Classical Dynamics: Example Sheet 4

Dr David Tong, November 2005

1. Verify the Jacobi identity for Poisson brackets:

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0 \tag{1}$$

2. A particle with mass m, position $\mathbf{x}$ and momentum $\mathbf{p}$ has angular momentum $\mathbf{L}=\mathbf{x}\times\mathbf{p}$. Evaluate $\{x_j,L_k\}$, $\{p_j,L_k\}$, $\{L_j,L_k\}$ and $\{L_i,\mathbf{L}^2\}$.

The Runge-Lenz vector is defined as

$$\mathbf{A}=\mathbf{p}\times\mathbf{L}-\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}=\mathbf{r}/|\mathbf{r}|$. Prove that $\{L_a,A_b\}=\epsilon_{abc}A_c$. If the system is described by the Hamiltonian $H=(p^2/2m)-1/r$ prove using Poisson brackets that $\mathbf{A}$ is conserved.

3 A particle of charge e moves in a background magnetic field $\mathbf{B}$. Show that

$$\{m\dot{r}_a,m\dot{r}_b\}=e\epsilon_{abc}B_c \quad , \quad \{m\dot{r}_a,r_b\}=-\delta_{ab}$$

A magnetic monopole is an particle which produces a radial magnetic field of the form

$$\mathbf{B}=g\,\frac{\hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}}$ is the unit vector in the $\mathbf{r}$ direction. Consider a charged particle moving in the background of the magnetic monopole. Define the generalised angular momentum $\mathbf{J}=m\mathbf{r}\times\dot{\mathbf{r}}-(eg/c)\hat{\mathbf{r}}$. Prove that $\{H,\mathbf{J}\}=0$. Why does this mean that $\mathbf{J}$ is conserved?

4. Prove that the following transformations are canonical:

(a) $P=\frac{1}{2}(p^2+q^2)$ and $Q=\tan^{-1}(q/p)$.

(b) $P=q^{-1}$ and $Q=pq^2$.

(c) $P=2\sqrt{q}\left(1+\sqrt{q}\cos p\right)\sin p$ and $Q=\log\left(1+\sqrt{q}\cos p\right)$

5. Prove that the following transformation is canonical for any constant λ

$$\begin{aligned} q_1&=Q_1\cos\lambda+P_2\sin\lambda \quad , \quad & q_2&=Q_2\cos\lambda+P_1\sin\lambda \\ p_1&=-Q_2\sin\lambda+P_1\cos\lambda \quad , \quad & p_2&=-Q_1\sin\lambda+P_2\cos\lambda \end{aligned} \tag{2}$$

If the original Hamiltonian is $H(q_i,p_i) = \frac{1}{2}(q_1^2+q_2^2+p_1^2+p_2^2)$, determine the new Hamiltonian $H(Q_i,P_i)$. Use this to solve for the dynamics under the constraint $Q_2 = P_2 = 0$.

6. A group of particles, all of the same mass m, have initial heights z and vertical momenta p lying in the rectangle $-a \le z \le a$ and $-b \le p \le b$. The particles fall freely in the Earth's gravitational field for a time t. Find the region in phase space in which they lie at time t and show by direct calculation that its area is still $4ab$.

7. A large, fixed number of non-interacting particles of mass m move in one-dimension in a potential $V(x) = \frac{1}{2}m\omega^2x^2$. At time t the number density of particles in (x,p) phase space is $f(x,p,t)$. Initially ω takes the value ω_1, and particles are injected so that the number density is constant $f = f_1$ for all particles whose energy is less than E_1. No particles of energy greater than E_1 are injected. How many particles are present?

The frequency of oscillation is now changed to a different value ω_2, so slowly that a particles final energy does not depend appreciably on the phase of that particle in its oscillation. Use the existence of an adiabatic invariant to show that the area of phase space occupied by the particles remains unchanged.

8. Explain what is meant by an *adiabatic invariant* for a mechanical system with one degree of freedom.

A light string passes through a small hole in the roof of an elevator compartment, and a small weight is attached at the lower end. Initially the elevator is at rest and the system behaves like a simple pendulum executing small oscillations. Discuss what happens if:

(a) if the elevator begins to move upwards with the string attached at the hole

(b) the string is slowly withdrawn through the roof.

9. Consider a system with Hamiltonian

$$H(p,q) = \frac{p^2}{2m} + \lambda\, q^{2n} \tag{3}$$

where λ is a positive constant and n is a positive integer. Show that the action variable I and energy E are related by

$$E = \lambda^{1/(n+1)} \left(\frac{n\pi I}{J_n}\right)^{2n/(n+1)} \left(\frac{1}{2m}\right)^{n/(n+1)} \tag{4}$$

where $J_n = \int_0^1 (1-x)^{1/2} x^{(1-2n)/2n}\, dx$.

Consider a particle moving in a potential $V(q) = \lambda q^4$. If λ varies slowly, show that the particle's total energy E is proportional to $\lambda^{1/3}$. Conversely, if λ is fixed, show that the period of the motion is proportional to $(\lambda E)^{-1/4}$.

10. A pulsar of mass m moves in a plane orbit around a luminous supergiant star with mass $M \gg m$. You may regard the supergiant to be fixed at the origin of a plane polar coordinate system (r, θ), and the neutron star to move under a central potential $V(r) = -GMm/r$. Construct the Hamiltonian for the motion, and show that p_θ and E are constants, where E is the total energy.

The neutron star is in a non-circular orbit with $E < 0$. Give an expression for the adiabatic invariant $J(E, p_\theta, M)$ associated with the radial motion. The supergiant is steadily losing mass in a radiatively driven wind. Show that over a long time $E \propto M^2$.

Eventually, the supergiant goes supernova, throwing off its outer layers on a short timescale and leaving behind a remnant black hole of mass $\frac{1}{2}M$. Explain why the theory of adiabatic invariants cannot be used to calculate the new orbit.

Note: You may find the following integral helpful:

$$\int_{r_1}^{r_2} \left\{ \left(1 - \frac{r_1}{r}\right)\left(\frac{r_2}{r} - 1\right) \right\}^{\frac{1}{2}} dr = \frac{\pi}{2}(r_1 + r_2) - \pi\sqrt{r_1 r_2} \tag{5}$$

where $0 < r_1 < r_2$.