$

with equality only if all the $\mathbf{r}_i$ lie on a line. If $\mathbf{c}$ is the a^{th} eigenvector of I then this result becomes $I_{ab}c^a c^b = I_a|\mathbf{c}|^2$ which tells us $I_a \geq 0$. $\square$

Example: The Rod

Consider the inertia tensor of a uniform rod of length l and mass M about its centre. The density of the rod is $\rho = M/l$. By symmetry, we have $I = \text{diag}(I_1, I_1, 0)$ where

$$I_1 = \int_{-l/2}^{l/2} \rho\, x^2\, dx = \frac{1}{12}Ml^2 \tag{3.23}$$

Example: The Disc

Now consider a uniform disc of radius r and mass M. We take the z axis to be perpendicular to the disc and measure I about its centre of mass. Again we know that $I = \mathrm{diag}(I_1, I_2, I_3)$. The density of the disc is $\rho = M/\pi r^2$, so we have

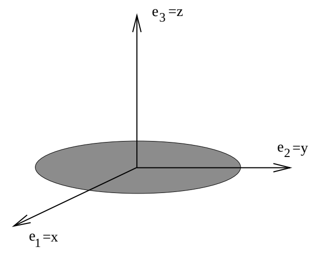

Figure 26:

$$I_1 = \int \rho y^2 d^2x \quad , \quad I_2 = \int \rho x^2 d^2x$$

so $I_1 = I_2$ by symmetry, while

$$I_3 = \int \rho(x^2 + y^2) d^2x$$

Therefore

$$I_3 = I_1 + I_2 = 2\pi\rho \int_0^r r'^3\, dr' = \tfrac{1}{2}Mr^2 \tag{3.24}$$

So the moments of inertia are $I_1 = I_2 = \frac{1}{4}Mr^2$ and $I_3 = \frac{1}{2}Mr^2$.

3.2.1 Parallel Axis Theorem

The inertia tensor depends on what point P in the body is held fixed. In general, if we know I about a point P it is messy to compute it about some other point P'. But it is very simple if P happens to coincide with the centre of mass of the object.

Claim: If P' is displaced by $\mathbf{c}$ from the centre of mass, then

$$(I_\mathbf{c})_{ab} = (I_{c.of.m})_{ab} + M(c^2\delta_{ab} - \mathbf{c}_a\mathbf{c}_b) \tag{3.25}$$

Proof:

$$\begin{aligned}(I_\mathbf{c})_{ab} &= \sum_i m_i \left\{(\mathbf{r}_i - \mathbf{c})^2\delta_{ab} - (\mathbf{r}_i - \mathbf{c})_a(\mathbf{r}_i - \mathbf{c})_b\right\} \qquad (3.26)\\ &= \sum_i m_i \left\{r_i^2\delta_{ab} - (\mathbf{r}_i)_a(\mathbf{r}_i)_b + [-2\mathbf{r}_i \cdot \mathbf{c}\delta_{ab} + (\mathbf{r}_i)_a\mathbf{c}_b + (\mathbf{r}_i)_b\mathbf{c}_a] + (c^2\delta_{ab} - \mathbf{c}_a\mathbf{c}_b)\right\}\end{aligned}$$

But the terms in square brackets that are linear in $\mathbf{r}_i$ vanish if $\mathbf{r}_i$ is measured from the centre of mass since $\sum_i m_i\mathbf{r}_i = 0$. $\square$

The term $M(c^2\delta_{ab} - \mathbf{c}_a\mathbf{c}_b)$ is the inertia tensor we would find if the whole body was concentrated at the centre of mass.

Example: The Rod Again

The inertia tensor of the rod about one of its ends is $I_1 = \frac{1}{12}Ml^2 + M(l/2)^2 = \frac{1}{3}Ml^2$.

Example: The Disc Again

Consider measuring the inertia tensor of the disc about a point displaced by $\mathbf{c} = (c, 0, 0)$ from the centre. We have

$$\begin{aligned} I_{\mathbf{c}} &= M \begin{pmatrix} \frac{1}{4}r^2 & & \\ & \frac{1}{4}r^2 & \\ & & \frac{1}{2}r^2 \end{pmatrix} + M \left[\begin{pmatrix} c^2 & & \\ & c^2 & \\ & & c^2 \end{pmatrix} - \begin{pmatrix} c^2 & & \\ & 0 & \\ & & 0 \end{pmatrix} \right] \\ &= M \begin{pmatrix} \frac{1}{4}r^2 & & \\ & \frac{1}{4}r^2 + c^2 & \\ & & \frac{1}{2}r^2 + c^2 \end{pmatrix} \end{aligned}$$

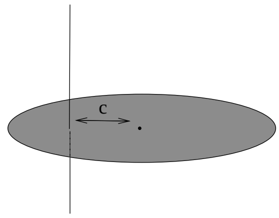

Figure 27:

3.2.2 Angular Momentum

The angular momentum $\mathbf{L}$ about a point P can also be described neatly in terms of the inertia tensor. We have

$$\begin{aligned} \mathbf{L} &= \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i \\ &= \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &= \sum_i m_i (r_i^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r}_i)\mathbf{r}_i) \\ &= I\boldsymbol{\omega} \end{aligned} \tag{3.27}$$

In the body frame, we can write $\mathbf{L} = L_a \mathbf{e}_a$ to get

$$L_a = I_{ab}\omega_b \tag{3.28}$$

where $\boldsymbol{\omega} = \omega_a \mathbf{e}_a$. Note that in general, $\boldsymbol{\omega}$ is *not* equal to $\mathbf{L}$: the spin of the body and its angular momentum point in different directions. This fact will lead to many of the peculiar properties of spinning objects.

3.3 Euler's Equations

So far we have been discussing the rotation of a body fixed at a point P. In this section we will be interested in the rotation of a free body suspended in space - for example, a satellite or the planets. Thankfully, this problem is identical to that of an object fixed at a point. Let's show why this is the case and then go on to analyse the motion.

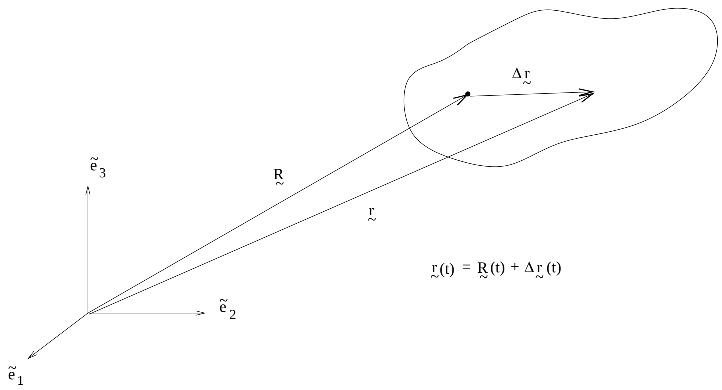

Figure 28:

The most general motion of a body is an overall translation superposed with a rotation. We could take this rotation to be about any point in the body (or, indeed, a point outside the body). But it is useful to consider the rotation to be about the center of mass. We can write the position of a particle in the body as

$$\mathbf{r}_i(t) = \mathbf{R}(t) + \Delta\mathbf{r}_i(t) \tag{3.29}$$

where $\Delta\mathbf{r}_i$ is the position measured from the centre of mass. Then examining the kinetic energy (which, for a free body, is all there is)

$$\begin{aligned} T &= \tfrac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i^2 \\ &= \sum_i m_i \left[\tfrac{1}{2}\dot{\mathbf{R}}^2 + \dot{\mathbf{R}}\cdot(\boldsymbol{\omega}\times\Delta\mathbf{r}_i) + \tfrac{1}{2}(\boldsymbol{\omega}\times\Delta\mathbf{r}_i)^2\right] \\ &= \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\boldsymbol{\omega}_a I_{ab}\boldsymbol{\omega}_b \end{aligned} \tag{3.30}$$

where we've used the fact that $\sum_i m_i \Delta\mathbf{r}_i = 0$. So we find that the dynamics separates into the motion of the centre of mass $\mathbf{R}$, together with rotation about the centre of mass. This is the reason that the analysis of the last section is valid for a free object.

3.3.1 Euler's Equations

From now on, we shall neglect the center of mass and concentrate on the rotation of the rigid body. Since the body is free, its angular momentum must be conserved. This gives us the vector equation

$$\frac{d\mathbf{L}}{dt} = 0 \tag{3.31}$$

Let's expand this in the body frame. we have

$$0 = \frac{d\mathbf{L}}{dt} = \frac{dL_a}{dt}\,\mathbf{e}_a + L_a\,\frac{d\mathbf{e}_a}{dt}$$
$$= \frac{dL_a}{dt}\,\mathbf{e}_a + L_a\,\boldsymbol{\omega}\times\mathbf{e}_a \tag{3.32}$$

This simplifies if we choose the body axes $\{\mathbf{e}_a\}$ to coincide with the the principal axes. Using $L_a = I_{ab}\omega_b$, we can then write $L_1 = I_1\omega_1$ and so on. The equations of motion (3.32) are now three non-linear coupled first order differential equations,

$$\begin{aligned} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) &= 0 \\ I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) &= 0 \\ I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) &= 0 \end{aligned} \tag{3.33}$$

These are *Euler's Equations.*

We can extend this analysis to include a torque $\boldsymbol{\tau}$. The equation of motion becomes $\dot{\mathbf{L}} = \boldsymbol{\tau}$ and we can again expand in the body frame along the principal axes to derive Euler's equations (3.33), now with the components of the torque on the RHS.

3.4 Free Tops

> "To those who study the progress of exact science, the common spinning-top is a symbol of the labours and the perplexities of men."
>
> *James Clerk Maxwell, no less*

In this section, we'll analyse the motion of free rotating bodies (known as free tops) using Euler's equation.

We start with a trivial example: the sphere. For this object, $I_1 = I_2 = I_3$ which means that the angular velocity $\boldsymbol{\omega}$ is parallel to the angular momentum $\mathbf{L}$. Indeed, Euler's equations tell us that ω_a is a constant in this case and the sphere continues to spin around the same axis you start it on. To find a more interesting case, we need to look at the next simplest object.

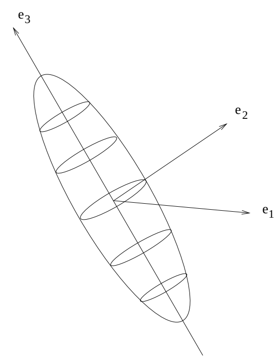

Figure 29:

3.4.1 The Symmetric Top

The symmetric top is an object with $I_1 = I_2 \neq I_3$. A typical example is drawn in figure 29. Euler's equations become

$$\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_1 - I_3) \\ I_2\dot{\omega}_2 &= -\omega_1\omega_3(I_1 - I_3) \\ I_3\dot{\omega}_3 &= 0 \end{aligned} \tag{3.34}$$

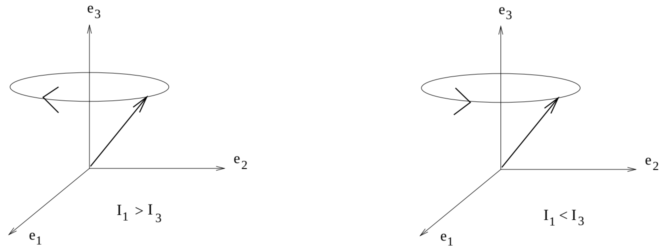

Figure 30: The precession of the spin: the direction of precession depends on whether the object is short and fat ($I_3 > I_1$) or tall and skinny ($I_3 < I_1$)

So, in this case, we see that ω_3, which is the spin about the symmetric axis, is a constant of motion. In contrast, the spins about the other two axes are time dependent and satisfy

$$\dot{\omega}_1 = \Omega\omega_2 \quad , \quad \dot{\omega}_2 = -\Omega\omega_1 \tag{3.35}$$

where

$$\Omega = \omega_3(I_1 - I_3)/I_1 \tag{3.36}$$

is a constant. These equations are solved by

$$(\omega_1, \omega_2) = \omega_0(\sin \Omega t, \cos \Omega t) \tag{3.37}$$

for any constant ω_0. This means that, in the body frame, the direction of the spin is not constant: it *precesses* about the $\mathbf{e}_3$ axis with frequency Ω. The direction of the spin depends on the sign on Ω or, in other words, whether $I_1 > I_3$ or $I_1 < I_3$. This is drawn in figure 30.

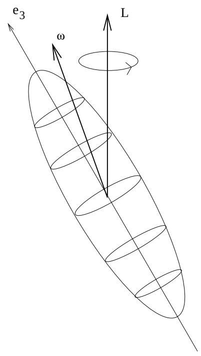

Figure 31:

What does this look like in an inertial frame? The angular momentum $\mathbf{L}$ is a fixed vector. But ω^3, and hence L_3, are also fixed which ensures that the angle between $\mathbf{e}_3$ and $\mathbf{L}$ doesn't change in time. Instead, $\mathbf{e}_3$ precesses around $\mathbf{L}$, while the body simultaneously spins such that $\boldsymbol{\omega}$ remains between $\mathbf{e}_3$ and $\mathbf{L}$. (If this is hard to visualise, then try searching YouTube for "free symmetric tops" or something similar and you will find videos like this.) The fact that $\boldsymbol{\omega}$ precesses around the body frame axis $\mathbf{e}_3$ is sometimes referred to as a wobble.

3.4.2 Example: The Earth's Wobble

The spin of the earth causes it to bulge at the equator so it is no longer a sphere but can be treated as a symmetric top. It is an oblate ellipsoid, with $I_3 > I_1$, and is spherical to roughly 1 part in 300, meaning

$$\frac{I_1 - I_3}{I_1} \approx -\frac{1}{300} \tag{3.38}$$

Of course, we know the magnitude of the spin ω_3: it is $\omega_3 = (1 \text{ day})^{-1}$. This information is enough to calculate the frequency of the earth's wobble; from (3.36), it should be

$$\Omega_{earth} = \frac{1}{300} \text{ day}^{-1} \tag{3.39}$$

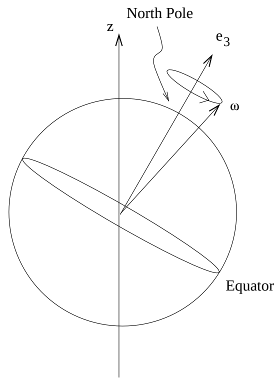

Figure 32:

This calculation was first performed by Euler in 1749 who predicted that the Earth completes a wobble every 300 days. Despite many searches, this effect wasn't detected until 1891 when Chandler re-analysed the data and saw a wobble with a period of 427 days. It is now known as the *Chandler wobble*. It is very small! The angular velocity $\boldsymbol{\omega}$ intercepts the surface of the earth approximately 10 metres from the North pole and precesses around it. More recent measurements place the frequency at 435 days, with the discrepancy between the predicted 300 days and observed 435 days due to the fact that the Earth is not a rigid body, but is flexible because of tidal effects. Less well understood is why these same tidal effects haven't caused the wobble to dampen and disappear completely. There are various theories about what keeps the wobble alive, from earthquakes to fluctuating pressure at the bottom of the ocean.

3.4.3 The Asymmetric Top: Stability

The most general body has no symmetries and $I_1 \neq I_2 \neq I_3 \neq I_1$. The rotational motion is more complicated but there is a simple result that we will describe here. Consider the case where the spin is completely about one of the principal axes, say $\mathbf{e}_1$. i.e.

$$\omega_1 = \Omega \quad , \quad \omega_2 = \omega_3 = 0 \tag{3.40}$$

This solves Euler's equations (3.33). The question we want to ask is: what happens if the spin varies slightly from this direction? To answer this, consider small perturbations about the spin

$$\omega_1 = \Omega + \eta_1 \quad , \quad \omega_2 = \eta_2 \quad , \quad \omega_3 = \eta_3 \tag{3.41}$$

where η_a, $a = 1, 2, 3$ are all taken to be small. Substituting this into Euler's equations and ignoring terms of order η^2 and higher, we have

$$\begin{aligned} I_1 \dot{\eta}_1 &= 0 \\ I_2 \dot{\eta}_2 &= \Omega \eta_3 (I_3 - I_1) \qquad (3.42) \\ I_3 \dot{\eta}_3 &= \Omega \eta_2 (I_1 - I_2) \qquad (3.43) \end{aligned}$$

We substitute the third equation into the second to find an equation for just one of the perturbations, say η_2,

$$I_2 \ddot{\eta}_2 = \frac{\Omega^2}{I_3}(I_3 - I_1)(I_1 - I_2)\eta_2 \equiv A\eta_2 \qquad (3.44)$$

The fate of the small perturbation depends on the sign of the quantity A. We have two possibilities

- $A < 0$: In this case, the disturbance will oscillate around the constant motion.
- $A > 0$: In this case, the disturbance will grow exponentially.

Examining the definition of A, we find that the motion is unstable if

$$I_2 < I_1 < I_3 \quad \text{or} \quad I_3 < I_1 < I_2 \qquad (3.45)$$

with all other motions stable. In other words, a body will rotate stably about the axis with the largest or the smallest moment of inertia, but not about the intermediate axis. Pick up a tennis racket and try it for yourself!

3.4.4 The Asymmetric Top: Poinsot Construction

The analytic solution for the general motion of an asymmetric top is rather complicated, involving Jacobian elliptic functions. But there's a nice geometrical way of viewing the motion due to Poinsot.

We start by working in the body frame. There are two constants of motion: the kinetic energy T and the magnitude of the angular momentum $\mathbf{L}^2$. In terms of the angular velocity, they are

$$\begin{aligned} 2T &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \qquad (3.46) \\ \mathbf{L}^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \qquad (3.47) \end{aligned}$$

Each of these equations defines an ellipsoid in $\boldsymbol{\omega}$ space. The motion of the vector $\boldsymbol{\omega}$ is constrained to lie on the intersection of these two ellipsoids. The first of these ellipsoids, defined by

$$\frac{I_1}{2T}\,\omega_1^2 + \frac{I_2}{2T}\,\omega_2^2 + \frac{I_3}{2T}\,\omega_3^2 = 1 \tag{3.48}$$

is known as the *inertia ellipsoid* (or, sometimes, the inertia quadric). If we fix the kinetic energy, we can think of this abstract ellipsoid as embedded within the object, rotating with it.

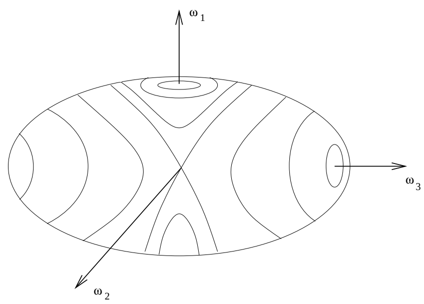

Figure 33:

The inertia ellipsoid is drawn in figure 33, where we've chosen $I_1 > I_2 > I_3$ so that the major axis is ω_3 and the minor axis is ω_1. The lines drawn on the figure are the intersection of the inertia ellipsoid with the other ellipsoid, defined by (3.47), for various values of $\mathbf{L}^2$. Since this has the same major and minor axes as the inertia ellipsoid (because $I_1^2 > I_2^2 > I_3^2$), the intersection lines are small circles around the ω_1 and ω_3 axes, but two lines passing through the ω_2 axis. For fixed T and $\mathbf{L}^2$, the vector $\boldsymbol{\omega}$ moves along one of the intersection lines. This provides a pictorial demonstration of the fact we learnt in the previous subsection: an object will spin in a stable manner around the principal axes with the smallest and largest moments of inertia, but not around the intermediate axis. The path that $\boldsymbol{\omega}$ traces on the inertia ellipsoid is known as the *polhode* curve. We see from the figure that the polhode curves are always closed, and motion in the body frame is periodic.

So much for the body frame. What does all this look like in the space frame? The vector $\mathbf{L}$ is a constant of motion. Since the kinetic energy $2T = \mathbf{L}\cdot\boldsymbol{\omega}$ is also constant, we learn that $\boldsymbol{\omega}$ must lie in a fixed plane perpendicular to $\mathbf{L}$. This is known as the *invariable plane*. The inertia ellipsoid touches the invariable plane at the point defined by the angular velocity vector $\boldsymbol{\omega}$. Moreover, the invariable plane is always tangent to the inertial ellipsoid at the point $\boldsymbol{\omega}$. To see this, note that the angular momentum can be written as

$$\mathbf{L} = \nabla_\omega T \tag{3.49}$$

where the gradient operator is in $\boldsymbol{\omega}$ space, i.e. $\nabla_\omega = (\partial/\partial\omega_1, \partial/\partial\omega_2, \partial/\partial\omega_3)$. But recall that the inertia ellipsoid is defined as a level surface of T, so equation (3.49) tells

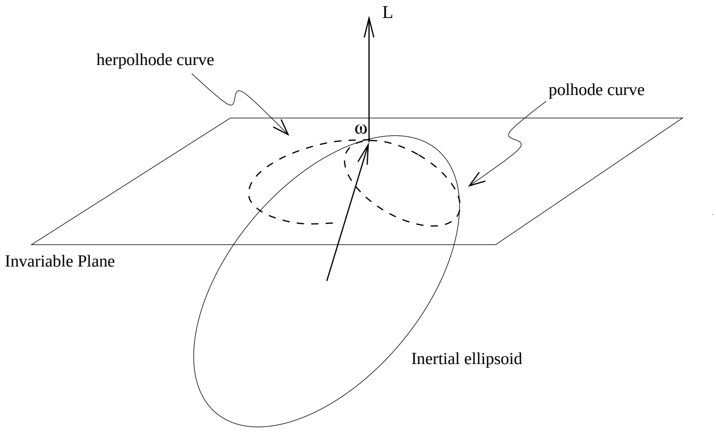

Figure 34: The inertia ellipsoid rolling around on the invariable plane, with the polhode and herpolhode curves drawn for a fixed time period.

us that the angular momentum **L** is always perpendicular to the ellipsoid. This, in turn, ensures that the invariable plane is always tangent to the ellipsoid. In summary, the angular velocity traces out two curves: one on the inertia ellipsoid, known as the polhode curve, and another on the invariable plane, known as the *herpolhode* curve. The body moves as if it is embedded within the inertia ellipsoid, which rolls around the invariable plane without slipping, with the center of the ellipsoid a constant distance from the plane. The motion is shown in figure 34. Unlike the polhode curve, the herpolhode curve does not necessarily close.

An Example: The Asteroid Toutatis

Astronomical objects are usually symmetric, but there's an important exception wandering around our solar system, depicted in figure[2] 35. This is the asteroid Toutatis. In September 2004 it passed the earth at a distance of about four times that to the moon. This is (hopefully!) the closest any asteroid will come for the next 60 years. The orbit of Toutatis is thought to be chaotic, which could potentially be bad news for Earth a few centuries from now. As you can see from the picture, its tumbling motion is complicated. It is aperiodic. The pictures show the asteroid at intervals of a day. The angular momentum vector **L** remains fixed and vertical throughout the motion. The angular velocity $\boldsymbol{\omega}$ traces out the herpolhode curve in the horizontal plane, perpendicular to **L**. The angular momentum vector $\boldsymbol{\omega}$ also traces out a curve over the asteroid's

[2]This picture was created by Scott Hudson of Washington State University and was taken from http://www.solarviews.com/eng/toutatis.htm where you can find many interesting facts about the asteroid.

