Quantum Field Theory: Example Sheet 1

Dr David Tong, October 2007

1. A string of length a, mass per unit length σ and under tension T is fixed at each end. The Lagrangian governing the time evolution of the transverse displacement $y(x,t)$ is

$$L = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right] \tag{1}$$

where x identifies position along the string from one end point. By expressing the displacement as a sine series Fourier expansion in the form

$$y(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a} \right) q_n(t) \tag{2}$$

show that the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} \right)^2 q_n^2 \right] . \tag{3}$$

Derive the equations of motion. Hence show that the string is equivalent to an infinite set of decoupled harmonic oscillators with frequencies

$$\omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a} \right) . \tag{4}$$

2. Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation Λ.

3. The motion of a complex field $\psi(x)$ is governed by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2 . \tag{5}$$

Write down the Euler-Lagrange field equations for this system. Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\delta\psi = i\alpha\psi \quad , \quad \delta\psi^* = -i\alpha\psi^* \tag{6}$$

Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by ψ.

4. Verify that the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{1}{2}m^2\phi_a\phi_a \tag{7}$$

for a triplet of real fields ϕ_a ($a = 1, 2, 3$) is invariant under the infinitesimal $SO(3)$ rotation by θ

$$\phi_a \rightarrow \phi_a + \theta\epsilon_{abc}n_b\phi_c \tag{8}$$

where n_a is a unit vector. Compute the Noether current j^μ. Deduce that the three quantities

$$Q_a = \int d^3x\ \epsilon_{abc}\,\dot{\phi}_b\phi_c \tag{9}$$

are all conserved and verify this directly using the field equations satisfied by ϕ_a.

5. A Lorentz transformation $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is such that it preserves the Minkowski metric $\eta_{\mu\nu}$, meaning that $\eta_{\mu\nu}x^\mu x^\nu = \eta_{\mu\nu}x'^\mu x'^\nu$ for all x. Show that this implies that

$$\eta_{\mu\nu} = \eta_{\sigma\tau}\Lambda^\sigma{}_\mu\Lambda^\tau{}_\nu\,. \tag{10}$$

Use this result to show that an infinitesimal transformation of the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{11}$$

is a Lorentz tranformation when $\omega^{\mu\nu}$ is antisymmetric: i.e. $\omega^{\mu\nu} = -\omega^{\nu\mu}$.

Write down the matrix form for $\omega^\mu{}_\nu$ that corresponds to a rotation through an infinitesimal angle θ about the x^3-axis. Do the same for a boost along the x^1-axis by an infinitesimal velocity v.

6. Consider the infinitesimal form of the Lorentz transformation derived in the previous question: $x^\mu \rightarrow x^\mu + \omega^\mu{}_\nu x^\nu$. Show that a scalar field transforms as

$$\phi(x) \rightarrow \phi'(x) = \phi(x) - \omega^\mu{}_\nu\, x^\nu\, \partial_\mu\phi(x) \tag{12}$$

and hence show that the variation of the Lagrangian density is a total derivative

$$\delta\mathcal{L} = -\partial_\mu(\omega^\mu{}_\nu x^\nu\,\mathcal{L}) \tag{13}$$

Using Noether's theorem deduce the existence of the conserved current

$$j^\mu = -\omega^\rho{}_\nu\, [T^\mu_\rho\, x^\nu] \tag{14}$$

The three conserved charges arising from spatial rotational invariance define the *total angular momentum* of the field. Show that these charges are given by,

$$Q_i = \epsilon_{ijk} \int d^3x\ \left(x^j T^{0k} - x^k T^{0j} \right) \tag{15}$$

Derive the conserved charges arising from invariance under Lorentz boosts. Show that they imply

$$\frac{d}{dt} \int d^3x\ (x^i\, T^{00}) = \text{constant} \tag{16}$$

and interpret this equation.

7. Maxwell's Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{17}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the 4-vector potential. Show that $\mathcal{L}$ is invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \xi \tag{18}$$

where $\xi = \xi(x)$ is a scalar field with arbitrary (differentiable) dependence on x.

Use Noether's theorem, and the spacetime translational invariance of the action, to construct the energy-momentum tensor $T^{\mu\nu}$ for the electromagnetic field. Show that the resulting object is neither symmetric nor gauge invariant. Consider a new tensor given by

$$\Theta^{\mu\nu} = T^{\mu\nu} - F^{\rho\mu}\, \partial_\rho A^\nu \tag{19}$$

Show that this object also defines four conserved currents. Moreover, show that it is symmetric, gauge invariant and traceless.

Comment: $T^{\mu\nu}$ and $\Theta^{\mu\nu}$ are both equally good definitions of the energy-momentum tensor. However $\Theta^{\mu\nu}$ clearly has the nicer properties. Moreover, if you couple Maxwell's Lagrangian to general relativity then it is $\Theta^{\mu\nu}$ which appears in Einstein's equations.

8. The Lagrangian density for a massive vector field C_μ is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 C_\mu C^\mu \tag{20}$$

where $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. Derive the equations of motion and show that when $m \neq 0$ they imply

$$\partial_\mu C^\mu = 0 \tag{21}$$

Further show that C_0 can be eliminated completely in terms of the other fields by

$$\partial_i \partial^i C_0 + m^2 C_0 = \partial^i \dot{C}_i \tag{22}$$

Construct the canonical momenta Π_i conjugate to C_i, $i = 1, 2, 3$ and show that the canonical momentum conjugate to C_0 is vanishing. Construct the Hamiltonian density $\mathcal{H}$ in terms of C_0, C_i and Π_i. (Note: Do not be concerned that the canonical momentum for C_0 is vanishing. C_0 is non-dynamical — it is determined entirely in terms of the other fields using equation (22)).

9. A class of interesting theories are invariant under the scaling of all lengths by

$$x^\mu \to (x')^\mu = \lambda\, x^\mu \quad \text{and} \quad \phi(x) \to \phi'(x) = \lambda^{-D}\phi(\lambda^{-1}x) \tag{23}$$

Here D is called the *scaling dimension* of the field. Consider the action for a real scalar field given by

$$S = \int d^4x \; \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g\phi^p \tag{24}$$

Find the scaling dimension D such that the derivative terms remain invariant. For what values of m and p is the scaling (23) a symmetry of the theory. How do these conclusions change for a scalar field living in an $(n+1)$-dimensional spacetime instead of a $3+1$-dimensional spacetime?

In $3+1$ dimensions, use Noether's theorem to construct the conserved current D^μ associated to scaling invariance.