Classical Dynamics: Example Sheet 2

Dr David Tong, October 2005

1. The linear triatomic molecule drawn in figure 1 consists of two identical outer atoms of mass m and a middle atom of mass M. It is a rough approximation to CO_2. The interactions between neighbouring atoms are governed by a complicated poten-

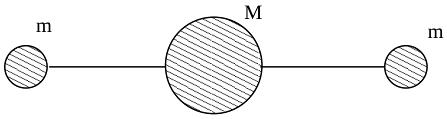

Figure 1: The linear triatomic molecule

tial $V(x_i - x_{i+1})$. If we restrict attention to motion in the x direction parallel to the molecule, the Lagrangian is

$$L = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}M\dot{x}_2^2 + \tfrac{1}{2}m\dot{x}_3^2 - V(x_1 - x_2) - V(x_2 - x_3) \tag{1}$$

where x_i is the position of the $i^{\rm th}$ particle. Define the equilibrium separation $r^0 = |x_i - x_{i+1}|$ of this system. Write down the equation describing small deviations from equilibrium in terms of the masses and the quantity

$$k = \left.\frac{\partial^2 V(r)}{\partial r^2}\right|_{r=r^0} \tag{2}$$

Show that the system has three normal modes and calculate the frequencies of oscillation of the system. One of these frequencies vanishes: what is the interpretation of this?

2. A pendulum consists of a mass m at the end of light rod of length l. The pivot of the pendulum is attached to a mass M which is free to slide without friction along a horizontal rail. Take the generalised coordinates to be the position x of the pivot and the angle θ that the pendulum makes with the vertical.

a. Write down the Lagrangian and derive the equations of motion.

b. Find the non-zero frequency of small oscillations around the stable equillibrium.

c. Now suppose a force acts on the the mass M causing it to travel with constant acceleration a in the positive x direction. Find the equilibrium angle θ of the pendulum.

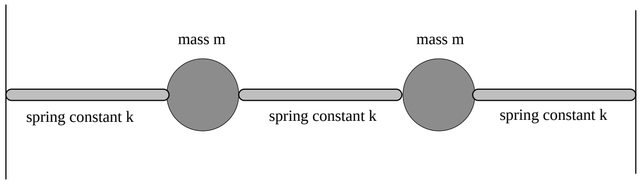

Figure 2: It's remarkably hard to draw curly springs on a computer.

3. Two equal masses m are connected to each other and to fixed points by three identical springs of force constant k as shown in figure 2. Write down the equations describing motion of the system in the direction parallel to the springs. Find the normal modes and their frequencies.

Suppose now that there are N equal masses joined by $N+1$ springs with fixed end points. Write down the equations of motion in matrix form. Find the normal mode frequencies. (**Hint:** To find the normal mode frequencies, you could first try the easier problem with "periodic boundary conditions" in which all masses lie on a circle with the first and last masses are identified)

4. Show that for any solid, the sum of any two principal moments of inertia is not less than the third. For what shapes is the sum of two equal to the third?

Calculate the moments of inertia of:

i) A uniform sphere of mass M, radius R about a diameter

ii) A hollow sphere of mass M, radius R about a diameter

iii) A uniform circular cone of mass M, height h and base radius R with respect to the principal axes whose origin is at the vertex of the cone.

iv) A solid uniform cylinder of radius r, height $2h$ and mass M about its centre of mass. For what height-to-radius ratio does the cylinder spin like a sphere?

v) A uniform ellipsoid of mass M, defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq R^2 \tag{3}$$

with respect to the (x, y, z) axes with origin at the centre of mass. (**Hint:** with a change of coordinates, you can reduce this problem to that of the solid sphere).

5. Four equal, uniform rods of mass m and length $2a$ are hinged together to form a rhombus $ABCD$. The point A is fixed, while C lies directly beneath it and is free to slide up and down. The whole system can rotate around the vertical. Let θ be the angle that AB makes with the vertical, and $\dot{\phi}$ be the angular velocity around the vertical, as shown in figure 3.

Find the Lagrangian for this system and show that there are two conserved constants of motion.

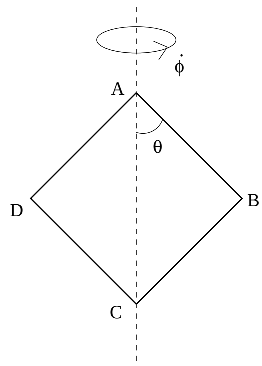

Figure 3: The rotating rhombus.

6. A hollow cylinder of radius R_2 rolls, without slipping, on a fixed cylinder of radius R_1 as shown in Figure 4. Denote the angle through the centre of the first cylinder and the point of contact by θ. Denote the angle of a marked point on upper cylinder with respect to a vertical axis by ϕ. Assume that the upper cylinder starts perched near the top at $\theta = 0$, and that it rolls without slipping, acted upon by gravity. Show that the constraint for small θ is

$$R_1\theta = R_2(\phi - \theta) \tag{4}$$

Is this constraint holonomic? Can the system be described by holonomic constraints for all θ? Write down the Lagrangian for the system assuming that this constraint holds.

(Remember that the cylinder has kinetic energy from the translation of its centre of mass, and from its spinning). Work out the equation of motion for θ. If the cylinder starts from rest at $\theta = 0$, show that it falls off the sphere at $\theta = \pi/3$.

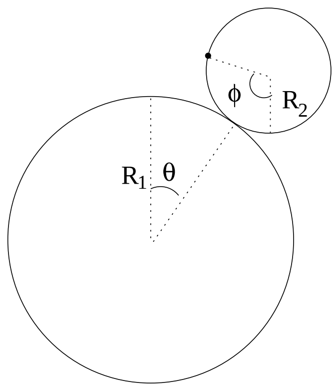

Figure 4: The relevant angles for the rotating cylinder. When does it fall off?

(**Note:** The question of when the cylinder falls off is not obviously captured by the Lagrangian you wrote down, which assumes the constraint (4) holds. You will have to revert to Newtonian thinking and consider the constraint forces at play, to solve this).