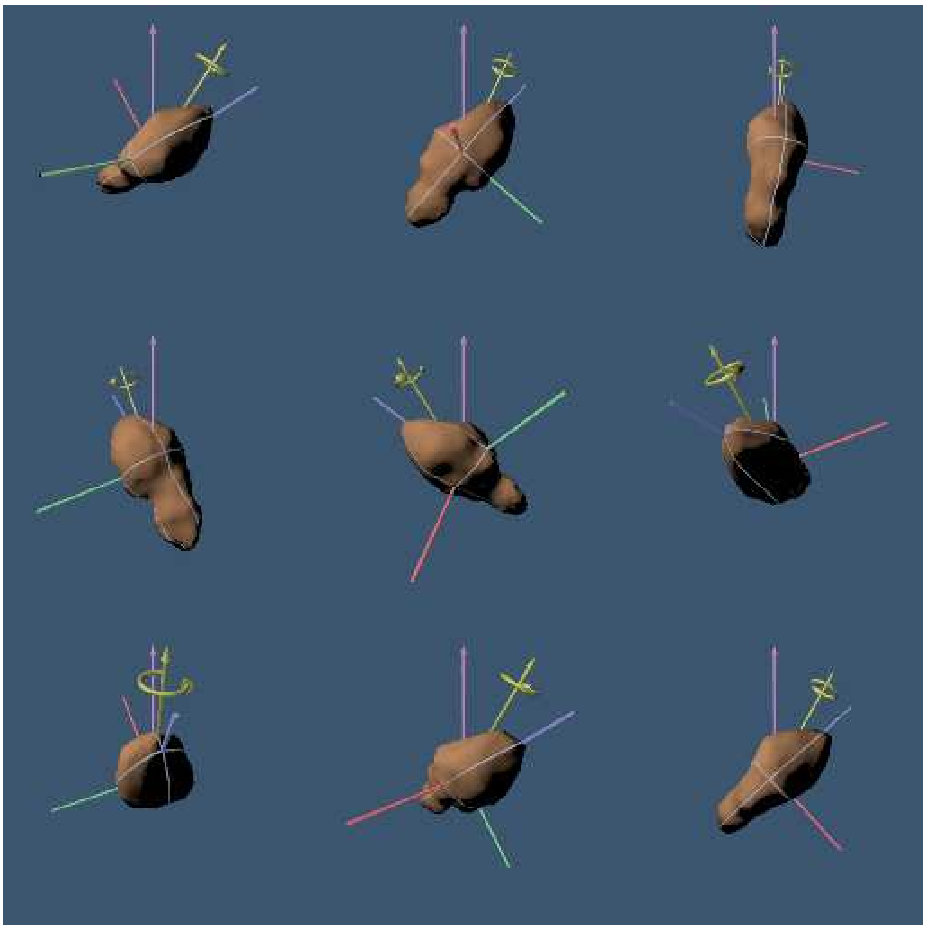

Figure 35: By Toutatis! The three principal axes are shown in red, green and blue (without arrows). The angular momentum $\mathbf{L}$ is the vertical, purple arrow. The angular velocity $\boldsymbol{\omega}$ is the circled, yellow arrow.

surface: this is the polhode curve. It has a period of 5.4 days which you can observe by noting that $\boldsymbol{\omega}$ has roughly the same orientation relative to the principal axes every five to six days.

However, there are further effects at play in a real object like Toutatis which is not spinning around a principal axis. Various stresses and strains lead to dissipation. This means that the angular velocity $\boldsymbol{\omega}$ does not quite follow the polhode curve in Figure 33. Instead it begins close to the major axis ω_3 and slowly spirals towards the minor axis ω_1. This is why we see so few wobbling asteroids.

3.5 Euler's Angles

So far we've managed to make quite a lot of progress working just with the angular velocity ω_a and we haven't needed to introduce an explicit parametrization of the configuration space C. But to make further progress we're going to need to do this. We will use a choice due to Euler which often leads to simple solutions.

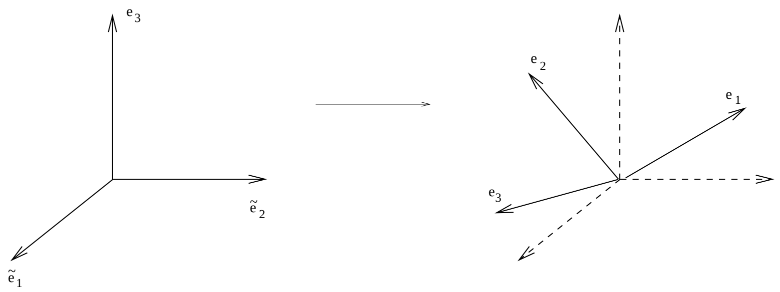

Figure 36: The rotation from space frame $\{\tilde{\mathbf{e}}_a\}$ to body frame $\{\mathbf{e}_a\}$.

A general rotation of a set of axis is shown in Figure 36. We'd like to construct a way of parameterizing such a rotation. The way to do this was first described by Euler:

Euler's Theorem:

An arbitrary rotation may be expressed as the product of 3 successive rotations about 3 (in general) different axes.

Proof: Let $\{\tilde{\mathbf{e}}_a\}$ be space frame axes. Let $\{\mathbf{e}_a\}$ be body frame axes. We want to find the rotation R so that $\mathbf{e}_a = R_{ab}\tilde{\mathbf{e}}_b$. We can accomplish this in three steps

$$\{\tilde{\mathbf{e}}_a\} \stackrel{R_3(\phi)}{\longrightarrow} \{\mathbf{e}'_a\} \stackrel{R_1(\theta)}{\longrightarrow} \{\mathbf{e}''_a\} \stackrel{R_3(\psi)}{\longrightarrow} \{\mathbf{e}_a\} \tag{3.50}$$

Let's look at these step in turn.

Step 1: Rotate by ϕ about the $\tilde{\mathbf{e}}_3$ axis. So $\mathbf{e}'_a = R_3(\phi)_{ab}\tilde{\mathbf{e}}_b$ with

$$R_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.51}$$

This is shown in Figure 37.

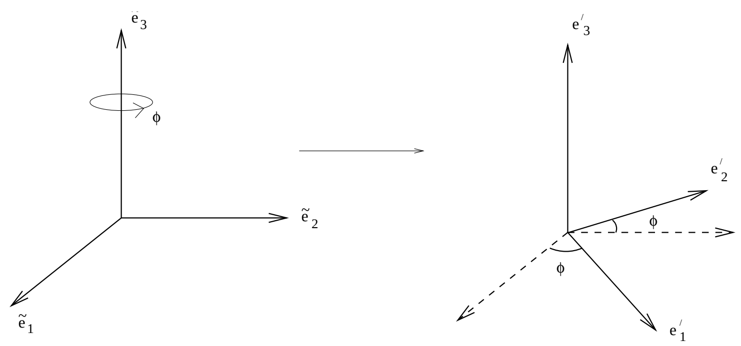

Figure 37: Step 1: Rotate around the space-frame axis $\tilde{\mathbf{e}}_3$.

Step 2: Rotate by θ about the *new* axis $\mathbf{e}'_1$. This axis $\mathbf{e}'_1$ is sometimes called the "line of nodes". We write $\mathbf{e}''_a = R_1(\theta)\mathbf{e}'_b$ with

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{3.52}$$

This is shown in Figure 38

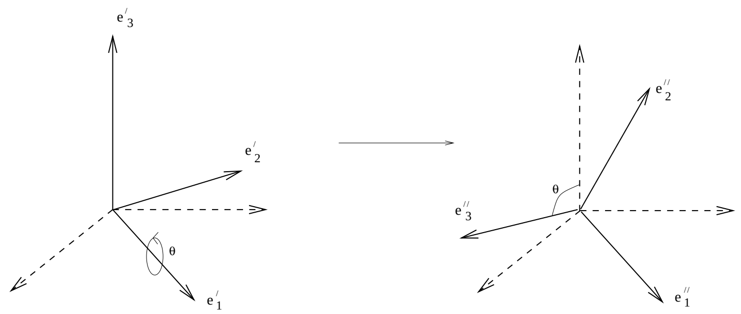

Figure 38: Step 2: Rotate around the new axis axis $\mathbf{e}'_1$.

Step 3: Rotate by ψ about the *new new* axis $\mathbf{e}''_3$ so $\mathbf{e}_a = R_3(\psi)_{ab}\mathbf{e}''_b$ with

$$R_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.53}$$

This is shown in Figure 39.

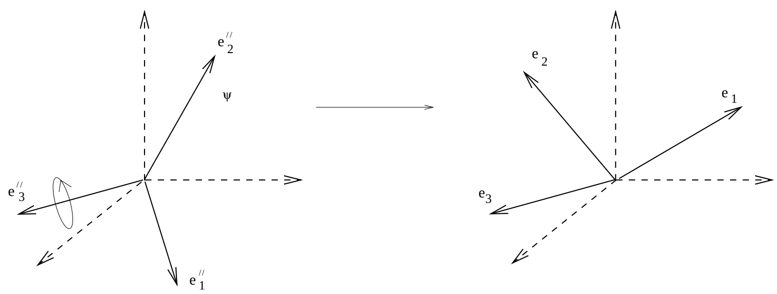

Figure 39: Step 3: Rotate around the latest axis $\mathbf{e}_3''$.

So putting it all together, we have

$$R_{ab}(\phi,\theta,\psi) = [R_3(\psi)R_1(\theta)R_3(\phi)]_{ab} \tag{3.54}$$

□

The angles ϕ,θ,ψ are the Euler angles. If we write out the matrix $R(\phi,\theta,\psi)$ longhand, it reads

$$R = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\sin\psi\cos\phi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Note: Recall that we may expand a vector $\mathbf{r}$ either in the body frame $\mathbf{r} = r_a\mathbf{e}_a$, or in the space frame $\mathbf{r} = \tilde{r}_a\tilde{\mathbf{e}}_a$. The above rotations can be equally well expressed in terms of the coordinates r_a rather than the basis $\{\mathbf{e}_a\}$: we have $\tilde{r}_b = r_a R_{ab}$. Be aware that some books choose to describe the Euler angles in terms of the coordinates r_a which they write in vector form. In some conventions this can lead to an apparent reversal in the ordering of the three rotation matrices.

3.5.1 Leonhard Euler (1707-1783)

As is clear from the section headings, the main man for this chapter is Euler, by far the most prolific mathematician of all time. As well as developing the dynamics of rotations, he made huge contributions to the fields of number theory, geometry, topology, analysis and fluid dynamics. For example, the lovely equation $e^{i\theta} = \cos\theta + i\sin\theta$ is due to Euler. In 1744 he was the first to correctly present a limited example of the calculus of variations (which we saw in section 2.1) although he generously gives credit to a rather botched attempt by his friend Maupertuis in the same year. Euler also invented much of the modern notation of mathematics: $f(x)$ for a function; e for exponential; π for, well, π and so on.

Euler was born in Basel, Switzerland and held positions in St Petersburg, Berlin and, after falling out with Frederick the Great, St Petersburg again. He was pretty much absorbed with mathematics day and night. Upon losing the sight in his right eye in his twenties he responded with: "Now I will have less distraction". Even when he went completely blind later in life, it didn't slow him down much as he went on to produce over half of his total work. The St Petersburg Academy of Science continued to publish his work for a full 50 years after his death.

3.5.2 Angular Velocity

There's a simple expression for the instantaneous angular velocity $\boldsymbol{\omega}$ in terms of Euler angles. To derive this, we could simply plug (3.54) into the definition of angular velocity (3.5). But this is tedious, and a little bit of thought about what this means physically will get us there quicker. Consider the motion of a rigid body in an infinitesimal time dt during which

$$(\psi, \theta, \phi) \to (\psi + d\psi, \theta + d\theta, \phi + d\phi) \tag{3.55}$$

From the definition of the Euler angles, the angular velocity must be of the form

$$\boldsymbol{\omega} = \dot{\phi}\, \tilde{\mathbf{e}}_3 + \dot{\theta}\, \mathbf{e}_1' + \dot{\psi}\, \mathbf{e}_3 \tag{3.56}$$

But we can express the first two vectors in terms of the body frame. They are

$$\begin{aligned} \tilde{\mathbf{e}}_3 &= \sin\theta \sin\psi\, \mathbf{e}_1 + \sin\theta\cos\psi\, \mathbf{e}_2 + \cos\theta\, \mathbf{e}_3 \\ \mathbf{e}_1' &= \cos\psi\, \mathbf{e}_1 - \sin\psi\, \mathbf{e}_2 \end{aligned} \tag{3.57}$$

from which we can express $\boldsymbol{\omega}$ in terms of the Euler angles in the body frame axis

$$\boldsymbol{\omega} = [\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]\mathbf{e}_1 + [\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi]\mathbf{e}_2 + [\dot{\psi} + \dot{\phi}\cos\theta]\mathbf{e}_3 \tag{3.58}$$

By playing a similar game, we can also express $\boldsymbol{\omega}$ in the space frame axis.

3.5.3 The Free Symmetric Top Revisited

In section 3.4 we studied the free symmetric top working in the *body frame* and found a constant spin ω_3 while, as shown in equation (3.37), ω_1 and ω_2 precess as

$$(\omega_1, \omega_2) = \omega_0(\sin\Omega t, \cos\Omega t) \quad \text{with} \quad \Omega = \omega_3 \frac{(I_1 - I_3)}{I_1} \tag{3.59}$$

But what does this look like in the *space frame*? Now that we have parametrised motion in the space frame in terms of Euler angles, we can answer this question. This is simplest if we choose the angular momentum $\mathbf{L}$ to lie along the $\tilde{\mathbf{e}}_3$ space-axis. Then because both $\mathbf{L}$ and its body-frame component $L_3 = I_3\omega_3$ are conserved, so too is the angle between them. But this angle is precisely θ. We learn that $\dot{\theta} = 0$.

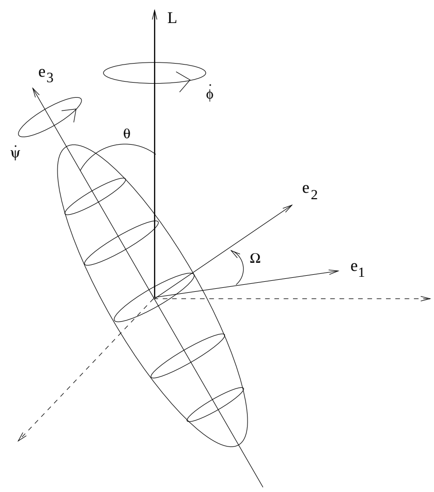

Figure 40: Euler angles for the free symmetric top when $\mathbf{L}$ coincides with $\tilde{\mathbf{e}}_3$

Next, we use the equations (3.58) to solve for $\phi(t)$ and $\psi(t)$. The solution is given by $\dot\psi = \Omega$. (This relation also follows by starting at the figure and thinking about how the object spins as it precesses around $\mathbf{L}$.) But we know from (3.58) that the expression for ω_3 (which, remember, is the component of $\boldsymbol{\omega}$ in the body frame) in terms of Euler angles is $\omega_3 = \dot\psi + \dot\phi\cos\theta$ so, substituting for $\Omega = \dot\psi$, we find the precession frequency

$$\dot\phi = \frac{I_3\omega_3}{I_1\cos\theta} \tag{3.60}$$

Staring at the figure, the relationship $\dot\psi = \Omega$ may be not be immediately clear.

An Example: The Wobbling Plate

The physicist Richard Feynman tells the following story:

> "*I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.*
>
> *I had nothing to do, so I start figuring out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate – two to one. It came out of a complicated equation!*
>
> *I went on to work out equations for wobbles. Then I thought about how the electron orbits start to move in relativity. Then there's the Dirac equation*

in electrodynamics. And then quantum electrodynamics. And before I knew it....the whole business that I got the Nobel prize for came from that piddling around with the wobbling plate."

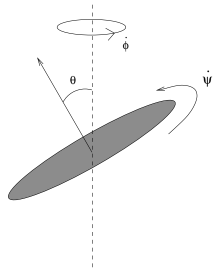

Figure 41:

Feynman was right about quantum electrodynamics. But what about the plate? We can look at this easily using what we've learnt. The spin of the plate is ω_3, while the precession, or wobble, rate is $\dot{\phi}$ which is given in (3.60). To calculate this, we need the moments of inertia for a plate. But we figured this out for the disc in Section 3.2 where we found that $I_3 = 2I_1$. We can use this to see that $\dot{\psi} = -\omega_3$ for this example and so, for slight angles θ, have

$$\dot{\phi} \approx -2\dot{\psi} \tag{3.61}$$

Or, in other words, the wobble rate is twice as fast as the spin of the plate. It's the opposite to how Feynman remembers!

There is another elegant and simple method you can use to see that Feynman was wrong: you can pick up a plate and throw it. It's hard to see that the wobble to spin ratio is exactly two. But it's easy to see that it wobbles faster than it spins.

3.6 The Heavy Symmetric Top

The "heavy" in the title of this section means that the top is acted upon by gravity. We'll deal only with a symmetric top, pinned at a point P which is a distance l from the centre of mass. This system is drawn in the figure. The principal axes are $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ and we have $I_1 = I_2$. From what we have learnt so far, it is easy to write down the Lagrangian:

$$\begin{aligned} L &= \tfrac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \tfrac{1}{2}I_3\omega_3^2 - Mgl\cos\theta \\ &= \tfrac{1}{2}I_1(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \tfrac{1}{2}I_3(\dot{\psi} + \cos\theta\,\dot{\phi})^2 - Mgl\cos\theta \end{aligned} \tag{3.62}$$

A quick examination of this equation tells us that both ψ and ϕ are ignorable coordinates. This gives us the constants of motion p_ψ and p_ϕ, where

$$p_\psi = I_3(\dot{\psi} + \cos\theta\,\dot{\phi}) = I_3\omega_3 \tag{3.63}$$

This is the angular momentum about the symmetry axis $\mathbf{e}_3$ of the top. The angular velocity $\boldsymbol{\omega}_3$ about this axis is simply called the *spin* of the top and, as for the free symmetric top, it is a constant. The other constant of motion is

$$p_\phi = I_1\sin^2\theta\,\dot{\phi} + I_3\cos\theta\,(\dot{\psi} + \dot{\phi}\cos\theta) \tag{3.64}$$

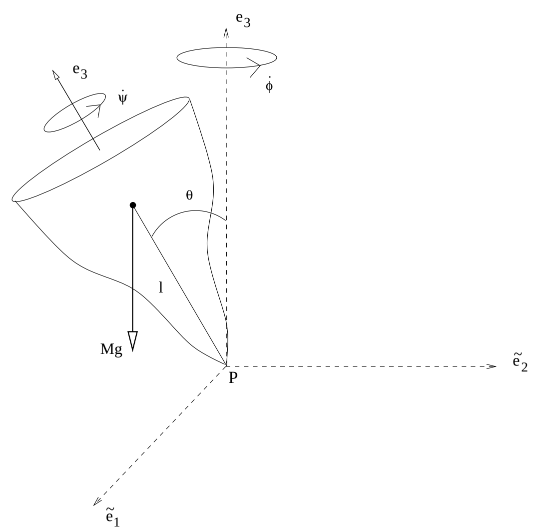

Figure 42: The heavy top with its Euler angles

As well as these two conjugate momenta, the total energy E is also conserved

$$E = T + V = \tfrac{1}{2} I_1(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \tfrac{1}{2} I_3 \omega_3^2 + Mgl\cos\theta \tag{3.65}$$

To simplify these equations, let's define the two constants

$$a = \frac{I_3\omega_3}{I_1} \quad \text{and} \quad b = \frac{p_\phi}{I_1} \tag{3.66}$$

Then we can write

$$\dot\phi = \frac{b - a\cos\theta}{\sin^2\theta} \tag{3.67}$$

and

$$\dot\psi = \frac{I_1 a}{I_3} - \frac{(b - a\cos\theta)\cos\theta}{\sin^2\theta} \tag{3.68}$$

So if we can solve $\theta = \theta(t)$ somehow, then we can always integrate these two equations to get $\phi(t)$ and $\psi(t)$. But first we have to figure out what θ is doing. To do this, let's define the "reduced energy" $E' = E - \frac{1}{2} I_3 \omega_3^2$. Then, since E and ω_3 are constant, so is E'. We have

$$E' = \tfrac{1}{2} I_1 \dot\theta^2 + V_{\rm eff}(\theta) \tag{3.69}$$

where the effective potential is

$$V_{\rm eff}(\theta) = \frac{I_1(b - a\cos\theta)^2}{2\sin^2\theta} + Mgl\cos\theta \tag{3.70}$$

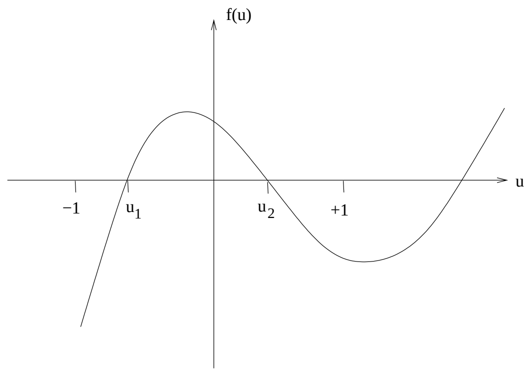

Figure 43:

So we've succeeded in getting an equation (3.69) purely in terms of θ. To simplify the analysis, let's define the new coordinate

$$u = \cos\theta \tag{3.71}$$

Clearly $-1 \leq u \leq 1$. We'll also define two further constants to help put the equations in the most concise form

$$\alpha = \frac{2E'}{I_1} \quad \text{and} \quad \beta = \frac{2Mgl}{I_1} \tag{3.72}$$

With all these redefinitions, the equations of motion (3.67), (3.68) and (3.69) can be written as

$$\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2 \equiv f(u) \tag{3.73}$$

$$\dot{\phi} = \frac{b - au}{1 - u^2} \tag{3.74}$$

$$\dot{\psi} = \frac{I_1 a}{I_3} - \frac{u(b - au)}{1 - u^2} \tag{3.75}$$

We could take the square root of equation (3.73) and integrate to reduce the problem to quadrature. The resulting integral is known as an "elliptic integral". But, rather than doing this, there's a better way to understand the physics qualitatively.

Note that the function $f(u)$ defined in (3.73) is a cubic polynomial that behaves as

$$f(u) \rightarrow \begin{cases} +\infty \text{ as } u \rightarrow \infty \\ -\infty \text{ as } u \rightarrow -\infty \end{cases} \tag{3.76}$$

and $f(\pm 1) = -(b \mp a)^2 \leq 0$. So if we plot the function $f(u)$, it looks like figure 43

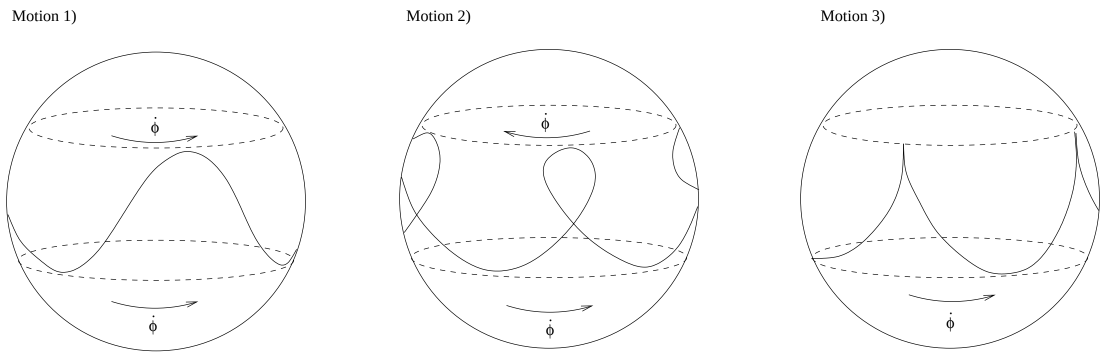

Figure 44: The three different types of motion depend on the direction of precession at the extremal points.

The physical range is $\dot{u}^2 = f(u) > 0$ and $-1 \leq u \leq 1$ so we find that, like in the spherical pendulum and central force problem, the system is confined to lie between the two roots of $f(u)$.

There are three possibilities for the motion depending on the sign of $\dot\phi$ at the two roots $u = u_1$ and $u = u_2$ as determined by (3.74). These are

- $\dot\phi > 0$ at both $u = u_1$ and $u = u_2$
- $\dot\phi > 0$ at $u = u_1$, but $\dot\phi < 0$ at $u = u_2$
- $\dot\phi > 0$ at $u = u_1$ and $\dot\phi = 0$ at $u = u_2$

The different paths of the top corresponding to these three possibilities are shown in figure 44. Motion in ϕ is called *precession* while motion in θ is known as *nutation.*

3.6.1 Letting the Top go

The last of these three motions is not as unlikely as it may first appear. Suppose we spin the top and let it go at some angle θ. What happens? We have the initial conditions

$$\begin{aligned} \dot\theta_{t=0} = 0 \quad &\Rightarrow \quad f(u_{t=0}) = 0 \\ &\Rightarrow \quad u_{t=0} = u_2 \\ \text{and} \qquad \dot\phi_{t=0} = 0 \quad &\Rightarrow \quad b - au_{t=0} = 0 \\ &\Rightarrow \quad u_{t=0} = \frac{b}{a} \end{aligned} \tag{3.77}$$

Remember also that the quantity

$$p_\phi = I_1\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta = I_3\omega_3\cos\theta_{t=0} \tag{3.78}$$

is a constant of motion. We now have enough information to figure out the qualitative motion of the top. Firstly, it starts to fall under the influence of gravity, so θ increases. But as the top falls, $\dot{\phi}$ must turn and increase in order to keep p_ϕ constant. Moreover, we also see that the direction of the precession $\dot{\phi}$ must be in the same direction as the spin ω_3 itself. What we get is motion of the third kind.

