

# Quantum Field Theory: Example Sheet 2

Dr David Tong, October 2007

1. A string has classical Hamiltonian given by

$$H = \sum_{n=1}^{\infty} \left( \frac{1}{2} p_n^2 + \frac{1}{2} \omega_n^2 q_n^2 \right) \quad (1)$$

where  $\omega_n$  is the frequency of the  $n$ th mode. (Compare this Hamiltonian to the Lagrangian (3) in Example Sheet 1. We have set the mass per unit length in that question to  $\sigma = 1$  to simplify some of the formulae a little). After quantization,  $q_n$  and  $p_n$  become operators satisfying

$$[q_n, q_m] = [p_n, p_m] = 0 \quad \text{and} \quad [q_n, p_m] = i\delta_{nm} \quad (2)$$

Introduce creation and annihilation operators  $a_n$  and  $a_n^\dagger$ ,

$$a_n = \sqrt{\frac{\omega_n}{2}} q_n + \frac{i}{\sqrt{2\omega_n}} p_n \quad \text{and} \quad a_n^\dagger = \sqrt{\frac{\omega_n}{2}} q_n - \frac{i}{\sqrt{2\omega_n}} p_n \quad (3)$$

Show that they satisfy the commutation relations

$$[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0 \quad \text{and} \quad [a_n, a_m^\dagger] = \delta_{nm} \quad (4)$$

Show that the Hamiltonian of the system can be written in the form

$$H = \sum_{n=1}^{\infty} \frac{1}{2} \omega_n (a_n a_n^\dagger + a_n^\dagger a_n) \quad (5)$$

Given the existence of a ground state  $|0\rangle$  such that  $a_n|0\rangle = 0$ , explain how, after removing the vacuum energy, the Hamiltonian can be expressed as

$$H = \sum_{n=1}^{\infty} \omega_n a_n^\dagger a_n \quad (6)$$

Show further that  $[H, a_n^\dagger] = \omega_n a_n^\dagger$  and hence calculate the energy of the state

$$|l_1, l_2, \dots, l_N\rangle = \left( a_1^\dagger \right)^{l_1} \left( a_2^\dagger \right)^{l_2} \dots \left( a_N^\dagger \right)^{l_N} |0\rangle \quad (7)$$

**2.** The Fourier decomposition of a real scalar field and its conjugate momentum in the Schrödinger picture is given by

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[ a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right] \quad (8)$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left[ a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right] \quad (9)$$

Show that the commutation relations

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \quad \text{and} \quad [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \quad (10)$$

imply that

$$[a_{\vec{p}}, a_{\vec{q}}] = [a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0 \quad \text{and} \quad [a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \quad (11)$$

**3.** Consider a real scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \quad (12)$$

Show that, after normal ordering, the conserved four-momentum  $P^\mu = \int d^3x T^{0\mu}$  takes the operator form

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a_{\vec{p}}^\dagger a_{\vec{p}} \quad (13)$$

where  $p^0 = E_{\vec{p}}$  in this expression. From this expression for  $P^\mu$  verify that if  $\phi(x)$  is now in the Heisenberg picture, then

$$[P^\mu, \phi(x)] = -i\partial^\mu \phi(x) \quad (14)$$

**4.** Show that in the Heisenberg picture,

$$\dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2 \phi(x) - m^2 \phi(x) \quad (15)$$

Hence show that the operator  $\phi(x)$  satisfies the Klein-Gordon equation.

**5.** Let  $\phi(x)$  be a real scalar field in the Heisenberg picture. Show that the relativistically normalized one-particle states  $|p\rangle = \sqrt{2E_{\vec{p}}} a_{\vec{p}}^\dagger |0\rangle$  satisfy

$$\langle 0 | \phi(x) | p \rangle = e^{-ip \cdot x} \quad (16)$$

6. In Example Sheet 1, you showed that the classical angular momentum of field is given by

$$Q_i = \epsilon_{ijk} \int d^3x (x^j T^{0k} - x^k T^{0j}) \quad (17)$$

Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian (12). Show that, after normal ordering, the quantum operator  $Q_i$  can be written as

$$Q_i = -i \epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}}^\dagger \left( p^j \frac{\partial}{\partial p_k} - p^k \frac{\partial}{\partial p_j} \right) a_{\vec{p}} \quad (18)$$

Hence confirm that the quanta of the scalar field have spin zero (i.e. a stationary one-particle state  $|\vec{p}=0\rangle$  has zero angular momentum).

7. The purpose of this question is to introduce you to non-relativistic quantum field theory. This is the only place you will encounter such a thing in this course. Consider the Lagrangian for a complex scalar field  $\psi$  given by

$$\mathcal{L} = +i\psi^* \partial_0 \psi - \frac{1}{2m} \nabla \psi^* \cdot \nabla \psi \quad (19)$$

Determine the equation of motion, the energy-momentum tensor and the conserved current arising from the symmetry  $\psi \rightarrow e^{i\alpha} \psi$ . Show that the momentum conjugate to  $\psi$  is  $i\psi^*$  and compute the classical Hamiltonian.

We now wish to quantize this theory. We will work in the Schrödinger picture. Explain why the correct commutation relations are

$$[\psi(\vec{x}), \psi(\vec{y})] = [\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})] = 0 \quad \text{and} \quad [\psi(\vec{x}), \psi^\dagger(\vec{y})] = \delta^{(3)}(\vec{x} - \vec{y}) \quad (20)$$

Expand the fields in a Fourier decomposition as

$$\begin{aligned} \psi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \\ \psi^\dagger(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \end{aligned} \quad (21)$$

Determine the commutation relations obeyed by  $a_{\vec{p}}$  and  $a_{\vec{p}}^\dagger$ . Why do we have only a single set of creation and annihilation operators  $a_{\vec{p}}$ ,  $a_{\vec{p}}^\dagger$  even though  $\psi$  is complex? What is the physical significance of this fact? Show that one particle states have the energy appropriate to a free non-relativistic particle of mass  $m$ .

8. Show that the time ordered product  $T(\phi(x_1)\phi(x_2))$  and the normal ordered product  $:\phi(x_1)\phi(x_2):$  are both symmetric under the interchange of  $x_1$  and  $x_2$ . Deduce that the Feynman propagator  $\Delta_F(x_1 - x_2)$  has the same symmetry property.

9. Verify Wick's theorem for the case of three scalar fields:

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = :\phi(x_1)\phi(x_2)\phi(x_3): + \phi(x_1)\Delta_F(x_2 - x_3) + \phi(x_2)\Delta_F(x_3 - x_1) + \phi(x_3)\Delta_F(x_1 - x_2) \quad (22)$$

10. Consider the scalar Yukawa theory given by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - M^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g \psi^* \psi \phi \quad (23)$$

Compute the amplitude for

- “Nucleon-Anti-Nucleon” annihilation  $\psi + \bar{\psi} \rightarrow \phi$  at order  $g$
- “Nucleon-Meson” scattering  $\phi + \psi \rightarrow \phi + \psi$  at order  $g^2$