Classical Dynamics: Final Exam

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Short Questions:

1. A particle of mass m_1 is restricted to move on a circle of radius R_1 in the plane $z = 0$. A second particle of mass m_2 is restricted to move on a circle of radius R_2 in the plane $z = c$. The two particles are connected by a spring resulting in the potential

$$V = \tfrac{1}{2}\omega^2 d^2$$

where d is the distance between the particles. Identify the two generalised coordinates and write down the Lagrangian of the system.

2. A free body has principal moments of intertia I_1, I_2 and I_3 and instantaneous angular velocity $\boldsymbol{\omega}$. Its motion is governed by Euler's equations

$$\begin{aligned}
I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) &= 0 \\
I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) &= 0 \\
I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) &= 0
\end{aligned}$$

Are the components of the angular velocity in this equation with respect to the body frame or space frame? Show that an asymmetric object spinning with $\omega_1 = \Omega$ and $\omega_2 = \omega_3 = 0$ is stable if and only if I_1 is either the largest or smallest moment of intertia.

3. Define the Poisson bracket between two functions $f(q_a, p_a)$ and $g(q_a, p_a)$ on phase space. If $f(q_a, p_a)$ has no explicit time dependence, prove that under the equations of motion for a Hamiltonian H,

$$\frac{df}{dt} = \{f, H\}$$

A particle with position $\mathbf{x}$ and momentum $\mathbf{p}$ has angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. Compute $\{p_a, L_b\}$ and $\{L_a, L_b\}$.

4. Define a canonical transformation for a one-dimensional system with coordinates $(q, p) \rightarrow (Q, P)$. Show that if the transformation is canonical then $\{Q, P\} = 1$. Find the values α and β such that the following transformations are canonical.

$$i) \qquad Q = pq^{\beta} \quad , \quad P = \alpha q^{-1}$$

$$ii) \qquad Q = \frac{\alpha p^\beta}{q} \quad , \quad P = q^2$$

$$iiii) \qquad Q = q^\alpha \cos(\beta p) \quad , \quad P = q^\alpha \sin(\beta p)$$

Long Questions:

1(i) The action for a system with N generalised coordinats q_a is given by

$$S = \int_{t_1}^{t_2} L(q_a, \dot{q}_a) \; dt$$

Derive Lagrange's equations from the principle of least action by considering all paths with fixed end points $\delta q_a(t_1) = \delta q_a(t_2) = 0$.

1(ii) A pendulum consists of a mass m at the end of light rod of length l. The pivot of the pendulum is attached to a mass M which is free to slide without friction along a horizontal rail. Take the generalised coordinates to be the position x of the pivot and the angle θ that the pendulum makes with the vertical.

a. Write down the Lagrangian and derive the equations of motion.

b. Find the non-zero frequency of small oscillations around the stable equillibrium.

c. Now suppose a force acts on the the mass M causing it to travel with constant acceleration a in the positive x direction. Find the equilibrium angle θ of the pendulum.

2(i) A one dimensional system undergoes periodic motion. Define the action variable I in terms of q and p. Prove that an orbit of energy E has period

$$T = 2\pi \frac{dI}{dE}$$

2(ii) A system has Hamiltonian

$$H(q,p) = \frac{p^2 + q^2}{\mu^2 - q^2}$$

where μ is a constant and $|q| < \mu$. Sketch the orbits in phase space for energies $E \gg 1$ and $E \ll 1$. Show that the action variable I is given in terms of the energy by

$$I = \frac{\mu^2}{2} \frac{E}{\sqrt{E+1}}$$

Hence show that for $E \gg 1$ the period of the orbit is $T \approx \mu^3\pi/2p_0$ where p_0 is the maximum value of the momentum obtained during the orbit.