3.6.2 Uniform Precession

Can we make the top precess with bobbing up and down? i.e. with $\dot{\theta} = 0$ and $\dot{\phi}$ constant. For this to happen, we would need the function $f(u)$ to have a single root u_0 lying in the physical range $-1 \leq u_0 \leq +1$. This root must satisfy,

$$\begin{aligned} f(u_0) &= (1-u_0^2)(\alpha - \beta u_0) - (b - au_0)^2 = 0 \\ \text{and} \quad f'(u_0) &= = -2u_0(\alpha - \beta u_0) - \beta(1-u_0^2) + 2a(b - au_0) = 0 \end{aligned} \tag{3.79}$$

Combining these, we find $\frac{1}{2}\beta = a\dot{\phi} - \dot{\phi}^2 u_0$. Substituting the definitions $I_1 a = I_3\omega_3$ and $\beta = 2Mgl/I_1$ into this expression, we find

$$Mgl = \dot{\phi}(I_3\omega_3 - I_1\dot{\phi}\cos\theta_0) \tag{3.80}$$

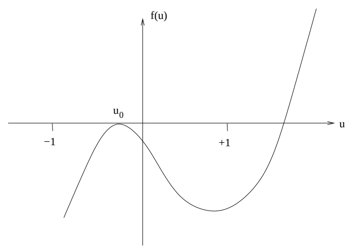

Figure 45:

The interpretation of this equation is as follows: for a fixed value of ω_3 (the spin of the top) and θ_0 (the angle at which you let it go), we need to give exactly the right push $\dot{\phi}$ to make the top spin without bobbing. In fact, since equation (3.80) is quadratic in $\dot{\phi}$, there are two frequencies with which the top can precess without bobbing.

Of course, these "slow" and "fast" precessions only exist if equation (3.80) has any solutions at all. Since it is quadratic, this is not guaranteed, but requires

$$\omega_3 > \frac{2}{I_3}\sqrt{MglI_1\cos\theta_0} \tag{3.81}$$

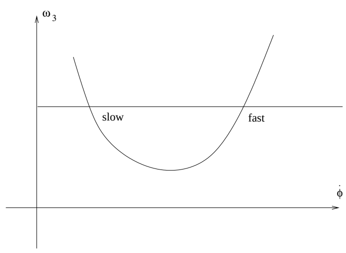

Figure 46:

So we see that, for a given θ_0, the top has to be spinning fast enough in order to have uniform solutions. What happens if it's not spinning fast enough? Well, the top falls over!

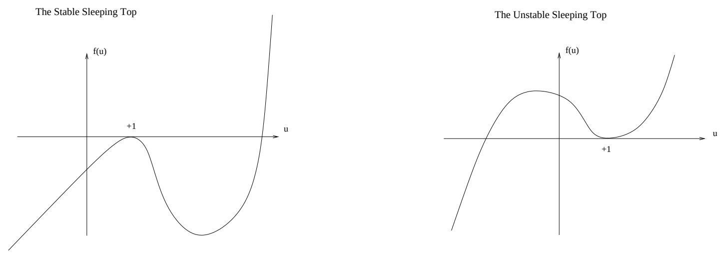

Figure 47: The function $f(u)$ for the stable and unstable sleeping top.

3.6.3 The Sleeping Top

Suppose we start the top spinning in an upright position, with

$$\theta = \dot{\theta} = 0 \tag{3.82}$$

When it spins upright, it is called a *sleeping top*. The question we want to answer is: will it stay there? Or will it fall over? From (3.73), we see that the function $f(u)$ must have a root at $\theta = 0$, or $u = +1$: $f(1) = 0$. From the definitions (3.66) and (3.72), we can check that $a = b$ and $\alpha = \beta$ in this situation and $f(u)$ actually has a double zero at $u = +1$,

$$f(u) = (1-u)^2(\alpha(1+u) - a^2) \tag{3.83}$$

The second root of $f(u)$ is at $u_2 = a^2/\alpha - 1$. There are two possibilities

1: $u_2 > 1$ or $\omega_3^2 > 4I_1Mgl/I_3^2$. In this case, the graph of $f(u)$ is drawn in first in figure 47. This motion is stable: if we perturb the initial conditions slightly, we will perturb the function $f(u)$ slightly, but the physical condition that we must restrict to the regime $f(u) > 0$ means that the motion will continue to be trapped near $u = 1$

2: $u_2 < 1$ or $\omega_3^2 < 4I_1Mgl/I_3^2$. In this case, the function $f(u)$ looks like the second figure of 47. Now the top is unstable; slight changes in the initial condition allow a large excursion.

In practice, the top spins upright until it is slowed by friction to $I_3\omega_3 = 2\sqrt{I_1Mgl}$, at which point it starts to fall and precess.

3.6.4 The Precession of the Equinox

The Euler angles for the earth are drawn in figure 48. The earth spins at an angle of $\theta = 23.5^o$ to the plane of its orbit around the sun (known as the plane of the elliptic).

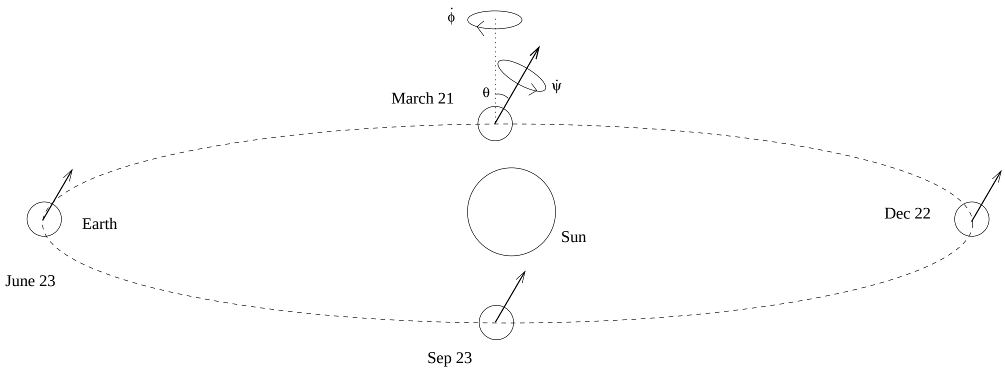

Figure 48: The precession of the earth.

The spin of the earth is $\dot{\psi} = (\mathrm{day})^{-1}$. This causes the earth to bulge at the equator so it is no longer a sphere, but rather a symmetric top. In turn, this allows the moon and sun to exert a torque on the earth which produces a precession $\dot{\phi}$. Physically this means that the direction in which the north pole points traces a circle in the sky and what we currently call the "pole star" will no longer be in several thousand years time. It turns out that this precession is "retrograde" i.e. opposite to the direction of the spin.

One can calculate the precession $\dot{\phi}$ of the earth due to the moon and sun using the techniques described in the chapter. But the calculation is rather long and we won't go over it in this course. (See the book by Hand and Finch if you're interested). Instead, we will use a different technique to calculate the precession of the earth: astrology.[3]

To compute the precession of the earth, the first fact we need to know is that Jesus was born in the age of Pisces. This doesn't mean that Jesus looked up Pisces in his daily horoscope (while scholars are divided over the exact date of his birth, he seems to exhibit many traits of a typical Capricorn) but rather refers to the patch of the sky in which the sun appears during the first day of spring. Known in astronomical terms as the "vernal equinox", this day of the year is defined by the property that the sun sits directly above the equator at midday. As the earth precesses, this event takes place at a slightly different point in its orbit each year, with a slightly different backdrop of stars as a result. The astrological age is defined to be the background constellation in which the sun rises during vernal equinox.

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[3]I learnt about this fact from John Baez' website where you can find lots of well written explanations of curiosities in mathematical physics: http://math.ucr.edu/home/baez/README.html.

It is easy to remember that Jesus was born in the age of Pisces since the fish was used as an early symbol for Christianity. The next fact that we need to know is that we're currently entering the age of Aquarius (which anyone who has seen the musical Hair will know). So we managed to travel backwards one house of the zodiac in 2,000 years. We've got to make it around 12 in total, giving us a precession time of $2,000 \times 12 = 24,000$ years. The actual value of the precession is $25,700$ years. Our calculation is pretty close considering the method!

The earth also undergoes other motion. The value of θ varies from 22.1^o to 24.5^o over a period of $41,000$ years, mostly due to the effects of the other planets. These also affect the eccentricity of the orbit over a period of 105,000 years.

3.7 The Motion of Deformable Bodies

Take a lively cat. (Not one that's half dead like Schrödinger's). Hold it upside down and drop it. The cat will twist its body and land sprightly on its feet. Yet it doesn't do this by pushing against anything and its angular momentum is zero throughout. If the cat were rigid, such motion would be impossible since a change in orientation for a rigid body necessarily requires non-vanishing angular momentum. But the cat isn't rigid (indeed, it can be checked that dead cats are unable to perform this feat) and bodies that can deform are able to reorient themselves without violating the conservation of angular momentum. In this section we'll describe some of the beautiful mathematics that lies behind this. I should warn you that this material is somewhat more advanced than the motion of rigid bodies. The theory described below was first developed in the late 1980s in order to understand how micro-organisms swim[4].

3.7.1 Kinematics

We first need to describe the configuration space $\mathcal{C}$ of a deformable body. We factor out translations by insisting that all bodies have the same center of mass. Then the configuration space $\mathcal{C}$ is the space of all shapes with some orientation.

Rotations act naturally on the space $\mathcal{C}$ (they simply rotate each shape). This allows us to define the smaller *shape space* $\tilde{\mathcal{C}}$ so that any two configurations in $\mathcal{C}$ which are related by a rotation are identified in $\tilde{\mathcal{C}}$. In other words, any two objects that have the same shape, but different orientation, are described by different points in $\mathcal{C}$, but the same point in $\tilde{\mathcal{C}}$. Mathematically, we say $\tilde{\mathcal{C}} \cong \mathcal{C}/SO(3)$.

[4]See A. Shapere and F. Wilczek, "*Geometry of Self-Propulsion at Low Reynolds Number*", J. Fluid Mech. **198** 557 (1989) . This is the same Frank Wilczek who won the 2004 Nobel prize for his work on quark interactions.

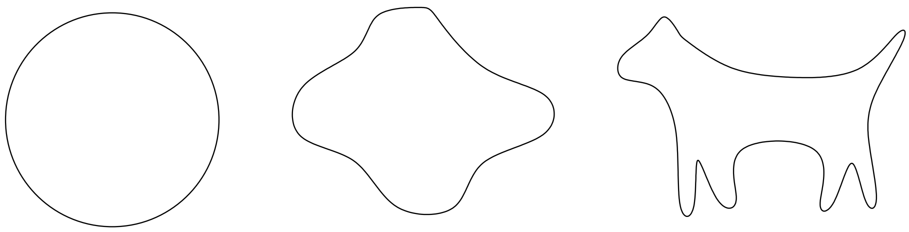

Figure 49: Three possible shapes of a deformable object.

We can describe this in more detail for a body consisting of N point masses, each with position $\mathbf{r}_i$. Unlike in section 3.1, we do not require that the distances between particles are fixed, i.e. $|\mathbf{r}_i - \mathbf{r}_j| \neq$ constant. (However, there may still be some restrictions on the $\mathbf{r}_i$). The configuration space $\mathcal{C}$ is the space of all possible configurations $\mathbf{r}_i$. For each different shape in $\mathcal{C}$, we pick a representative $\tilde{\mathbf{r}}_i$ with some, fixed orientation. It doesn't matter what representative we choose — just as long as we pick one. These variables $\tilde{\mathbf{r}}_i$ are coordinates on the space shape $\tilde{\mathcal{C}}$. For each $\mathbf{r}_i \in \mathcal{C}$, we can always find a rotation matrix $R \in SO(3)$ such that

$$\mathbf{r}_i = R\,\tilde{\mathbf{r}}_i \tag{3.84}$$

As in section 3.1, we can always do this to continuous bodies. In this case, the configuration space $\mathcal{C}$ and the shape space $\tilde{\mathcal{C}}$ may be infinite dimensional. Examples of different shapes for a continuously deformable body are shown in figure 49.

We want to understand how an object rotates as it changes shape keeping its angular momentum fixed (for example, keeping $\mathbf{L} = 0$ throughout). The first thing to note is that we can't really talk about the rotation between objects of different shapes. (How would you say that the the third object in figure 49 is rotated with respect to the first or the second?). Instead, we should think of an object moving through a sequence of shapes before returning to its initial shape. We can then ask if there's been a net rotation. As the object moves through its sequence of shapes, the motion is described by a time dependent $\tilde{\mathbf{r}}_i(t)$, while the corresponding change through the configuration space is

$$\mathbf{r}_i(t) = R(t)\,\tilde{\mathbf{r}}(t) \tag{3.85}$$

where the 3×3 rotation matrix $R(t)$ describes the necessary rotation to go from our fixed orientation of the shape $\tilde{\mathbf{r}}$ to the true orientation. As in section 3.1.1, we can define the 3×3 anti-symmetric matrix that describes the instantaneous angular velocity of the object. In fact, it will for once prove more useful to work with the "convective angular velocity" defined around equation (3.10)

$$\Omega = R^{-1}\,\frac{dR}{dt} \tag{3.86}$$

This angular velocity is non-zero due to the changing shape of the object, rather than the rigid rotation that we saw before. Let's do a quick change of notation and write coordinates on the shape space $\tilde{\mathcal{C}}$ as x^A, with $A = 1,\ldots,3N$ instead of in vector notation $\tilde{\mathbf{r}}_i$, with $i = 1,\ldots,N$. Then, since Ω is linear in time derivatives, we can write

$$\Omega = \Omega_A(x)\,\dot{x}^A \tag{3.87}$$

The component $\Omega_A(x)$ is the 3×3 angular velocity matrix induced if the shape changes from x^A to $x^A + \delta x^A$. It is independent of time: all the time dependence sits in the $\dot{x}^A$ factor which tells us how the shape is changing. The upshot is that for each shape $x \in \tilde{C}$, we have a 3×3 anti-symmetric matrix Ω_A associated to each of the $A = 1,\ldots,3N$ directions in which the shape can change.

However, there is an ambiguity in defining the angular velocity Ω. This comes about because of our arbitrary choice of reference orientation when we picked a representative $\tilde{\mathbf{r}}_i \in \tilde{\mathcal{C}}$ for each shape. We could quite easily have picked a different orientation,

$$\tilde{\mathbf{r}}_i \to S(x^A)\,\tilde{\mathbf{r}}_i \tag{3.88}$$

where $S(x^A)$ is a rotation that, as the notation suggests, can vary for each shape x^A. If we pick this new set of representative orientations, then the rotation matrix R defined in (3.85) changes: $R(t) \to R(t)\,S^{-1}(x^A)$. Equation (3.86) then tells us that the angular velocity also change as

$$\Omega_A \to S\,\Omega_A\,S^{-1} + S\,\frac{\partial S^{-1}}{\partial x^A} \tag{3.89}$$

This ambiguity is related to the fact that we can't define the meaning of rotation between two different shapes. Nonetheless, we will see shortly that when we come to compute the net rotation of the same shape, this ambiguity will disappear, as it must. Objects such as Ω_A which suffer an ambiguity of form (3.89) are extremely important in modern physics and geometry. They are known as non-abelian *gauge potentials* to physicists, or as *connections* to mathematicians.

3.7.2 Dynamics

So far we've learnt how to describe the angular velocity Ω of a deformable object. The next step is to see how to calculate Ω. We'll now show that, up to the ambiguity described in (3.89), the angular velocity Ω is specified by the requirement that the angular momentum $\mathbf{L}$ of the object is zero.

$$\begin{aligned}\mathbf{L} &= \sum_i m_i\, \mathbf{r}_i \times \dot{\mathbf{r}}_i \\ &= \sum_i m_i \left[(R\tilde{\mathbf{r}}_i) \times (R\dot{\tilde{\mathbf{r}}}_i) + (R\tilde{\mathbf{r}}_i) \times (\dot{R}\tilde{\mathbf{r}}_i)\right] = 0 \end{aligned} \tag{3.90}$$

In components this reads

$$L_a = \epsilon_{abc} \sum_i m_i \left[R_{bd}R_{ce}(\tilde{\mathbf{r}}_i)_d(\dot{\tilde{\mathbf{r}}}_i)_e + R_{bd}\dot{R}_{ce}(\tilde{\mathbf{r}}_i)_d(\tilde{\mathbf{r}}_i)_e\right] = 0 \tag{3.91}$$

The vanishing $\mathbf{L} = 0$ is enough information to determine the following result:

Claim: The 3×3 angular velocity matrix $\Omega_{ab} = R^{-1}_{ac}\dot{R}_{cb}$ is given by

$$\Omega_{ab} = \epsilon_{abc}\, \tilde{I}^{-1}_{cd}\, \tilde{L}_d \tag{3.92}$$

where $\tilde{I}$ is the instantaneous inertia tensor of the shape described by $\tilde{\mathbf{r}}_i$,

$$\tilde{I}_{ab} = \sum_i m_i((\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_i)\delta_{ab} - (\tilde{\mathbf{r}}_i)_a(\tilde{\mathbf{r}}_i)_b) \tag{3.93}$$

and $\tilde{L}_a$ is the apparent angular momentum

$$\tilde{L}_a = \epsilon_{abc} \sum_i m_i\, (\tilde{\mathbf{r}}_i)_b(\dot{\tilde{\mathbf{r}}}_i)_c \tag{3.94}$$

Proof: We start by multiplying L_a by ϵ_{afg}. We need to use the fact that if we multiply two ϵ-symbols, we have $\epsilon_{abc}\epsilon_{afg} = (\delta_{bf}\delta_{cg} - \delta_{bg}\delta_{cf})$. Then

$$\begin{aligned}\epsilon_{afg}L_a = \sum_i m_i\, &\left[R_{fd}R_{ge}(\tilde{\mathbf{r}}_i)_d(\dot{\tilde{\mathbf{r}}}_i)_e - R_{gd}R_{fe}(\tilde{\mathbf{r}}_i)_d(\dot{\tilde{\mathbf{r}}}_i)_e\right. \\ &\left. -R_{gd}\dot{R}_{fe}(\tilde{\mathbf{r}}_i)_d(\tilde{\mathbf{r}}_i)_e + R_{fd}\dot{R}_{ge}(\tilde{\mathbf{r}}_i)_d(\tilde{\mathbf{r}}_i)_e\right] = 0 \end{aligned} \tag{3.95}$$

Now multiply by $R_{fb}R_{gc}$. Since R is orthogonal, we known that $R_{fb}R_{fd} = \delta_{bd}$ which, after contracting a bunch of indices, gives us

$$R_{fb}R_{gc}\epsilon_{afg}L_a = \sum_i m_i\left[(\tilde{\mathbf{r}}_i)_b(\dot{\tilde{\mathbf{r}}}_i)_c - (\tilde{\mathbf{r}}_i)_c(\dot{\tilde{\mathbf{r}}}_i)_b - \Omega_{bd}(\tilde{\mathbf{r}}_i)_c(\tilde{\mathbf{r}}_i)_d + \Omega_{cd}(\tilde{\mathbf{r}}_i)_b(\tilde{\mathbf{r}}_i)_d\right] = 0$$

This is almost in the form that we want, but the indices aren't quite contracted in the right manner to reproduce (3.92). One can try to play around to get the indices working right, but at this stage it's just as easy to expand out the components explicitly. For example, we can look at

$$\begin{aligned}\tilde{L}_1 &= \sum_i m_i\,\left[(\tilde{\mathbf{r}}_i)_2(\dot{\tilde{\mathbf{r}}}_i)_3 - ((\tilde{\mathbf{r}}_i)_3(\dot{\tilde{\mathbf{r}}}_i)_2\right]\\ &= \sum_i m_i\,[\Omega_{21}(\tilde{\mathbf{r}}_i)_3(\tilde{\mathbf{r}}_i)_1 + \Omega_{23}(\tilde{\mathbf{r}}_i)_3(\tilde{\mathbf{r}}_i)_3 - \Omega_{31}(\tilde{\mathbf{r}}_i)_2(\tilde{\mathbf{r}}_i)_1 - \Omega_{32}(\tilde{\mathbf{r}}_i)_2(\tilde{\mathbf{r}}_i)_2]\\ &= \tilde{I}_{11}\Omega_{23} + \tilde{I}_{12}\Omega_{31} + \tilde{I}_{13}\Omega_{12} = \tfrac{1}{2}\,\epsilon_{abc}\tilde{I}_{1a}\Omega_{bc}\end{aligned} \tag{3.96}$$

where the first equality is the definition of $\tilde{L}_1$, while the second equality uses our result above, and the third equality uses the definition of $\tilde{I}$ given in (3.93). There are two similar equations, which are summarised in the formula

$$\tilde{L}_a = \tfrac{1}{2}\,\epsilon_{bcd}\tilde{I}_{ab}\Omega_{cd} \tag{3.97}$$

Multiplying both sides by $\tilde{I}^{-1}$ gives us precisely the claimed result (3.92). This concludes the proof. $\Box$.

To summarise: a system with no angular momentum that can twist and turn and change its shape has an angular velocity (3.92) where $\tilde{\mathbf{r}}_i(t)$ is the path it chooses to take through the space of shapes. This is a nice formula. But what do we do with it? We want to compute the net rotation R as the body moves through a sequence of shapes and returns to its starting point at a time T later. This is given by solving (3.86) for R. The way to do this was described in section 3.1.2. We use *path ordered exponentials*

$$R = \tilde{P}\exp\left(\int_0^T \Omega(t)\,dt\right) = \tilde{P}\exp\left(\oint\,\Omega_A\,dx^A\right) \tag{3.98}$$

The path ordering symbol $\tilde{P}$ puts all matrices evaluated at later times to the right. (This differs from the ordering in section 3.1.2 where we put later matrices to the left. The difference arises because we're working with the angular velocity $\Omega = R^{-1}\dot{R}$ instead of the angular velocity $\omega = \dot{R}R^{-1}$). In the second equality above, we've written the exponent as an integral around a closed path in shape space. Here time has dropped out. This tells us an important fact: it doesn't matter how quickly we perform the change of shapes — the net rotation of the object will be the same.

In particle physics language, the integral in (3.98) is called a "Wilson loop". We can see how the rotation fares under the ambiguity (3.87). After some algebra, you can find that the net rotation R of an object with shape x^A is changed by

$$R \to S(x^A)\,R\,S(x^A)^{-1} \tag{3.99}$$

This is as it should be: the S^{-1} takes the shape back to our initial choice of standard orientation; the matrix R is the rotation due to the change in shape; finally S puts us back to the new, standard orientation. So we see that even though the definition of the angular velocity is plagued with ambiguity, when we come to ask physically meaningful questions — such as how much has a shape rotated — the ambiguity disappears. However, if we ask nonsensical questions — such as the rotation between two different shapes — then the ambiguity looms large. In this manner, the theory contains a rather astonishing new ingredient: it lets us know what are the sensible questions to ask! Quantities for which the ambiguity (3.87) vanishes are called *gauge invariant*.

In general, it's quite hard to explicitly compute the integral (3.98). One case where it is possible is for infinitesimal changes of shape. Suppose we start with a particular shape x_A^0, and move infinitesimally in a loop in shape space:

$$x_A(t) = x_A^0 + \alpha_A(t) \tag{3.100}$$

Then we can Taylor expand our angular velocity components,

$$\Omega_A(x(t)) = \Omega_A(x^0) + \left.\frac{\partial \Omega_A}{\partial x^B}\right|_{x^0} \alpha_B \tag{3.101}$$

Expanding out the rotation matrix (3.98) and taking care with the ordering, one can show that

$$\begin{aligned} R &= 1 + \frac{1}{2} F_{AB} \oint \alpha_A \dot{\alpha}_B \, dt + \mathcal{O}(\alpha^3) \\ &= 1 + \frac{1}{2} \int F_{AB} \, dA_{AB} + \mathcal{O}(\alpha^3) \end{aligned} \tag{3.102}$$

where F_{AB} is anti-symmetric in the shape space indices A and B, and is a 3×3 matrix (the $a, b = 1, 2, 3$ indices have been suppressed) given by

$$F_{AB} = \frac{\partial \Omega_A}{\partial x^B} - \frac{\partial \Omega_B}{\partial x^A} + [\Omega_A, \Omega_B] \tag{3.103}$$

It is known as the *field strength* to physicists (or the *curvature* to mathematicians). It is evaluated on the initial shape x_A^0. The second equality in (3.102) gives the infinitesimal rotation as the integral of the field strength over the area traversed in shape space. This field strength contains all the information one needs to know about the infinitesimal rotations of objects induced by changing their shape.

One of the nicest things about the formalism described above is that it mirrors very closely the mathematics needed to describe the fundamental laws of nature, such as the strong and weak nuclear forces and gravity. They are all described by "non-abelian gauge theories", with an object known as the gauge potential (analogous to Ω_A) and an associated field strength.

4. The Hamiltonian Formalism

We'll now move onto the next level in the formalism of classical mechanics, due initially to Hamilton around 1830. While we won't use Hamilton's approach to solve any further complicated problems, we will use it to reveal much more of the structure underlying classical dynamics. If you like, it will help us understands what questions we should ask.

4.1 Hamilton's Equations

Recall that in the Lagrangian formulation, we have the function $L(q_i, \dot{q}_i, t)$ where q_i $(i = 1, \ldots, n)$ are n generalised coordinates. The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{4.1}$$

These are n 2^{nd} order differential equations which require $2n$ initial conditions, say $q_i(t = 0)$ and $\dot{q}_i(t = 0)$. The basic idea of Hamilton's approach is to try and place q_i and $\dot{q}_i$ on a more symmetric footing. More precisely, we'll work with the n *generalised momenta* that we introduced in section 2.3.3,

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad\qquad i = 1, \ldots, n \tag{4.2}$$

so $p_i = p_i(q_j, \dot{q}_j, t)$. This coincides with what we usually call momentum only if we work in Cartesian coordinates (so the kinetic term is $\frac{1}{2}m_i\dot{q}_i^2$). If we rewrite Lagrange's equations (4.1) using the definition of the momentum (4.2), they become

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{4.3}$$

The plan will be to eliminate $\dot{q}_i$ in favour of the momenta p_i, and then to place q_i and p_i on equal footing.

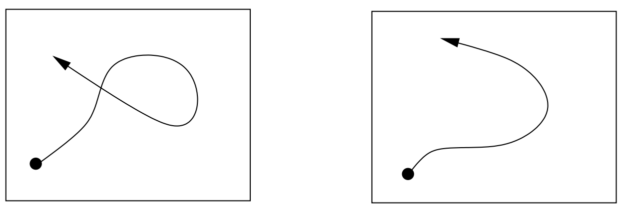

Figure 50: Motion in configuration space on the left, and in phase space on the right.

Let's start by thinking pictorially. Recall that $\{q_i\}$ defines a point in n-dimensional *configuration space* C. Time evolution is a path in C. However, the *state* of the system is defined by $\{q_i\}$ and $\{p_i\}$ in the sense that this information will allow us to determine the state at all times in the future. The pair $\{q_i, p_i\}$ defines a point in $2n$-dimensional *phase space*. Note that since a point in phase space is sufficient to determine the future evolution of the system, paths in phase space can never cross. We say that evolution is governed by a *flow* in phase space.

An Example: The Pendulum

Consider a simple pendulum. The configuration space is clearly a circle, S^1, parameterised by an angle $\theta \in [-\pi, \pi)$. The phase space of the pendulum is a cylinder $R \times S^1$, with the R factor corresponding to the momentum. We draw this by flattening out the cylinder. The two different types of motion are clearly visible in the phase space flows.

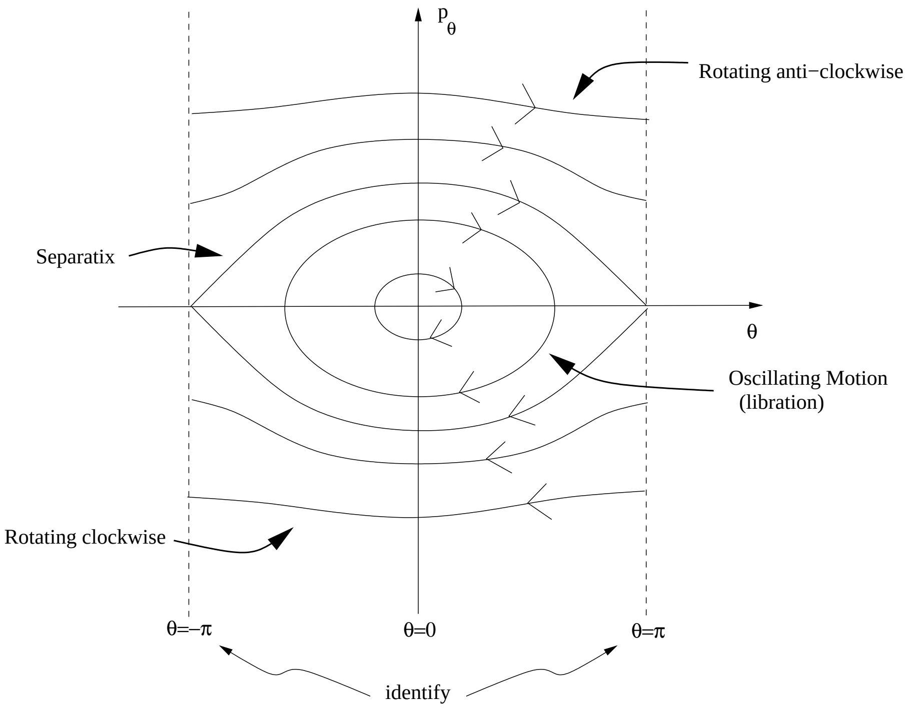

Figure 51: Flows in the phase space of a pendulum.

For small θ and small momentum, the pendulum oscillates back and forth, motion which appears as an ellipse in phase space. But for large momentum, the pendulum swings all the way around, which appears as lines wrapping around the S^1 of phase space. Separating these two different motions is the special case where the pendulum starts upright, falls, and just makes it back to the upright position. This curve in phase space is called the *separatix*.

4.1.1 The Legendre Transform

We want to find a function on phase space that will determine the unique evolution of q_i and p_i. This means it should be a function of q_i and p_i (and not of $\dot{q}_i$) but must contain the same information as the Lagrangian $L(q_i, \dot{q}_i, t)$. There is a mathematical trick to do this, known as the Legendre transform.

To describe this, consider an arbitrary function $f(x, y)$ so that the total derivative is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{4.4}$$

Now define a function $g(x, y, u) = ux - f(x, y)$ which depends on three variables, x, y and also u. If we look at the total derivative of g, we have

$$dg = d(ux) - df = u\,dx + x\,du - \frac{\partial f}{\partial x} dx - \frac{\partial f}{\partial y} dy \tag{4.5}$$

At this point u is an independent variable. But suppose we choose it to be a specific function of x and y, defined by

$$u(x, y) = \frac{\partial f}{\partial x} \tag{4.6}$$

Then the term proportional to dx in (4.5) vanishes and we have

$$dg = x\,du - \frac{\partial f}{\partial y} dy \tag{4.7}$$

Or, in other words, g is to be thought of as a function of u and y: $g = g(u, y)$. If we want an explicit expression for $g(u, y)$, we must first invert (4.6) to get $x = x(u, y)$ and then insert this into the definition of g so that

$$g(u, y) = u\,x(u, y) - f(x(u, y), y) \tag{4.8}$$

This is the Legendre transform. It takes us from one function $f(x, y)$ to a different function $g(u, y)$ where $u = \partial f/\partial x$. The key point is that we haven't lost any information. Indeed, we can always recover $f(x, y)$ from $g(u, y)$ by noting that

$$\left.\frac{\partial g}{\partial u}\right|_y = x(u, y) \qquad \text{and} \qquad \left.\frac{\partial g}{\partial y}\right|_u = \frac{\partial f}{\partial y} \tag{4.9}$$

which assures us that the inverse Legendre transform $f = (\partial g/\partial u)u - g$ takes us back to the original function.

The geometrical meaning of the Legendre transform is captured in the diagram. For fixed y, we draw the two curves $f(x,y)$ and ux. For each slope u, the value of $g(u)$ is the maximal distance between the two curves. To see this, note that extremising this distance means

$$\frac{d}{dx}\left(ux - f(x)\right) = 0 \quad \Rightarrow \quad u = \frac{\partial f}{\partial x} \tag{4.10}$$

This picture also tells us that we can only apply the Legendre transform to convex functions for which this maximum exists. Now, armed with this tool, let's return to dynamics.

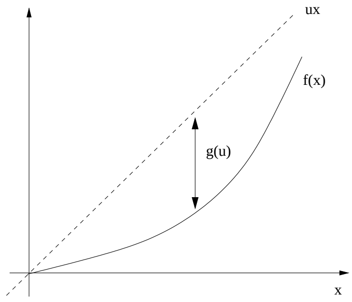

Figure 52:

4.1.2 Hamilton's Equations

The Lagrangian $L(q_i, \dot{q}_i, t)$ is a function of the coordinates q_i, their time derivatives $\dot{q}_i$ and (possibly) time. We define the *Hamiltonian* to be the Legendre transform of the Lagrangian with respect to the $\dot{q}_i$ variables,

$$H(q_i, p_i, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \tag{4.11}$$

where $\dot{q}_i$ is eliminated from the right hand side in favour of p_i by using

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = p_i(q_j, \dot{q}_j, t) \tag{4.12}$$

and inverting to get $\dot{q}_i = \dot{q}_i(q_j, p_j, t)$. Now look at the variation of H:

$$\begin{aligned} dH &= (dp_i\,\dot{q}_i + p_i\,d\dot{q}_i) - \left(\frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial L}{\partial t}dt\right) \\ &= dp_i\,\dot{q}_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial t}dt \end{aligned} \tag{4.13}$$

but we know that this can be rewritten as

$$dH = \frac{\partial H}{\partial q_i}\,dq_i + \frac{\partial H}{\partial p_i}\,dp_i + \frac{\partial H}{\partial t}\,dt \tag{4.14}$$

So we can equate terms. So far this is repeating the steps of the Legendre transform. The new ingredient that we now add is Lagrange's equation which reads $\dot{p}_i = \partial L/\partial q_i$. We find

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned} \tag{4.15}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{4.16}$$

These are *Hamilton's equations*. We have replaced n 2nd order differential equations by $2n$ 1st order differential equations for q_i and p_i. In practice, for solving problems, this isn't particularly helpful. But, as we shall see, conceptually it's very useful!

4.1.3 Examples

1) A Particle in a Potential

Let's start with a simple example: a particle moving in a potential in 3-dimensional space. The Lagrangian is simply

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(\mathbf{r}) \tag{4.17}$$

We calculate the momentum by taking the derivative with respect to $\dot{\mathbf{r}}$

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} \tag{4.18}$$

which, in this case, coincides with what we usually call momentum. The Hamiltonian is then given by

$$H = \mathbf{p}\cdot\dot{\mathbf{r}} - L = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{r}) \tag{4.19}$$

where, in the end, we've eliminated $\dot{\mathbf{r}}$ in favour of $\mathbf{p}$ and written the Hamiltonian as a function of $\mathbf{p}$ and $\mathbf{r}$. Hamilton's equations are simply

$$\begin{aligned} \dot{\mathbf{r}} &= \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}\mathbf{p} \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{r}} = -\nabla V \end{aligned} \tag{4.20}$$

which are familiar: the first is the definition of momentum in terms of velocity; the second is Newton's equation for this system.

2) A Particle in an Electromagnetic Field

We saw in section 2.5.7 that the Lagrangian for a charged particle moving in an electromagnetic field is

$$L = \tfrac{1}{2}m\dot{\mathbf{r}}^2 - e\left(\phi - \dot{\mathbf{r}}\cdot\mathbf{A}\right) \tag{4.21}$$

From this we compute the momentum conjugate to the position

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + e\mathbf{A} \tag{4.22}$$

which now differs from what we usually call momentum by the addition of the vector potential $\mathbf{A}$. Inverting, we have

$$\dot{\mathbf{r}} = \frac{1}{m}(\mathbf{p} - e\mathbf{A}) \tag{4.23}$$

So we calculate the Hamiltonian to be

$$\begin{aligned} H(\mathbf{p}, \mathbf{r}) &= \mathbf{p} \cdot \dot{\mathbf{r}} - L \\ &= \frac{1}{m}\mathbf{p} \cdot (\mathbf{p} - e\mathbf{A}) - \left[\frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 - e\phi + \frac{e}{m}(\mathbf{p} - e\mathbf{A}) \cdot \mathbf{A}\right] \\ &= \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\phi \end{aligned} \tag{4.24}$$

Now Hamilton's equations read

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}(\mathbf{p} - e\mathbf{A}) \tag{4.25}$$

while the $\dot{\mathbf{p}} = -\partial H/\partial \mathbf{r}$ equation is best expressed in terms of components

$$\dot{p}_a = -\frac{\partial H}{\partial r_a} = -e\frac{\partial \phi}{\partial r_a} + \frac{e}{m}(p_b - eA_b)\frac{\partial A_b}{\partial r_a} \tag{4.26}$$

To show that this is equivalent to the Lorentz force law requires some rearranging of the indices, but it's not too hard.

An Example of the Example

Let's illustrate the dynamics of a particle moving in a magnetic field by looking at a particular case. Imagine a uniform magnetic field pointing in the z-direction: $\mathbf{B} = (0, 0, B)$. We can get this from a vector potential $\mathbf{B} = \nabla \times \mathbf{A}$ with

$$\mathbf{A} = (-By, 0, 0) \tag{4.27}$$

This vector potential isn't unique: we could choose others related by a gauge transform as described in section 2.5.7. But this one will do for our purposes. Consider a particle moving in the (x, y)-plane. Then the Hamiltonian for this system is

$$H = \frac{1}{2m}(p_x + eBy)^2 + \frac{1}{2m}p_y^2 \tag{4.28}$$

From which we have four, first order differential equations which are Hamilton's equations

$$\dot{p}_x = 0$$

$$\begin{aligned}\dot{x} &= \frac{1}{m}(p_x + eBy) \\ \dot{p}_y &= -\frac{eB}{m}(p_x + eBy) \\ \dot{y} &= \frac{p_y}{m}\end{aligned} \tag{4.29}$$

If we add these together in the right way, we find that

$$p_y + eBx = a = \text{ const.} \tag{4.30}$$

and

$$p_x = m\dot{x} - eBy = b = \text{ const.} \tag{4.31}$$

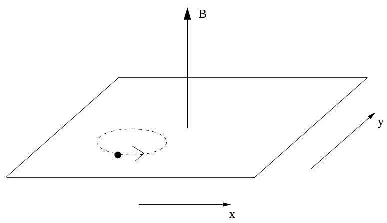

Figure 53:

which is easy to solve: we have

$$\begin{aligned}x &= \frac{a}{eB} + R\sin\left(\omega(t - t_0)\right) \\ y &= -\frac{b}{eB} + R\cos\left(\omega(t - t_0)\right)\end{aligned} \tag{4.32}$$

with a, b, R and t_0 integration constants. So we see that the particle makes circles in the (x, y)-plane with frequency

$$\omega = \frac{eB}{m} \tag{4.33}$$

This is known as the *cyclotron frequency.*

4.1.4 Some Conservation Laws

In Section 2, we saw the importance of conservation laws in solving a given problem. The conservation laws are often simple to see in the Hamiltonian formalism. For example,

Claim: If $\partial H/\partial t = 0$ (i.e. H does not depend on time explicitly) then H itself is a constant of motion.

Proof:

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} \\ &= -\dot{p}_i\dot{q}_i + \dot{q}_i\dot{p}_i + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t}\end{aligned} \tag{4.34}$$

Claim: If an ignorable coordinate q doesn't appear in the Lagrangian then, by construction, it also doesn't appear in the Hamiltonian. The conjugate momentum p_q is then conserved.

Proof

$$\dot{p}_q = \frac{\partial H}{\partial q} = 0 \tag{4.35}$$

4.1.5 The Principle of Least Action

Recall that in section 2.1 we saw the principle of least action from the Lagrangian perspective. This followed from defining the action

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt \tag{4.36}$$

Then we could derive Lagrange's equations by insisting that $\delta S = 0$ for all paths with fixed end points so that $\delta q_i(t_1) = \delta q_i(t_2) = 0$. How does this work in the Hamiltonian formalism? It's quite simple! We define the action

$$S = \int_{t_1}^{t_2} (p_i \dot{q}_i - H) dt \tag{4.37}$$

where, of course, $\dot{q}_i = \dot{q}_i(q_i, p_i)$. Now we consider varying q_i and p_i *independently*. Notice that this is different from the Lagrangian set-up, where a variation of q_i automatically leads to a variation of $\dot{q}_i$. But remember that the whole point of the Hamiltonian formalism is that we treat q_i and p_i on equal footing. So we vary both. We have

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \left\{ \delta p_i\, \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial p_i}\, \delta p_i - \frac{\partial H}{\partial q_i}\, \delta q_i \right\} dt \\ &= \int_{t_1}^{t_2} \left\{ \left[\dot{q}_i - \frac{\partial H}{\partial p_i} \right] \delta p_i + \left[-\dot{p}_i - \frac{\partial H}{\partial q_i} \right] \delta q_i \right\} dt + [p_i \delta q_i]_{t_1}^{t_2} \end{aligned} \tag{4.38}$$

and there are Hamilton's equations waiting for us in the square brackets. If we look for extrema $\delta S = 0$ for all δp_i and δq_i we get Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{4.39}$$

Except there's a very slight subtlety with the boundary conditions. We need the last term in (4.38) to vanish, and so require only that

$$\delta q_i(t_1) = \delta q_i(t_2) = 0 \tag{4.40}$$

while δp_i can be free at the end points $t = t_1$ and $t = t_2$. So, despite our best efforts, q_i and p_i are not quite symmetric in this formalism.

Note that we could simply impose $\delta p_i(t_1) = \delta p_i(t_2) = 0$ if we really wanted to and the above derivation still holds. It would mean we were being more restrictive on the types of paths we considered. But it does have the advantage that it keeps q_i and p_i on a symmetric footing. It also means that we have the freedom to add a function to consider actions of the form

$$S = \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q,p) + \frac{dF(q,p)}{dt} \right) \tag{4.41}$$

so that what sits in the integrand differs from the Lagrangian. For some situations this may be useful.

4.1.6 What's Your Name, Man? William Rowan Hamilton (1805-1865)

The formalism described above arose out of Hamilton's interest in the theory of optics. The ideas were published in a series of books entitled "Theory of Systems of Rays", the first of which appeared while Hamilton was still an undergraduate at Trinity College, Dublin. They also contain the first application of the Hamilton-Jacobi formulation (which we shall see in Section 4.7) and the first general statement of the principal of least action, which sometimes goes by the name of "Hamilton's Principle".

Hamilton's genius was recognised early. His capacity to soak up classical languages and to find errors in famous works of mathematics impressed many. In an unprecedented move, he was offered a full professorship in Dublin while still an undergraduate. He also held the position of "Royal Astronomer of Ireland", allowing him to live at Dunsink Observatory even though he rarely did any observing. Unfortunately, the later years of Hamilton's life were not happy ones. The woman he loved married another and he spent much time depressed, mired in drink, bad poetry and quaternions[5].

4.2 Liouville's Theorem

We've succeeded in rewriting classical dynamics in terms of first order differential equations in which each point in phase space follows a unique path under time evolution. We speak of a *flow* on phase space. In this section, we'll look at some of the properties of these flows

[5]The "What's your name, man?" joke is not mine. I stole it from the outrageously talented Tim Blais, whose entertaining and scholarly account of the life of Hamilton can be seen on YouTube.

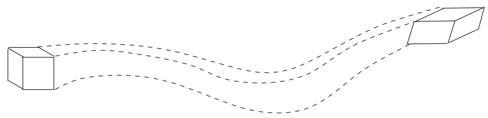

Figure 54: An infinitesimal volume element of phase space evolving in time.

Liouville's Theorem: Consider a region in phase space and watch it evolve over time. Then the shape of the region will generically change, but Liouville's theorem states that the volume remains the same.

Proof: Let's consider an infinitesimal volume moving for an infinitesimal time. We start in a neighbourhood of the point (q_i, p_i) in phase space, with volume

$$V = dq_1 \dots dq_n dp_1 \dots dp_n \tag{4.42}$$

Then in time dt, we know that

$$q_i \to q_i + \dot{q}_i dt = q_i + \frac{\partial H}{\partial p_i} dt \equiv \tilde{q}_i \tag{4.43}$$

and

$$p_i \to p_i + \dot{p}_i dt = p_i - \frac{\partial H}{\partial q_i} dt \equiv \tilde{p}_i \tag{4.44}$$

So the new volume in phase space is

$$\tilde{V} = d\tilde{q}_1 \dots d\tilde{q}_n d\tilde{p}_1 \dots d\tilde{p}_n = (\det \mathcal{J})\, V \tag{4.45}$$

where $\det \mathcal{J}$ is the Jacobian of the transformation defined by the determinant of the $2n \times 2n$ matrix

$$\mathcal{J} = \begin{pmatrix} \partial \tilde{q}_i / \partial q_j & \partial \tilde{q}_i / \partial p_j \\ \partial \tilde{p}_i / \partial q_j & \partial \tilde{p}_i / \partial p_j \end{pmatrix} \tag{4.46}$$

To prove the theorem, we need to show that $\det \mathcal{J} = 1$. First consider a single degree of freedom (i.e. $n = 1$). Then we have

$$\det \mathcal{J} = \det \begin{pmatrix} 1 + (\partial^2 H / \partial p \partial q) dt & (\partial^2 H / \partial p^2)\, dt \\ -(\partial^2 H / \partial q^2)\, dt & 1 - (\partial^2 H / \partial q \partial p)\, dt \end{pmatrix} = 1 + \mathcal{O}(dt^2) \tag{4.47}$$

which means that

$$\frac{d(\det \mathcal{J})}{dt} = 0 \tag{4.48}$$

so that the volume remains constant for all time. Now to generalise this to arbitrary n, we have

$$\det \mathcal{J} = \det \begin{pmatrix} \delta_{ij} + (\partial^2 H/\partial p_i \partial q_j) dt & (\partial^2 H/\partial p_i \partial p_j)\, dt \\ -(\partial^2 H/\partial q_i \partial q_j)\, dt & \delta_{ij} - (\partial^2 H/\partial q_i \partial p_j)\, dt \end{pmatrix} \tag{4.49}$$

To compute the determinant, we need the result that $\det(1+\epsilon M) = 1 + \epsilon \,\mathrm{Tr} M + \mathcal{O}(\epsilon^2)$ for any matrix M and small ϵ. Then we have

$$\det \mathcal{J} = 1 + \sum_i \left(\frac{\partial^2 H}{\partial p_i \partial q_i} - \frac{\partial^2 H}{\partial q_i \partial p_i} \right) dt + \mathcal{O}(dt^2) = 1 + \mathcal{O}(dt^2) \tag{4.50}$$

and we're done. $\square$

4.2.1 Liouville's Equation

So how should we think about the volume of phase space? We could consider an ensemble (or collection) of systems with some density function $\rho(p, q, t)$. We might want to do this because

- We have a single system but don't know the exact state very well. Then ρ is understood as a probability parameterising our ignorance and

$$\int \rho(q, p, t) \prod_i dp_i dq_i = 1 \tag{4.51}$$

- We may have a large number N of identical, non-interacting systems (e.g. $N = 10^{23}$ gas molecules in a jar) and we really only care about the averaged behaviour. Then the distribution ρ satisfies

$$\int \rho(q, p, t) \prod_i dq_i dp_i = N \tag{4.52}$$

In the latter case, we know that particles in phase space (i.e. dynamical systems) are neither created nor destroyed, so the number of particles in a given "comoving" volume is conserved. Since Liouville tells us that the volume elements $dpdq$ are preserved, we have $d\rho/dt = 0$. We write this as

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \\ &= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial H}{\partial q_i} = 0 \end{aligned} \tag{4.53}$$

Rearranging the terms, we have,

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i} \tag{4.54}$$

which is *Liouville's equation.*

Notice that Liouville's theorem holds whether or not the system conserves energy. (i.e. whether or not $\partial H/\partial t = 0$). But the system must be described by a Hamiltonian. For example, systems with dissipation typically head to regions of phase space with $\dot{q}_i = 0$ and so do not preserve phase space volume.

The central idea of Liouville's theorem – that volume of phase space is constant – is somewhat reminiscent of quantum mechanics. Indeed, this is the first of several occasions where we shall see ideas of quantum physics creeping into the classical world. Suppose we have a system of particles distributed randomly within a square $\Delta q \Delta p$ in phase space. Liouville's theorem implies that if we evolve the system in any Hamiltonian manner, we can cut down the spread of positions of the particles only at the cost of increasing the spread of momentum. We're reminded strongly of Heisenberg's uncertainty relation, which is also written $\Delta q \Delta p = \text{constant}$.

While Liouville and Heisenberg seem to be talking the same language, there are very profound differences between them. The distribution in the classical picture reflects our ignorance of the system rather than any intrinsic uncertainty. This is perhaps best illustrated by the fact that we can evade Liouville's theorem in a real system! The crucial point is that a system of classical particles is really described by collection of points in phase space rather than a continuous distribution $\rho(q, p)$ as we modelled it above. This means that if we're clever we can evolve the system with a Hamiltonian so that the points get closer together, while the spaces between the points get pushed away. A method for achieving this is known as stochastic cooling and is an important part of particle collider technology. In 1984 van der Meer won the the Nobel prize for pioneering this method.

4.2.2 Time Independent Distributions

Often in physics we're interested in probability distributions that don't change explicitly in time (i.e. $\partial \rho/\partial t = 0$). There's an important class of these of the form,

$$\rho = \rho(H(q, p)) \tag{4.55}$$

To see that these are indeed time independent, look at

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i}$$

$$= \frac{\partial \rho}{\partial H}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial \rho}{\partial H}\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} = 0 \tag{4.56}$$

A very famous example of this type is the *Boltzmann distribution*

$$\rho = \exp\left(-\frac{H(q,p)}{kT}\right) \tag{4.57}$$

for systems at a temperature T. Here k is the Boltzmann constant.

For example, for a free particle with $H = \mathbf{p}^2/2m$, the Boltzmann distribution is $\rho = \exp(-m\dot{\mathbf{r}}^2/2kT)$ which is a Gaussian distribution in velocities.

An historically more interesting example comes from looking at a free particle in a magnetic field, so $H = (\mathbf{p} - e\mathbf{A})^2/2m$ (where we've set the speed of light $c = 1$ for simplicity). Then the Boltzmann distribution is

$$\rho = \exp\left(-\frac{H(q,p)}{kT}\right) = \exp\left(-\frac{m\dot{\mathbf{r}}^2}{2kT}\right) \tag{4.58}$$

which is again a Gaussian distribution of velocities. In other words, the distribution in velocities is independent of the magnetic field. But this is odd: the magnetism of solids is all about how the motion of electrons is affected by magnetic fields. Yet we've seen that the magnetic field doesn't affect the velocities of electrons. This is known as the Bohr-van Leeuwen paradox: there can be no magnetism in classical physics! This was one of the motivations for the development of quantum theory.

4.2.3 Poincaré Recurrence Theorem

We now turn to work of Poincaré from around 1890. The following theorem applies to systems with a *bounded* phase space (i.e. of finite volume). This is not an uncommon occurrence. For example, if we have a conserved energy $E = T + V$ with $T > 0$ and $V > 0$ then the accessible phase space is bounded by the spatial region $V(\mathbf{r}) \leq E$. With this in mind, we have

Theorem: Consider an initial point P in phase space. Then for any neighbourhood D_0 of P, there exists a point $P' \in D_0$ that will return to D_0 in a finite time.

Proof: Consider the evolution of D_0 over a finite time interval T. Hamilton's equations provide a map $D_0 \mapsto D_1$ shown in figure 55. By Liouville's theorem, we know that $\text{Vol}(D_0) = \text{Vol}(D_1)$, although the shapes of these two regions will in general be different. Let D_k be the region after time kT where k is an integer. Then there must exist integers k and k' such that the intersection of D_k and $D_{k'}$ is not empty:

$$D_k \cap D_{k'} \neq \phi \tag{4.59}$$

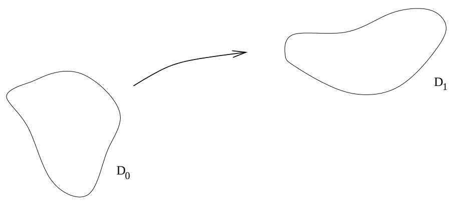

Figure 55: The Hamiltonian map in a time step T.

(If this isn't true then the total volume $\bigcup_{k=0}^{\infty} D_k \to \infty$ but, by assumption, the phase space volume is finite). Take $k' > k$ such that $\omega_{k,k'} = D_k \cap D_{k'} \neq \phi$. But since the Hamiltonian mapping $D_k \to D_{k+1}$ is invertible, we can track backwards to find $\omega_{0,k'-k} = D_0 \cap D_{k'-k} \neq 0$.

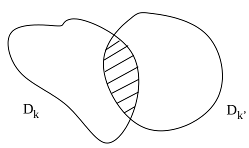

Figure 56:

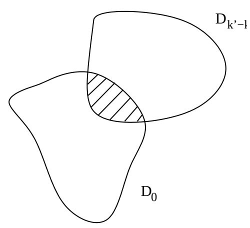

Figure 57:

So some point $P' \in D_0$ has returned to D in $k' - k$ time steps T. □

What does the Poincaré recurrence theorem mean? Consider gas molecules all in one corner of the room. If we let them go, they fill the room. But this theorem tells us that if we wait long enough, they will all return once more to the corner of the room. The trick is that the Poincaré recurrence time for this to happen can easily be longer than the lifetime of the universe!

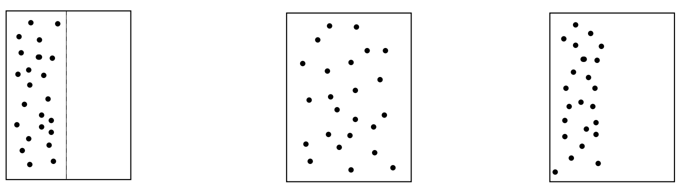

Figure 58: Eventually all the air molecules in a room will return to one corner.

Question: Where's your second law of thermodynamics now?!

4.3 Poisson Brackets

In this section, we'll present a rather formal, algebraic description of classical dynamics which makes it look almost identical to quantum mechanics! We'll return to this analogy later in the course.

We start with a definition. Let $f(q, p)$ and $g(q, p)$ be two functions on phase space. Then the *Poisson bracket* is defined to be

$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{4.60}$$

Since this is a kind of weird definition, let's look at some of the properties of the Poisson bracket to get a feel for it. We have

- $\{f, g\} = -\{g, f\}$.
- linearity: $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$ for all $\alpha, \beta \in \mathbf{R}$.
- Leibniz rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g$ which follows from the chain rule in differentiation.
- Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$. To prove this you need a large piece of paper and a hot cup of coffee. Expand out all 24 terms and watch them cancel one by one.

What we've seen above is that the Poisson bracket $\{\, ,\, \}$ satisfies the same algebraic structure as matrix commutators $[\, ,\,]$ and the differentiation operator d. This is related to Heisenberg's and Schrödinger's viewpoints of quantum mechanics respectively. (You may be confused about what the Jacobi identity means for the derivative operator d. Strictly speaking, the Poisson bracket is like a "Lie derivative" found in differential geometry, for which there is a corresponding Jacobi identity).

The relationship to quantum mechanics is emphasised even more if we calculate

$$\begin{aligned} \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned} \tag{4.61}$$

We'll return to this in section 4.8.

Claim: For any function $f(q, p, t)$,

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \tag{4.62}$$

Proof:

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t} \\ &= -\frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial f}{\partial t} \\ &= \{f, H\} + \frac{\partial f}{\partial t}\end{aligned} \tag{4.63}$$

Isn't this a lovely equation! One consequence is that if we can find a function $I(p, q)$ which satisfy

$$\{I, H\} = 0 \tag{4.64}$$

then I is a constant of motion. We say that I and H *Poisson commute*. As an example of this, suppose that q_i is ignorable (i.e. it does not appear in H) then

$$\{p_i, H\} = 0 \tag{4.65}$$

which is the way to see the relationship between ignorable coordinates and conserved quantities in the Poisson bracket language.

Note that if I and J are constants of motion then $\{\{I, J\}, H\} = \{I, \{J, H\}\} + \{\{I, H\}, J\} = 0$ which means that $\{I, J\}$ is also a constant of motion. We say that the constants of motion form a closed algebra under the Poisson bracket.

4.3.1 An Example: Angular Momentum and Runge-Lenz

Consider the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ which, in component form, reads

$$L_1 = r_2 p_3 - r_3 p_2 \quad , \quad L_2 = r_3 p_1 - r_1 p_3 \quad , \quad L_3 = r_1 p_2 - r_2 p_1 \tag{4.66}$$

and let's look at the Poisson bracket structure. We have

$$\begin{aligned}\{L_1, L_2\} &= \{r_2 p_3 - r_3 p_2, r_3 p_1 - r_1 p_3\} \\ &= \{r_2 p_3, r_3 p_1\} + \{r_3 p_2, r_1 p_3\} \\ &= -r_2 p_1 + p_2 r_1 \;=\; L_3\end{aligned} \tag{4.67}$$

So if L_1 and L_2 are conserved, we see that L_3 must also be conserved. Or, in other words, the whole vector $\mathbf{L}$ is conserved if any two components are. Similarly, one can show that

$$\{L^2, L_i\} = 0 \tag{4.68}$$

where $L^2 = \sum_i L_i^2$. This should all be looking familiar from quantum mechanics.

Another interesting object is the (Hermann-Bernoulli-Laplace-Pauli-) Runge-Lenz vector, defined as

$$\mathbf{A} = \frac{1}{m}\,\mathbf{p}\times\mathbf{L} - \hat{\mathbf{r}} \tag{4.69}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$. This vector satisfies $\mathbf{A}\cdot\mathbf{L} = 0$. If you're willing to spend some time playing with indices, it's not hard to derive the following expressions for the Poisson bracket structure

$$\{L_a, A_b\} = \epsilon_{abc}A_c \qquad , \qquad \{A_a, A_b\} = -\frac{2}{m}\left(\frac{\mathbf{p}^2}{2m} - \frac{1}{r}\right)\epsilon_{abc}L_c \tag{4.70}$$

The last of these equations suggests something special might happen when we consider the familiar Hamiltonian $H = \mathbf{p}^2/2m - 1/r$ so that the Poisson bracket becomes

$$\{A_a, A_b\} = -\frac{2H}{m}\epsilon_{abc}L_c \tag{4.71}$$

Indeed, for this choice of Hamiltonian is a rather simple to show that

$$\{H, \mathbf{A}\} = 0 \tag{4.72}$$

So we learn that the Hamiltonian with $-1/r$ potential has another constant of motion $\mathbf{A}$ that we'd previously missed! The fact that $\mathbf{A}$ is conserved can be used to immediately derive Kepler's elliptical orbits: dotting $\mathbf{A}$ with $\hat{\mathbf{r}}$ yields $\hat{\mathbf{r}}\cdot\mathbf{A} + 1 = \mathbf{L}^2/r$ which is the equation for an ellipse. Note that the three constants of motion, $\mathbf{L}$, $\mathbf{A}$ and H form a closed algebra under the Poisson bracket.

Noether's theorem tells us that the conservation of $\mathbf{L}$ and H are related to rotational symmetry and time translation respectively. One might wonder whether there's a similar symmetry responsible for the conservation of $\mathbf{A}$. It turns out that there is: the Hamiltonian has a hidden $SO(4)$ symmetry group. You can read more about this in Goldstein.

4.3.2 An Example: Magnetic Monopoles

We've seen in the example of section 4.1.3 that a particle in a magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ is described by the Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}(\mathbf{r})\right)^2 = \frac{m}{2}\,\dot{\mathbf{r}}^2 \tag{4.73}$$

where, as usual in the Hamiltonian, $\dot{\mathbf{r}}$ is to be thought of as a function of $\mathbf{r}$ and $\mathbf{p}$. It's a simple matter to compute the Poisson bracket structure for this system: it reads

$$\{m\dot{r}_a, m\dot{r}_b\} = e\,\epsilon_{abc}B_c \qquad , \qquad \{m\dot{r}_a, r_b\} = -\delta_{ab} \tag{4.74}$$

Let's now use this to describe a postulated object known as a magnetic monopole. It's a fact that all magnets ever discovered are dipoles: they have both a north and south pole. Chop the magnet in two, and each piece also has a north and a south pole. Indeed, this fact is woven into the very heart of electromagnetism when formulated in terms of the gauge potential $\mathbf{A}$. Since we define $\mathbf{B} = \nabla \times \mathbf{A}$, we immediately have one of Maxwell's equations,

$$\nabla \cdot \mathbf{B} = 0 \tag{4.75}$$

which states that any flux that enters a region must also leave. Or, in other words, there can be no magnetic monopole. Such a monopole would have a radial magnetic field,

$$\mathbf{B} = g\,\frac{\mathbf{r}}{r^3} \tag{4.76}$$

which doesn't satisfy (4.75) since it gives rise to a delta function on the right-hand side. So if magnetic monopoles have never been observed, and are forbidden by Maxwell's equations, why are we interested in them?! The point is that every theory that goes beyond Maxwell's equations and tries to unify electromagnetism with the other forces of Nature predicts magnetic monopoles. So there's reason to suspect that, somewhere in the universe, there may be particles with a radial magnetic field given by (4.76).

What happens if an electron moves in the background of a monopole? It's tricky to set up the Lagrangian as we don't have a gauge potential $\mathbf{A}$. (Actually, one can work with certain singular gauge potentials but we won't go there). However, we can play with the Poisson brackets (4.74) which contain only the magnetic field. As an application, consider the generalised angular momentum,

$$\mathbf{J} = m\mathbf{r} \times \dot{\mathbf{r}} - ge\hat{\mathbf{r}} \tag{4.77}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$. For $g = 0$, this expression reduces to the usual angular momentum. It is a simple matter to show using (4.74) that in the background of the magnetic monopole the Hamiltonian $H = \frac{1}{2}m\dot{\mathbf{r}}^2$ and $\mathbf{J}$ satisfy

$$\{H, \mathbf{J}\} = 0 \tag{4.78}$$

which guarantees that $\mathbf{J}$ is a constant of motion. What do we learn from this? Since $\mathbf{J}$ is conserved, we can look at $\hat{\mathbf{r}} \cdot \mathbf{J} = -eg$ to learn that the motion of an electron in the background of a magnetic monopole lies on a cone of angle $\cos\theta = eg/J$ pointing away from the vector $\mathbf{J}$.

4.3.3 An Example: The Motion of Vortices

The formal structure of Poisson brackets that we've introduced here can be employed even when it's not obvious that we're talking about coordinates and momenta. To illustrate this, consider the rather odd motion of line vortices moving in a plane. For n vortices with positions $\mathbf{r}_i = (x_i, y_i)$, each with strength γ_i, the equations of motion are

$$\begin{aligned} \gamma_i \dot{x}_i &= -\sum_{j \neq i} \gamma_i \gamma_j \frac{y_i - y_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} \\ \gamma_i \dot{y}_i &= +\sum_{j \neq i} \gamma_i \gamma_j \frac{x_i - x_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} \end{aligned} \tag{4.79}$$

where there is no sum over i on the left hand side of these equations. Notice that these are first order equations for the position variables, rather than for position and momentum. How can we cast this dynamics in a Hamiltonian framework? The trick is to consider one of the positions as a "canonical momentum". We consider the Hamiltonian

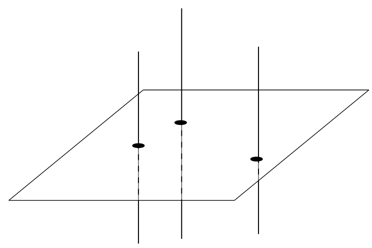

Figure 59:

$$H = -\sum_{i<j} \gamma_i \gamma_j \, \log |\mathbf{r}_i - \mathbf{r}_j| \tag{4.80}$$

which is endowed with the Poisson bracket structure

$$\{f, g\} = \sum_{i=1}^{n} \frac{1}{\gamma_i} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right) \tag{4.81}$$

In particular, we have

$$\{x_i, y_j\} = \frac{\delta_{ij}}{\gamma_i} \tag{4.82}$$

Using this, we can check that Hamilton's equations reproduce the equations of motion,

$$\begin{aligned} \dot{x}_i &= \{x_i, H\} = \frac{1}{\gamma_i} \frac{\partial H}{\partial y_i} \\ \dot{y}_i &= \{y_i, H\} = -\frac{1}{\gamma_i} \frac{\partial H}{\partial x_i} \end{aligned} \tag{4.83}$$

The system has a number of conserved quantities. Firstly, there is the "total momentum",

$$P_x = \sum_i \gamma_i y_i \quad , \quad P_y = -\sum_i \gamma_i x_i \tag{4.84}$$

which satisfy $\{P_x, H\} = \{P_y, H\} = 0$, ensuring that they are conserved quantities. We also have $\{P_x, P_y\} = \sum_i \gamma_i$ and the right hand side, being constant, is trivially conserved.

The other conserved quantity is the "total angular momentum",

$$J = -\tfrac{1}{2}\sum_{i=1}^{n} \gamma_i (x_i^2 + y_i^2) \tag{4.85}$$

which again satisfies $\{J, H\} = 0$, ensuring it is conserved. The full algebra of the conserved quantities includes $\{P_x, J\} = -P_y$ and $\{P_y, J\} = P_x$, so the system closes (meaning we get back something we know on the right hand side). In fact, one can show that H, J and $(P_x^2 + P_y^2)$ provide three mutually Poisson commuting conserved quantities.

So what is the resulting motion of a bunch of vortices? For two vortices, we can simply solve the equations of motion to find,

$$\begin{aligned} x_1 - x_2 &= R \sin\left(\frac{\omega}{R^2}(t - t_0)\right) \\ y_1 - y_2 &= R \cos\left(\frac{\omega}{R^2}(t - t_0)\right) \end{aligned} \tag{4.86}$$

where R is the separation between the vortices and $\omega = (\gamma_1 + \gamma_2)/R^2$. So we learn that two vortices orbit each other with frequency inversely proportional to the square of their separation.

For three vortices, it turns out that there is a known solution which is possible because of the three mutually Poisson commuting conserved quantities we saw above. We say the system is "integrable". We'll define this properly shortly. For four or more vortices, the motion is chaotic[6].

You may think that the Poisson bracket structure $\{x, y\} \neq 0$ looks a little strange. But it also appears in a more familiar setting: a charged particle moving in a magnetic field $\mathbf{B} = (0, 0, B)$. We saw this example in section 4.1.3, where we calculated

$$p_x = m\dot{x} - \frac{eB}{mc}\, y \tag{4.87}$$

For large magnetic fields the second term in this equation dominates, and we have $p_x \approx -eBy/mc$. In this case the Poisson bracket is

$$\{x, p_x\} = 1 \quad \Rightarrow \quad \{x, y\} \approx -\frac{mc}{eB} \tag{4.88}$$

[6]For more details on this system, see the review H. Aref, "*Integrable, Chaotic, and Turbulent Vortex Motion in Two-Dimensional Flows*", Ann. Rev. Fluid Mech. **15** 345 (1983).

This algebraic similarity between vortices and electrons is a hot topic of current research: can we make vortices do similar things to electrons in magnetic fields? For example: will vortices in a Bose-Einstein condensate form a fractional quantum Hall state? This is currently an active area of research.

4.4 Canonical Transformations

There is a way to write Hamilton's equations so that they look even more symmetric. Define the $2n$ vector $\mathbf{x} = (q_1, \ldots, q_n, p_1, \ldots, p_n)^T$ and the $2n \times 2n$ matrix J,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.89}$$

where each entry is itself an $n \times n$ matrix. Then with this notation, Hamilton's equations read

$$\dot{\mathbf{x}} = J \frac{\partial H}{\partial \mathbf{x}} \tag{4.90}$$

Now remember that in the Lagrangian formalism we made a big deal about the fact that we could change coordinates $q_i \to Q_i(q)$ without changing the form of the equations. Since we've managed to put q_i and p_i on an equal footing in the Hamiltonian formalism, one might wonder if its possible to make an even larger class of transformations of the form,

$$q_i \to Q_i(q,p) \quad \text{and} \quad p_i \to P_i(q,p) \tag{4.91}$$

The answer is yes! But not all such transformations are allowed. To see what class of transformations leaves Hamilton's equations invariant, we use our new symmetric form in terms of $\mathbf{x}$ and write the transformation as

$$x_i \to y_i(x) \tag{4.92}$$

Note that we'll continue to use the index i which now runs over the range $i = 1, \ldots, 2n$. We have

$$\dot{y}_i = \frac{\partial y_i}{\partial x_j} \dot{x}_j = \frac{\partial y_i}{\partial x_j} J_{jk} \frac{\partial H}{\partial y_l} \frac{\partial y_l}{\partial x_k} \tag{4.93}$$

or, collating all the indices, we have

$$\dot{\mathbf{y}} = (\mathcal{J}\, J\, \mathcal{J}^T) \frac{\partial H}{\partial \mathbf{y}} \tag{4.94}$$

where $\mathcal{J}_{ij} = \partial y_i / \partial x_j$ is the Jacobian that we met in section 4.2. We see that Hamilton's equations are left invariant under any transformation whose Jacobian $\mathcal{J}$ satisfies

$$\mathcal{J}\, J\, \mathcal{J}^T = J \qquad \Rightarrow \qquad \frac{\partial y_i}{\partial x_j}\, J_{jk}\, \frac{\partial y_l}{\partial x_k} = J_{il} \tag{4.95}$$

The Jacobian $\mathcal{J}$ is said to be *symplectic* if this holds. A change of variables with a symplectic Jacobian is said to be a *canonical transformation.*

There is a nice method to construct canonical transformations using "generating functions" which we will mention in section 4.4.3. Before we get to this, let's look at some uses. We start by proving a theorem relating canonical transformations with Poisson brackets.

Theorem: The Poisson bracket is invariant under canonical transformations. Conversely, any transformation which preserves the Poisson bracket structure so that

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0 \quad \text{and} \quad \{Q_i, P_j\} = \delta_{ij} \tag{4.96}$$

is canonical.

Proof: Let's start by showing that the Poisson bracket is invariant under canonical transformations. Consider two functions $f(x_i)$ and $g(x_i)$. Then,

$$\begin{aligned} \{f, g\} &= \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \\ &= \frac{\partial f}{\partial x_i}\, J_{ij}\, \frac{\partial g}{\partial x_j} \end{aligned} \tag{4.97}$$

So if $x \to y(x)$, we have

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial y_k}\, \mathcal{J}_{ki} \tag{4.98}$$

and, assuming the transformation is canonical, the Poisson bracket becomes

$$\{f, g\} = \frac{\partial f}{\partial y_k}\, \mathcal{J}_{ki}\, J_{ij}\, \mathcal{J}_{lj}\, \frac{\partial g}{\partial y_l} = \frac{\partial f}{\partial y_k}\, J_{kl}\, \frac{\partial g}{\partial y_l} \tag{4.99}$$

This means that we can compute our Poisson brackets in any coordinates related by a canonical transformation. Now let's show the converse. Go back to the notation (q_i, p_i) and the new coordinates $(Q_i(q,p), P_i(q,p))$. The Jacobian is given by

$$\mathcal{J}_{ij} = \begin{pmatrix} \partial Q_i / \partial q_j & \partial Q_i / \partial p_j \\ \partial P_i / \partial q_j & \partial P_i / \partial p_j \end{pmatrix} \tag{4.100}$$

If we now compute $\mathcal{J}J\mathcal{J}^T$ in components, we get

$$(\mathcal{J}J\mathcal{J}^T)_{ij} = \begin{pmatrix} \{Q_i, Q_j\} & \{Q_i, P_j\} \\ \{P_i, Q_j\} & \{P_i, P_j\} \end{pmatrix} \tag{4.101}$$

So whenever the Poisson bracket structure is preserved, the transformation is canonical. □

Example

In the next section we'll see several non-trivial examples of canonical transformations which mix up q and p variables. But for now let's content ourselves with reproducing the coordinate changes that we had in section 2. Consider a change of coordinates of the form

$$q_i \to Q_i(q) \tag{4.102}$$

We know that Lagrange's equations are invariant under this. But what transformation do we have to make on the momenta

$$p_i \to P_i(q, p) \tag{4.103}$$

so that Hamilton's equations are also invariant? We write $\Theta_{ij} = \partial Q_i / \partial q_j$ and look at the Jacobian

$$\mathcal{J}_{ij} = \begin{pmatrix} \Theta_{ij} & 0 \\ \partial P_i/\partial q_j & \partial P_i/\partial p_j \end{pmatrix} \tag{4.104}$$

in order for the transformation to be canonical, we require $\mathcal{J}J\mathcal{J}^T = J$. By expanding these matrices out in components, we see that this is true if

$$P_i = (\Theta^{-1})_{ji} p_j \tag{4.105}$$

This is as we would expect, for it's equivalent to $P_i = \partial L / \partial \dot{Q}_i$. Note that although $Q_i = Q_i(q)$ only, $P_i \neq P_i(p)$. Instead, the new momentum P_i depends on both q and p.

4.4.1 Infinitesimal Canonical Transformations

Consider transformations of the form

$$\begin{aligned} q_i \to Q_i &= q_i + \alpha F_i(q, p) \\ p_i \to P_i &= p_i + \alpha E_i(q, p) \end{aligned} \tag{4.106}$$

where α is considered to be infinitesimally small. What functions $F_i(q,p)$ and $E_i(q,p)$ are allowed for this to be a canonical transformation? The Jacobian is

$$\mathcal{J}_{ij} = \begin{pmatrix} \delta_{ij} + \alpha\, \partial F_i/\partial q_j & \alpha\, \partial F_i/\partial p_j \\ \alpha\, \partial E_i/\partial q_j & \delta_{ij} + \alpha\, \partial E_i/\partial p_j \end{pmatrix} \tag{4.107}$$

so the requirement that $\mathcal{J} J \mathcal{J}^T = J$ gives us

$$\frac{\partial F_i}{\partial q_j} = -\frac{\partial E_i}{\partial p_j} \tag{4.108}$$

which is true if

$$F_i = \frac{\partial G}{\partial p_i} \quad \text{and} \quad E_i = -\frac{\partial G}{\partial q_i} \tag{4.109}$$

for some function $G(q,p)$. We say that G *generates* the transformation.

This discussion motivates a slightly different way of thinking about canonical transformations. Suppose that we have a one-parameter family of transformations,

$$q_i \to Q_i(q,p;\alpha) \quad \text{and} \quad p_i \to P_i(q,p;\alpha) \tag{4.110}$$

which are canonical for all $\alpha \in \mathbf{R}$ and have the property that $Q_i(q,p;\alpha=0) = q_i$ and $P_i(q,p;\alpha=0) = p_i$. Up until now, we've been thinking of canonical transformations in the "passive" sense, with the (Q_i, P_i) labelling the same point in phase space as (q_i, p_i), just in different coordinates. But a one-parameter family of canonical transformations can be endowed with a different interpretation, namely that the transformations take us from one point in the phase space (q_i, p_i) to another point in the same phase space $(Q_i(q,p;\alpha), P_i(q,p;\alpha))$. In this "active" interpretation, as we vary the parameter α we trace out lines in phase space. Using the results (4.106) and (4.109), the tangent vectors to these lines are given by,

$$\frac{dq_i}{d\alpha} = \frac{\partial G}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{d\alpha} = -\frac{\partial G}{\partial q_i} \tag{4.111}$$

But these look just like Hamilton's equations, with the Hamiltonian replaced by the function G and time replaced by the parameter α. What we've found is that every one-parameter family of canonical transformations can be thought of as "Hamiltonian flow" on phase space for an appropriately chosen "Hamiltonian" G. Conversely, time evolution can be thought of as a canonical transformation for the coordinates

$$(q_i(t_0), p_i(t_0)) \to (q_i(t), p_i(t)) \tag{4.112}$$

generated by the Hamiltonian. Once again, we see the link between time and the Hamiltonian.

As an example, consider the function $G = p_k$. Then the corresponding infinitesimal canonical transformation is $q_i \to q_i + \alpha\delta_{ik}$ and $p_i \to p_i$, which is simply a translation. We say that translations of q_k are generated by the conjugate momentum $G = p_k$.

4.4.2 Noether's Theorem Revisited

Recall that in the Lagrangian formalism, we saw a connection between symmetries and conservation laws. How does this work in the Hamiltonian formulation?

Consider an infinitesimal canonical transformation generated by G. Then

$$\begin{aligned} \delta H &= \frac{\partial H}{\partial q_i}\,\delta q_i + \frac{\partial H}{\partial p_i}\,\delta p_i \\ &= \alpha\,\frac{\partial H}{\partial q_i}\,\frac{\partial G}{\partial p_i} - \alpha\,\frac{\partial H}{\partial p_i}\,\frac{\partial G}{\partial q_i} + \mathcal{O}(\alpha^2) \\ &= \alpha\,\{H, G\} \end{aligned} \tag{4.113}$$

The generator G is called a *symmetry* of the Hamiltonian if $\delta H = 0$. This holds if

$$\{G, H\} = 0 \tag{4.114}$$

But we know from section 4.3 that $\dot{G} = \{G, H\}$. We have found that if G is a symmetry then G is conserved. Moreover, we can reverse the argument. If we have a conserved quantity G, then we can always use this to generate a canonical transformation which is a symmetry.

4.4.3 Generating Functions

There's a simple method to construct canonical transformations between coordinates (q_i, p_i) and (Q_i, P_i). Consider a function $F(q, Q)$ of the original q_i's and the final Q_i's. Let

$$p_i = \frac{\partial F}{\partial q_i} \tag{4.115}$$

After inverting, this equation can be thought of as defining the new coordinate $Q_i = Q_i(q, p)$. But what is the new canonical momentum P? We'll show that it's given by

$$P_i = -\frac{\partial F}{\partial Q_i} \tag{4.116}$$

The proof of this is a simple matter of playing with partial derivatives. Let's see how it works in an example with just a single degree of freedom. (It generalises trivially to the case of several degrees of freedom). We can look at the Poisson bracket

$$\{Q, P\} = \left.\frac{\partial Q}{\partial q}\right|_p \left.\frac{\partial P}{\partial p}\right|_q - \left.\frac{\partial Q}{\partial p}\right|_q \left.\frac{\partial P}{\partial q}\right|_p \tag{4.117}$$

At this point we need to do the playing with partial derivatives. Equation (4.116) defines $P = P(q, Q)$, so we have

$$\left.\frac{\partial P}{\partial p}\right|_q = \left.\frac{\partial Q}{\partial p}\right|_q \left.\frac{\partial P}{\partial Q}\right|_q \quad \text{and} \quad \left.\frac{\partial P}{\partial q}\right|_p = \left.\frac{\partial P}{\partial q}\right|_Q + \left.\frac{\partial Q}{\partial q}\right|_p \left.\frac{\partial P}{\partial Q}\right|_q \tag{4.118}$$

Inserting this into the Poisson bracket gives

$$\{Q, P\} = -\left.\frac{\partial Q}{\partial p}\right|_q \left.\frac{\partial P}{\partial q}\right|_Q = \left.\frac{\partial Q}{\partial p}\right|_q \frac{\partial^2 F}{\partial q \partial Q} = \left.\frac{\partial Q}{\partial p}\right|_q \left.\frac{\partial p}{\partial Q}\right|_q = 1 \tag{4.119}$$

as required. The function $F(q, Q)$ is known as a generating function of the first kind.

There are three further types of generating function, related to the first by Legendre transforms. Each is a function of one of the original coordinates and one of the new coordinates. You can check that the following expression all define canonical transformations:

$$\begin{aligned} F_2(q, P): \qquad & p_i = \frac{\partial F_2}{\partial q_i} \quad \text{and} \quad Q_i = \frac{\partial F_2}{\partial P_i} \\ F_3(p, Q): \qquad & q_i = -\frac{\partial F_3}{\partial p_i} \quad \text{and} \quad P_i = -\frac{\partial F_3}{\partial Q_i} \\ F_4(p, P): \qquad & q_i = -\frac{\partial F_4}{\partial p_i} \quad \text{and} \quad Q_i = \frac{\partial F_4}{\partial P_i} \end{aligned} \tag{4.120}$$

4.5 Action-Angle Variables

We've all tried to solve problems in physics using the wrong coordinates and seen what a mess it can be. If you work in Cartesian coordinates when the problem really requires, say, spherical polar coordinates, it's always possible to get to the right answer with enough perseverance, but you're really making life hard for yourself. The ability to change coordinate systems can drastically simplify a problem. Now we have a much larger set of transformations at hand; we can mix up q's and p's. An obvious question is: Is this useful for anything?! In other words, is there a natural choice of variables which makes solving a given problem much easier. In many cases, there is. They're called "angle-action" variables.

4.5.1 The Simple Harmonic Oscillator

We'll start this section by doing a simple example which will illustrate the main point. We'll then move on to the more general theory. The example we choose is the simple harmonic oscillator. Notice that as our theory gets more abstract, our examples get easier!

We have the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \tag{4.121}$$

so that Hamilton's equations are the familiar

$$\dot{p} = -m\omega^2 q \quad \text{and} \quad \dot{q} = \frac{p}{m} \tag{4.122}$$

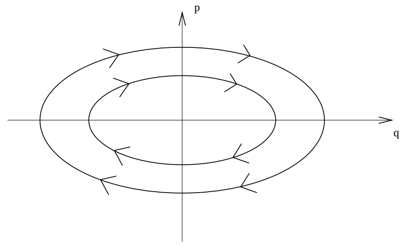

Figure 60:

which has the rather simple solution

$$q = A\cos(\omega(t - t_0)) \quad \text{and} \quad p = -m\omega A\sin(\omega(t - t_0)) \tag{4.123}$$

where A and t_0 are integration constants. The flows in phase space are ellipses as shown in the figure.

Now let's do a rather strange change of variables in which we use our freedom to mix up the position and momentum variables. We write

$$(q, p) \to (\theta, I) \tag{4.124}$$

where you can think of θ is our new position coordinate and I our new momentum coordinate. The transformation we choose is:

$$q = \sqrt{\frac{2I}{m\omega}}\,\sin\theta \quad \text{and} \quad p = \sqrt{2Im\omega}\,\cos\theta \tag{4.125}$$

It's an odd choice, but it has advantages! Before we turn to these, let's spend a minute checking that this is indeed a canonical transformation. There's two ways to do this and we'll do both:

1) We can make sure that the Poisson brackets are preserved. In fact, it's easier to work backwards and check that $\{q, p\} = 1$ in (θ, I) coordinates. In other words, we need to show that

$$\{q, p\}_{(\theta,I)} \equiv \frac{\partial q}{\partial\theta}\frac{\partial p}{\partial I} - \frac{\partial q}{\partial I}\frac{\partial p}{\partial\theta} = 1 \tag{4.126}$$

To confirm this, let's substitute the transformation (4.125),

$$\begin{aligned} \{q, p\}_{(\theta,I)} &= \left\{\sqrt{\frac{2I}{m\omega}}\sin\theta, \sqrt{2Im\omega}\cos\theta\right\}_{(\theta,I)} \\ &= 2\left\{\sqrt{I}\sin\theta, \sqrt{I}\cos\theta\right\}_{(\theta,I)} = 1 \end{aligned} \tag{4.127}$$

where the final equality follows after a quick differentiation. So we see that the transformation (4.125) is indeed canonical.

2) The second way to see that the transformation is canonical is to prove that the Jacobian is symplectic. Let's now check it this way. We can calculate

$$\mathcal{J} = \begin{pmatrix} \partial\theta/\partial q & \partial\theta/\partial p \\ \partial I/\partial q & \partial I/\partial p \end{pmatrix} = \begin{pmatrix} (m\omega/p)\cos^2\theta & -(m\omega q/p^2)\cos^2\theta \\ m\omega q & p/m\omega \end{pmatrix} \tag{4.128}$$

from which we can calculate $\mathcal{J}J\mathcal{J}^T$ and find that it is equal to J as required.

So we have a canonical transformation in (4.125). But what's the point of doing this? Let's look at the Hamiltonian in our new variables.

$$H = \frac{1}{2m}(2m\omega I)\sin^2\theta + \frac{1}{2}m\omega^2\frac{2I}{m\omega}\cos^2\theta = \omega I \tag{4.129}$$

so the Hamiltonian doesn't depend on the variable θ! This means that Hamilton's equations read

$$\dot{\theta} = \frac{\partial H}{\partial I} = \omega \quad \text{and} \quad \dot{I} = -\frac{\partial H}{\partial\theta} = 0 \tag{4.130}$$

We've managed to map the phase space flow onto a cylinder parameterised by θ and I so that the flows are now all straight lines as shown in the figure. The coordinates (θ, I) are examples of *angle-action variables.*

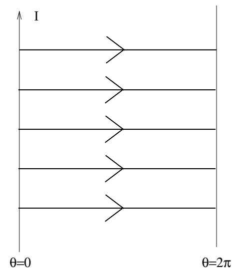

Figure 61:

4.5.2 Integrable Systems

In the above example, we saw that we could straighten out the flow lines of the simple harmonic oscillator with a change of variables, so that the motion in phase space became trivial. It's interesting to ask if we can we do this generally? The answer is: only for certain systems that are known as *integrable.*

Suppose we have n degrees of freedom. We would like to find canonical transformations

$$(q_i, p_i) \to (\theta_i, I_i) \tag{4.131}$$

such that the Hamiltonian becomes $H = H(I_1, \ldots, I_n)$ and doesn't depend on θ_i. If we can do this, then Hamilton's equations tell us that we have n conserved quantities I_i, while

$$\dot{\theta}_i = \frac{\partial H}{\partial I_i} = \omega_i \tag{4.132}$$

where ω_i is independent of θ (but in general depends on I) so that the solutions are simply $\theta_i = \omega_i t$. Whenever such a transformation exists, the system is said to be *integrable.* For bounded motion, the θ_i are usually scaled so that $0 \leq \theta_i < 2\pi$ and the coordinates (θ_i, I_i) are called *angle-action variables.*

Liouville's Theorem on Integrable Systems: There is a converse statement. If we can find n mutually Poisson commuting constants of motion $I_1, \ldots, I_n$ then this implies the existence of angle-action variables and the system is integrable. The requirement of Poisson commutation $\{I_i, I_j\} = 0$ is the statement that we can view the I_i as canonical momentum variables. This is known as Liouville's theorem. (Same Liouville, different theorem). A proof can be found in the book by Arnold.

Don't be fooled into thinking all systems are integrable. They are rather special and precious. It remains an active area of research to find and study these systems. But many – by far the majority – of systems are not integrable (chaotic systems notably among them) and don't admit this change of variables. Note that the question of whether angle-action variables exist is a *global* one. Locally you can always straighten out the flow lines; it's a question of whether you can tie these straight lines together globally without them getting tangled.

Clearly the motion of a completely integrable system is restricted to lie on $I_i =$ constant slices of the phase space. A theorem in topology says that these surfaces must be tori ($S^1 \times \ldots \times S^1$) known as the *invariant tori*.

4.5.3 Action-Angle Variables for 1d Systems

Let's see how this works for a 1d system with Hamiltonian

$$H = \frac{p^2}{2m} + V(q) \tag{4.133}$$

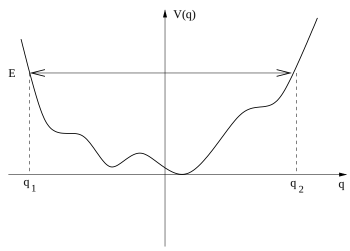

Figure 62:

Since H itself is a constant of motion, with $H = E$ for some constant E throughout the motion, the system is integrable. We assume that the motion is bounded so that $q_1 \leq q \leq q_2$ as shown in the figure. Then the motion is periodic, oscillating back and forth between the two end points, and the motion in phase space looks something like the figure 63. Our goal is to find a canonical transformation to variables θ and I that straightens out this flow to look like the second figure in the diagram.

So what are I and θ? Since I is a constant of motion, it should be some function of the energy or, alternatively,

$$H = H(I) = E \tag{4.134}$$

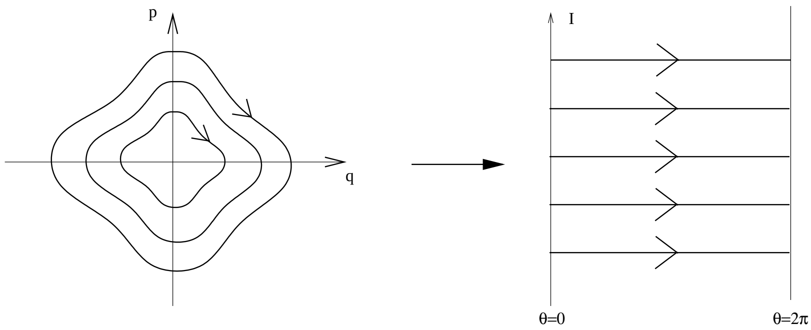

Figure 63: Can we straighten out the flow lines in phase space?

But which choice will have as its canonical partner $\theta \in [0, 2\pi)$ satisfying

$$\dot{\theta} = \frac{\partial H}{\partial I} = \frac{\partial E}{\partial I} \equiv \omega \tag{4.135}$$

for a constant ω which is the frequency of the orbit?

Claim: The correct choice for I is

$$I = \frac{1}{2\pi} \oint p \, dq \tag{4.136}$$

which is the area of phase space enclosed by an orbit (divided by 2π) and is a function of the energy only.

Proof: Since the Hamiltonian is conserved, we may write the momentum as a function of q and E:

$$p = \sqrt{2m}\,\sqrt{E - V(q)} \tag{4.137}$$

We know that for this system $p = m\dot{q}$ so we have

$$dt = \sqrt{\frac{m}{2}} \frac{dq}{\sqrt{E - V(q)}} \tag{4.138}$$

Integrating over a single orbit with period $T = 2\pi/\omega$, we have

$$\begin{aligned}\frac{2\pi}{\omega} &= \sqrt{\frac{m}{2}} \oint \frac{dq}{\sqrt{E - V(q)}} \\ &= \oint \sqrt{2m} \left(\frac{d}{dE} \sqrt{E - V(q)} \right) dq \end{aligned} \tag{4.139}$$

At this point we take the differentiation d/dE outside the integral. This isn't obviously valid since the path around which the integral is evaluated itself changes with energy E. Shortly we'll show that this doesn't matter. For now, let's assume that this is valid and continue to find

$$\begin{aligned}\frac{2\pi}{\omega} &= \frac{d}{dE}\oint \sqrt{2m}\sqrt{E-V(q)}\,dq \\ &= \frac{d}{dE}\oint p\,dq \\ &= 2\pi\,\frac{dI}{dE}\end{aligned} \tag{4.140}$$

where in the last line, we've substituted for our putative action variable I. Examining our end result, we have found that I does indeed satisfy

$$\frac{dE}{dI} = \omega \tag{4.141}$$

where ω is the frequency of the orbit. This is our required result, but it remains to show that we didn't miss anything by taking d/dE outside the integral. Let's think about this. We want to see how the area enclosed by the curve changes under a small shift in energy δE. Both the curve itself and the end points $q_1 \leq q \leq q_2$ vary as the energy shifts. The latter change by $\delta q_i = (dV(q_i)/dq)^{-1}\,\delta E$. Allowing the differential d/dE to wander inside and outside the integral is tantamount to neglecting the change in the end points. The piece we've missed is the small white region in the figure. But these pieces are of order δE^2. To see this, note that order δE pieces are given by

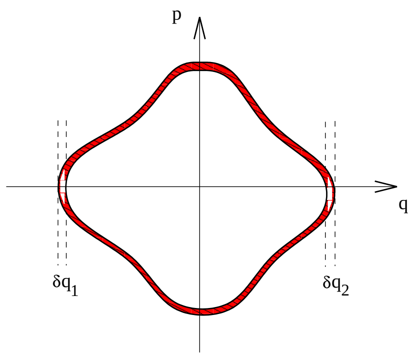

Figure 64:

$$\int_{q_i+\delta q_i}^{q_i} \sqrt{2m}\sqrt{E-V(q)}\,dq \approx \sqrt{2m}\sqrt{E-V(q)}\,\left(\frac{\partial V}{\partial q}\right)^{-1}\delta E \tag{4.142}$$

evaluated at the end point $q = q_i$. They vanish because $E = V(q_i)$ at the end points. This completes the proof. $\square$

This tells us that we can calculate the period of the orbit ω by figuring out the area enclosed by the orbit in phase space as a function of the energy. Notice that we can do this without ever having to work out the angle variable θ (which is a complicated function of q and p) which travels with constant speed around the orbit (i.e. satisfies $\theta = \omega t$).

In fact, it's not too hard to get an expression for θ by going over the above analysis for a small part of the period. It follows from the above proof that

$$t = \frac{d}{dE}\int p\,dq \tag{4.143}$$

but we want a θ which obeys $\theta = \omega t$. We see that we can achieve this by taking the choice

$$\theta = \omega\,\frac{d}{dE}\int p\,dq = \frac{dE}{dI}\,\frac{d}{dE}\int p\,dq = \frac{d}{dI}\int p\,dq \tag{4.144}$$

Because E is conserved, all 1d systems are integrable. What about higher dimensional systems? If they are integrable, then there exists a change to angle-action variables given by

$$\begin{aligned} I_i &= \frac{1}{2\pi}\oint_{\gamma_i}\sum_j p_j\,dq_j \\ \theta_i &= \frac{\partial}{\partial I_i}\int_{\gamma_i}\sum_j p_j\,dq_j \end{aligned} \tag{4.145}$$

where the γ_i are the periods of the invariant tori.

4.5.4 Action-Angle Variables for the Kepler Problem

Perhaps the simplest integrable system with more than one degree of freedom is the Kepler problem. This is a particle of mass m moving in three dimensions, subject to the potential

$$V(\mathbf{r}) = -\frac{k}{r}$$

We solved this already back in the Dynamics and Relativity course. Recall that we can use the conservation of the (direction of) angular momentum to restrict dynamics to a two-dimensional plane. We'll work in polar coordinates (r,ϕ) in this spatial plane. The associated momenta are $p_r = m\dot{r}$ and $p_\phi = mr^2\dot{\phi}$. The Hamiltonian is

$$H = \frac{1}{2m}p_r^2 + \frac{1}{2mr^2}p_\phi^2 - \frac{k}{r} \tag{4.146}$$

There are two action variables, one associated to the radial motion and one associated to the angular motion. The latter is straightforward: it is the angular momentum itself

$$I_\phi = \frac{1}{2\pi}\int_0^{2\pi} p_\phi d\phi = p_\phi$$

The action variable for the radial motion is more interesting. We can calculating it by using the fact that the total energy, E, and the angular momentum I_ϕ are both conserved. Then, rearranging (4.146), we have

$$p_r^2 = 2m\left(E + \frac{k}{r}\right) - \frac{I_\phi^2}{r^2}$$

and the action variable is

$$I_r = \frac{1}{2\pi}\oint p_r dr = \frac{1}{2\pi}2\int_{r_{\rm min}}^{r_{\rm max}} p_r dr = \frac{1}{2\pi}2\int_{r_{\rm min}}^{r_{\rm max}}\sqrt{2m\left(E+\frac{k}{r}\right) - \frac{I_\phi^2}{r^2}}dr$$

Here $r_{\rm min}$ and $r_{\rm max}$ are, respectively, the closest and furthest distance to the origin. (If you try to picture this in space, you'll need to recall that in the Kepler problem the origin sits on the focus of the ellipse, rather than the centre; this means that the smallest and furthest distance are opposite each other on the orbit).The factor of 2 in the second equality comes because a complete cycle goes from $r_{\rm min}$ to $r_{\rm max}$ and back again. To do this integral, you'll need the result

$$\int_{r_{\rm min}}^{r_{\rm max}}\sqrt{\left(1-\frac{r_{\rm min}}{r}\right)\left(\frac{r_{\rm max}}{r}-1\right)} = \frac{\pi}{2}\left(r_{\rm min}+r_{\rm max}\right) - \pi\sqrt{r_{\rm min}r_{\rm max}}$$

Using this, we find

$$I_r = \sqrt{\frac{m}{2|E|}}k - I_\phi$$

Or, re-arranging,

$$E = -\frac{mk^2}{2(I_r+I_\phi)^2} \qquad (4.147)$$

There's something rather nice lurking in this result. The energy is the same as the Hamiltonian in this case and we can use it to compute the speed at which the angular variables change. This follows from Hamilton's equations,

$$\dot\theta_r = \frac{\partial H}{\partial I_r} \quad \text{and} \quad \dot\theta_\phi = \frac{\partial H}{\partial I_\phi}$$

Here $\theta_\phi = \phi$ while θ_r is some complicated function of r. But we see from (4.147) that the Hamiltonian is symmetric in I_r and I_ϕ. This means that the frequency at which the particle completes a ϕ cycle is the same frequency with which it completes a θ_r cycle. But that's the statement that the orbit is closed: when you go around 2π in space, you come back to the same r value. The existence of closed orbits is a unique feature of the $1/r$ potential. The calculation reveals the underlying reason for this.

4.6 Adiabatic Invariants

Consider a 1d system with a potential $V(q)$ that depends on some parameter λ. If the motion is bounded by the potential then it is necessarily periodic. We want to ask what happens if we *slowly* change λ over time. For example, we may slowly change the length of a pendulum, or the frequency of the harmonic oscillator.

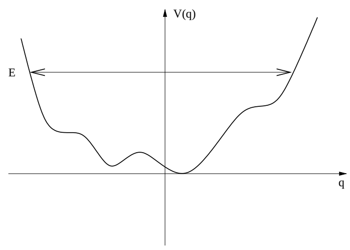

Figure 65:

Since we now have $\lambda = \lambda(t)$, the energy is not conserved. Rather $E = E(t)$ where

$$\dot{E} = \frac{\partial H}{\partial \lambda}\dot{\lambda} \tag{4.148}$$

But there are combinations of E and λ which remain (approximately) constant. These are called *adiabatic invariants* and the purpose of this section is to find them. In fact, we've already come across them: we'll see that the adiabatic invariants are the action variables of the previous section.

For the 1d system, the Hamiltonian is

$$H = \frac{p^2}{2m} + V(q;\lambda(t)) \tag{4.149}$$

and we claim that the adiabatic invariant is

$$I = \frac{1}{2\pi}\oint p\,dq \tag{4.150}$$

where the path in phase space over which we integrate now depends on time and is given by $p = \sqrt{2m}\sqrt{E(t) - V(q;\lambda(t))}$. The purpose of this section is to show that I is indeed an adiabatic invariant. At the same time we will also make clearer what we mean when we say that λ must change slowly.

Let's start by thinking of I as a function of the energy E and the parameter λ. As we vary either of these, I will change. We have,

$$\dot{I} = \left.\frac{\partial I}{\partial E}\right|_{\lambda}\dot{E} + \left.\frac{\partial I}{\partial \lambda}\right|_{E}\dot{\lambda} \tag{4.151}$$

where the subscripts on the partial derivatives tell us what variable we're keeping fixed. For an arbitrary variation of E and λ, this equation tells us that I also changes. But, of course, E and λ do not change arbitrarily: they are related by (4.148). The point of the adiabatic invariant is that when $\dot{E}$ and $\dot{\lambda}$ are related in this way, the two terms in (4.151) approximately cancel out. We can deal with each of these terms in turn. The first term is something we've seen previously in equation (4.141) which tells us that,

$$\left.\frac{\partial I}{\partial E}\right|_\lambda = \frac{1}{\omega(\lambda)} = \frac{T(\lambda)}{2\pi} \tag{4.152}$$

where $T(\lambda)$ is the period of the system evaluated at fixed λ. The second term in (4.151) tells us how the path changes as λ is varied. For example, two possible paths for two different λ's are shown in the figure and the change in I is the change in the area of under the two curves. We have

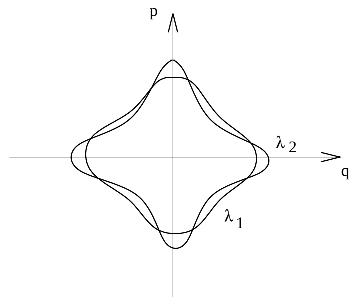

Figure 66:

$$\left.\frac{\partial I}{\partial \lambda}\right|_E = \frac{1}{2\pi}\left.\frac{\partial}{\partial \lambda}\right|_E \oint p dq = \frac{1}{2\pi}\oint \left.\frac{\partial p}{\partial \lambda}\right|_E dq = \frac{1}{2\pi}\int_0^{T(\lambda)} \left.\frac{\partial p}{\partial \lambda}\right|_E \left.\frac{\partial H}{\partial p}\right|_\lambda dt' \tag{4.153}$$

where, in the second equality, we have neglected a contribution arising from the fact that the path around which we integrate changes as λ changes. But this contribution can be safely ignored by the same argument given around (4.142).

We can get a simple expression for the product of partial derivatives by differentiating the Hamiltonian and remembering what depends on what. We have the expression $H(q,p,\lambda) = E$ where, in the left-hand side we substitute $p = \sqrt{2m}\sqrt{E(t) - V(q;\lambda(t))}$. Then differentiating with respect to λ, keeping E (and q) fixed, we have

$$\left.\frac{\partial H}{\partial \lambda}\right|_p + \left.\frac{\partial H}{\partial p}\right|_\lambda \left.\frac{\partial p}{\partial \lambda}\right|_E = 0 \tag{4.154}$$

So substituting this into (4.153) we have

$$\left.\frac{\partial I}{\partial \lambda}\right|_E = -\frac{1}{2\pi}\int_0^{T(\lambda)} \left.\frac{\partial H}{\partial \lambda}\right|_E dt' \tag{4.155}$$

So putting it all together, we have the time variation of I given by

$$\dot{I} = \left[T(\lambda)\left.\frac{\partial H}{\partial \lambda}\right|_E - \left(\int_0^{T(\lambda)} \left.\frac{\partial H}{\partial \lambda}\right|_E dt'\right)\right]\frac{\dot{\lambda}}{2\pi} \tag{4.156}$$

where, in the first term, we've replaced $\dot{E}$ with the expression (4.148). Now we're almost done. So far, each term on the right-hand side is evaluated at a given time t or, correspondingly for a given $\lambda(t)$. The two terms look similar, but they don't cancel! But we have yet to make use of the fact that the change in λ is *slow*. At this point we can clarify what we mean by this. The basic idea is that the speed at which the particle bounces backwards and forwards in the potential is much faster than the speed at which λ changes. This means that the particle has performed many periods before it notices any appreciable change in the potential. This means that if we compute averaged quantities over a single period,

$$\langle A(\lambda)\rangle = \frac{1}{T}\int_0^T A(t,\lambda)\,dt \tag{4.157}$$

then inside the integral we may treat λ as if it is effectively constant. We now consider the time averaged motion $\langle \dot{I}\rangle$. Since λ can be taken to be constant over a single period, the two terms in (4.156) do now cancel. We have

$$\langle \dot{I}\rangle = 0 \tag{4.158}$$

This is the statement that I is an adiabatic invariant: for small changes in λ, the averaged value of I remains constant[7].

The adiabatic invariants played an important role in the early history of quantum mechanics. You might recognise the quantity I as the object which takes integer values according to the old 1915 Bohr-Sommerfeld quantisation condition

$$\frac{1}{2\pi}\oint p\,dq = n\hbar \qquad n\in\mathbf{Z} \tag{4.159}$$

The idea that adiabatic invariants and quantum mechanics are related actually predates the Bohr-Somerfeld quantisation rule. In the 1911 Solvay conference Einstein answered a question of Lorentz: if the energy is quantised as $E=\hbar n\omega$ where $n\in\mathbf{Z}$ then what happens if ω is changed slowly? Lorentz' worry was that integers cannot change slowly – only by integer amounts. Einstein's answer was not to worry: E/ω remains constant. These days the idea of adiabatic invariants in quantum theory enters into the discussion of quantum computers.

An Example: The Simple Harmonic Oscillator

We saw in section 4.5 that for the simple harmonic oscillator we have $I=E/\omega$. So if we change ω slowly, then the ratio E/ω remains constant. This was Einstein's 1911 point. In fact, for the SHO it turns out that there is an exact invariant that remains constant no matter how quickly you change ω and which, in the limit of slow change, goes over to I. This exact invariant is

$$J=\frac{1}{2}\left[\frac{q^2}{g(t)^2}+(g(t)\dot{q}-q\dot{g}(t))^2\right] \tag{4.160}$$

[7] The proof given above is intuitive, but begins to creak at the seams when pushed. A nice description of these issues, together with a more sophisticated proof using generating functions for canonical transformations is given in in the paper "The Adiabatic Invariance of the Action Variable in Classical Dynamics" by C.G.Wells and S.T.Siklos which can be found at http://arxiv.org/abs/physics/0610084.

where $g(t)$ is a function satisfying the differential equation

$$\ddot{g} + \omega^2(t) g - \frac{1}{g^3} = 0 \tag{4.161}$$

4.6.1 Adiabatic Invariants and Liouville's Theorem

There's a way to think of adiabatic invariants using Liouville's theorem. Consider first a series of systems, all described by a Hamiltonian with fixed parameter λ. We set off each system with the same energy E or, equivalently, the same action I, but we start them with slightly different phases θ. This means that their dynamics is described by a series of dots, all chasing each other around a fixed curve as shown in the figure. Now let's think about how this train of dots evolves under the Hamiltonian with time dependent $\lambda(t)$. Recall that Liouville's theorem states that the area of phase space is invariant under any Hamiltonian evolution. This holds whether or not $\partial H/\partial t = 0$, so is still valid for the time dependent Hamiltonian with $\lambda(t)$. One might be tempted to say that we're done since all the words sound right: Liouville's theorem implies that the area is conserved which is also the statement that our adiabatic invariant I doesn't change with time. But this is a little too fast! Liouville's theorem says the area of a distribution of particles in phase space is conserved, not the area enclosed by a perimeter ring of particles. Indeed, Liouville's theorem holds for any variation $\lambda(t)$, not just for adiabatic variations. For a fast change of $\lambda(t)$, there is nothing to ensure that the particles that had the same initial energy, but different phases, would have the same final energy and we lose the interpretation of a ring of dots in phase space enclosing some area.

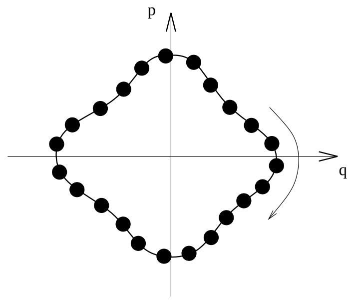

Figure 67:

The missing ingredient is the "*adiabatic principle*". In this context it states that for a suitably slow change of the parameter λ, all the systems in the same orbit, with the same energy, are affected in the same manner. If this holds, after some time the dots in phase space will still be chasing each other around another curve of constant energy E'. We can now think of a distribution of particles filling the area I inside the curve. As λ varies slowly, the area doesn't change *and* the outer particles remain the outer particles, all with the same energy. Under these circumstances, Liouville's theorem implies the adiabatic invariant I is constant in time.

4.6.2 An Application: A Particle in a Magnetic Field

We saw in Section 4.1 that a particle in a constant magnetic field $\mathbf{B} = (0, 0, B)$ makes circles with Larmor frequency $\omega = eB/mc$ and a radius R, which depends on the energy of the particle. But what happens if B is slowly varying over space? i.e. $B = B(x, y)$, but with

$$\partial_i B \ll R \tag{4.162}$$

so that the field is roughly constant over one orbit.

In this example, there is no explicit time dependence of the Hamiltonian so we know that the Hamiltonian itself is an exact constant of motion. For a particle in a constant magnetic field we can calculate H of an orbit by substituting the solutions (4.32) into the Hamiltonian (4.28). We find

$$H = \frac{1}{2}m\omega^2 R^2 = \frac{e^2 R^2 B^2}{2mc^2} \tag{4.163}$$

This quantity is conserved. But what happens to the particle? Does it drift to regions with larger magnetic field B, keeping H constant by reducing the radius of the orbit? Or to regions with smaller B with larger orbits?

We can answer this by means of the adiabatic invariant. We can use this because the motion of the particle is periodic in space so the particle sees a magnetic field which varies slowly over time. The adiabatic invariant is

$$I = \frac{1}{2\pi}\oint p\,dq \tag{4.164}$$

which is now to be thought of as a path integral along the orbit of the electron. We evaluate this on the solution for the uniform magnetic field (4.32)

$$\begin{aligned} I &= \frac{1}{2\pi}\int_0^T (p_x\dot{x} + p_y\dot{y})\,dt \\ &= \frac{1}{2\pi}\int_0^T \big(bR\omega\,\cos\omega t + m\omega^2 R^2 \sin^2\omega t\big)\,dt \\ &= \frac{m\omega R^2}{2\pi}\int_0^{2\pi} \sin^2\theta\,d\theta \end{aligned} \tag{4.165}$$

Setting $\omega = eB/mc$, we see that the adiabatic invariant I is proportional to $(e/c)BR^2$. Since the electric charge e and the speed of light c are not things we can change, we find that BR^2 is constant over many orbits. But as $H \sim B^2R^2$ is also conserved, both the magnetic field B seen by the particle and the radius of the orbit R must be individually conserved. This means the particle can't move into regions of higher or lower magnetic fields: it must move along constant field lines[8].

[8]For results that go beyond the adiabatic approximations, see the paper by the man: E. Witten "*A Slowly Moving Particle in a Two-Dimensional Magnetic Field*", Annals of Physics **120** 72 (1979).

Finally, there's a simple physical way to see that the particle indeed drifts along lines of constant magnetic field. As the particle makes its little Larmor circles, it feels a slightly stronger force when it's, say, at the top of its orbit where the field is slightly larger, compared to when its at the bottom. This net force tends to push the particle to regions of weaker or stronger magnetic field. But we've seen through the use of adiabatic invariants that this isn't possible. The slow drift of the particle acts such that it compensates for this small force, keeping the particle on constant field lines.

There's a slight variant of the above set-up which allows you to trap charged particles using magnetic fields. Consider the particle making its little Larmor circles in the (x, y) plane, but also moving in the z direction and take a magnetic field that's constant in the (x, y)-plane, but ever increasing in the z-direction. The energy of the particle is given by,

$$H = \frac{1}{2}m\dot{z}^2 + \frac{e^2R^2B^2}{2mc^2} = \frac{1}{2}m\dot{z}^2 + \frac{IemB}{mc} \tag{4.166}$$

Both $H > 0$ and $I > 0$ are constant in time. This ensures that there exists a value of the magnetic field $B > 0$ at which we necessarily have $\dot{z} = 0$ and the particle can go no further. At this stage it turns round and goes back again. By creating a magnetic field that increases at two ends, charged particles can be made to bounce back and forth in the z direction, while executing their little circles in the (x, y)-plane. It is this mechanism that traps charged particles in magnetic loops emitted from the sun and is ultimately responsible for solar flares.

4.6.3 Hannay's Angle

Consider a particle bouncing around, with periodic motion, in an arbitrary potential. There are many parameters λ_a describing the shape of the potential. As we slowly vary the λ_a the path in phase space changes although, as we have seen, the area enclosed by the path remains the same. After some time $t_{\rm long}$ (which, by the assumption of adiabiticity, is much longer than the period T of a single orbit) we return to the original parameters so that $\lambda_a(t_{\rm long}) = \lambda_a(0)$. The question we want to ask is: how has the phase angle θ changed?

For any fixed λ_a, the velocity of the angle variable is $\dot{\theta} = \partial H/\partial I = \omega(I, \lambda_a)$. As we slowly vary the parameters, the particle is spinning around its phase space orbits. When we return we therefore expect that the phase has been shifted by $\int \omega dt$. Which is true. But it turns out that there is another, more subtle, contribution to the phase shift as well. We'll now see where this comes from.

As the parameters change, we can write the change in the angle θ as

$$\dot{\theta} = \frac{\partial H}{\partial I} + \frac{\partial \theta}{\partial \lambda_a}\,\dot{\lambda}_a \tag{4.167}$$

Which looks simple enough. But there's a problem. The second term is not well defined. For each set of parameters λ_a we have different action angle variables $I(\lambda_a)$ and $\theta(\lambda_a)$. But there's nothing that stops choosing a different origin $\theta = 0$ for each choice of the parameters. In other words, we could always redefine

$$\theta(\lambda_a) \to \theta(\lambda_a) + \beta(\lambda_a) \tag{4.168}$$

where we shift by a different constant β for each λ_a. What this means is that it doesn't really make any sense to compare the angle variable for different parameters λ_a. This makes the second term — which tells us how the angle variable changes as we change the parameters — ambiguous. One might think this means that we can just ignore it. Or, more precisely, we could choose the shifts β so that the angle variables are defined in such a way that the second term vanishes. But it turns out that this isn't possible. Let's see why. The point is that it does make sense to compare the angle variable for the same parameters λ_a. After such a time $t_{\rm long}$, we have

$$\theta(t_{\rm long}) = \theta(0) + \int_0^{t_{\rm long}} \omega\, dt + \Delta\theta \tag{4.169}$$

The term $\int \omega dt$ is the dynamic term that we anticipated above, arising from the fact that θ is continually making orbits around the curve in phase space. It depends on the time $t_{\rm long}$ that we took to make the change. The other term that we call $\Delta\theta$ is more interesting. From (4.167) it is given by

$$\Delta\theta = \int_0^{t_{\rm long}} \left\langle \frac{\partial \theta}{\partial \lambda_a} \right\rangle \dot{\lambda}_i\, dt = \oint_C \left\langle \frac{\partial \theta}{\partial \lambda_a} \right\rangle d\lambda_a \tag{4.170}$$

where we've used the fact that the change in λ_a is adiabatic to replace the integrand with its average over one period of the orbit. The final expression above means an integration over the curve C that the system traces in parameter space. We see that $\Delta\theta$ is independent of the time $t_{\rm long}$ taken to make the change. However, it does depend on the path that we took through the space of all possible potentials. It is known as the "Hannay angle". Note that the Hannay angle is invariant under the ambiguity (4.169) even though the quantity $\partial\theta/\partial\lambda_a$ that appears in the integrand isn't. This idea of integrating quantities around loops is an example of "holonomy", an important concept in modern physics.

Rather surprisingly, the Hannay angle was first discovered only in 1984. The history is interesting. First Berry discovered a similar phase for the wavefunction in quantum mechanics (now known as the Berry phase). Many physicists were shocked that such a simple and profound idea as Berry's phase had lain hidden in the formulation of quantum mechanics for 50 years and it set off a flurry of theoretical and experimental research. Soon after this, Hannay showed that an analogous phase had lain undiscovered in classical mechanics for 150 years! Although, in certain examples in celestial mechanics, the phase $\Delta\theta$ had been correctly calculated, the general theory lying behind it had not been appreciated. We now describe this theory.

The first step is to use a higher dimensional version of Stokes' theorem to express the contour integral (4.170) as a surface integral

$$\Delta\theta = \int_S \left(\frac{\partial}{\partial\lambda_a} \left\langle \frac{\partial\theta}{\partial\lambda_b} \right\rangle - \frac{\partial}{\partial\lambda_b} \left\langle \frac{\partial\theta}{\partial\lambda_a} \right\rangle \right) dA_{ab} \tag{4.171}$$

where S is a surface in parameter space bounded by the curve C and dA_{ab} is the infinitesimal surface element.

Claim: The Hannay angle can be written as

$$\Delta\theta = \frac{d}{dI} \int_S W_{ab}\, dA_{ab} \tag{4.172}$$

where the anti-symmetric matrix W_{ab} (known mathematically as a 2-form) is given by

$$W_{ab} = \left\langle \frac{\partial\theta}{\partial\lambda_a}\frac{\partial I}{\partial\lambda_b} - \frac{\partial\theta}{\partial\lambda_b}\frac{\partial I}{\partial\lambda_a} \right\rangle \tag{4.173}$$

Proof: To start with let's think about the averaging procedure a little more. In equation (4.157) we wrote $\langle A\rangle$ as a time average, integrating over a single period. We could equally as well write it as an angle average,

$$\langle A\rangle = \oint A(I,\theta)\, d\theta = \int A(q',p')\, \delta(I'-I)\frac{dq'dp'}{2\pi} \tag{4.174}$$

where in the second equality we integrate over all of phase space and insert the delta function $\delta(I'-I)$ to restrict the integral to the orbit over the curve with action I. It's this latter formula that we'll use. This allows us to write,

$$\frac{\partial}{\partial\lambda_a} \left\langle \frac{\partial\theta}{\partial\lambda_b} \right\rangle - \frac{\partial}{\partial\lambda_b} \left\langle \frac{\partial\theta}{\partial\lambda_a} \right\rangle = \left[\frac{\partial}{\partial\lambda_a} \int \frac{\partial\theta}{\partial\lambda_b} - \frac{\partial}{\partial\lambda_b} \int \frac{\partial\theta}{\partial\lambda_a} \right] \delta(I'-I)\, \frac{dq'dp'}{2\pi}$$

$$
\begin{aligned}
&= \int \left[\frac{\partial \theta}{\partial \lambda_b} \frac{\partial \delta}{\partial \lambda_a} - \frac{\partial \theta}{\partial \lambda_a} \frac{\partial \delta}{\partial \lambda_b} \right] \frac{dq' dp'}{2\pi} \\
&= \int \left[\frac{\partial \theta}{\partial \lambda_b} \frac{\partial I'}{\partial \lambda_a} - \frac{\partial \theta}{\partial \lambda_a} \frac{\partial I'}{\partial \lambda_b} \right] \frac{\partial \delta}{\partial I'} \frac{dq' dp'}{2\pi} \\
&= -\frac{d}{dI} \int \left[\frac{\partial \theta}{\partial \lambda_b} \frac{\partial I'}{\partial \lambda_a} - \frac{\partial \theta}{\partial \lambda_a} \frac{\partial I'}{\partial \lambda_b} \right] \delta(I' - I) \frac{dq' dp'}{2\pi} \\
&= \frac{d}{dI} W_{ab} \qquad (4.175)
\end{aligned}
$$

which completes the proof. I haven't included any examples here of the Hannay angle: some simple ones can be found in the original literature[9] and more advanced applications can be found by googling "Hannay angles". For the pendulum example, in which the length of the pendulum varies, the Hannay angle vanishes. This is because there is only one parameter to vary, while a non-trivial $\Delta\theta$ occurs only if we make a non-trivial loop C in parameter space.

4.7 The Hamilton-Jacobi Equation

In this section we will describe yet another viewpoint on classical dynamics, known as Hamilton-Jacobi theory. It will tie together several concepts that we've met so far. Recall from section 2.1 the principle of least action. We define the action

$$S = \int_0^T L(q_i, \dot{q}_i, t)\, dt \qquad (4.176)$$

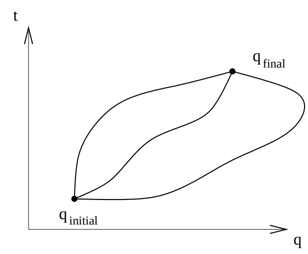

Figure 68:

which we evaluate for *all* paths $q(t)$ with fixed end points

$$q_i(0) = q_i^{\text{initial}} \quad , \quad q_i(T) = q_i^{\text{final}} \qquad (4.177)$$

Then the true path taken is an extremum of the action: $\delta S = 0$.

Now let's change perspective a little. Consider the action evaluated only along the *true* path $q_i^{\text{classical}}(t)$ and define

$$W(q_i^{\text{initial}}, q_i^{\text{final}}, T) = S[q_i^{\text{classical}}(t)] \qquad (4.178)$$

While S is a functional on any path, W is to be considered as a function of the initial and final configurations q_i^{initial} and q_i^{final} as well as the time T it takes to get between them.

[9] J. Hannay, "*Angle Variable Holonomy in Adiabatic Excursion of an Integrable Hamiltonian*" J. Phys A, **18** 221 (1985); M. Berry, "*Classical Adiabatic Angles and Quantal Adiabatic Phase*" J. Phys A, **18** 15 (1985).

Now let's ask what happens if we keep $q_i^{\rm initial}$ fixed but vary the end point $q_i^{\rm final}$. We can go back to the analysis of section 2.1 to see that when the action is varied it looks like

$$\delta S = \int_0^T dt \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \delta q_i(t) + \left[\frac{\partial L}{\partial \dot{q}_i}\,\delta q_i(t)\right]_0^T \tag{4.179}$$

If we evaluate this on the classical path the first term vanishes. We're left with

$$\frac{\partial W}{\partial q_i^{\rm final}} = \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{t=T} = p_i^{\rm final} \tag{4.180}$$

The next thing we want to compute is $\partial W/\partial T$. Let's start by considering a classical path with fixed initial configuration $q_i^{\rm initial}$. We'll let the path run on a little longer than before, so $T \to T + \delta T$. Then we have

$$\frac{dW}{dT} = \frac{\partial W}{\partial T} + \frac{\partial W}{\partial q_i^{\rm final}}\,\dot{q}_i^{\rm final} = \frac{\partial W}{\partial T} + p_i^{\rm final}\dot{q}_i^{\rm final} \tag{4.181}$$

But this total derivative is easily calculated since $dS/dT = L$, or

$$\frac{dW}{dT} = L(q_i^{\rm classical}(T), \dot{q}_i^{\rm classical}(T), T) = L(q_i^{\rm final}, \dot{q}_i^{\rm final}, T) \tag{4.182}$$

So we arrive at the equation,

$$\begin{aligned}\frac{\partial W}{\partial T} &= -\left(p_i^{\rm final}\dot{q}_i^{\rm final} - L(q_i^{\rm final}, \dot{q}_i^{\rm final}, T)\right)\\ &= -H(q_i^{\rm final}, p_i^{\rm final}, T)\end{aligned} \tag{4.183}$$

At this stage, the only time in the game is T and the only position in the game is $q_i^{\rm final}$. So we can simply drop the word "final", and relabel $T \to t$. We have found ourselves a time dependent function on configuration space $W = W(q_i, t)$ which satisfies

$$\frac{\partial W}{\partial q_i} = p_i \qquad \text{and} \qquad \frac{\partial W}{\partial t} = -H(q_i, p_i, t) \tag{4.184}$$

or, substituting the first into the second, we have

$$\frac{\partial W}{\partial t} = -H(q_i, \partial W/\partial q_i, t) \tag{4.185}$$

This is the *Hamilton-Jacobi Equation*.

We've shown how a solution to the Hamilton-Jacobi equation can be constructed by looking at the classical action of paths which reach a point q_i at time T starting from some initial reference point $q_i^{\rm initial}$. The starting point $q_i^{\rm initial}$ can be considered integration constants. In fact, there are more general solutions to the Hamilton-Jacobi equation, although all are related to the classical action in a similar way.

Suppose we find a solution to (4.185). What do we do with it? We're now armed with some time-dependent function $W(q_i, t)$ on configuration space. We combine this with the first of Hamilton's equations which reads

$$\dot{q}_i = \left.\frac{\partial H}{\partial p_i}\right|_{p_i = \partial W/\partial q_i} \tag{4.186}$$

where, on the right-hand-side, we've replaced every appearance of the momenta p_i by a function of the coordinates using $p_i = \partial W/\partial q_i$. What we're left with is n first-order differential equations for the evolution of q_i. In this manner the function W determines the path of the classical system: start it off at a point in configuration space and W can be considered as a real valued classical wavefunction which tells it how to evolve. What we need to show is that the evolution dictated by (4.186) does indeed satisfy the equations of motion. In other words, we should prove that the second of Hamilton's equations, $\dot{p}_i = -\partial H/\partial q_i$, is satisfied. We have

$$\dot{p}_i = \frac{d}{dt}\left(\frac{\partial W}{\partial q_i}\right) = \frac{\partial^2 W}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 W}{\partial t \partial q_i} \tag{4.187}$$

But differentiating the Hamilton-Jacobi equation (4.185) with respect to q_i, we see that we can rewrite the right-hand-side of this equation using

$$\frac{\partial^2 W}{\partial t \partial q_i} = -\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial p_j}\frac{\partial^2 W}{\partial q_i \partial q_j} = -\frac{\partial H}{\partial q_i} - \dot{q}_j \frac{\partial^2 W}{\partial q_i \partial q_j} \tag{4.188}$$

So that (4.187) becomes $\dot{p}_i = -\partial H/\partial q_i$ as required.

Let's see what we've done. We're used to dealing with second order differential equations for the time evolution on configuration space (i.e. Lagrange's equations) and first order differential equations for time evolution on phase space (Hamilton's equations). But the Hamilton-Jacobi approach allows us to incorporate n of the integration constants in the function $W(q_i, t)$ so that we're left solely with first order differential equations on configuration space given by (4.186).

When we have conservation of the Hamiltonian, so $\partial H/\partial t = 0$, there is solution of the Hamilton-Jacobi equation of a particularly simple form. Define

$$W(q_i, t) = W^0(q_i) - Et \tag{4.189}$$

for some constant E. Then the time dependence drops out and we get the equation

$$H(q_i, \partial W^0/\partial q_i) = E \tag{4.190}$$

W^0 is known as *Hamilton's principal function.* The special property of this solution to the Hamilton-Jacobi equation is that every path in configuration space determined by the function W_0 has the same energy E. With a little thought, we can envisage how to construct solutions to (4.190). Start with a co-dimension one surface in configuration space which we will specify to be a surface of constant W_0. (Co-dimension one means that the surface has dimension $(n-1)$: it splits the configuration space in two). At any point in this surface, the potential energy $V(q)$ is determined. Since $p_i = \partial W_0/\partial q_i$, the momentum is perpendicular to the surface and in the direction of increasing W_0. Its magnitude is fixed by requiring that the total energy is E. But this magnitude then tells us the position of the next surface of constant W_0 (with incremental increase). In this manner, it should be clear that, in multi-dimensional configuration spaces there are many solutions to (4.190). It should also be clear that something singular happens to W_0 in regions where $V(q_i) = 0$. Finally, we note that even when $\partial H/\partial t = 0$, there exist other solutions W to (4.185) which encode families of trajectories on configuration space which have different energies.

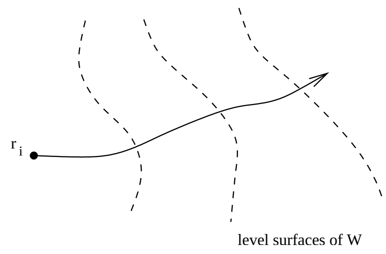

Figure 69:

4.7.1 Action and Angles from Hamilton-Jacobi

For the majority of examples the Hamilton-Jacobi approach doesn't give a particularly useful way for solving a problem; its utility really lies in the structure it reveals about classical dynamics. So rather than go through the gymnastics of solving a complicated problem using this method, let us focus on a rather simple example which which illustrates connections between the different ideas we've seen. A system with a single degree of freedom has Hamiltonian

$$H = \frac{p^2}{2m} + V(q) \tag{4.191}$$

Since the Hamiltonian is time independent, energy is conserved and the solution to the Hamilton-Jacobi equation has a single integration constant, let's call it β, which is necessarily some function of the energy. In the above discussion we were a little lax about showing these integration constants explicitly, but let's do it now: we'll write $W = W(q, t; \beta)$ with $\beta = \beta(E)$. Now we ask a somewhat strange question: suppose we try to perform a canonical transformation from (q, p) to new coordinates (α, β) such that β is the new momentum. What is the new coordinate α?

Since we wish the change of coordinates to be canonical, we must be able to write $q = q(\alpha, \beta)$ and $p = p(\alpha, \beta)$ such that

$$\{q, p\}_{(\alpha,\beta)} \equiv \frac{\partial q}{\partial \alpha}\frac{\partial p}{\partial \beta} - \frac{\partial q}{\partial \beta}\frac{\partial p}{\partial \alpha} = 1 \tag{4.192}$$

Using $p = \partial W/\partial q$ and remembering what all depends on what all ($W = W(q, \beta)$ and $q = q(\alpha, \beta)$ and $p = p(\alpha, \beta)$) we can write this as,

$$\{q, p\}_{(\alpha,\beta)} = \frac{\partial q}{\partial \alpha}\left(\frac{\partial^2 W}{\partial \beta \partial q} + \frac{\partial^2 W}{\partial q^2}\frac{\partial q}{\partial \beta}\right) - \frac{\partial q}{\partial \beta}\frac{\partial^2 W}{\partial q^2}\frac{\partial q}{\partial \alpha} = \frac{\partial q}{\partial \alpha}\frac{\partial}{\partial q}\left(\frac{\partial W}{\partial \beta}\right) \tag{4.193}$$

and we find that the transformation is canonical if we take $\alpha = \partial W/\partial \beta$. Note the nice symmetry here: we have a solution $W(q, t; \beta)$ to the Hamilton Jacobi equation and we can think in terms of canonical coordinates (q, p) or alternatively (α, β) where

$$p = \frac{\partial W}{\partial q} \quad , \quad \alpha = \frac{\partial W}{\partial \beta} \tag{4.194}$$

The function W is an example of a generating function of the second kind (4.120).

So what to do with this? Let's look at some examples. Take $\beta = E$, so that our new momentum variable is the energy itself. What is the canonical coordinate? If we write $W(q, t; E) = W_0(q, E) - Et$ then the coordinate canonically dual to E is

$$\alpha = \frac{\partial W_0}{\partial E}(q, E) - t \tag{4.195}$$

Taking the time dependence over the left-hand-side, we see that α has the interpretation of $-t_0$, the initial starting time. This tells us that we may parameterise every trajectory in a one-dimensional system in terms of the energy and starting time of the path, and that these are canonical variables. Again we see the dual relationship between energy and time. Note that both E and t_0 are independent of time; we've found canonical variables for which neither the coordinate nor the momentum vary along the path.

As another example consider the case of $\beta = I$, our action variable of Section 4.5. What is the canonical coordinate α in this case? We expect that it will be related to the angle variable θ. To see this, we use the fact that W is the classical action to write

$$W_0 = \int L\,dt + Et = \int (L + H)dt = \int \dot{q}p\,dt = \int p\,dq \tag{4.196}$$

So we have that

$$\alpha = \frac{\partial W}{\partial \beta} = \frac{d}{dI}\int p\,dq - \frac{dE}{dI}t = \theta - \omega t \tag{4.197}$$

where we've used our expression (4.144) for the angle variable, as well as the equation (4.141) for the frequency of motion ω. So we see that α is not quite equal to θ, but is shifted by a term linear in time. In fact this means that α itself does not change in time. Once again, we've arrived at a way to parameterise the orbits of motion by canonical variables which do not themselves change with time. In fact, in most presentations, this is the starting motivation for the Hamilton-Jacobi approach to classical dynamics and, even for higher dimensional systems, the function W can be thought of as a way to generate new, time independent, canonical variables. More discussion on the relationship between canonical transformations, angle-action variables and the Hamilton-Jacobi formulation can be found in the book by Hand and Finch, or deep within Goldstein.

4.8 Quantum Mechanics

One of the primary reasons for studying the rather formal aspects of classical mechanics discussed in this course is to make contact with quantum mechanics. For this reason, in this last section of the course we will illustrate the connection between the classical and quantum world and point out a few analogies that might just make the quantum behaviour look a little less weird. (Just a little less: after all, it really is weird!)

In classical mechanics the state of a system is described by a point (q_i, p_i) in phase space. In quantum mechanics the state is described by a very different object: a complex valued wavefunction $\psi(q)$ over the configuration space. The observables are operators on the space of wavefunctions. The standard representations of the position operator $\hat{q}_i$ and momentum operator $\hat{p}_i$ are

$$\begin{aligned} \hat{q}_i\psi(q) &= q_i\psi(q) \\ \hat{p}_i\psi(q) &= i\hbar\frac{\partial\psi}{\partial q_i} \end{aligned} \tag{4.198}$$

which leads to the well known Heisenberg commutation relations

$$\begin{aligned} [\hat{p}_i, \hat{p}_j] &= 0 \\ [\hat{q}_i, \hat{q}_j] &= 0 \\ [\hat{q}_i, \hat{p}_j] &= i\hbar\,\delta_{ij} \end{aligned} \tag{4.199}$$

Of course, we've already seen something very familiar in section 4.3 on Poisson brackets as summarised in equation (4.61). Although the bilinear, antisymmetric operators $[\,,\,]$ and $\{\,,\,\}$ act on very different spaces, they carry the same algebraic structure. Heuristically the relations (4.61) and (4.199) both represent the mathematical fact that momentum p_i generates infinitesimal translations of q_i: in classical mechanics we saw this in when we studied infinitesimal canonical transformations in section 4.4.1; in quantum mechanics it follows from the representation (4.198) and Taylor's expansion.

In general the map between a classical system and a quantum system goes via the Poisson brackets and was formulated by Dirac:

$$\{\ ,\ \}_{\text{classical}} \leftrightarrow -\frac{i}{\hbar}[\ ,\ ]_{\text{quantum}} \tag{4.200}$$

This prescription for going between the classical and quantum theories is known as *canonical quantisation*. It also gives rise to the quantum equations of motion. In the Poisson bracket language, we have seen that the classical equation of motion for an arbitrary function $f(q,p)$ is

$$\dot{f} = \{f, H\} \qquad \longrightarrow \qquad i\hbar\dot{\hat{f}} = [\hat{f}, \hat{H}] \tag{4.201}$$

which is the equation of motion in the Heisenberg picture, in which the time dependence is assigned to the operator rather than the state.

While a great physicist, Dirac was never much of a storyteller. It shows in the following anecdote recounting his graduate student days:

> "*I went back to Cambridge at the beginning of October 1925, and resumed my previous style of life, intense thinking about these problems during the week and relaxing on Sunday, going for a long walk in the country alone. The main purpose of these long walks was to have a rest so that I would start refreshed on the following Monday.*
>
> *It was during one of the Sunday walks in October 1925, when I was thinking about this $(uv - vu)$, in spite of my intention to relax, that I thought about Poisson brackets. I remembered something which I had read up previously, and from what I could remember, there seemed to be a close similarity between a Poisson bracket of two quantities and the commutator. The idea came in a flash, I suppose, and provided of course some excitement, and then came the reaction "No, this is probably wrong".*
>
> *I did not remember very well the precise formula for a Poisson bracket, and only had some vague recollections. But there were exciting possibilities there, and I thought that I might be getting to some big idea. It was really a very disturbing situation, and it became imperative for me to brush up on*

my knowledge of Poisson brackets. Of course, I could not do that when I was right out in the countryside. I just had to hurry home and see what I could find about Poisson brackets.

I looked through my lecture notes, the notes that I had taken at various lectures, and there was no reference there anywhere to Poisson brackets. The textbooks which I had at home were all too elementary to mention them. There was nothing I could do, because it was Sunday evening then and the libraries were all closed. I just had to wait impatiently through that night without knowing whether this idea was really any good or not, but I still think that my confidence gradually grew during the course of the night.

The next morning I hurried along to one of the libraries as soon as it was open, and then I looked up Poisson brackets in Whitackers Analytical Dynamics*, and I found that they were just what I needed.*"

4.8.1 Hamilton, Jacobi, Schrödinger and Feynman

While the Poisson bracket structure of quantum mechanics dovetails nicely with Heisenberg's approach, the Hamilton-Jacobi equation is closely tied to Schrödinger's wave equation. Let's first recall what Schrödinger's equation looks like for a one-dimensional system with a Hamiltonian operator $\hat{H} = \hat{p}^2/2m + V(\hat{q})$ acting on wavefunctions $\psi(q)$,

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial q^2} + V(q)\psi \tag{4.202}$$

where we have used the representation (4.198) for the position and momentum operators. To see the relationship of this equation to classical dynamics we decompose the wavefunction into the modulus and phase,

$$\psi(q,t) = R(q,t)\, e^{iW(q,t)/\hbar} \tag{4.203}$$

where R and W are both real functions. We know that R is related to the probability P for finding a particle at position q at time t: $P(q,t) = |\psi(q,t)|^2 = R(q,t)^2$. But what is the interpretation of the phase W? Let's substitute this decomposition of ψ into the Schrödinger equation to find

$$i\hbar\left[\frac{\partial R}{\partial t} + \frac{iR}{\hbar}\frac{\partial W}{\partial t}\right] = -\frac{\hbar^2}{2m}\left[\frac{\partial^2 R}{\partial q^2} + \frac{2i}{\hbar}\frac{\partial R}{\partial q}\frac{\partial W}{\partial q} - \frac{R}{\hbar^2}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{iR}{\hbar}\frac{\partial^2 W}{\partial q^2}\right] + VR$$

At this stage, we take the classical limit $\hbar \to 0$. Or, more precisely, we consider a situation with

$$\hbar \left| \frac{\partial^2 W}{\partial q^2} \right| \ll \left| \frac{\partial W}{\partial q} \right| \tag{4.204}$$

which can be understood physically as the requirement that the de Broglie wavelength of the particle is much smaller than any other length scale around. Either way, collecting together the terms above to leading order in $\hbar$ we find

$$\frac{\partial W}{\partial t} + \frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + V(q) = \mathcal{O}(\hbar) \tag{4.205}$$

which we recognise as the Hamilton-Jacobi equation (4.185). So in the classical limit the phase of the wavefunction is understood as the classical action of the path taken by the particle.

Finally, let us finish on the same topic that we started: the principle of least action. Recall from section 2.1 that we can determine the true path of a system by assigning a number, called the action S, to every possible path. The equations of motion are then equivalent to insisting that the true path is an extremum of S. But what about all the other paths? Do they play any role in Nature? The answer is that, in the quantum world, they do. Suppose a particle is observed to be at position q_i at time $t = 0$. Then the probability P that it will later be observed to be at position q_f at time $t = T$ is encapsulated in the wavefunction $\psi(q_f, T)$. The Feynman path integral formula for the wavefunction is

$$\psi(q_f, T) = N \int_{q_i}^{q_f} \mathcal{D}q(t)\, e^{iS[q(t)]/\hbar} \tag{4.206}$$

The N here is just a normalisation constant to ensure that probabilities add up to one: i.e. $\int |\psi(q)|^2 dq = 1$. The tricky part of this formula is the integral: it is a sum over all possible paths from $q = q_i$ at time 0 to $q = q_f$ at time T. These paths are weighted with their action. It's as if the particle really does take every possible path, but with a particular phase. In the limit $\hbar \to 0$, the phases oscillate wildly for any path away from the classical equation of motion $\delta S = 0$ and they cancel out in the integral. But for situations where $\hbar$ is important, the other paths are also important.

Let's prove that the wavefunction defined by (4.206) satisfies the Schrödinger equation. Firstly we need to understand this integral over paths a little better. We do this by splitting the path into n small segments, each ranging over a small time $\delta t = t/n$.

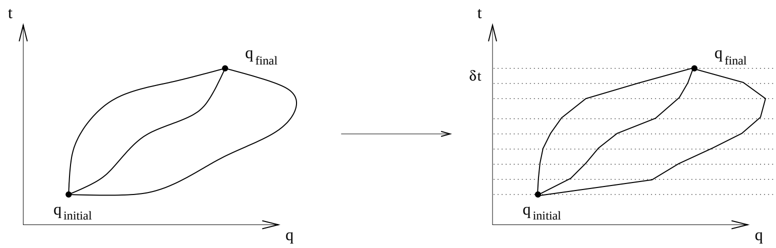

Figure 70: Discretising the paths.

Then we define

$$\int \mathcal{D}q(t) = \lim_{n\to\infty} \prod_{k=1}^{n} \int_{-\infty}^{+\infty} \frac{dq_k}{C} \tag{4.207}$$

where q_k is the position of the particle at time $t = k\delta t$. In this expression C is a constant that we're going to figure out shortly that will be required to make sense of this infinite number of integrals. In any given segment, we treat the path as straight lines as shown in the figure and replace the action with the appropriate quantity,

$$S = \int_0^T dt\ \left(\frac{1}{2}m\dot{q}^2 - V(q)\right) \to \sum_{k=1}^{n} \left(\frac{m}{2}\frac{(q_{k+1}-q_k)^2}{\delta t} - \delta t\, V\left(\frac{q_{k+1}+q_k}{2}\right)\right)$$

Then to prove that ψ defined in (4.206) satisfies Schrödinger's equation, let's consider adding a single extra time step to look at the wavefunction at time $t + \delta t$. We can Taylor expand the left hand side of (4.206) happily

$$\psi(q_f, T + \delta t) = \psi(q_f, T) + \frac{\partial \psi}{\partial T}\,\delta t + \mathcal{O}(\delta t^2) \tag{4.208}$$

while the right hand side requires us to do one extra integral over the penultimate position of the path q'. But the first n integrals simply give back the original wavefunction, now evaluated at q'. We get

$$\int_{-\infty}^{+\infty} \frac{dq'}{C}\ \exp\left[\frac{im}{2\hbar}\frac{(q_f - q')^2}{\delta t} - \frac{i}{\hbar}\delta t\, V\left(\frac{q_f + q'}{2}\right)\right] \psi(q', t) \tag{4.209}$$

The term in the exponent means that the integral oscillates wildly whenever q' is far from q_f and these regions of the integral will all cancel out. We can therefore Taylor expand around $(q_f - q')$ to rewrite this as

$$\int_{-\infty}^{+\infty} \frac{dq'}{C}\ \exp\left[\frac{im}{2\hbar}\frac{(q_f - q')^2}{\delta t}\right] \left(1 - \frac{i\delta t}{\hbar}V(q_f) + \ldots\right)$$
$$\left(1 + (q' - q_f)\frac{\partial}{\partial q_f} + \tfrac{1}{2}(q' - q_f)^2 \frac{\partial^2}{\partial q_f^2} + \ldots\right) \psi(q_f, T) \tag{4.210}$$

At this stage we do the integral over q'. We'll use the formulae for Gaussian integration

$$\int dy\, e^{-ay^2} = \sqrt{\frac{\pi}{a}} \quad , \quad \int dy\, ye^{-ay^2} = 0 \quad , \quad \int dy\, y^2 e^{-ay^2} = \frac{1}{2a}\sqrt{\frac{\pi}{a}} \tag{4.211}$$

Then equating the lefthand side (4.208) with the righthand side (4.210), we have

$$\psi(q_f, T) + \frac{\partial \psi}{\partial T}\,\delta t = \frac{1}{C}\sqrt{\frac{2\pi\hbar\delta t}{-im}}\left[1 - \frac{i\delta t}{\hbar}V(q_f) + \frac{i\hbar\delta t}{2m}\frac{\partial^2}{\partial q_f^2} + \mathcal{O}(\delta t^2)\right]\psi(q_f, T)$$

At this stage we see what the constant C has to be to make sense of this whole calculation: we should take

$$C = \sqrt{\frac{2\pi\hbar\delta t}{-im}} \tag{4.212}$$

so that $C \to 0$ as $\delta t \to 0$. Then the terms of order $\mathcal{O}(\delta t^0)$ agree. Collecting the terms of order $\mathcal{O}(\delta t)$, and replacing the time T at the end point with the general time t, we see that we have

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial q_f^2} + V(q_f)\psi = \hat{H}\psi \tag{4.213}$$

and we recover Schrödinger's equation as promised.

4.8.2 Nambu Brackets

Throughout this section, we've seen that several of the structures appearing in quantum mechanics were anticipated, in some form, within the framework of classical dynamics. You just need to know where to look. One might wonder whether classical mechanics also contains other structures which will prove to be important in future discoveries. Or, alternatively, whether there are ways to extend the framework of classical dynamics that hints at new ways to formulate the laws of physics. In this section, I'll briefly describe such an extension due to Nambu in 1973. I should confess immediately that there's no known use for Nambu's formalism! And it's not at all clear that one will be found! But then again, maybe it holds the clue that will prove crucial in the search for the ideas beyond the known laws of Nature.

We've seen that the Hamiltonian framework deals with canonical pairs of coordinates and momenta (q_i, p_i) with $i = 1, \ldots, n$. Nambu's idea was to extend this to triplets of objects (q_i, p_i, r_i) with $i = 1, \ldots, n$. We don't say what this extra variable r_i is: just that it is necessary to define the state of a system. This means that the phase space has dimension $3n$. The Nambu bracket, which replaces the Poisson bracket, acts on three functions f, g and h in phase space,

$$\{f,g,h\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\frac{\partial h}{\partial r_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial r_i}\frac{\partial h}{\partial p_i} + \frac{\partial f}{\partial r_i}\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i}$$
$$- \frac{\partial f}{\partial r_i}\frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i} + \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial r_i}\frac{\partial h}{\partial q_i} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i}\frac{\partial h}{\partial r_i} \tag{4.214}$$

This satisfies similar properties to the Poisson bracket, including linearity and

- Anti-symmetry: $\{f,g,h\} = -\{g,f,h\} = \{g,h,f\}$.
- Leibniz: $\{fg,h,l\} = f\{g,h,l\} + \{f,h,l\}g$.
- "Jacobi": $\{\{f,g,h\},l,m\} + \{h,\{f,g,l\},m\} + \{h,l,\{f,g,m\}\} = \{f,g,\{h,l,m\}\}$.

In order to specify time evolution, we need two "Hamiltonians". We call them $H(q,p,r)$ and $G(q,p,r)$. Then any function f over phase space evolves as

$$\frac{df}{dt} = \{f,G,H\} \tag{4.215}$$

In particular, the new version Hamilton's equations read

$$\begin{aligned}
\dot{q}_i &= \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial r_i} - \frac{\partial G}{\partial r_i}\frac{\partial H}{\partial p_i} \\
\dot{p}_i &= \frac{\partial G}{\partial r_i}\frac{\partial H}{\partial q_i} - \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial r_i} \\
\dot{r}_i &= \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i}
\end{aligned} \tag{4.216}$$

where there's no sum over i on the right-hand side of these equations. By the anti-symmetry of the Nambu bracket, we learn that both H and G are conserved (as long as neither have explicit time dependence).

Many of the key features of classical dynamics are retained in Nambu's formalism. For example, Liouville's theorem still holds. (This was Nambu's original motivation for suggesting this framework). Similarly, canonical transformations can be defined as a change of variables $q_i \to Q_i(q,p,r)$ and $p_i \to P_i(q,p,r)$ and $r_i \to R_i(q,p,r)$ such that the Nambu bracket structure is preserved, for example

$$\{Q_i, P_j, R_k\} = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \tag{4.217}$$

together with similar equations involving other combinations of Q's, P's and R's. "Hamilton's" equations (4.216) are invariant under these canonical transformations.

The Nambu bracket provides a generalisation of classical dynamics. But can we quantise it? In other words, can we find some operators which reproduce the Nambu bracket structure (up to a factor of $\hbar$ and perhaps an i) in much the same way that the usual quantum theory ties in with the Poisson bracket? This turns out to be pretty tricky. In particular, it seems difficult to keep all three conditions: anti-symmetry, Leibniz and Jacobi. Perhaps this suggests that the correct mathematical structure has not yet been uncovered. Perhaps it suggests that the Nambu bracket is just not useful!

Chances are that you won't ever have any use for the Nambu bracket. But you never know. Perhaps one day, like Dirac, you'll return from wandering the fields around Cambridge and desperately need to recall this concept. But, unlike Dirac, you'll be able to find a reference in your lecture notes (the notes you had taken at various lectures).